ON THE ASYMPTOTIC THEORY OF FIXED-WIDTH SEQUENTIAL CONFIDENCE INTERVALS FOR THE MEAN

By Y. S. Chow[1] and Herbert Robbins[2]

Purdue University and Columbia University

1. Introduction. Let $x_1, x_2, \cdots$ be a sequence of independent observations from some population. We want to find a confidence interval of prescribed width $2d$ and prescribed coverage probability α for the unknown mean μ of the population. If the variance σ^2 of the population is known, and if d is small compared to σ^2, this can be done as follows. For any $n \geqq 1$ define

$$\bar{x}_n = n^{-1}\sum_1^n x_i, \qquad I_n = [\bar{x}_n - d, \bar{x}_n + d],$$

and choose a to satisfy

$$(2\pi)^{-\frac{1}{2}}\int_{-a}^{a} e^{-u^2/2}\,du = \alpha.$$

Then for a sample size n determined by

$$n = \text{smallest integer} \geqq (a^2\sigma^2)/d^2, \tag{1}$$

the interval I_n has coverage probability

$$P(\mu \,\varepsilon\, I_n) = P(\sqrt{n}|\bar{x}_n - \mu|/\sigma \leqq d\sqrt{n}/\sigma).$$

Since (1) implies that $\lim_{d\to 0} (d^2 n)/(a^2\sigma^2) = 1$, it follows from the central limit theorem that

$$\lim_{d\to 0} P(\mu \,\varepsilon\, I_n) = (2\pi)^{-\frac{1}{2}}\int_{-a}^{a} e^{-u^2/2}\,du = \alpha.$$

We shall be concerned with the case in which the nature of the population, and hence σ^2, is unknown, so that no fixed sample size method is available. Define

$$v_n = n^{-1}\sum_1^n (x_i - \bar{x}_n)^2 + n^{-1} \qquad (n \geqq 1), \tag{2}$$

let $a_1, a_2, \cdots$ be any sequence of positive constants such that $\lim_{n\to\infty} a_n = a$, and define

$$N = \text{smallest } k \geqq 1 \quad \text{such that} \quad v_k \leqq (d^2k)/a_k^2. \tag{3}$$

The object of the present note is to prove the following

Theorem. *Under the sole assumption that* $0 < \sigma^2 < \infty$,

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(4) $$\lim_{d\to 0} (d^2N)/(a^2\sigma^2) = 1 \quad a.s.,$$

(5) $$\lim_{d\to 0} P(\mu \in I_N) = \alpha \qquad (\textit{asymptotic ``consistency''}),$$

(6) $$\lim_{d\to 0} (d^2EN)/(a^2\sigma^2) = 1. \qquad (\textit{asymptotic ``efficiency''}).$$

REMARKS.

1. In case the distribution function of the x_i is continuous, Definition (2) can be replaced by, e.g.,

(7) $$v_n = n^{-1}\sum_1^n (x_i - \bar{x}_n)^2.$$

2. As will become evident from the proof, N in (3) could be defined as the smallest (or the smallest odd, etc.) integer $\geqq n_0$ such that the indicated inequality holds, where n_0 is any fixed positive integer.

2. Proof of the theorem.

LEMMA 1. *Let y_n $(n = 1, 2, \cdots)$ be any sequence of random variables such that $y_n > 0$ a.s., $\lim_{n\to\infty} y_n = 1$ a.s., let $f(n)$ be any sequence of constants such that*

$$f(n) > 0, \qquad \lim_{n\to\infty} f(n) = \infty, \qquad \lim_{n\to\infty} f(n)/f(n-1) = 1,$$

and for each $t > 0$ define

(8) $$N = N(t) = \textit{smallest } k \geqq 1 \quad \textit{such that} \quad y_k \leqq f(k)/t.$$

Then N is well-defined and non-decreasing as a function of t,

(9) $$\lim_{t\to\infty} N = \infty \quad a.s., \qquad \lim_{t\to\infty} EN = \infty,$$

and

(10) $$\lim_{t\to\infty} f(N)/t = 1 \quad a.s.$$

PROOF. (9) is easily verified. To prove (10) we observe that for $N > 1$, $y_N \leqq f(N)/t < [f(N)/f(N-1)]y_{N-1}$, whence (10) follows as $t \to \infty$.

LEMMA 2. *If the conditions of Lemma 1 hold and if also $E(\sup_n y_n) < \infty$, then*

(11) $$\lim_{t\to\infty} Ef(N)/t = 1.$$

PROOF. Let $z = \sup_n y_n$; then $Ez < \infty$. Choose m such that $f(n)/f(n-1) \leqq 2$, $(n > m)$. Then for $N > m$

$$f(N)/t = [f(N)f(N-1)]/[f(N-1)t] < 2y_{N-1} < 2z.$$

Hence for $t \geqq 1$,

(12) $$f(N)/t \leqq 2z + f(1) + \cdots + f(m).$$

(11) follows from (10), (12), and Lebesgue's dominated convergence theorem.

PROOF OF (4) AND (5). Set

(13) $$y_n = v_n/\sigma^2 = (1/n\sigma^2)\left(\sum_1^n (x_i - \bar{x}_n)^2 + 1\right),$$

(14) $$f(n) = (na^2)/a_n^2, \qquad t = (a^2\sigma^2)/d^2;$$

then (3) can be written as

$$N = N(t) = \text{smallest } k \geqq 1 \quad \text{such that} \quad y_k \leqq f(k)/t.$$

By Lemma 1,

$$(15) \qquad 1 = \lim_{t\to\infty} f(N)/t = \lim_{d\to 0} (d^2N)/(a^2\sigma^2) \quad \text{a.s.},$$

which proves (4). Now

$$P(\mu \,\varepsilon\, I_N) = P(|x_1 + \cdots + x_N - N\mu|/\sigma\sqrt{N} \leqq d\sqrt{N}/\sigma).$$

By (15), $d\sqrt{N}/\sigma \to a$ and $N/t \to 1$ in probability as $t \to \infty$; it follows from a result of Anscombe [1] that as $t \to \infty$,

$$(x_1 + \cdots + x_N - N\mu)/\sigma\sqrt{N} \sim N(0, 1).$$

Hence

$$\lim_{t\to\infty} P(\mu \,\varepsilon\, I_N) = (2\pi)^{-\frac{1}{2}}\int_{-a}^{a} e^{-u^2/2}\, du = \alpha,$$

which proves (5).

It remains to prove (6). This is an immediate consequence of Lemma 2 whenever the distribution of the x_i is such that

$$(16) \qquad E\{\sup_n (n^{-1}\textstyle\sum_1^n (x_i - \bar{x}_n)^2\} < \infty,$$

for then

$$(17) \qquad \lim_{t\to\infty} [Ef(N)]/t = 1,$$

and from the fact that the function $f(n)$ defined by (14) is $n + o(n)$ it follows from (17) that

$$1 = \lim_{t\to\infty} EN/t = \lim_{d\to 0} (d^2EN)/(a^2\sigma^2).$$

For (16) to hold it would suffice for the fourth moment of the x_i to be finite; however, we shall in the following prove that (6) holds without such a restriction. For this we need

LEMMA 3. *If the conditions of Lemma 1 hold, if* $\lim_{n\to\infty} f(n)/n = 1$, *if for* N *defined by* (8),

$$(18) \qquad EN < \infty \ (\textit{all } t > 0), \qquad \limsup_{t\to\infty} E(Ny_N)/EN \leqq 1,$$

and if there exists a sequence of constants $g(n)$ *such that*

$$g(n) > 0, \qquad \lim_{n\to\infty} g(n) = 1, \qquad y_n \geqq g(n)y_{n-1},$$

then

$$(19) \qquad \lim_{t\to\infty} EN/t = 1.$$

PROOF. For any $0 < \epsilon < 1$ choose m so that

$$f(n-1) \geqq (1-\epsilon)f(n)$$

$$f(n-1) \geqq (1-\epsilon)n \qquad \text{for } n \geqq m$$
$$g(n) \geqq 1-\epsilon$$

and $E(Ny_N) \leqq (1+\epsilon)EN$ for $t \geqq m$. On the set $A = \{N \geqq m\}$ it follows that

$$[(1-\epsilon)^2/t]N^2 = (1-\epsilon)N\cdot(1-\epsilon)N/t \leqq g(N)Nf(N-1)/t < g(N)Ny_{N-1} \leqq Ny_N .$$

Hence

$$[(1-\epsilon)^2/t](\textstyle\int_A N)^2 \leqq [(1-\epsilon)^2/t]\int_A N^2 \leqq \int_A Ny_N \leqq E(Ny_N),$$
$$[(1-\epsilon)^2/t]\textstyle\int_A N \leqq E(Ny_N)/\int_A N,$$
$$[(1-\epsilon)^2/t](EN-m) \leqq E(Ny_N)/(EN-m).$$

From (9) and (18) it follows that

$$(1-\epsilon)^2 \limsup_{t\to\infty} EN/t \leqq \limsup_{t\to\infty} E(Ny_N)/(EN) \leqq 1,$$

so that

$$\limsup_{t\to\infty} EN/t \leqq 1. \tag{20}$$

Now let $y_n' = \min(1, y_n)$. Then

$$0 < y_n' \leqq 1, \qquad y_n' \leqq y_n, \qquad \lim_{n\to\infty} y_n' = 1 \quad \text{a.s.}$$

Define

$$N' = N'(t) = \text{smallest } k \geqq 1 \quad \text{such that} \quad y_k' \leqq f(k)/t.$$

From Lemma 2, since $\sup_n (y_n') \leqq 1$,

$$1 = \lim_{t\to\infty} [Ef(N)]/t = \lim_{t\to\infty} (EN')/t.$$

But since $y_n' \leqq y_n$, $N' \leqq N$, and hence $EN' \leqq EN$. Thus

$$\liminf_{t\to\infty} (EN)/t \geqq \liminf_{t\to\infty} (EN')/t = 1,$$

which, with (20), proves (19).

Proof of (6). Fix $t > 0$, choose m such that $f(n)/t \geqq 1 (n \geqq m)$, choose $\delta > 0$ such that $(n-1)f(n-1) \geqq \delta n^2 (n \geqq 2)$, and define for any $r \geqq m$, $M = \min(N, r)$. By Wald's theorem for cumulative sums,

$$E(\textstyle\sum_1^M (x_i - \mu)^2) = EM\cdot E(x_i-\mu)^2 = EM\cdot\sigma^2.$$

Hence by (13),

$$\begin{aligned} E(My_M) &= (1/\sigma^2)E(\textstyle\sum_1^M (x_i - \bar{x}_M)^2 + 1) \\ &\leqq (1/\sigma^2)E(\textstyle\sum_1^M (x_i-\mu)^2 + 1) = EM + (1/\sigma^2). \end{aligned} \tag{21}$$

Put $g(n) = (n-1)/n$, $(n \geqq 2)$; then

$y_n \geqq (1/n\sigma^2)\sum_1^{n-1}(x_i - \bar{x}_{n-1})^2 + (1/n\sigma^2) = [(n-1)/n]y_{n-1} = g(n)y_{n-1}$.

Hence

$$E(My_M) \geqq \int_{\{N>r\}} ry_r + \int_{\{N\leqq r\}} Ny_N \geqq [rf(r)/t]P(N > r) + \int_{\{2\leqq N\leqq r\}} Ny_N$$
$$\geqq rP(N > r) + \int_{\{2\leqq N\leqq r\}} [Ng(N)f(N-1)]/t$$
$$\geqq rP(N > r) + (\delta/t)\int_{\{2\leqq N\leqq r\}} N^2.$$

Hence by (21),

$$\int_{\{N\leqq r\}} N \geqq (\delta/t)\int_{\{2\leqq N\leqq r\}} N^2 - (1/\sigma^2) \geqq (\delta/t)\left(\int_{\{2\leqq N\leqq r\}} N\right)^2 - (1/\sigma^2),$$

and letting $r \to \infty$ it follows that

$$EN = \lim_{r\to\infty} \int_{\{N\leqq r\}} N < \infty,$$

which is the first part of (18). Again by Wald's theorem,

$$E(Ny_N) \leqq EN + (1/\sigma^2),$$

so by (9),

$$\limsup_{t\to\infty} [E(Ny_N)]/(EN) \leqq 1,$$

which is the second part of (18). All the conditions of Lemma 3 therefore hold, and hence

$$1 = \lim_{t\to\infty} EN/t = \lim_{d\to 0} (d^2EN)/(a^2\sigma^2),$$

which is (6). This completes the proof of the theorem of Section 1. As to Remark 1 following the theorem, it is clear that the only purpose of the term n^{-1} in (2) is to ensure that $y_n = v_n/\sigma^2 > 0$ a.s., this fact having been used in the proof of Lemma 1 to guarantee that $N \to \infty$ a.s. as $t \to \infty$. If the distribution function of the x_i is continuous the definition (7) is equally good, the only change being that the term $1/\sigma^2$ in the proof of (6) disappears.

The method used in this note is a modification of that used in [3] to prove the elementary renewal theorem. The theorem in this note has been proved when the x_i are $N(\mu, \sigma^2)$ by Stein [6], Anscombe [1], [2], and Gleser, Robbins, and Starr [4]. Some numerical computations for a slightly modified procedure have been made by Ray [5] who, apparently misled by having considered too few values of d, doubts the validity of (5) in his case. Extensive numerical computations in the $N(\mu, \sigma^2)$ case have been made by Starr and will soon be available. They indicate, for example, that for $\alpha = .95$ the lower bound for all $d > 0$ of $P(\bar{x}_N - d \leqq \mu \leqq \bar{x}_N + d)$, where N is the smallest odd integer $k \geqq 3$ such that

$$(k-1)^{-1}\sum_1^k (x_i - \bar{x}_k)^2 \leqq (d^2k)/a_k^2,$$

is about .929 if the values a_k are taken from the t-distribution with $(k-1)$ degrees of freedom.

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