# ABSTRACTS OF PAPERS

*(Abstracts of papers presented at the Annual Meeting, Amherst, Massachusetts, August 26–29, 1964. Additional abstracts appeared in earlier issues.)*

### 40. **Some Properties of the Dependence Capacity of a Stochastic Process.** J. G. Baldwin, Research Triangle Institute.

Let $T$ be an index set and $\hat{\xi} = \{\xi_t \mid t \varepsilon T\}$ be a stochastic process. For every finite set of random variables $\xi_{t_1}, \cdots, \xi_{t_n}$ let $p_{\xi_{t_1} v \cdots v \xi_{t_n}}$ be their joint distribution and $p_{\xi_{t_1}} x \cdots x p_{\xi_{t_n}}$ the correspondingly induced product distribution. We set $U_{\hat{\xi}}(t_1, \cdots, t_n) = \int \beta \, dp_{\xi_{t_1} v \cdots v \xi_{t_n}}$ if $p_{\xi_{t_1} v \cdots v \xi_{t_n}} \ll p_{\xi_{t_1}} x \cdots x p_{\xi_{t_n}}$ ($\beta$ being the Random-Nikodym density) and $U_{\hat{\xi}}(t_1, \cdots, t_n) = \infty$ otherwise. $U_{\hat{\xi}}(t_1, \cdots, t_n)$ vanishes if and only if $\xi_{t_1}, \cdots, \xi_{t_n}$ are independent. $U_{\hat{\xi}}$, considered as a function on the ring $R(T)$ of finite subsets of $T$ (setting $U_{\hat{\xi}}(N) = 0$ if $N$ is empty or contains one element), is a non-negative monotonic increasing function enjoying the property, $U_{\hat{\xi}}(E \cup F) + U_{\hat{\xi}}(E \cap F) \geqq U_{\hat{\xi}}(E) + U_{\hat{\xi}}(F)$, of being a monotone capacity of at least the second order and hence is called the dependence capacity of the process. Among other properties of $U_{\hat{\xi}}$ the following are of particular interest: (1) A necessary and sufficient condition that $\hat{\xi}$ be Markovian ($T$ being a subset of the real line) is that for each set $\{t_1, \cdots, t_n\}$

$$U_{\hat{\xi}}(t_1, \cdots, t_n) = \sum_{k=1}^{(n-1)} U_{\hat{\xi}}(t_k, t_{k+1}).$$

(2) If $\hat{\xi}$ is Markovian, then for every process $\hat{\eta}$ with $U_{\hat{\xi}}(s, t) = U_{\hat{\eta}}(s, t)$ for all $s, t \varepsilon T$, $U_{\hat{\xi}} \leqq U_{\hat{\eta}}$. Hence the quantity $\mu(t_1, \cdots, t_n) = U_{\hat{\eta}}(t_1, \cdots, t_n) - \sum_{k=1}^{n-1} U_{\hat{\eta}}(t_k, t_{k+1})$ is a measure of the "Markovity" of the process $\hat{\eta}$ on the set $\{t_1, \cdots, t_n\}$ and for a sequence of variables $\eta_1, \eta_2, \cdots$ the quantity $\bar{\mu}(\hat{\eta}) = \lim_{t \to \infty} t^{-1} \mu(\{1, \cdots, t\})$ is meaningful as the mean Markovity of the process. (3) If $U_{\hat{\xi}}$ is bounded on $R(T)$ then any two variables $\xi_s, \xi_t$ become asymptotically independent as $|s - t| \to \infty$.

### 41. **Best Sequential Tests of an Identification Hypothesis in a Degenerate Case** (Preliminary Report). S. Blumenthal, T. Christie and M. Sobel, Rutgers University and University of Minnesota.

Given $k$ populations and two distribution functions having densities $f(x)$ and $g(x)$ respectively, we assume that $k - 1$ populations have density $f(x)$ and one has density $g(x)$. The problem is to identify which population has density $g(x)$, using a sequential procedure. We require that no matter which pairing is true, the probability of a correct selection must exceed a given constant $P$. We consider the degenerate case where the ratio $f(x)/g(x)$ can take only the values $0, 1, \infty$. We examine both procedures which take only one observation at each sampling stage and those which take observations from all contending populations (non-contenders being eliminated from sampling) at each stage. For either type of sampling, we find an entire family of tests which achieve a minimum average sample size subject to the given constraints. These procedures use randomization to decide whether to stop or to continue sampling. Among these, there is one which minimizes the variance of the sample size. This is a pseudo-truncated, non-randomized (except at one stage) procedure.

### 42. **On a Multivariate Fisher's Z** (Preliminary Report). T. Cacoullos, University of Minnesota.

Let $R = (r_{ij})$ denote the sample correlation matrix obtained from $N$ observations on a normally distributed random vector $X = (X_1, \cdots, X_p)$ with population correlation matrix

$\bar{R} = (\rho_{ij})$. Let $Z_{ij} = \frac{1}{2} \log (1 + r_{ij})/(1 - r_{ij})$, $\zeta_{ij} = \frac{1}{2} \log (1 + \rho_{ij})/(1 - \rho_{ij})$. It is shown that the $Z_{ij}^{*} = (N - 1)^{\frac{1}{2}} (Z_{ij} - \zeta_{ij})$ $i \neq j$, $i, j = 1, \cdots, p$, are asymptotically jointly normally distributed. Unfortunately the covariance between $Z_{ij}^{*}$ and $Z_{ks}^{*}$, in addition to $\rho_{ij}$ and $\rho_{ks}$, involves as nuisance parameters the $\rho_{hg}$ with $h, g = i, j, k, s$. Some applications are also discussed.

43. **Multi-Decision Sequential Procedures** (Preliminary Report). H. T. David and William D. Lawing, Iowa State University.

Sobel and Wald (Ann. Math. Statist. **20**) and other subsequent authors discuss a three-decision procedure which may be interpreted as follows: an initial stage, based on a triangular stop region, decides between ($H_1$ or $H_2$) and ($H_2$ or $H_3$); a terminal stage, based on an SPRT, then decides between the two surviving hypotheses. This procedure may be extended to the multi-decision case in several ways. For example, a four decision procedure may be based on two successive triangular stages, followed by an SPRT stage, or on one initial triangular stage, followed by two successive SPRT stages. Complete or several-point O.C. functions analogous to the five-point O.C. function of sequential analysis may be evaluated; these O.C. functions are exact for the case of zero excess.

44. **A Model for Phage Attachment to Bacteria with Death and Reproduction.** Joseph M. Gani and T. Nagai, Michigan State University and Kyushu University.

In a bacterial colony of initial size $n_0$ subject to attack by $\nu_0$ bacteriophage, suppose that at any time $t > 0$, there are $n$ uninfected bacteria, $N$ bacteria with one or more phages attached to them, and $\nu$ unattached phages.

Approximate deterministic equations for these can be derived

$$dn/dt = -\lambda n\nu + \beta n$$

$$dN/dt = \lambda n\nu - \mu N$$

$$d/dt = -\lambda n\nu - \alpha\lambda N\nu + \mu KN$$

where $\lambda$, $\alpha\lambda$ are phage attachment rates to uninfected and infected bacteria respectively ($\alpha < 1$), $\beta$ is the bacterial reproduction rate, $\mu$ the bacterial death rate from phage infection, and $K$ the number of phages produced when an infected bacterium dies.

These equations are discussed, and assuming a solution for the $\nu$ can be found, whether exact or approximate, it is shown that a stochastic model may be set up for $n$, $N$, such that their joint probability generating function can be obtained.

45. **Bayesian Estimation in Multivariate Analysis.** Seymour Geisser, National Institute of Arthritis and Metabolic Diseases.

The Bayes approach to Multivariate Analysis taken previously by Geisser and Cornfield (*J. Roy. Statist. Soc. Ser. B* **2** 368–376) is extended and given a more comprehensive treatment. Posterior joint and marginal densities are derived for vector means, linear combinations of means; simple and partial variances; simple, partial and multiple correlation coefficients. Also discussed are the posterior distributions of the canonical correlations and of the principal components.

For the general multivariate linear hypothesis, it is demonstrated that the joint Bayesian posterior region for the elements of the regression matrix is equivalent to the usual confidence region for these parameters. The joint predictive density of a set of future observations generated by the linear hypothesis is obtained thus enabling one to specify the probability that a set of future observations will be contained in a particular region.

### 46. On the Local and the Asymptotic Minimax Tests of a Multivariate Problem. Narayan Chandra Giri, Cornell University.

Let $X = (X_1 \cdots X_p)'$ be normally distributed with mean $\xi = (\xi_1 \cdots \xi_p)'$ and non singular covariance matrix $\Sigma$. Giri (1961) has found the likelihood ratio test of the hypothesis $H_0 : \xi = 0$, against the alternative $H_1 : \xi_1 = \cdots = \xi_q = 0$ and $\xi'\Sigma^{-1}\xi = \delta > 0$ (for arbitrary positive $\delta$) when $q < p$ and $\Sigma$ is unknown. Nothing is known about the optimum properties of this test. In this paper a test has been suggested which is locally minimax as $\delta \to 0$ for testing $H_0$ against $H_1$. It has also been shown that the Hotelling's $T^2$ test is asymptotically minimax as $\delta \to \infty$ for this problem.

### 47. Scale Parameter Estimation from the Order Statistics of Unequal Gamma Components. R. Gnanadesikan, E. Lauh and M. B. Wilk, Bell Telephone Laboratories, Inc.

If $X$ is a gamma random variable with shape parameter $\eta$ and scale parameter $\lambda$, then $X/\eta$ is here referred to as a shape-scaled gamma. From observations on a collection of shape-scaled gammas, each having a possibly different (but known) shape parameter and all having the same (unknown) scale parameter, one may define the order statistics of such a sample. The present paper is concerned with the maximum likelihood estimation of the scale parameter when given certain subsets of the order statistics, which are regarded as statistically conditioned in various ways. Tables are provided to facilitate obtaining the estimate. Some examples of application are presented.

### 48. On the Compound Testing Problem for Two Specified Distributions. J. F. Hannan and J. R. van Ryzin, Michigan State University and Argonne National Laboratory.

Consider the following compound decision problem. Let $X = (X_1, \cdots, X_n)$ be $n$ independent random variables, where $X_k$ is distributed as $P_{\theta_k}$, $\theta_k \varepsilon \Omega = \{0, 1\}$. Observe $X$ and decide whether $\theta_k = 0$ or 1, $k = 1, \cdots, n$. The loss for wrong decision is $b(1 - \theta_k) + a\theta_k$, $a > 0, b > 0$, and zero for correct decision. The risk is defined as the average of the $n$ component risks. For the decision procedure $t(X) = (t_1(X), \cdots, t_n(X))$ and the parameter $\theta = (\theta_1, \cdots, \theta_n)$ in the set $\Omega_n$ of all $2^n$ binary vectors, the risk is denoted by $R(t, \theta)$. Define $\theta' = n^{-1} \sum_{k=1}^{n} \theta_k$, $\mu = aP_1 + bP_0$, and $Z = bdP_0/d\mu$. The procedure $t_{\theta'}(X) = (t_{\theta'}(X_1), \cdots, t_{\theta'}(X_n))$, where $t_{\theta'}(X_k) = 0$ or 1 as $Z(X_k) \geqq \theta'$ or $< \theta'$ for $Z(X_k) \varepsilon (0, 1)$ and $t_{\theta'}(X_k) = 1 - Z(X_k)$ for $Z(X_k) = 0$ or 1, is a Bayes procedure against $(1 - \theta', \theta')$ on $\Omega$ having risk $\phi(\theta') = R(t_{\theta'}, \theta)$. Let $h$ be any function in $L_3(P_i)$ such that $\int h dP_i = i$ for $i = 0, 1$. Define $\bar{h} = n^{-1} \sum_{k=1}^{n} h(X_k)$ and the decision rule $t^*$ defined by substituting $\bar{h}$ for $\theta'$ in $t_{\theta'}$. Then the following theorem holds:

Theorem. *There exists a constant $C_1$ such that $R(t^*, \theta) - \phi(\theta') \leqq C_1 n^{-\frac{1}{2}}$ for all $n$ and $\theta \varepsilon \Omega_n$. Higher order bounds for $R(t^*, \theta) - \phi(\theta')$ are attained in two other theorems by successively stronger assumptions on the induced distributions $P_i Z^{-1}$.*

### 49. A New Approach to Time Series with Mixed Spectra. George R. Hext, Stanford University.

The simplest time series having a "mixed spectrum" is the sum of a sine-wave "signal" and stationary "noise"; that is, we assume $X(t) = C \cos(\omega_s t + \varphi) + \eta(t)$. In this paper we estimate the unknowns $C$, $\omega_s$, $\varphi$ and $f_n(\omega)$, the spectral density function (sdf) of the noise $\eta(t)$. A sample of $T$ observations is analysed using several truncated spectral windows $K(\lambda;\, \omega,\, M)$ having the same basic structure but different bandwidths. At or near $\omega_s$ the sample s.d.f. $f_X(\omega;\, K,\, M)$ will have a marked peak, whose height will vary with the bandwidth of the window, and thus with $M$, the truncation point. In fact, $E[f_X(\omega_s\, ;\, K,\, M)] \simeq f_n(\omega_s) + \frac{1}{4}KC^2M$, where $K$ is a constant depending on the basic window structure. Thus if we evaluate the sample s.d.f. for several values of $M$, we may estimate $\omega_s$ from the peaks; then the estimates for $\frac{1}{2}C^2$ and $f_n(\omega_s)$ follow from a regression of the peak heights on $M$, notwithstanding the high correlation between the different spectral estimates at $\omega_s$. Finally for large $T$ the asymptotic variance of the estimate for the "signal power", $\frac{1}{2}C^2$, is the same here as it would be if $\omega_s$ were known and the power estimated from the regression of $X(t)$ on $t$.

## 50. Efficiency of Some Rank-Based Estimates for Normal Samples of Moderate Size. J. L. Hodges, Jr., University of California, Berkeley.

Hodges and Lehmann [*Ann. Math. Statist.* **34** (1963) 603] proposed, as an estimate for the center of a symmetric population, the median of the means of all pairs of sample values. Among the attractive features of this estimate is its high asymptotic efficiency (.955) in the normal case. This paper reports on sampling experiments for samples of moderate size drawn from a normal population; the experiments and certain other considerations suggest that the efficiency is similarly high for small or moderate sample sizes. A main drawback of the estimate is that it is not easy to compute. A class of simplified estimates is introduced, one of which is the median of the means of symmetric pairs of order statistics. The random samples indicate that this estimate, which is simpler to compute, has about the same efficiency for moderate normal samples as that considered above. A concept of robustness against gross errors is proposed and illustrated.

## 51. On the Line Graph of a Symmetric Balanced Incomplete Block Design. A. J. Hoffman and D. K. Ray-Chaudhuri, IBM Research Center.

We consider only undirected graphs with at most one edge between two vertices and no edge joining a vertex to itself. If $G$ is an undirected graph with $N$ vertices, its adjacency matrix $A = A(G)$ is an $(N \times N)$ matrix $a_{ij} = 1$ if $i$ and $j$ are adjacent vertices and $a_{ij} = 0$, otherwise. The line graph $L(G)$ of a graph $G$ is the graph whose vertices are the edges of $G$ with two vertices of $L(G)$ if and only if the corresponding edges of $G$ are adjacent. A symmetric balanced incomplete block (SBIB) design with parameters $(v, k, \lambda)$ can be conceived as a bipartite graph $\Pi(v, k, \lambda)$ on $v + v$ vertices, each vertex having valence $k$, with any two vertices in the same part adjacent to exactly $\lambda$ vertices of the other part. It is seen that the line graph $L(\Pi)$ is a regular connected graph on $vk$ vertices and its adjacency matrix has $2k - 2, -2, k - 2 \pm (k - \lambda)^{\frac{1}{2}}$ as its distinct eigenvalues. We raise the question: If $H$ is a regular connected graph on $vk$ vertices with $2k - 2, -2, k - 2 \pm (k - \lambda)^{\frac{1}{2}}$ as distinct eigenvalues of the adjacency matrix, is $H$ isomorphic to some $L(\Pi(v, k, \lambda))$? The answer is yes, unless $v = 4$, $k = 3$, $\lambda = 2$ in which case there is exactly one exception.

## 52. Applications of the Characterizations of Distributions to Tests of Fit, Randomness, and Independence. Robert V. Hogg, University of Iowa.

Substitutions of equivalent hypotheses for statistical hypotheses which concern fit, when unknown parameters are present, are considered. These substitutions make it possible to avoid the use of estimates of the unknown parameters in the test statistics. For example, let $Y_1, Y_2, \cdots, Y_n$ be the order statistics of a random sample $X_1, X_2, \cdots, X_n$ from a certain distribution. Let $Z_1 = Y_n - Y_{n-1}$, $Z_2 = 2(Y_{n-1} - Y_{n-2})$, $\cdots$, $Z_{n-1} = (n-1)(Y_2 - Y_1)$ and $Z_n = nY_1$; and let $W_1 = Z_1/W_{n-1}$, $W_2 = (Z_1 + Z_2)/W_{n-1}, \cdots, W_{n-2}$

$= (Z_1 + \cdots + Z_{n-2})/W_{n-1}$ and $W_{n-1} = Z_1 + \cdots + Z_{n-1}$. Certain characterization theory shows that the following three hypotheses are equivalent: $(H_1)$ the underlying distribution is exponential with location parameter $A$ and spread parameter $B$; $(H_2)$ $Z_1, \cdots, Z_{n-1}, Z_n$ are mutually independent; and $(H_3)$ the vector $(W_1, \cdots, W_{n-2})$ and $W_{n-1}$ are independent and $W_1, \cdots, W_{n-2}$ are distributed like the order statistics of a sample of size $n - 2$ from the uniform distribution (0, 1). Then, for instance, this last part of $(H_3)$ can be easily tested by either a Kolmogorov-Smirnov type statistic or a Cramér-von Mises type statistic without estimating $A$ and $B$ first. Another example is to replace $(H_1)$ by $(H_4)$, which is that $U_{ij} = \min(X_i, X_j)$ and $V_{ij} = |X_i - X_j|$ are independent for all $i \neq j$. Now the problem is the construction of a test of that independence $(H_4)$ from the $n(n-1)/2$ pairs $(U_{ij}, V_{ij})$. It should be noted, however, that not all $n(n-1)/2$ pairs are independent because, for instance, $U_{ij}$ and $U_{ik}$ are dependent. This complicates this test of independence.

## 53. $\phi$-Distinguishability of Families of Distributions. H. S. KONIJN, City College, New York.

Berger and Wald discussed distinctness of hypotheses and later the corresponding families of distribution have been called distinguishable. This paper proposes a number of extensions of these notions to functionals $\phi$ defined over the families. Thus, for all $\phi(F)$ in $R_k$, if, for some $j$, $\phi_j(F) < \phi_j(F')$ for $F \varepsilon \mathfrak{F}_1$ and $F' \varepsilon \mathfrak{F}_2$, the families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are called $\phi$-distinguishable if there is a function $h$ from the sample space to subsets of $R_1$ such that for each $F \varepsilon \mathfrak{F}_1$, $F' \varepsilon \mathfrak{F}_2$ we have $\mathcal{E}\{h(x) \mid F\} < \mathcal{E}\{h(x) \mid F'\}$, $\mathcal{E}\{h(x) \mid F\} < \phi_j(F')$, and $\phi_j(F) < \mathcal{E}\{h(x) \mid F'\}$; $\phi$-distinguishable by confidence intervals of given confidence coefficient if the subjects are restricted to such confidence intervals, etc. (Here for any sets $S$, $S' \varepsilon R_1$, we say $S < S'$ if each point of $S$ is less than any point of $S'$; and, for a random $S$, $\mathcal{E}S = \{y \varepsilon R_1 : \mathcal{E}U = y, U \varepsilon S\}$.) Subsequently, we consider translation families and give some theorems about $\phi$-indistinguishability in these families.

## 54. Unbiased Coin-Tosses with a Biased Coin. JAMES A. LECHNER, Westinghouse Electric Corp.

The problem of reducing or eliminating the (known or unknown) bias in a binary random-number generator is considered. New (to the author) methods are compared with several older ones discussed by Horton, Smith, and Walsh in Ann. Math. Statist. **19 20**. Optimality is discussed for the case wherein the bias is to be completely removed. (The definition of optimality in other cases is subject to wide limits of choice.)

## 55. Exact Lower Confidence Bounds for Reliability When Failure Times Have the Two-Parameter Weibull Distribution. NANCY R. MANN, Rocketdyne, North American Aviation, Inc.

Consider a size $n$ random sample of ordered failure times, $0 \leqq T_1 \leqq T_2 \leqq \cdots \leqq T_r \leqq \cdots \leqq T_n$, censored by sample number $r \leqq n$. Let the distribution from which the sample is drawn be specified by the two-parameter. Weibull density, $f_{\delta,b}(t) = (1/\delta b)(t/\delta)^{1/b-1} \exp[-(t/\delta)^{1/b}]$, $t > 0$, $f_{\delta,b}(t) = 0$, $t < 0$; $\delta, b > 0$. A family of exact unbiased scale-invariant tests of $b = b_0$ against one and two-sided alternatives is obtained from the distribution of $\log Z_r = \log(T_r/T_1)$, which has been derived by the author and tabled for $2 \leqq n \leqq 40$, $2 \leqq r \leqq n$. Unless $r$ is very large and close to $n$ this statistic is shown to be preferable for testing $b = b_0$ (against all alternatives) to a statistic based on all the ordered sample times, $1/n(r-1)[-(n-1) + (T_2/T_1)^{1/b_0} + (T_3/T_1)^{1/b_0} + \cdots + (n-r+1)(T_r/T_1)^{1/b_0}]$, which is distributed as $F[2(r-1), 2]$ when $b = b_0$. Suppose it is desired to obtain a lower confidence bound for $R$, the reliability at a specified mission time $t_m$. Let $\bar{b}$ be an upper confidence bound for $b$, based on $Z_r$; and suppose $\bar{b}$ is obtained at confidence level $1 - \alpha$. Let $\theta = \delta^{1/b}$ and $\hat{\theta} = 1/r(\sum_{i=1}^{r-1} t_i^{1/b} + (n - r + 1)t_r^{1/b})$. When $b$ is known, a uniformly most accurate lower confidence bound for $R$ at level $1 - \alpha'$ may be based on the random variable $r\hat{\theta}/t_m^{1/b}$. For $t_m$ sufficiently small, $r\hat{\theta}/t_m^{1/b}$ is monotonically decreasing in $b$, and by a result of Lukacs [Third Berkeley Symposium, Volume II], $r\hat{\theta}$ is independent of $Z_r$. Thus an exact lower confidence bound for $R$ at level $(1 - \alpha)(1 - \alpha')$ may be obtained by substituting $\bar{b}$ for $b$ in $r\hat{\theta}/t_m^{1/b}$.

## 56. Some Remarks of the "Joint-Ranking" Procedure. K. L. Mehra, University of Alberta.

For testing the equality of several ($K$) treatments on the basis of $\binom{K}{2}$ independent samples of paired-observations viz., $(x_{il}, x_{jl})$ $l = 1, 2, \cdots, N_{ij}$ for each pair $(i, j)(1 \leqq i < j \leqq K)$, a family of rank-order statistics were proposed in an earlier paper (to appear in *Ann. Math. Statist.*; see abstract *Ann. Math. Statist.* **34** 683), based on the ranks in a "joint-ranking" of the $N = \sum_{i<j} N_{ij}$ absolute differences $|Z_l^{(ij)}| = |X_{il} - X_{jl}|$, $l = 1, 2, \cdots, N_{ij}$ $(1 \leqq i < j \leqq K)$. One observes, however, that given any joint-ranking statistic $L_N(\xi_N, \xi)$ there exists a statistic $L_N^*(\xi_N, \xi)$, constructed similarly but based now on "separate-rankings" of absolute $Z$'s for each pair $(i, j)$ is as Pitman-efficient as $L_N(\xi_N, \xi)$. The question of preference between the two procedures—"joint-ranking" or "separate-rankings"—was partially investigated in the earlier paper. In this paper this question is investigated from the standpoint of Bahadur-efficiency (Bahadur, *Ann. Math. Statist.* **31** 276–295). It is shown, under certain conditions on the rank-scores, that for testing against shift in location the "joint-ranking" procedure is more (Bahadur) efficient than the "separate-rankings" procedure provided (among other conditions) the underlying symmetric continuous distribution possesses a unimodal density. Similar conclusions also hold for Lehmann's distribution-free alternatives.

## 57. On Confidence Bounds Associated with MANOVA and Nonindependence Between Two Sets of Variates. G. S. Mudholkar, University of Rochester.

A theory of confidence bounds on certain parametric functions, and their 'partials', associated with a number of problems in multivariate normal analysis has been developed by S. N. Roy and his associates in a number of publications over the past ten years. In this paper we have obtained confidence bounds on the members, and their 'partials', of a class of meaningful parametric functions associated with the two problems mentioned in the title. Roy's confidence bounds have been shown to be particular cases of these bounds. Furthermore, our confidence bounds include bounds which can be associated with Hotelling's trace criterion for MANOVA.

## 58. A Model for the Distribution of Individuals by Species in an Environment (Preliminary Report). John William McCloskey, Michigan State University.

A statistical model for the distribution of individuals by specie with the following basic properties is to be developed: (1) The number of individuals in each specie in the environment is infinite. (2) The number of species present in the environment is infinite. (3) In any sample taken from the model the number of individuals in the sample from each specie is Poisson with mean proportional to the "intensity" of the specie.

Discription of the model: to each of the species of the environment an intensity $X_i$ is assigned such that $0 < X_i < \infty$ and the sum of these intensities will be $X$, the total intensity of the environment. $X$ is assumed to be finite with probability one and to have probability density $g(x)$. Define the function $f(x)$ on $[0, \infty)$ such that for $0 \leqq a < b < \infty$ the number of species with intensity in the interval $[a, b)$ is Poisson with mean $\int_a^b f(x)dx$.

The form of the functions $f$ and $g$ is determined and maximum likelihood estimate taken of their parameters using data previously gathered from a biological environment.

The model will then be simulated on the Control Data 3600 computer and a sample taken from the simulated model.

The simulated data and the biological data will then be compared.

### 59. A Note on the Distinct Score Sequences of a Round Robin Tournament (Preliminary Report). T. V. Narayana and J. Sarangi, University of Alberta.

Let $(s_1, s_2, \cdots, s_n)$ be a vector of non-negative integers satisfying (i) $s_1 \leqq s_2 \leqq \cdots \leqq s_n$ (ii) $s_1 + s_2 + \cdots + s_k \geqq \binom{k}{2}$ for $k = 2, 3, \cdots, n-1, s_1 + \cdots + s_n = \binom{n}{2}$. Landau has shown that (ii) is a necessary and sufficient condition for the vector $(s_1, s_2, \cdots, s_n)$ to be the scores of $n$ players in a round-robin tournament. (No ties are allowed in individual games and the winner scores 1 and the loser 0.) We denote the set of all vectors $(s_1, \cdots, s_n)$ satisfying (i) and (ii) by $[*, *, \cdots, *]_n$ where there are $n$ asterisks within the brackets. Using an obvious notation, we shall denote, for example, by $[*, \cdots, k]_n$ the set of all vectors satisfying (i), (ii) and having $s_n = k$. The main results of this note are: (1) A simple algorithm to write all members of the set $[*, *, \cdots, *]_n$ without duplication is given. (2) Letting $\#(S)$ denote the number of elements of the set $S$, it is proved that for $n \geqq 3$,

$$\#[1, *, \cdots, *, k]_n \geqq \#[0, *, \cdots, *, k]_n .$$

### 60. The Single Server Queue with Poisson Input and Semi-Markov Service Times: I and II. Marcel F. Neuts, Purdue University.

I

In this paper a single server queue is considered with Poisson input and an $m$-state semi-Markov sequence of service times. This essentially amounts to considering $m$ types of customers, whereby customer types form a Markov chain. It is shown that the successive busy periods, together with the type of the first customer to arrive during each busy period, form an imbedded semi-Markov sequence. Its matrix of transition probabilities is calculated. The virtual waiting time process is studied next; it leads to a system of integro-differential equations, which are a matrix version of Takàcs' equation for the case of renewal service times. The existence of a limiting distribution for the virtual waiting time in the stable case is shown. The analysis relies heavily on the theory of non-negative matrices. The characteristics of the processes related to the queue are expressed in terms of the roots of the following determinental equation: det $[zI - w\Psi(s + \lambda - \lambda z)] = 0$ where $\Psi_{ij}(s)$ is the L.S. transform of the massfunction $Q_{ij}(x)$ for the semi-Markov process and $\lambda$ is the arrival rate. It is indicated that the case of renewal service times and also the case of bulk service with fixed bulk size are special cases of this model.

II

This paper is the sequel to a paper with identical title ($I$). In it the queue length at the times of departures of successive customers is studied, as well as the queue length distribution in continuous time. Some further results are obtained on the output process and on the waiting time of the $n$th customer, assuming that service is in order of arrival.

All equations, for the transient as well as for the stationary case, are matrix versions of Takàcs' results for the queue with renewal service times. The queueing process with bulk service and fixed bulk size is obtained by a very special choice of the semi-Markov sequence of service times.

### 61. A Characterization of the Geometric Distribution. V. R. RAO UPPULURI, Oak Ridge National Laboratory.

Let $X_1, X_2, \cdots X_n$ be $n$ independent identically distributed random variables each with range $0, 1, 2, \cdots$, and let $Y = \min (X_1, X_2, \cdots X_n)$.

PROPOSITION. (1) *If each* $X_i$ *has the geometric distribution given by* $P[X = x] = (1 - q) q^x, 0 < q < 1, x = 0, 1, 2 \cdots$, *then* $Y$ *hrs the geometric distribution given by* $P[Y = y] = (1 - q^n)(q^n)^y, y = 0, 1, 2, \cdots$. *And conversely,* (2) *if* $Y$ *has the geometric distribution given by* $P[Y = y] = (1 - r)r^y, 0 < r < 1, y = 0, 1, 2, \cdots$, *then each* $X_i$ *has the geometric distribution given by* $P[X = x] = [1 - r^{1/n}][r^{1/n}]^x, x = 0, 1, 2 \cdots$.

### 62. The Model Number of Successes in Independent Trials (Preliminary Report). STEPHEN M. SAMUELS, Purdue University.

Let $S_n = X_1 + X_2 + \cdots + X_n$ be a sum of independent Bernoulli random variables with $P\{X_i = 1\} = p_i = 1 - P\{X_i = 0\}$. Several results are obtained, including the following: If, for some integer $k$, $k - (n - k + 2)^{-1} < E(S_n) < k + (k + 2)^{-1}$, then $k$ is the mode of $S_n$. If, on the other hand, $k + (k + 2)^{-1} < E(S_n) < k + 1 - (n - k + 1)^{-1}$, then the mode can be either $k$ or $k + 1$. If, in addition, the $p_i$'s satisfy $E(S_n) + \max_{i=1,\cdots,n} p_i < k + 1$, then $k$ is the mode.

The latter result has been applied by the author in connection with the problem of finding the least upper bound for $P\{Y_n \geqq y\}$ where $Y_n$ is the sum of $n$ independent non-negative random variables with prescribed means and $y$ is a prescribed constant.

### 63. On an Upper Bound for the Maximum Number of Points No Four on One Plane in $PG(r, 2)$ (Preliminary Report). ESTHER SEIDEN, Michigan State University.

The maximum number of points no four on one plane plays an important role in the theory of design of experiments and error correcting códes. An upper bound is obtained for $r \geqq 3$. For $r \leqq 6$ an exact value is also evaluated. The upper bound is attained for $r \leqq 5$.

### 64. A Characterization of the Logarithmic and Geometric Distributions. V. SESHADRI, McGill University.

The logarithmic series distribution is defined for positive integers as $P(X = x) = \alpha\theta^x/x$ where $\alpha = -1/\log (1 - \theta)$ and $0 < \theta < 1$. Let now $X$ be a multiple of $k$, a positive integer. If $i$ is any positive integer $P(X = ki/X = 0 (\mathrm{mod}\ k))$ is also a logarithmic series distribution with the parameter $\theta^k$. This property is found to be true for the geometric distribution as well. Conversely if for any positive integer $i$ and a fixed integer $k$ (positive) $P(X = ki/X = 0(\mathrm{mod}\ k))$ is a logarithmic (geometric) distribution then the distribution of $X$ can be shown to be logarithmic (geometric).

*(Abstracts presented at the European Regional Meeting, Berne, Switzerland, September 14–16, 1964. Additional abstracts appeared in earlier issues.)*

**15. Asymptotic Efficiency of Certain Distance Criteria** (Preliminary Report). F. C. ANDREWS, University of Oregon and Mathematisch Centrum, Amsterdam.

Asymptotic relative efficiencies of tests based upon the integral of the squared differences between two empirical distribution functions are considered. Such tests are applicable to the two sample problem, $c$-sample problem, and to the problem of testing independence in bivariate populations with certain classes of alternatives. As the sample sizes increase to infinity, the limiting distribution of each test statistic is shown to have a noncentral von Mises' distribution in the neighborhood of the null hypothesis. This limiting distribution has the same form as that of an infinite series of weighted, independent, non-central chi-square random variables. Some applications concerning a test proposed by Lehmann (*Ann. Math. Statist.* (1951) **22** 165–179; (1953) **24** 23–44) are discussed.

**16. Ranking and Selection Problems of Uniform Distributions.** D. R. BARR and M. HASEEB RIZVI, Aerospace Research Laboratories.

Consider $k \geqq 2$ uniform distributions with unknown parameters $\theta_i$ $(i = 1, 2, \cdots, k)$ and let $0 \leqq \theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$ be the ordered $\theta$ values. This paper proposes a decision procedure $R_t$ for selecting $t < k$ distributions with $t$ largest parameters such that Pr $\{CS \mid R_t, 0 < \theta_{[k-t]}/\theta_{[k-t+1]} \leqq \rho^* < 1\} \geqq P^*$ and a procedure $R$ for selecting a nonempty subset containing a distribution with parameter $\theta_{[k]}$ such that Pr $\{CS \mid R\} \geqq P^*$; here CS (correct selection) is defined obviously for each problem and $P^*$ and $\rho^*$ are specified constants. Let $Y_i$ be the largest item from the $i$th sample of size $n$. Then $R_t$ ranks $Y_i$ and selects the distributions with $t$ largest $Y_i$ as ones having $t$ largest $\theta_i$ and $R$ selects a random sized subset by placing the $i$th distribution in it if and only if $Y_i \geqq \rho \max Y_i$. The requirement that $R_t(R)$ satisfy the basic probability condition determines $n$. $R_t$ is minimax (wrt a simple loss function) and most economical. The supremum of the expected size of the subset selected by $R$ is obtained; also, $R$ possesses a certain monotonicity property. The analogous problem of the smallest parameter is also treated. All these results are extended to a broader family of non-regular distributions.

**17. Covariance Estimation for Linear Time Series Under Minimal Assumptions.** F. EICKER, Institut fuer Angewandte Mathematik der Universitaet.

Let the real linear time series $\{x_t; t = \cdots, -1, 0, 1, \cdots\}$ be represented in the usual fashion by an infinite moving average $x_t = \sum_{j=-\infty}^{\infty} c_j \xi_{t-j}$. The real coefficients $c_j$ are quadratically summable, i.e. the sequence $\{c_j\}$ is an element of the Hilbert space $\ell^2$. The real valued random sequence $\{\xi_t\}$ is orthonormal. The covariance estimators are the sample covariances denoted by $R_{kn}$. Conditions are derived in terms of the sequence $\{\xi_t\}$ only in order that the $R_{kn}$ are consistent in the sense of convergence in absolute mean: (i) $E(|R_{kn} - R_k|) \to 0$ as $n \to \infty$. $R_k$ is the $k$th covariance. It is proved that (i) holds for all integers $k$ under the following condition (ii): the sample covariances $r_{kn} = n^{-1} \sum_{t=1}^{n} \xi_t \xi_{t+k}$ estimate consistently in absolute mean the covariances $\delta_{k0} = 1$ for $k = 0$, and $=0$ for $k \neq 0$ of $\{\xi_t\}$. Condition (ii) is necessary if the theorem is to hold for the class of all linear time series $\{x_t\}$ formed with one and the same random sequence $\{\xi_t\}$ and all sequences of $\ell^2$. This requirement corresponds to practical estimation situations where the sequence $\{R_k\}$ is unknown. Consequently condition (ii) is minimal in the sense that it cannot be relaxed. A similar theorem holds when convergence in absolute mean is replaced by stochastic convergence. Thus $r_{kn} \to \delta_{k0}$ i.p. for all $k$ is necessary and sufficient for $R_{kn} \to R_k$ i.p. for all $\{c_j\} \varepsilon \ell^2$ and all $k$.

## 18. Variance of Mean Estimators when Some Observations Are Stragglers. FRIEDRICH GEBHARDT, University of Connecticut.

Let $X_1, X_2, \cdots, X_n$ be independent random variables with mean $m$. The variance of various estimators for the mean is computed numerically for each combination of the following conditions: (a) $n = 6$ or $n = 10$; (b) none, one, or two variables have the variance $9\sigma^2$ or $36\sigma^2$ while all other variables have the variance $\sigma^2$; (c) the variables have a normal or a logistic distribution or the cdf $C(\sigma)/[\sigma^4 + (x - m)^4]$. $\sigma$ and $m$ are unknown. The estimators include Bayesian estimators for normal distributions, discarding one or more observations if they deviate too far from the sample mean, Winsorizing one or two observations. The results suggest that discarding observations that deviate very far from the sample mean and Winsorizing observations that deviate moderately should be a strategy whose variance is not much greater than that of the Bayes estimators and that is rather unsensible against changes in the distribution parameters.

## 19. Some Remarks on Ballot Problems. F. GOBEL, Mathematical Centre, Amsterdam. (Introduced by P. van der Laan.)

A construction by J. L. Hodges (*Biometrika* **42** 261–262) for a combinatorial proof of the symmetric version of the Chung-Feller theorem will be used to extend this symmetric case in such a way that the homogeneity of the distribution concerned is preserved.

Some recurrence relations are given for the number of paths below an increasing but otherwise arbitrary function. If this function is linear, a result is obtained that is closely related to a formula by Pólya for the sum

$$\textstyle\sum_{j=0}^{\infty} \binom{a+bj}{j} x^j.$$

## 20. Certain General Properties of Ordered Least Squares Estimates of Location and Scale Parameters. ZAKKULA GOVINDARAJULU, Case Institute of Technology.

Some upper bounds for the variances of ordered least squares estimates of (i) location parameter, (ii) scale parameter, and (iii) both location and scale parameters of a distribution, are derived. These bounds are valid even if estimates are based on a random sample which is subject to any general censoring. The estimates achieving these upper bounds are also derived. These results constitute generalizations of results due to Lloyd (*Biometrika* **39** (1952) 88–95) and Downtown (*Biometrika* **40** (1953) 457–458, and *Ann. Math. Statist.* **25** (1954) 303–316). Lower bounds for the variances of these ordered least squares estimates, analogous to Cramér-Rao lower bounds are also considered. Computation of these lower bounds might be complicated when the sample is censored.

## 21. An Interpretation of Negative and Other Unorthodox Probabilities. JOHN H. HALTON, Brookhaven National Laboratory.

Optimisation procedures for variance-reduction in sampling schemes often result in a formal probability-density taking negative values. A general interpretation of such distributions, in terms of a set of several independent sample-spaces, is established, which retains the reduction of variance and laws of large numbers.

## 22. A Generalization of the Martingale Convergence Theorem. SØREN JOHANSEN, University of Copenhagen.

Let $(\varphi_n, n \geqq 1)$ be a sequence of finite $\sigma$-additive setfunctions defined on a nondecreasing sequence of $\sigma$-fields of a probability space. If $\sum_{n=1}^{\infty} | \varphi_{n+1} - \varphi_n | \Omega < +\infty$, then $\varphi_0 = \lim \varphi_n$ exists and is bounded on the union field, and $d\varphi_n/dP \rightarrow d\varphi/dP$ a.s. $[P]$, where $\varphi$ is the $\sigma$-additive setfunction "nearest" to $\varphi_0$. This theorem is proved by the method applied by E. Sparre-Andersen and B. Jessen (Danske Vid. Selsk. Mat.-Fys. medd. 25 No. 5. 1948) extended to the case, where the limiting setfunction $\varphi_0$ is only finitely additive. See also S. Johansen and J. Karush (*Ann. Math. Statist.* **34** 1120) for the application of this method to semimartingales.

## 23. Iterated Tests for the Equality of Mean Vectors from Multivariate Normal Populations (Preliminary Report). P. R. KRISHNAIAH, Aerospace Research Laboratories.

Consider $k$ $p$-variate normal populations with mean vectors $\underline{\mu}_1, \cdots, \underline{\mu}_k$ and a common covariance matrix $\Sigma$. Let the total hypothesis $H:\underline{\mu}_1 = \cdots = \underline{\mu}_k$ be expressed as $H = \bigcap_{i=1}^{q} \bigcap_{j=1}^{p} H_{ij}$, where $H_{ij} : \sum_{r=1}^{k} b_{ir} \mu_{rj} = 0$, $\mu_{rj}$ is $r$th population mean on $j$th variate and $b_{ir}$'s are known constants subject to the restrictions $\sum_{r=1}^{k} b_{ir} = 0$ and $\sum_{r=1}^{k} b_{ir}^2 = 1$. Also, let $H_{ij}^*$ denote the conditional hypothesis $H_{ij}$ given $\bigcap_{s=1}^{i-1} \bigcap_{t=1}^{p} H_{st}$ and $\bigcap_{t=1}^{j-1} H_{it}$. In this paper, iterative procedures are proposed to test $H$ against one-sided and two-sided alternatives. These test procedures are based upon testing $H_{11}^*, \cdots, H_{1p}^*, H_{21}^*, \cdots, H_{2p}^*, \cdots, H_{q1}^*, \cdots, H_{qp}^*$ sequentially against the proper alternatives. The testing is terminated with the rejection of $H$ at the stage where $H_{ij}^*$ is rejected the first time. The hypothesis $H$ is accepted if all $H_{ij}^*$'s are accepted.

## 23a. On Optimal Diffusion Processes. PETR MANDL, Czechoslovak Academy of Sciences. (Introduced by L. Schmetterer.)

Let the diffusion coefficient $a$ and the coefficient of local shift $b$ of a homogenous, one-dimensional, non-stopped, diffusion process on a finite interval $[r_0, r_1]$ be functions of the space variable $x$ and the control variable $z$. The behaviour of the trajectory on the boundaries $r_i$ is given by Feller's general boundary condition. If the trajectory is for the time $dt$ in position $x_0$ and $z$ takes the value $z_0$, then the cost $c(x_0, z_0)\, dt$ arises. Further every jump from the boundary $r_i$ into the position $x$ causes the cost $\nu_i(x)$. Let $\omega(x)$ be the value of $z$ in position $x$, $\theta(\omega)$ the mean cost per unit of time associated with the control $\omega$ and let $\hat{\theta} = \inf \theta(\omega)$. Then $\hat{\theta}$ is the unique value of the parameter $\theta$, for which the equation $dw/dx + \min [b(x, z)w + c(x, z) - \theta]/a(x, z) = 0$ has a solution satisfying two relations derived from the boundary condition for the process.

## 24. The Effective Use of Quadratic Programming Methods in Some Statistical Problems. PETER E. NÜESCH, Kantonale Oberrealschule, Zürich.

In many testing situations (e.g. testing equality of means versus ordered alternatives, multivariate extensions of one-sided tests of location) the following problem has to be solved in order to obtain the likelihood-ratio tests statistic: Minimize the quadratic function $f(x) = 2qx + x'Cx$ subject to $x \geqq 0$, where $C$ is a positive definite $p \times p$ matrix, $q$ is a $1 \times p$ matrix of constants. For this "reduced" quadratic programming problem ("reduced" since the additional constraint $Ax = b$ is missing), a *closed* solution can be given (in contradiction to a remark in: Bartholomew, D. J. (1961). A test of homogeneity of means under restricted alternatives. *J. Roy. Statist. Soc. Ser. B* **23** 239–281.) The solution is obtained by modifying the usual stepwise procedure that leads to the solution of the quadratic programming problem.

### 25. On the Selection of Independent Variables in a Regression Equation. J. Oosterhoff, Mathematical Centre, Amsterdam. (Introduced by W. R. van Zwet.)

In many applications of multiple regression theory it may be of interest to reduce the number of $m$ independent variables to a smaller subset of size $k$, say. If a sample is given, one wants to choose an optimal subset of size $k$ (associated with a maximum sum of squares due to regression). The methods of forward selection or backward elimination are often used to obtain such subsets. It is shown that the subsets yielded by these methods may be far from optimal, even if both methods lead to identical results for all $k$.

### 26. A Sequential Signed Rank Procedure. E. A. Parent, USN, Stanford University.

Let $X_1, X_2, \cdots X_m, X_{m+1}, \cdots$ be a sequence of independent observations where $X_i$, $i = 1, 2, \cdots m$ are distributed according to $F(x)$ and $X_i$, $i \geqq m + 1$ are distributed according to $G(x)$, $F$ and $G$ each absolutely continuous cdf's. For $m$ unknown, it is desired to detect when the change from $F$ to $G$ occurs. The *sequential signed rank* of each observation is defined as $Z_n = \text{sgn}(X_n) \cdot \text{sequential rank}(X_n)$ where $\text{sgn}(X_n) = 1$ if $X_n \geqq 0$ and $-1$ if $X_n < 0$, and sequential rank $(X_n)$ = rank of $|X_n|$ among $|X_1|, \cdots, |X_n|$. Distribution theory for $Z_n$ is obtained with special emphasis on the case where $G = h(F)$ and $F$ belongs to a wide class of cdf's (which includes symmetric cdf's). Some stopping rules based on the sequence $\{Z_n\}$ are proposed.

### 27. Sampling With or Without Replacement? J. N. K. Rao, Graduate Research Center of the South West.

In simple random sampling with replacement, several writers have shown that estimators based only on the distinct units in the sample exist which are more efficient than the estimators based on all the units in the sample. By taking the cost of $a$ with replacement sample as proportional to the number of distinct units in the sample, Seth and Rao (*Sankhyá, Ser. A* (1964)) have shown that for the same expected cost the usual estimator in simple random sampling without replacement is more efficient than any estimator belonging to a class of estimators based only on the distinct units if the square of the coefficient of variation is less than the number of units in the population, $N$—one seldom encounters in practice populations for which coefficient of variation is greater than $N^{\frac{1}{2}}$. In this paper these results are extended to multi-stage sampling. For unequal probability sampling, in the special case where the units can be grouped wrt selection probabilities, $p_i$, such that the units in a group have the same $p$-value, it is shown that for the same expected cost Stevens' (*J. Roy. Statist. Soc. Ser. B*, 1958) estimator in sampling without replacement is more efficient than Pathak's (*Sankhyá*, Ser. A (1962) 315) estimator based only on the distinct units in sampling with replacement.

### 28. Exact Power of Some Rank Tests. P. van der Laan, Mathematical Centre, Amsterdam.

Let $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$ be independently distributed with cumulative distribution functions $F(x)$ and $G(y)$ respectively. For testing the hypothesis $H_0 : F(x) \equiv G(x)$ against $H_1 : F(x - \lambda) \equiv G(x)$ one of the following distribution-free two-sample tests can be used: Wilcoxon, van der Waerden, or Terry. The objects of this investigation are: 1°. To derive the exact power functions of these tests for small sample sizes against parametric one and two sided alternatives of exponential and rectangular populations. 2°. To compare the power functions of these tests. This is done for $m = n = 3, 4, 5, 6$ and for some significance levels.

## 29. Convex Transformations: A New Approach to Skewness and Kurtosis. W. R. van Zwet, Mathematisch Centrum, Amsterdam.

Based on increasing convex transformations of random variables and on increasing concave-convex transformations of symmetrically distributed random variables, one may define two weak-order relations (called $c$- and $s$-ordering respectively) for distribution functions (cf. *Ann. Math. Statist.* **33** 1484). These $c$- and $s$-orderings appear to order distributions according to skewness and kurtosis; in fact they seem to be much better suited to express our intuitive ideas of what is meant by these concepts than the classical measures of standardized third and fourth central moments. A number of characterizations of these order-relations in terms of inequalities for expected values and odd moments of order statistics are given. The orderings can successfully be applied to describe the effect of skewness and kurtosis in such divergent fields as small sample inequalities for expected values of order statistics in terms of quantiles, the relative asymptotic efficiency of Wilcoxon's test to the normal scores test, the behaviour of Student's test under nonstandard conditions and the relative efficiency of sample median to sample mean.

*(Abstracts to be presented at the Central Regional Meeting, Chicago, Illinois, December 28–29, 1964.)*

## 1. Minimal Augmentation of Orthogonal Main-Effect Plans to Remove Bias From Two-Factor Interactions. Cuthbert Daniel, Private Consultant.

Orthogonal main-effect plans are specifications for the levels of each of several factors in a set of experiments or runs, which permit orthogonal and unbiased estimation of all the fixed parameters in a linear model equation. The term was first used by S. Addelman and O. Kempthorne, who have published a large collection of these plans. The model equation may be wrong in the sense that a much better representation of the data can be secured by adding a small number of two-factor interaction (2fi) terms. Some of these terms may have been confounded with main-effect estimates in the original plan. This paper offers methods for finding the smallest sets of additional runs that will permit: (a) separation of single or chosen main-effect estimates from chosen 2fi, or (b) separation of all main-effect estimates from all 2fi bias, or (c) either a. or b. above, plus estimation of a chosen set of 2fi, unconfounded with any other 2fi.

The methods used are direct extensions of those in the paper: "Sequences of fractional replicates in the $2^{p-q}$ series", *J. Amer. Statist. Assoc.* **57** 403–429, by C. Daniel.

## 2. The Convex Hull of a Random Set of Points. Bradley Efron, Stanford University.

Let $H_N$ be the convex hull of $N$ independent and identically distributed random points in the plane. Renyi and Sulanke (Zeit. Wahr. **2** 1–11 1950, 75–84) have given an integral expression for the expected number of vertices of $H_N$. In this paper similar formulas are derived for the expected area, perimeter, and probability content of $H_N$, and also for the properties of the convex hull of random points in three dimensions. These formulas are particularly simple when the underlying distribution is normal or uniform over an ellipse (or ellipsoid). The expression for the expected area is easily recognized as an extension of the well-known formula for the expected range in the one-dimensional case.

### 3. An Alternative Efficiency for the One Sample Wilcoxon. Jerome H. Klotz, Harvard University.

A large sample efficiency for the one sample Wilcoxon is obtained by letting the type one error $\alpha$ approach zero while keeping the type two error $\beta$ fixed. This efficiency, defined by Cochran and applied to the sign test by Bahadur, was derived for the Wilcoxon using a limit theorem of Feller (*Trans. Amer. Math. Soc.* **54** (1943) pp. 361–372). For a fixed alternative, the normal approximation is used for the large sample type two error. Using the representation of the statistic as a sum of independent non identically distributed random variables under the hypothesis, the exponential rate of convergence, of the type one error, to zero was derived from the above theorem. Comparison of this rate of convergence with that of other tests gives relative efficiency values. In particular, for the $\bar{x}$ test and normal alternatives with mean $\mu$, variance 1 we obtain $e(\mu) = [2r(2\gamma pq + p^2 - \frac{1}{2}) - \log(\cos hr)]/[\mu^2/2]$ where $p = 1 - q$ is the probability of a positive observation, $\gamma$ is the probability that a positive observation is greater in magnitude than a negative one, and $r$ is defined by $\int_0^1 u \tan h(ru)\, du = 2\gamma pq + p^2 - \frac{1}{2}$. The limit of the above expression, as $\mu$ approaches the hypothesis zero, equals the Pitman limiting efficiency $3/\pi$.

### 4. Solution of the Compound Decision Problem with $m \times n$ Finite Loss Matrix. J. R. van Ryzin, Argonne National Laboratory.

Simultaneous consideration of $N$ independent statistical decision problems having identical generic structure constitutes a compound decision problem. The risk is defined as the average risk of the component problems. This paper treats the case in which each component problem involves making one of $n$ decisions given that the observation for that problem came from one of $m$ distributions $\{P_1, \cdots, P_m\}$. A procedure is given which depends on data from all $N$ problems and whose risk is bounded from above by $cN^{-\frac{1}{2}}$ plus the risk of a procedure which in each component problem is Bayes against the $m$-point empirical distribution, $\xi_N$, of parameter values from the $N$ problems. The constant $c$ is independent of the $N$-fold vector of parameter values. The procedure is a strengthening (in order of bound) and an extension of a result of Hannan and Robbins (*Ann. Math. Statist.* **26** (1955), 37–51) for $m = n = 2$. The decision procedure is obtained by substituting estimates $\hat{\xi}_N$ of $\xi_N$ in a componentwise Bayes solution with respect to $\xi_N$. A solution of the estimation problem for $\hat{\xi}_N$ is also included.

### 5. Comparing Distances Between Multivariate Normal Populations. M. S. Srivastava, University of Toronto.

Let $\Pi_i$ be $p$-variate normal populations with unknown mean $\mu_i$ and common covariance matrix $\Delta$. Let $\delta_i^2 = (\mu_i - \mu_0)' \Delta^{-1} (\mu_i - \mu_0)$ denote the Mahalanolius generalized distance between $\Pi_i$ and $\Pi_0$. On the basis of a sample of size $n$ from each $\Pi_i$, a population $\Pi_j$; $j = 1, 2, \cdots k$ with minimum $\delta_j^2$ is to be selected. Let $d_j$ denote the decision of selecting $\Pi_j$. The following results have been obtained for simple loss function.

Theorem 1. *For known $\Delta$, the decision rule: take decision $d_j$ if $D_j^2 \equiv (\bar{X}_j - \bar{X}_0)' \cdot \Delta^{-1}(\bar{X}_j - \bar{X}_0) = \min(D_1^2, \cdots, D_k^2)$, is admissible in the whole class of procedures; $\bar{X}_i$ denotes the sample mean of the $i$th population.*

Theorem 2. *For unknown $\Delta$, the decision rule: take decision $d_j$ if $T_j^{*2} \equiv (\bar{X}_j - \bar{X}_0)' \cdot$*

$(A^{-1} + S^*)^{-1} (\bar{X}_j - \bar{X}_0) = \min (T_1^{*2}, \cdots, T_k^{*2})$, *is admissible in the class of translation invariant procedures*; *A known* pos. def.

THEOREM 3. *For unknown* $\Delta$ *and invariant procedures (invariant under translation and* n.s. *linear transformations), A of Theorem 2 drops off.*

$$S^* = S + a\Sigma Y_j Y_j' + b\Sigma Y_i Y_j'; \; Y_j = \bar{X}_j - \bar{X}_0,$$

$a$ and $b$ are known constants.

### 6. On a Multivariate Slippage Problem. M. S. SRIVASTAVA, University of Toronto.

Let $S_0, S_1, \cdots, S_k$, be independently Wishart distributed each with mean $n'\Delta_i$ and $n' = (n - 1)$ degrees of freedom. Suppose we have the $k + 1$ hypotheses, $H_0:\Delta_1 = \cdots = \Delta_k = \Delta_0$, $H_i:\Delta_1 = \cdots = \Delta_{i-i} = \Delta_{i+i} \equiv \cdots = \Delta_k = \Delta_0$, and $\Delta_i = a\Delta 0$, $a$ known and $>0$; $i = 1, 2, \cdots, k$.

Suppose $H_i$ is true for exactly one $i \varepsilon (0, 1, \cdots k)$. The problem is to select one of these $(k + 1)$ hypotheses, subject to the restriction, (a) that if all the covariances are equal, $H_0$ is accepted with probability $1 - \alpha$, $\alpha$ is a pre-assigned number. We shall further restrict, (b) to procedures invariant under the additive and triangular group of transformations. Then with symmetric loss function and under the restrictions (a) and (b), the following procedure is admissible.

Accept $H_0$ if $\min_{1 \leqq j \leqq k} \prod_{\delta=1}^p [C_{\delta\delta}^{(j)}]^{2\delta-p-1-n''} - a^{n'/2} \prod_{\delta=1}^p [C_{\delta\delta}^{(0)}]^{2\delta-p-1-n''} \geqq L_\alpha$; Accept $H_i$ if $H_0$ is rejected and $\prod_{\delta=1}^p [C_{\delta\delta}^{(i)}]^{2\delta-p-1-n''} < \prod_{\delta=1}^p [C_{\delta\delta}^{(j)}]^{2\delta-p-1-n''}$ for all $j \neq i$, $j = 1, 2, \cdots k$, where $L_\alpha$ is a constant to be determined by the condition (a), $C^{(i)}C^{(i)\prime} = I + \sum_{\beta=1}^k U_\beta + (a - 1)U_i$, $U_\beta = T^{-1}S_\beta T^{-1\prime}$, $a = 1$ for $H_0$, $S_0 = TT'$, $C^{(i)}$ and $T$ are lower triangular matrices with positive ($>0$) diagonal elements, and $n'' = n'(k + 1)$.

If we take simple loss function, the above procedure is uniformly most powerful amongst the class of all symmetric invariant (restriction $b$) procedures.

*(Abstracts not connected with any meeting of the Institute.)*

### 1. Estimation of Parameters of the Logistic Distribution in Censored Samples. MIR M. ALI, A. B. M. LUTFUL KABIR, and A. K. M. EHSANES SALEH, University of Western Ontario.

The problem of estimating the location and scale parameters of the logistic distribution in censored samples is studied for large sample sizes. It has been shown that for estimating the mean $\mu$ when $\alpha\%$ is censored from the left and $(1 - \beta)\%$ from the right, the optimum symmetric spacing of the quantiles is given by $\lambda i = \alpha + (\alpha - \beta)i/(k + 1)$, where $k$ is the number of quantiles on which the estimation is based. The efficiency of the estimate compared to the best linear estimate based on $k$ optimum quantiles in complete sample has been tabulated for $k = 1(1)10$. The equation for optimum spacing for the estimation of the scale parameter in complete sample has been obtained, and some studies on the optimum spacing in censored samples have been made.

### 2. Estimating the Parameters of the Exponential Distribution from a Censored Sample Based on Suitably Chosen Quantiles. MIR M. ALI, and A. K. M. EHSANES SALEH, University of Western Ontario.

The present study concerns the determination of the optimal set of order statistics for a given integer $k$ (where $k$ is less than the number of observations in the censored sample), in estimating the parameters of the exponential distribution when the sample is censored. The study is based on the asymptotic theory of quantiles and under the Type II censoring procedure. The following results are proved in the course of the study: (i) For a fixed integer $k$, there exists a unique set of optimum order statistics for the simultaneous estimation of the location and the scale parameters in both singly (either on the left or on the right) and doubly censored samples. (ii) For a fixed integer $k$, there exists a unique set of optimum order statistics for the estimation of the scale parameter (assuming the location parameter to be known) in singly (either on the left or on the right) censored samples. For $k = 2(1)4$ and proportion of censoring on the right $1 - \beta = .05(.05).40$, a table has been prepared for the estimation of the scale parameter (assuming the location parameter to be known) furnishing the coefficients of the estimator and the spacings corresponding to the optimum order statistics. For $k = 2(1)4$ and equal proportions of censoring on both sides from .05 to .25 at steps of .05, a similar table has been prepared for the simultaneous estimation of the location and the scale parameters.

## 3. The Probability Integral and the Percentage Points of the Distribution of Range from Logistic Samples. P. H. Dave and B. K. Shah, University of Baroda and Yale University. (Introduced by J. J. Anscombe.)

Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ be an ordered sample from '$n$' independent and identically distributed random variables from the standard logistic distribution whose cumulative density function (cdf) is $[1 + \exp(-\beta x)]^{-1}$, where $\beta = (3/\pi)^{\frac{1}{2}}$. Let $w_n = x_{(n)} - x_{(1)}$ be the range depending upon the sample size. The exact expressions for mean and variance of the distribution of range is obtained here for $n = 2(2)10$. These results were applied in estimating simple and unbiased estimates of mean and standard deviation, based on mid range and semi range. The efficiencies for these estimates were also computed for $n = 2(2)10$.

The probability integral of the distribution of range $w_n$ in logistic samples of size $n$, i.e. $\Pr\{W_n \geqq w_n\}$, is computed for $n = 2(1)10$ and for $w_n = 0.00(.05)7.00$. The usual lower and upper percentage points of the distribution of range is also calculated for $n = 2(1)10$.

## 4. The Asymptotic Relative Efficiency of Goodness-of-Fit Tests Against Scalar Alternatives. J. W. Gelzer and Ronald Pyke, The Boeing Company and University of Washington.

The purpose of this paper is to study the ARE of the one and two-sided Kolmogorov-Smirnov tests when used to test $H_0:\theta = 0$ against $H_1:\theta > 0$ where the random sample has a distribution function of the form $F_\theta(x) = F(x\theta + x)$ for some specified distribution function $F$. Following the methods of D. Quade (unpublished Ph.D. Thesis, 1959, University of North Carolina) on explicit representation of the ARE is given for these scalar alternatives. This expression involves knowing the probability of Brownian Motion staying between two specified functions (usually non-linear) which is determined by $F$. In the case where $F$ is a symmetric distribution about zero, a new test is introduced, namely the one-sided Kolmogorov-Smirnov test applied to the absolute values of the observations (which form a sufficient statistic in this case). Some values of the ARE and asymptotic power of these tests is computed when $F$ is uniform, double exponential and Normal. These figures show that the test based on the sufficient statistic is by far the best among the three.

## 5. A Poisson Traffic Model for the Intersection of a Single-Lane Road with a Two-Lane Road. Rudy Gideon and Ronald Pyke, University of Washington.

Consider a two-lane road which is interesected on one side by a single-lane secondary road. A single car waiting on the secondary road may merge into either the nearest or the farthest lane. It is assumed that the traffic in each lane is independent of the other lanes and that the inter-arrival times of cars at the intersection in their respective lanes is exponential. The main purpose of this paper is to study the queue size on the secondary road where the secondary road has a special right turn lane which allows cars to merge into the nearest lane of the main road. The problem is approached by first setting up a four-state Markov Renewal process to describe the traffic on the main road. Next the merging process on the secondary road is described. The event that the queue is empty is studied, and conditions are stated under which this event is recurrent or transient. Finally, the quantities which occur in the conditions for recurrence of an empty queue are derived explicitly for a one-car right turn lane.

### 6. The Use of the Concept of a Future Observation in a Goodness-of-Fit Test (Preliminary Report). Irwin Guttman, University of Wisconsin.

A procedure for the following age-old problem is given. "On the basis of $n$ independent observations ($n$ assumed large) $\mathbf{x} = (x_1, \cdots, x_n)$, test whether the sample has come from a population whose pdf (or pf) is $f(x \mid \theta)$, where $\theta$ may be vector valued."

The procedure, which is Bayesian in outlook is as follows:

(i) Determine the posterior distribution of $\theta$ given the sample $\mathbf{x}$, taking as prior $p(\theta)$ the locally uniform distribution as advocated by Jeffreys, Savage, Box and Tiao, etc. (i.e., we are assuming that *a priori* we are in total ignorance re $\theta$).

Denoting this posterior as $p(\theta \mid x)$, then we have $p(\theta \mid \mathbf{x}) = Cp(\theta) \prod_{i=1}^n f(x_i \mid \theta)$, where $C^{-1} = \int p(\theta) \prod_{i=1}^n f(x_i \mid \theta)\, d\theta$. Note that $p(\theta \mid \mathbf{x})$ is peculiarly sensitive to the function $f(x \mid \theta)$.

(ii) Determine the distribution of a future observation $x$, given $\mathbf{x}$, viz $h(x \mid \mathbf{x}) = \int f(x \mid \theta) p(\theta \mid x)\, d\theta = C \int f(x \mid \theta) p(\theta) \prod_{i=1}^n f(x_i \mid \theta)\, d\theta$.

Now if the "hypothesis" that $f(x \mid \theta)$ is the population pdf (or pf) is correct, the behaviour of the sample should be compatible with that of $h(x \mid \mathbf{x})$. Thus, we may group the sample $x$ into various categories, e.g. if $X$ is continuous, we may group the components of $\mathbf{x}$ into $k$ cells with cell boundaries say $(a_{i-1}, a_i)$, $i = 1, 2, \cdots, k$, and determine the compatibility of the relative frequencies of these $k$ cells with $p_i = \int_{a_{i-1}}^{a_i} h(x \mid \mathbf{x})\, dx$, using the Chi-Square test statistic.

Examples involving the Poisson and Normal are given as well as a discussion of the degrees of freedom for the Chi-Square statistic.

### 7. On the Structural Information Contained in the Output of $GI/G/\infty$. D. G. Kendall and T. Lewis, Churchill College, Cambridge and University College, London.

The output of $GI/G/\infty$ can be interpreted as a delayed renewal process, with $dA$ as the distribution generating the renewal process and $dB$ as the delay distribution. Suppose we have available one complete record of the output of such a system, together with the associated permutation $P$ linking arrival-order to departure-order. It is shown (with the aid of a theorem of the brothers Reisz) that $dA$ is uniquely determined by this information, and that $dB$ is all but uniquely determined (only its location remaining in doubt).

### 8. Some Remarks on the Regression in the Multivariate Poisson Distribution. (Preliminary Report). D. M. Mahamunulu, University of Minnesota.

Assuming $X = (X_1, \cdots, X_p)$ follows a $p$-variate ($p > 2$) Poisson distribution (H. Teicher, *Skand. Aktuarietidskr.* **37** 1–9), the regression of $X_p$ on $X_1, \cdots, X_{p-1}$ is obtained.

Two sets of sufficient conditions for the linearity of this regression are obtained. It has been found that, for some combination of values of the regression variables, the regression is linear even when neither of the two sets of sufficient conditions is satisfied.

## 9. **Note on Ranking with Three Populations.** (Preliminary Report). D. M. Mahamunulu, University of Minnesota.

An experimenter's goal is to choose a subset of two populations from three normal populations with common known variance $\sigma^2$, such that the chosen subset includes the "best" population (population with largest mean). The procedure is as follows—take samples of size $n$ from all the three populations and select those populations which gave the two largest sample means. The problem of the choice of $n$, so that the procedure achieves the goal meeting the usual PCS requirements, is solved. The goal set forth here, is less stringent than the goal (2) of Bechhofer (*Ann. Math. Statist.* **25** 16–39). The percentage saving in the sample size due to the relaxation of Bechhofer's goal to the present one has been found. A table, giving $d(= n^{\frac{1}{2}}\delta^* \mid \sigma)$ values and percentage saving has been prepared. The same problem, when $\sigma$ is unknown, can be handled by the above procedure by defining $\delta^*$ in $\sigma$-units.

## 10. **Estimation in Markov Renewal Processes.** Erin Moore and Ronald Pyke, The Boeing Company and University of Washington.

A finite state Markov Renewal process (MRP) is determined by a matrix $(Q_{ij})$, $(i, j = 1, 2, \cdots, m)$ of mass functions for which $Q_{ij}(0) = 0$ and $\sum_{j=1}^{m} Q_{ij}(+\infty) = 1$. For the process, $Q_{ij}(x)$ represents the probability that the next transition will occur during $(0, x]$ and will be into state $j$ given that state $i$ has just been entered. Set $p_{ij} = Q_{ij}(+\infty)$ and $F_{ij}(x) = Q_{ij}(x)/p_{ij}$ if $p_{ij} \neq 0$. Suppose one observes the process over the interval, $(0, t]$. Consider the "natural" estimator for $Q_{ij}(x)$ defined by $\hat{Q}_{ij}(x) = [N_{ij}(t)/N_i(t)]\hat{F}_{ij}(x{:}t)$, where $N_{ij}(t)$ is the number of $i \rightarrow j$ transitions during $(0, t]$, $N_i(t) = \sum_{j=1}^{m} N_{ij}(t)$ and $\hat{F}_{ij}(x{:}t)$ is the empirical distribution function based on the $N_{ij}(t)$ sample inter-transition times between states $i$ and $j$. It is proved that $\sup_x |\hat{Q}_{ij}(x) - Q_{ij}(x)| \rightarrow 0$ (a.s.) and that the finite dimensional distributions of the processes $\{t^{\frac{1}{2}}[\hat{Q}_{ij}(x) - Q_{ij}(x)]{:}x \geqq 0\}$ converge to those of a Normal process. The density function of an MRP over $(0, t]$ is derived and the maximum likelihood estimators of $p_{ij}$ and $F_{ij}(x)$ are derived in several cases of interest.

## 11. **The Robbins-Isbell Two-Armed-Bandit Problem with Finite Memory.** Ronald Pyke and Carter V. Smith, University of Washington and The Boeing Company.

In this problem two coins with respective probabilities $p_1$ and $p_2$ of heads are tossed according to a rule which depends only upon the history of the outcomes of the $r$ preceding tosses. In this paper, a collection of rules $\{R_{r,s}{:}1 \leqq s \leqq r - 2\}$ is defined for memory length $r > 2$. The Isbell rule is $R_{r,1}$. It is proved in this paper that $R_{r,s+1}$ is a better rule than $R_{r,s}$ if the rules are compared by means of their limiting frequencies of heads, as is done by Robbins *Proc. Nat. Acad. Sci.* **42** 920–923 and by Isbell *Ann. Math. Statist.* **30** 606–610. A more general class of rules is also defined. This class of rules is constructed by using all possible coding arrangements of the $4^r$ states in order to give the rule an essential memory of length about $2^{r-2}$. A specific one of these rules is conjectured to be the uniformly best rule.

## 12. Testing the Independence of Regression Disturbances. K. Koteswara Rao, University of Minnesota.

Let $y = X\beta + u$ be a normal independent linear regression model. Define the residual random vector $\hat{u} = [I - X(X'X)^{-1}X']y$. It has been shown that there exist matrices $A$ and $B$ and a constant $c$ such that $c[\hat{u}'A\hat{u}/\hat{u}'B\hat{u}]$ is a von Neumann ratio. This ratio has been used to construct a test statistic to test the independence of the disturbances $u$ $(= u_1, \cdots, u_n)$. The power of the test is being studied.

## 13. Stochastic Behavior of Interarrival Times with Different Distributions for a Modified Queueing System $M/G/I$. V. M. Sehgal, Wayne State University.

Takamatsu (*Ann. Inst. Statist. Math.* **15** 73–78) considers the stochastic behavior of the interarrival times for a modified queueing system $M/G/I$. He assumes the distribution of the interarrival times to be an exponential distribution. In this paper we extend some results of Takamatsu by considering that the interarrival times have different kinds of distributions, Gamma, Erlangian, Homogeneous recurrent, Palm's, etc.

## 14. Further Results for the Queueing System $GI/M/1$. D. N. Shanbhag, Karnatak University. (Introduced by B. R. Bhat.)

Let $t_n$ be the time at which the ($n_0$th) arrival takes place and $Q_n$ be the queue length at $t_{n+1} - 0$ in the queueing system $GI/M/1$ with (a) the inter-arrival time distribution as $dB(t)$ $(0 < t < \infty)$, (b) the service time distribution as $\lambda \exp(-\lambda t)\, dt$ $(0 < t < \infty)$, and (c) one server, who serves only one customer at a time. Defining $\Pr\{Q_n = m, Q_r \geqq i$ for $r = 1, 2, \cdots, n-1, t \leqq t_{n+1} - t_1 < t + dt \mid Q_0 = k\} = dP(t; n, k, m, i)$ because of the lack of memory property of the negative exponential distribution and the law of total probability, the following identities have been established (1) $dP(t; n, i, m, i+1) = dP(t; n, o, m-i, 1)$ $(n \geqq m - i \geqq 1, i \geqq 0)$, & (2) $dB_n(t) = dP(t; n, 0, 0, 0) + \sum_{i=1}^{n} dP(t; n, 0, i, 0)$ $(n \geqq 0)$ where $B_n(t)$ is the $n$-fold convolution of $B(t)$ with itself and $dB_0(t)$ is Dirac's delta function. From the law of total probability and (1), (3) $dP(t; n, 0, i, 0) = \sum_{m=0}^{n} \int_{\tau=0}^{t} dP(\tau; m, 0, i-1, 0)\, dP(t-\tau; n-m, 0, 1, 1)$ $(i \geqq 1, n \geqq 0)$, which from (2) gives (4) $dP(t; n, 0, 0, 0) = dB_n(t) - \sum_{m=0}^{n} \int_{\tau=0}^{t} dB_m(\tau)\, dP(t-\tau; n-m, 0, 1, 1)$ $(n \geqq 0)$. As $dP(t; n, 0, 1, 1)$ has been derived by the author (*Austral. J. Statist.* **5** 57–61), $dP(t; n, 0, 0, 0)$ can be derived. Since $dP(t; n, k, m, 1)$ for $k \geqq 1$, $m \geqq 1$, $n + k - m \geqq 0$, and $n \geqq 0$ can be derived by a method used by Prabhu and Bhat (*Operations Res.* **11** 380–86), using (4) and the results established by the author (1963) referred above, almost all results which are important in the analysis of the present queueing system can be derived using probabilistic arguments only.