SAMPLE SEQUENCES OF MAXIMA

By James Pickands III

Virginia Polytechnic Institute

1. Introduction and summary. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent, identically distributed random variables with common distribution function F. Let $Z_n = \max\{X_1, X_2, \cdots X_n\}$.

Conditions for the stability and relative stability of such sequences with the various modes of convergence have been given by Geffroy [3], and Barndorff-Nielsen [1]. The principal result of this paper is Theorem 2.1, which is an analogue for maxima of the law of the iterated logarithm for sums (Loève [6] pages 260–1).

In Section 3, it is indicated that the theorem is satisfied by a wide class of distributions, and specific forms are given for the normal and exponential distributions.

2. The result.

Lemma 2.1. *Let b_n be such that*

$$1 - F(b_n) = Cn^{-1}(\log n)^{-\alpha}, 0 < C < \infty.$$

Then $Z_n > b_n$ *infinitely often* (i.o.) *with probability one* iff $\alpha \leqq 1$, *and* $Z_n \leqq b_n$ i.o. iff $\alpha \geqq 0$.

Proof. Clearly $Z_n > b_n$ i.o. iff $X_n > b_n$ i.o. Since the X_i are independently and identically distributed, and by the Borel zero-one criterion (Loève [6], page 228), $Z_n > b_n$ i.o. with probability one iff $\sum_{n=2}^{\infty}(1 - F(b_n)) = \infty$. But, clearly, this is so iff $\alpha \leqq 1$.

Note that if $\alpha = 0$, $P\{Z_n \leqq b_n\} = (1 - C/n)^n \to e^{-C}$, as $n \to \infty$. Hence $Z_n \leqq b_n$ i.o. with probability one, if $\alpha \geqq 0$. Clearly

$$\begin{aligned} -\log P\{Z_n \leqq b_n\} &= -n \log F(b_n) \\ &= -n \log(1 - C(\log n)^{-\alpha}/n) \\ &= C(\log n)^{-\alpha} + O(n^{-1}(\log n)^{-2\alpha}). \end{aligned}$$

So

$$P\{Z_n \leqq b_n\} = \exp(-C(\log n)^{-\alpha})(1 + o(1)) \tag{2.1}$$

as $n \to \infty$. Now suppose $\alpha < 0$. Since $Z_n > b_n$ i.o., if in addition $Z_n \leqq b_n$ i.o., infinitely many of the events $E_n \equiv \{Z_{n-1} > b_{n-1}, Z_n \leqq b_n\}$ occur. To prove the lemma, then, it is sufficient to show that $P\{E_n\}$ is summable. Clearly

$$\begin{aligned} F^{n-1}(b_n) &= F^n(b_n)(1 - C(\log n)^{|\alpha|}/n)^{-1} \\ &= F^n(b_n)(1 + Cn^{-1}(\log n)^{|\alpha|} + O(n^{-2}(\log n)^{2|\alpha|})) \end{aligned}$$

Received 6 September 1966.

as $n \to \infty$. But

$$(2.2)\quad P\{E_n\} \leqq P\{b_{n-1} < Z_{n-1} \leqq b_n\} = F^n(b_n) - F^{n-1}(b_{n-1}) + Cn^{-1}(\log n)^{|\alpha|}F^n(b_n)(1 + o(1)),$$

as $n \to \infty$. Clearly $\sum_{n=2}^{\infty} (F^n(b_n) - F^{n-1}(b_{n-1})) = \lim_{n\to\infty} F^n(b_n) - F(b_1) < \infty$. So it is sufficient to show that the last term on the right hand side of (2.2) is summable. By (2.1), this is proportional to

$$c_n = n^{-1}(\log n)^{|\alpha|} \exp(-C(\log n)^{|\alpha|})(1 + O(1))$$

as $n \to \infty$, which is summable if $\alpha < 0$. This proves the lemma.

Let

$$(2.3)\quad \psi(x) \equiv -\log(1 - F(x)).$$

This is a non-decreasing function which goes from zero to infinity, as x increases. Let

$$(2.4)\quad Q(y) \equiv \inf\{x : \psi(x) > y\}.$$

That is, $Q(y)$ is the inverse function for $\psi(x)$.

If the distribution functions $F(x)$ is such that

$$(2.5)\quad \lim_{n\to\infty} F^n(a_n x + b_n) = \exp(-e^{-x}),$$

for a pair of sequences $\{a_n\}$ and $\{b_n\}$ with $a_n > 0$, we say the distribution function $F(x)$ lies in the domain of attraction of the double exponential distribution function. Following Gumbel [5], we will simply say that $F(x)$ is of the "exponential type." According to Gnedenko [4], this is true iff

$$(2.6)\quad \lim_{n\to\infty} n(1 - F(a_n x + b_n)) = e^{-x}.$$

By (2.3), above, this can be rewritten

$$(2.7)\quad \lim_{n\to\infty} \psi_n(x) = x,$$

where $\psi_n(x) = \psi(a_n x + b_n) - \log n$. Let

$$(2.8)\quad Q(y:s) = \inf\{x : \psi_n(x) > y\}$$

where $s = \log n$. Then $Q(y:s) = (Q(s + y) - \beta(s))/\alpha(s)$, where $\alpha(s)$ and $\beta(s)$ are a_n and b_n respectively. Equation (2.7) can then be rewritten

$$(2.9)\quad \lim_{s\to\infty} Q(y:s) = y.$$

Suppose it is required that, in addition to (2.5),

$$(2.10)\quad \lim_{n\to\infty} na_n F^{n-1}(a_n x + b_n) f(a_n x + b_n) = \exp -(x + e^{-x}).$$

That is, in (2.5), the limit of the derivative is the derivative of the limit. Clearly by differentiating in (2.9), a sufficient condition is that

$$\lim_{s\to\infty} \partial Q(y:s)/\partial y = \lim_{s\to\infty} Q'(s + y)/\alpha(s) = 1.$$

If, in addition, it is assumed that the density $f(x)$ is itself differentiable and hence continuous we can assume without loss of generality that $\alpha(s) \equiv Q'(s)$, and $\beta(s) \equiv Q(s)$. Then $Q(y{:}s) = \int_s^{s+y} Q'(t)\,dt/Q'(s)$. A sufficient condition, then, is that

$$\lim_{s\to\infty} \sup_{s \leqq t \leqq s+y} |(Q'(t)/Q'(s) - 1| = 0.$$

Equivalently

$$\lim_{s\to\infty} \sup_{s \leqq t \leqq s+y} |\log Q'(t) - \log Q'(s)| = 0.$$

So, a sufficient condition is that

$$\text{(2.11)} \qquad \lim_{s\to\infty} \partial \log Q'(s)/\partial s = 0.$$

The following lemma has thus been proved.

Lemma 2.2. *Let $F(x)$ be twice differentiable for all sufficiently large x, then equations* (2.5) *and* (2.10) *hold, provided equation* (2.11) *does.*

Clearly a possible choice of sequence a_n and b_n, $a_n > 0$ is given by $\psi(b_n) = \log n$, and $a_n = 1/\psi'(b_n)$, since it can be required that $\psi_n(0) = 0$, $\psi_n'(0) = 1$, for all n. Equivalently $1 - F(b_n) = 1/n$, and $a_n = (nf(b_n))^{-1}$. Consider the condition, (2.11), which can be rewritten, $\lim_{s\to\infty} (Q''(s)/Q'(s)) = 0$, or equivalently $\lim_{s\to\infty} (dQ'(s)/dQ(s)) = 0$. By the definition (2.4),

$$\text{(2.13)} \qquad Q'(s) = (1/\psi'(x)),$$

and so $dQ'(s)/dQ(s) = d(1/\psi'(x))/dx$, which gives the result of von Mises ([7], page 285). That is, a sufficient condition for equations (2.5) and (2.10) is

$$\text{(2.14)} \qquad \lim_{x\to\infty} d(1/\psi'(x))/dx = 0,$$

where $\psi(x)$ is given by eq. (2.3).

Theorem 2.1. *If*

$$\text{(2.15)} \qquad \mathit{lim}_{x\to\infty} (\log \psi(x))\partial(1/\psi'(x))/\partial x = 0,$$

then equations (2.5) *and* (2.10) *hold, and in addition,*

$$\text{(2.16)} \qquad P\{\liminf_{n\to\infty} (Z_n - b_n)/a_n \log\log n = 0,$$

$$\limsup_{n\to\infty} (Z_n - b_n)/a_n \log\log n = 1\} = 1,$$

where a_n and b_n are any pair of sequences, satisfying equation (2.5).

Proof. Clearly (2.15) implies the von Mises condition (2.14), and so there exist sequences a_n and b_n with $a_n > 0$, satisfying (2.5) and (2.10). For (2.16), by Lemma 2.1, it is sufficient to prove that

$$\lim_{n\to\infty} n(\log n)^\theta(1 - F(b_n + a_n\theta \log\log n)) = 1,$$

for all θ in some open interval (a, b), which includes the closed interval [0, 1]. Equivalently, using the definition (2.3), it is sufficient to show that

$$\psi(b_n + a_n\theta \log\log n) - \log n - \theta \log\log n \to 0,$$

as $n \to \infty$, for all $\theta \varepsilon (a, b)$. In other words the trajectory of points $(x_n, \psi_n(x_n))$ is asymptotic to the straight line $y = x$, where $x_n = \theta \log \log n$. Using the definition (2.8), it is sufficient that the family of points $(s, Q(\theta \log s{:}s))$ be asymptotic to the same straight line. But this is true, iff

$$(Q(s + \theta \log s) - Q(s))/Q'(s) = \int_s^{s+\theta \log s} (Q'(t)/Q'(s))\, dt \to 0,$$

as $s \to \infty$. Clearly it is sufficient that

$$\lim_{s\to\infty} \sup_{s \leqq t \leqq s+\theta \log s} |(Q'(t)/Q'(s)) - 1| = 0,$$

or equivalently, that

$$\lim_{s\to\infty} \sup_{s \leqq t \leqq s+\theta \log s} |\log Q'(t) - \log Q'(s)| = 0.$$

Therefore, it is sufficient that

$$\lim_{s\to\infty} (\log s)\partial \log Q'(s)/\partial s = 0.$$

From the definitions (2.3) and (2.4) and the relationship (2.13), it follows that this is equivalent to the condition (2.15).

Let a_n, and b_n, with $a_n' > 0$ be any other sequence satisfying (2.5) and (2.10). It follows from a result due to Gnedenko [4] that $\lim_{n\to\infty} (a_n/a_n') = 1$, and $\lim_{n\to\infty} (b_n - b_n')/a_n = 0$. Clearly, then, the result (2.16) still holds with a_n', and b_n' replacing a_n and b_n. The theorem is proved.

3. Discussion and examples. Condition (2.15) is slightly stronger than the von Mises condition (2.14), but it is satisfied by the principal distributions of the exponential type, in particular the normal, lognormal, and gamma distributions. To see this, it is sufficient to note that the condition (2.15) is satisfied, if either

$$\lim_{x\to\infty} (\psi(x) - (\alpha \log x + x^\beta)) = 0, \qquad -\infty < \alpha < \infty, \quad \beta > 0,$$

or

$$\lim_{x\to\infty} (\psi(x) - (\alpha \log \log x + (\log x)^\beta)) = 0, \quad -\infty < \alpha < \infty, \quad \beta > 1,$$

where $\psi(x)$, given by equation (2.3), is twice differentiable for sufficiently large. x. If

$$\lim_{x\to\infty} (\psi(x) - \log x(\log \log x)) = 0,$$

the von Mises condition is satisfied, but ours, (2.15), is not. It is not known whether Theorem 2.1 holds for all distributions of the exponential type. Nor is it known, whether the same principle, or an analogous one holds for distributions not of the exponential type, although it is conjectured that one does not.

If X is normal with mean zero and variance one, Cramér ([2], page 374) gives $a_n = (2 \log n)^{-\frac{1}{2}}$ and $b_n = (2 \log n)^{\frac{1}{2}} - (2 \log n)^{-\frac{1}{2}}(\log \log n + \log 4\pi)/2$. By Theorem 2.1, then,

$$P\{\liminf_{n\to\infty} (2 \log n)^{\frac{1}{2}}(Z_n - (2 \log n)^{\frac{1}{2}})/\log \log n = -\tfrac{1}{2},$$
$$\limsup_{n\to\infty} (2 \log n)^{\frac{1}{2}}(Z_n - (2 \log n)^{\frac{1}{2}})/\log \log n = \tfrac{1}{2}\} = 1.$$

For the exponential distribution

$$P\{\liminf_{n\to\infty} (Z_n - \log n)/\log\log n = 0, \quad \limsup_{n\to\infty} (Z_n - \log n)/\log\log n = 1\} = 1.$$

The author thanks the referee for many helpful comments.

REFERENCES

[1] BARNDORFF-NIELSEN, OLE (1963). On the limit behavior of extreme order statistics. *Ann. Math. Statist.* **34** 992–1002.

[2] CRAMÉR, HARALD (1951). *Mathematical Methods of Statistics*. Princeton Univ. Press.

[3] GEFFORY, JEAN (1958). Contribution a La Theorie des Valeurs extremes. Thesis, University of Paris.

[4] GNEDENKO, B. V. (1943). Sur La Distribution du Terme Maximum D'Une Series Aleatoire. *Ann. Math.* **2** 423–453.

[5] GUMBEL, E. J. (1958). *Statistics of Extremes*. Columbia Univ. Press.

[6] LOÈVE, M. (1963). *Probability Theory*. Van Nostrand Co., New York.

[7] VON MISES, RICHARD (1954). *Selected Papers of Richard von Mises*. American Mathematical Society, Providence.