ESTIMABILITY OF VARIANCE COMPONENTS FOR THE TWO-WAY CLASSIFICATION WITH INTERACTION

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1. Introduction and summary. Graybill and Hultquist [2] have defined a variance component to be estimable if there exists a quadratic function of the observations having expectation equal to the component. This definition (extended to functions of variance components) will be used in the present paper in investigating certain aspects of the estimability of linear functions of variance components for the two-way completely-random classification with unequal numbers of observations in the subclasses. It is obvious that at least some functions of the variance components are not estimable for certain sets of subclass numbers.

The objectives underlying the present paper were (1) to derive, for the two-way classification, necessary and sufficient conditions which must be satisfied by the subclass numbers in order for linear functions of the variance components to be estimable and (2) to determine, for the same classification, those sets of subclass numbers for which two commonly-used variance-component estimation procedures, Methods 1 and 3 of Henderson [3], yield unbiased estimates of the components or linear functions of the components.

Observations y_{ijk} are taken as having the linear model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk},$$

with $i = 1, \cdots, a; j = 1, \cdots, b$; and $k = 1, \cdots, n_{ij}$. μ is a general mean, the α_i and the β_j are main effects, the γ_{ij} are interaction effects, and the ϵ_{ijk} are residual effects. μ is regarded as fixed while the $\alpha_i, \beta_j, \gamma_{ij}$, and ϵ_{ijk} are taken to be mutually-independent random variables with zero means and variances $\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2$, and σ_ϵ^2. The total number of filled subclasses (subclasses such that $n_{ij} \geqq 1$) will be denoted by p. It is assumed that $a \geqq b$, which can be done without loss of generality.

Methods 1 and 3 (of Henderson) for estimating variance components are based on the analyses of variance given in Tables 1 and 2 respectively where, letting $n_{i\cdot} = \sum_j n_{ij}$, $n_{\cdot j} = \sum_i n_{ij}$, and $n_{\cdot\cdot} = \sum_i n_{i\cdot} = \sum_j n_{\cdot j}$ and using ordinary notation for means, $R_0 = \sum_{ijk} y_{ijk}^2$, $R_\mu = n_{\cdot\cdot}\bar{y}_{\cdot\cdot\cdot}^2$, $R_\alpha = \sum_i n_{i\cdot}\bar{y}_{i\cdot\cdot}^2$, $R_\beta = \sum_j n_{\cdot j}\bar{y}_{\cdot j\cdot}^2$, and $R_\gamma = \sum_{ij} n_{ij}\bar{y}_{ij\cdot}^2$. Also, taking the $b \times 1$ vector $\hat{\boldsymbol{\beta}}$ to be any solution to

$$\mathbf{W}\hat{\boldsymbol{\beta}} = \mathbf{q} \tag{1}$$

where the elements of the $b \times b$ matrix $\mathbf{W}$ are

$$w_{jj} = n_{\cdot j} - \textstyle\sum_i (n_{ij}^2/n_{i\cdot}), \qquad j = 1, \cdots, b,$$

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TABLE 1

Source	d.f.	Sum of Squares
α-classes	$a - 1$	$r_1 = R_\alpha - R_\mu$
β-classes	$b - 1$	$r_2 = R_\beta - R_\mu$
Interaction	$p - a - b + 1$	$r_3 = R_\gamma - R_\alpha - R_\beta + R_\mu$
Residual	$n.. - p$	$r_4 = R_0 - R_\gamma$
Total	$n.. - 1$	$R_0 - R_\mu$

TABLE 2

Source	d.f.	Sum of Squares
α-classes	$a + m - b$	$t_1 = R_{\alpha\beta} - R_\beta$
β-classes	m	$t_2 = R_{\alpha\beta} - R_\alpha$
Interaction	$p - a - m$	$t_3 = R_\gamma - R_{\alpha\beta}$
Residual	$n.. - p$	$t_4 = R_0 - R_\gamma$

and

$$w_{jk} = -\textstyle\sum_i (n_{ij}n_{ik}/n_{i\cdot}), \qquad j \neq k = 1, \cdots, b,$$

and the elements of the $b \times 1$ vector $\mathbf{q}$ are

$$q_j = n_{\cdot j}\bar{y}_{\cdot j\cdot} - \textstyle\sum_i n_{ij}\bar{y}_{i\cdot\cdot}, \qquad j = 1, \cdots, b,$$

$$R_{\alpha\beta} = R_\alpha + \hat{\boldsymbol{\beta}}'\mathbf{q}.$$

The system (1) represents the β_j-equations of the normal equations for the two-way classification without interaction after absorption of the $\mu + \alpha_i$ equations. m is defined to be the rank of $\mathbf{W}$.

Letting $\mathbf{r}' = (r_1, r_2, r_3, r_4)$, $\mathbf{t}' = (t_1, t_2, t_3, t_4)$, and $\boldsymbol{\sigma}' = (\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2, \sigma_\epsilon^2)$ and taking E to be the expectation operator, we have $E\{\mathbf{r}\} = \mathbf{C}\boldsymbol{\sigma}$ and $E\{\mathbf{t}\} = \mathbf{D}\boldsymbol{\sigma}$; where

$$\mathbf{C} = \begin{Vmatrix} n.. - \theta_3 & \theta_1 - \theta_4 & \theta_1 - \theta_5 & a - 1 \\ \theta_2 - \theta_3 & n.. - \theta_4 & \theta_2 - \theta_5 & b - 1 \\ \theta_3 - \theta_2 & \theta_4 - \theta_1 & n.. - \theta_1 - \theta_2 + \theta_5 & p - a - b + 1 \\ 0 & 0 & 0 & n.. - p \end{Vmatrix}$$

and

$$\mathbf{D} = \begin{Vmatrix} n.. - \theta_2 & 0 & \theta_1 - \theta_2 + \theta_6 & a + m - b \\ 0 & n.. - \theta_1 & \theta_6 & m \\ 0 & 0 & n.. - \theta_1 - \theta_6 & p - a - m \\ 0 & 0 & 0 & n.. - p \end{Vmatrix}$$

with $\theta_1 = \sum_{ij} (n_{ij}^2/n_{i\cdot})$, $\theta_2 = \sum_{ij} (n_{ij}^2/n_{\cdot j})$, $\theta_3 = (1/n_{\cdot\cdot}) \sum_i n_{i\cdot}^2$, $\theta_4 = (1/n_{\cdot\cdot}) \sum_j n_{\cdot j}^2$, $\theta_5 = (1/n_{\cdot\cdot}) \sum_{ij} n_{ij}^2$, and

$$\theta_6 = \sum_j w_{jj}^* \sum_u n_{u,\kappa_j}^2 - 2 \sum_{jx} w_{jx}^* \sum_u (n_{u,\kappa_j} n_{u,\kappa_x}^2 / n_{u\cdot}) + \sum_{jx} w_{jx}^* \sum_{iu} (n_{i,\kappa_j} n_{i,\kappa_x} n_{iu}^2 / n_{i\cdot}^2);$$

$\mathbf{W}^*$, a matrix whose jxth element is w_{jx}^*, is the inverse of an $m \times m$ matrix, with jxth element w_{κ_j,κ_x}, formed by deleting from $\mathbf{W}$ $b - m$ columns and the corresponding rows (the columns to be deleted are chosen in such a way that the remaining columns are linearly independent).

The usual problem in variance component estimation is to estimate g linear functions of the variance components $\boldsymbol{\lambda}_i'\boldsymbol{\sigma}$, $i = 1, \cdots, g$, where $\boldsymbol{\lambda}_i$ is a 4×1 vector whose jth element is the constant λ_{ij}.

The Method 1 estimate of $\boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ will be taken to be $\boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*$ where $\hat{\boldsymbol{\sigma}}^*$ is any solution to

$$\mathbf{C}\hat{\boldsymbol{\sigma}} = \mathbf{r} \tag{2}$$

subject to the restriction that those elements of $\hat{\boldsymbol{\sigma}}$, corresponding to the columns of $\mathbf{C}$ remaining after selection of a set of some ρ_c ($\equiv$ the rank of $\mathbf{C}$) linearly-independent columns, are set equal to zero. Similarly, for Method 3, the estimate of $\boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ will be taken to be $\boldsymbol{\lambda}_i'\tilde{\boldsymbol{\sigma}}^*$ where $\tilde{\boldsymbol{\sigma}}^*$ is any solution to

$$\mathbf{D}\tilde{\boldsymbol{\sigma}} = \mathbf{t} \tag{3}$$

subject to the restriction that those elements of $\tilde{\boldsymbol{\sigma}}$, corresponding to the columns of $\mathbf{D}$ remaining after selection of a set of some ρ_d ($\equiv$ the rank of $\mathbf{D}$) linearly-independent columns, are set equal to zero.

It will be made clear in Sections 3 and 4 that solutions to (2) and (3) always exist.

Definition 1. A Method 1 estimate of $\boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ is said to exist if and only if $E\{\boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$.

Definition 2. A Method 3 estimate of $\boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ is said to exist if and only if $E\{\boldsymbol{\lambda}_i'\tilde{\boldsymbol{\sigma}}^*\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$.

The above definitions for the Method 1 estimate and the existence of a Method 1 estimate are motivated by the following easily-proven observations: (i) If $\mathbf{b}$ is a 4×1 vector such that $E\{\mathbf{b}'\hat{\boldsymbol{\sigma}}^*\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$, then $\mathbf{b}'\hat{\boldsymbol{\sigma}}^* = \boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*$; (ii) $E\{\boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ if and only if there exists a 4×1 vector $\mathbf{a}$ such that $E\{\mathbf{a}'\mathbf{r}\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$; (iii) $E\{\boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ if and only if there exists, for each possible solution $\hat{\boldsymbol{\sigma}}$ of the unrestricted equations (2), a 4×1 vector $\mathbf{b}$ such that $E\{\mathbf{b}'\hat{\boldsymbol{\sigma}}\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$; and (iv) if $E\{\boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*\} = \boldsymbol{\lambda}_i'\boldsymbol{\sigma}$, then for each possible solution $\hat{\boldsymbol{\sigma}}$ of the unrestricted equations (2) $\boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}} = \boldsymbol{\lambda}_i'\hat{\boldsymbol{\sigma}}^*$. The definitions for the Method 3 estimate and the existence of a Method 3 estimate are motivated by similar observations.

It is easy to show that a Method 1 or 3 estimate of $\boldsymbol{\lambda}_i'\boldsymbol{\sigma}$ exists if and only if $\boldsymbol{\lambda}_i'$ is an element of the space spanned by the rows of $\mathbf{C}$ or $\mathbf{D}$, respectively. Thus, the maximum-possible number of elements in a set of linearly-independent linear functions of the elements of $\boldsymbol{\sigma}$ for which there exist Method 1 (or 3) estimates is exactly ρ_c (or ρ_d).

Define conditions 1–6 as follows:

condition 1: $a \geqq 2$; condition 2: $b \geqq 2$; condition 3: $a > b$;
condition 4: $p > a$ or, equivalently, $m > 0$; condition 5: $p > a + m$;
condition 6: $n.. > p$.

If condition 1, 2, 3, 4, 5, or 6 is unsatisfied, $a = 1, b = 1, a = b, p = a$ and $m = 0$, $p = a + m$, or $n.. = p$, respectively. The equivalence of the two condition 4's will become clear in Section 4. Let $\mathbf{c}_i$ and $\mathbf{d}_i$, $i = 1, \cdots, 4$, denote the ith rows of $\mathbf{C}$ and $\mathbf{D}$; let $\mathbf{c}^i$ and $\mathbf{d}^i$, $i = 1, \cdots, 4$, represent the ith columns of the two matrices; and take $\mathbf{0}$ to be a vector or matrix of appropriate dimension having all zero elements.

The major results of the present paper are summarized in Theorems 1 and 2.

TABLE 3

Line	Set of Satisfied Conditions	Set of Unsatisfied Conditions	ϕ = Maximum-Possible Number of Elements in a Set of Lin.-Ind. Est. Lin. Functions of the Components	Properties of Equations (2)	ϕ Linearly-Independent Estimable Sums
1	1, 2, 4, 6		4		$\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2, \sigma_\epsilon^2$
2	1, 2, 3, 6	4, 5	3	$r_2 = -r_3$ $\mathbf{c}_2 = -\mathbf{c}_3$ $\mathbf{c}^1 = \mathbf{c}^3$	$\sigma_\alpha^2 + \sigma_\gamma^2, \sigma_\beta^2, \sigma_\epsilon^2$
3	1, 2, 6	3, 4, 5	2	$r_1 = r_2 = -r_3$ $\mathbf{c}_1 = \mathbf{c}_2 = -\mathbf{c}_3$ $\mathbf{c}^1 = \mathbf{c}^2 = \mathbf{c}^3$	$\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\gamma^2, \sigma_\epsilon^2$
4	1, 3, 6	2, 4, 5	2	$r_2 = r_3 = 0$ $\mathbf{c}_2 = \mathbf{c}_3 = \mathbf{0}$ $\mathbf{c}^1 = \mathbf{c}^3, \mathbf{c}^2 = \mathbf{0}$	$\sigma_\alpha^2 + \sigma_\gamma^2, \sigma_\epsilon^2$
5	6	1, 2, 3, 4, 5	1	$r_1 = r_2 = r_3 = 0$ $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}_3 = \mathbf{0}$ $\mathbf{c}^1 = \mathbf{c}^2 = \mathbf{c}^3 = \mathbf{0}$	σ_ϵ^2
6	1, 2, 4	6	3	$r_4 = 0, \mathbf{c}_4 = \mathbf{0}$ $\mathbf{c}^3 = \mathbf{c}^4$	$\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2 + \sigma_\epsilon^2$
7	1, 2, 3	4, 5, 6	2	$r_4 = 0, r_2 = -r_3$ $\mathbf{c}_4 = \mathbf{0}, \mathbf{c}_2 = -\mathbf{c}_3$ $\mathbf{c}^1 = \mathbf{c}^3 = \mathbf{c}^4$	$\sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\epsilon^2, \sigma_\beta^2$
8	1, 2	3, 4, 5, 6	1	$r_4 = 0, \mathbf{c}_4 = \mathbf{0}$ $r_1 = r_2 = -r_3$ $\mathbf{c}_1 = \mathbf{c}_2 = -\mathbf{c}_3$ $\mathbf{c}^1 = \mathbf{c}^2 = \mathbf{c}^3 = \mathbf{c}^4$	$\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\gamma^2 + \sigma_\epsilon^2$
9	1, 3	2, 4, 5, 6	1	$r_2 = r_3 = r_4 = 0$ $\mathbf{c}_2 = \mathbf{c}_3 = \mathbf{c}_4 = \mathbf{0}$ $\mathbf{c}^1 = \mathbf{c}^3 = \mathbf{c}^4, \mathbf{c}^2 = \mathbf{0}$	$\sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\epsilon^2$
10		1, 2, 3, 4, 5, 6	0	$\mathbf{r} = \mathbf{0}, \mathbf{C} = \mathbf{0}$	

TABLE 4

Line	Set of Satisfied Conditions	Set of Unsatisfied Conditions	ρ_d = Rank of D-Matrix	Properties of Equations (3)	ρ_d Linearly-Independent Sums For Which Method-3 Estimates Exist
1	1, 2, 4, 5, 6		4		$\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2, \sigma_\epsilon^2$
2	1, 3, 6	4, 5	2	$t_2 = t_3 = 0$ $\mathbf{d}_2 = \mathbf{d}_3 = 0$ $\mathbf{d}^1 = \mathbf{d}^3, \mathbf{d}^2 = 0$	$\sigma_\alpha^2 + \sigma_\gamma^2, \sigma_\epsilon^2$
3	6	3, 4, 5	1	$t_1 = t_2 = t_3 = 0$ $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d}_3 = 0$ $\mathbf{d}^1 = \mathbf{d}^2 = \mathbf{d}^3 = 0$	σ_ϵ^2
4	1, 2, 4, 6	5	3	$t_3 = 0, \mathbf{d}_3 = 0$ $\mathbf{d}^1 + \mathbf{d}^2 = \mathbf{d}^3$	$\sigma_\alpha^2 + \sigma_\gamma^2,$ $\sigma_\beta^2 + \sigma_\gamma^2, \sigma_\epsilon^2$
5	1, 2, 4, 5	6	3	$t_4 = 0, \mathbf{d}_4 = 0$ $\mathbf{d}^3 = \mathbf{d}^4$	$\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2 + \sigma_\epsilon^2$
6	1, 3	4, 5, 6	1	$t_2 = t_3 = t_4 = 0$ $\mathbf{d}_2 = \mathbf{d}_3 = \mathbf{d}_4 = 0$ $\mathbf{d}^1 = \mathbf{d}^3 = \mathbf{d}^4$ $\mathbf{d}^2 = 0$	$\sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\epsilon^2$
7		3, 4, 5, 6	0	$\mathbf{t} = 0, \mathbf{D} = 0$	
8	1, 2, 4	5, 6	2	$t_3 = t_4 = 0$ $\mathbf{d}_3 = \mathbf{d}_4 = 0$ $\mathbf{d}^1 + \mathbf{d}^2 = \mathbf{d}^3 = \mathbf{d}^4$	$\sigma_\alpha^2 + \sigma_\gamma^2 + \sigma_\epsilon^2,$ $\sigma_\beta^2 + \sigma_\gamma^2 + \sigma_\epsilon^2$

THEOREM 1. *When certain of the conditions* 1–6 *are satisfied and others unsatisfied as given on line k of Table* 3; *then ϕ ($\equiv$ the maximum-possible number of elements in a set of linearly-independent estimable linear functions of the variance components = the rank of the matrix* $\mathbf{C}$), *certain properties of the equations* (2), *and ϕ linearly-independent estimable sums of the variance components for which Method*-1 *estimates exist are also as given on line k, $k = 1, \cdots, 10$.*

THEOREM 2. *When certain of the conditions* 1–6 *are satisfied and others unsatisfied as given on line k of Table* 4; *then ρ_d, certain properties of the equations* (3), *and ρ_d linearly-independent sums for which Method*-3 *estimates exist are also as given on line k, $k = 1, \cdots, 8$.*

Theorem 1 will be proved in Sections 2 and 3, while Theorem 2 will be proved in Section 4.

Note that specifying whether certain of the conditions 1–6 are satisfied or unsatisfied determines a subset of the set of all possible sequences $\{n_{ij}\}$ ($i = 1, \cdots, a; j = 1, \cdots, b$), and that the subsets so determined by the lines of Table 3 (or Table 4) are mutually-exclusive and exhaustive. Then note that all linear functions of the linearly-independent sums associated with a set of satisfied conditions listed on a given line of Table 3 (or Table 4) can also be expressed as linear functions of the sums associated with any other set of conditions containing the first as a subset.

It follows that the set of satisfied conditions listed on a line of Table 3 comprises a sufficient set of conditions for the estimability of all linear functions of the linearly-independent estimable sums listed on the same line. This same set of conditions is also sufficient for the existence of Method-1 estimates of all linear functions of these sums.

To determine (from Table 3) necessary conditions for the estimability of (or for the existence of a Method-1 estimate of) a linear function of the variance components, two successive steps are followed: (i) ascertain from Table 3 the collection of all sets of conditions that are sufficient for the estimability of the function and (ii) eliminate from that collection those sets having proper subsets in the collection. It is necessary for estimability (or for existence of a Method-1 estimate) that all the conditions in at least one of the sets remaining after step (ii) be satisfied.

In the same manner, Table 4 can be used to determine necessary and sufficient conditions for the existence of Method-3 estimates of linear functions of variance components.

Three interesting points stand out: (1) A Method-1 estimate of a linear function of the components exists for all sequences $\{n_{ij}\}$ $(i = 1, \cdots, a; j = 1, \cdots, b)$ for which the function is estimable, (2) Method-3 estimates of some linear functions of the components do not exist for certain sets of sequences for which the functions are estimable and for which Method-1 estimates exist (an example will be presented in Section 5), and (3) Method-1 estimates of linear functions of the components may exist even when the interaction "sum of squares" in Table 1, which is not really a sum of squares, has negative degrees of freedom.

In addition to the above results, the paper contains an algorithm that can be used to determine the rank of the matrix $\mathbf{W}$ and to ascertain which rows and columns of $\mathbf{W}$ to delete in order to obtain an $m \times m$ matrix of full rank. This algorithm is useful in establishing necessary and sufficient conditions for the existence of Method-3 estimates.

The results mentioned above are useful in applying Monte Carlo techniques to the problem of establishing certain properties of Methods 1 and 3 variance-component estimators when the subclass numbers are assumed to have distributional properties. Such an approach ordinarily involves the repeated generation of sets of subclass numbers and sample data and computation of variance-component estimates for each set. If these operations are being carried out on a computer as is generally the case, generalized inverse methods for obtaining solutions for the systems of equations (1), (2), or (3) and for determining linearly-estimable functions of the variance components may not be satisfactory because of computer rounding errors in computing the entries in the $\mathbf{C}$, $\mathbf{D}$, or $\mathbf{W}$ matrices or the $\mathbf{r}$, $\mathbf{t}$, or $\mathbf{q}$ vectors and/or in the actual solving of the equations. However, methods based on the use of the conditions and algorithm presented in this paper are not affected by these difficulties.

The algorithm is also useful (for reasons similar to those given above) when the assumed model is the two-way classification with no interaction and with fixed α_i and β_j, one objective then being to determine and estimate linearly-estimable functions of the β_j's.

2. Estimability of linear functions of the variance components. A linear function of the variance components is said to be estimable if there exist constants $\pi_{ijk,uxz}$; $i, u = 1, \cdots, a; j, x = 1, \cdots, b; k = 1, \cdots, n_{ij}; z = 1, \cdots, n_{ux}$, such that the expected value of

$$\sum_{ijk} \sum_{uxz} \pi_{ijk,uxz} y_{ijk} y_{uxz} \tag{4}$$

is equal to the function. The expected value of the quadratic form (4) is

$$e_\mu \mu^2 + e_\alpha \sigma_\alpha^2 + e_\beta \sigma_\beta^2 + e_\gamma \sigma_\gamma^2 + e_\epsilon \sigma_\epsilon^2,$$

where

$$\begin{aligned} e_\epsilon &= \sum_{ijk} \pi_{ijk,ijk}, \\ e_\gamma &= \sum_{ijk} \sum_z \pi_{ijk,ijz} = e_\epsilon + \sum_{ijk} \sum_{z \neq k} \pi_{ijk,ijz}, \\ e_\beta &= \sum_{ijk} \sum_{uz} \pi_{ijk,ujz} = e_\gamma + \sum_{ijk} \sum_{u \neq i} \sum_z \pi_{ijk,ujz}, \\ e_\alpha &= \sum_{ijk} \sum_{xz} \pi_{ijk,ixz} = e_\gamma + \sum_{ijk} \sum_{x \neq j} \sum_z \pi_{ijk,ixz}, \\ e_\mu &= \sum_{ijk} \sum_{uxz} \pi_{ijk,uxz} = e_\gamma + (e_\alpha - e_\gamma) + (e_\beta - e_\gamma) \\ &\quad + \sum_{ijk} \sum_{u \neq i} \sum_{x \neq j} \sum_z \pi_{ijk,uxz}. \end{aligned}$$

The proofs of the results concerning estimability specified by the ten lines of Table 3 and set forth in Theorem 1 are straight-forward and similar. The proof for the result specified by line 6 of the table is typical. If condition 6 is unsatisfied, $e_\gamma = e_\epsilon$ for any choice of the $\pi_{ijk,uxz}$. If, in addition, conditions 1, 2, and 4 are satisfied, a linear function of the components, say $\boldsymbol{\lambda}_i' \boldsymbol{\sigma}$, is estimable if and only if $\boldsymbol{\lambda}_i'$ is of the form

$$(e_\alpha, e_\beta, e_\epsilon, e_\epsilon) = e_\alpha(1, 0, 0, 0) + e_\beta(0, 1, 0, 0) + e_\epsilon(0, 0, 1, 1),$$

where e_α, e_β, and e_ϵ are arbitrarily chosen constants. Thus, when conditions 1, 2, 4, and 6 are as given on line 6 of Table 3, $\phi = 3$, one choice of ϕ linearly independent estimable sums is σ_α^2, σ_β^2, and $\sigma_\gamma^2 + \sigma_\epsilon^2$, and a linear function of the components is estimable if and only if it can be expressed as a linear function of these three sums.

3. Existence of Method-1 estimates. In determining those sets of sequences $\{n_{ij}\}(i = 1, \cdots, a; j = 1, \cdots, b)$ for which Method-1 estimates of linear functions of the variance components exist, the following three lemmas are used.

LEMMA 1. *$n_{..} - \theta_3 > 0$ if $a \geqq 2$ and $= 0$ if $a = 1$, and $n_{..} - \theta_4 > 0$ if $b \geqq 2$ and $= 0$ if $b = 1$.*

LEMMA 2. *$n_{..} - \theta_1 > 0$ if $p > a$ and $= 0$ if $p = a$, and $n_{..} - \theta_2 > 0$ if $p > b$ and $= 0$ if $p = b$.*

LEMMA 3. *$n_{..} - \theta_3 - \theta_4 + \theta_5 > 0$ if $a \geqq 2$ and $b \geqq 2$ and $= 0$ otherwise.*

Lemma 1 becomes obvious when $n_{..} - \theta_3$ and $n_{..} - \theta_4$ are rewritten as $(1/n_{..}) \sum_i \sum_{u \neq i} n_{i.} n_{u.}$ and $(1/n_{..}) \sum_j \sum_{x \neq j} n_{.j} n_{.x}$, respectively. Likewise, Lemma 2 becomes clear when $n_{..} - \theta_1$ and $n_{..} - \theta_2$ are rewritten as $\sum_i (1/n_{i.})(n_{i.}^2 - \sum_j n_{ij}^2)$ and $\sum_j (1/n_{.j})(n_{.j}^2 - \sum_i n_{ij}^2)$. The proof of Lemma 3 consists of noting that $n_{..} - \theta_3 - \theta_4 + \theta_5$ equals $(1/n_{..}) \sum_{ij} n_{ij} \sum_{u \neq i} \sum_{x \neq j} n_{ux}$.

The result (concerning the existence of Method-1 estimates) specified by line 1 of Table 3 and set forth in Theorem 1 is: If conditions 1, 2, 4, and 6 are satisfied, then **C** is of full rank and Method-1 estimates exist for σ_α^2, σ_β^2, σ_γ^2, and σ_ϵ^2. To prove this result it suffices to show that, when conditions 1, 2, 4, and 6 are satisfied, the determinant of **C** is nonzero; since, when det $\mathbf{C} \neq 0$, the solution to the system of equations (2) is uniquely given by

$$\hat{\boldsymbol{\sigma}}^* = \mathbf{C}^{-1}\mathbf{r}, \quad \text{and} \quad E\{\hat{\boldsymbol{\sigma}}^*\} = \mathbf{C}^{-1}E\{\mathbf{r}\} = \boldsymbol{\sigma}.$$

Det **C** can be put in the form

$$(n.. - \theta_1)(n.. - \theta_2)(n.. - \theta_3 - \theta_4 + \theta_5)(n.. - p).$$

It is clear from Lemmas 2 and 3 that, when conditions 1, 2, 4, and 6 are satisfied, det $\mathbf{C} > 0$.

The result specified by line 2 of Table 3 is: If conditions 1, 2, 3, and 6 are satisfied and conditions 4 and 5 are unsatisfied, then **C** has rank 3, $r_2 = -r_3$, $\mathbf{c}_2 = -\mathbf{c}_3$, $\mathbf{c}^1 = \mathbf{c}^3$, and Method-1 estimates exist for $\sigma_\alpha^2 + \sigma_\gamma^2$, σ_β^2, and σ_ϵ^2. That $r_2 = -r_3$, $\mathbf{c}_2 = -\mathbf{c}_3$, and $\mathbf{c}^1 = \mathbf{c}^3$ when the conditions are as specified is obvious from inspection of the elements of **r** and **C**. Taking $\mathbf{C}^*$ to be the 3×3 matrix formed by deleting the 3rd row and the 3rd column of **C**, we find by use of Lemmas 1 and 2 that, when the conditions are as specified,

$$\det \mathbf{C}^* = (n.. - \theta_2)(n.. - \theta_4)(n.. - p) > 0$$

and consequently **C** has rank 3. An appropriate solution $\hat{\boldsymbol{\sigma}}^*$ to the system of equations (2) is obtained by taking $\hat{\sigma}_3^2 = 0$ and

$$\begin{Vmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \\ \hat{\sigma}_4^2 \end{Vmatrix} = \mathbf{C}^{*-1} \begin{Vmatrix} r_1 \\ r_2 \\ r_4 \end{Vmatrix}.$$

Since

$$E \begin{Vmatrix} r_1 \\ r_2 \\ r_4 \end{Vmatrix} = \mathbf{C}^* \begin{Vmatrix} \sigma_\alpha^2 + \sigma_\gamma^2 \\ \sigma_\beta^2 \\ \sigma_\epsilon^2 \end{Vmatrix},$$

$E\{\hat{\boldsymbol{\sigma}}^{*\prime}\} = (\sigma_\alpha^2 + \sigma_\gamma^2, \sigma_\beta^2, 0, \sigma_\epsilon^2)$, and the result specified by line 2 of Table 3 is proved.

Proofs for the results (concerning the existence of Method-1 estimates) specified by lines 3–10 of Table 3 can be constructed by employing the same approach used in proving the result specified by line 2.

4. Existence of Method-3 estimates. We begin by defining an algorithm for partitioning the set G of all integers k, $k = 1, 2, \cdots, b$, into u^* mutually-exclusive subsets. These subsets, whose construction is based in a simple way on the sequence $\{n_{ij}\}(i = 1, \cdots, a; j = 1, \cdots, b)$, can be used to find the value of m and to determine dependencies among the set of equations (1). The algorithm is also used in the proof of Remark 3 which is, in turn, used in proving Theorem 2.

ALGORITHM. Taking G_0 to be the empty set,

$$G_u = \sum_{x=0}^{z_u-1} G_{ux}, \qquad u = 1, \cdots, u^*;$$

where G_{u0} is any element of $G - \sum_{j=0}^{u-1} G_j$; G_{ux} is the set consisting of those elements k of G not contained in $\sum_{j=0}^{u-1} G_j$ or $\sum_{h=0}^{x-1} G_{uh}$ for which $n_{ik}n_{if} > 0$ for some i, $i = 1, \cdots, a$, and for some $f \varepsilon G_{u,x-1}$; z_u is the first integer g such that $G_{ug} = G_0$; and $u^* =$ the first integer g such that $\sum_{j=1}^{g} G_j = G$.

REMARK 1. *$m = \sum_{j=1}^{u^*} (m_j - 1)$, where $m_j =$ the number of elements in the set G_j.*

PROOF. The normal equations for the two-way classification without interaction, with fixed mean μ, and with fixed main effects α_i, $i = 1, \cdots, a$, and β_j, $j = 1, \cdots, b$, are

$$\mathbf{S}\hat{\boldsymbol{\delta}} = \mathbf{y}, \tag{5}$$

where

$$\mathbf{S} = \begin{Vmatrix} n_{\cdot\cdot} & \mathbf{s}_{\mu\alpha}' & \mathbf{s}_{\mu\beta}' \\ \mathbf{s}_{\mu\alpha} & \mathbf{S}_{\alpha\alpha} & \mathbf{S}_{\alpha\beta} \\ \mathbf{s}_{\mu\beta} & \mathbf{S}_{\alpha\beta}' & \mathbf{S}_{\beta\beta} \end{Vmatrix}, \quad \hat{\boldsymbol{\delta}} = \begin{Vmatrix} \hat{\mu} \\ \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\beta}} \end{Vmatrix}, \quad \mathbf{y} = \begin{Vmatrix} n_{\cdot\cdot}\bar{y}_{\cdot\cdot\cdot} \\ \mathbf{y}_\alpha \\ \mathbf{y}_\beta \end{Vmatrix}.$$

$\mathbf{s}_{\mu\alpha}$ is an $a \times 1$ vector with entries $n_{i\cdot}$, $i = 1, \cdots, a$; $\mathbf{s}_{\mu\beta}$ is a $b \times 1$ vector with entries $n_{\cdot j}$, $j = 1, \cdots, b$; $\mathbf{S}_{\alpha\alpha}$ is an $a \times a$ matrix with diagonal elements $n_{i\cdot}$, $i = 1, \cdots, a$, and with off-diagonal elements all zero; $\mathbf{S}_{\beta\beta}$ is a $b \times b$ matrix with diagonal elements $n_{\cdot j}$, $j = 1, \cdots, b$, and with off-diagonal elements also all zero; $\mathbf{S}_{\alpha\beta}$ is an $a \times b$ matrix with ijth element n_{ij}, $i = 1, \cdots, a, j = 1, \cdots, b$; $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ are $a \times 1$ and $b \times 1$ vectors, respectively; $\mathbf{y}_\alpha$ is an $a \times 1$ vector with entries $n_{i\cdot}\bar{y}_{i\cdot\cdot}$, $i = 1, \cdots, a$; and $\mathbf{y}_\beta$ is a $b \times 1$ vector with entries $n_{\cdot j}\bar{y}_{\cdot j\cdot}$, $j = 1, \cdots, b$.

Let $\boldsymbol{\psi}' = (\mu, \boldsymbol{\alpha}', \boldsymbol{\beta}')$, $\mathbf{n}' = (\eta_\mu, \mathbf{n}_\alpha', \mathbf{n}_\beta')$, and $\mathbf{v}' = (\nu_\mu, \mathbf{v}_\alpha', \mathbf{v}_\beta')$; where the $a \times 1$ vector $\boldsymbol{\alpha}$ has elements α_i, $i = 1, \cdots, a$; the $b \times 1$ vector $\boldsymbol{\beta}$ has elements β_j, $j = 1, \cdots, b$; $\mathbf{n}$ is an $a + b + 1$ vector of known constants with $a \times 1$ and $b \times 1$ subvectors $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$, and $\mathbf{v}$ is an $a + b + 1$ vector with $a \times 1$ and $b \times 1$ subvectors $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$.

By Theorem 11.1 in Graybill [1], the linear combination $\mathbf{n}'\boldsymbol{\psi}$ (where the elements of $\boldsymbol{\psi}$ are regarded as fixed effects) is linearly estimable if and only if there exists a solution for $\mathbf{v}$ in the equations

$$\mathbf{S}\mathbf{v} = \mathbf{n}. \tag{6}$$

When $\mathbf{n}_\mu = 0$ and $\mathbf{n}_\alpha = \mathbf{0}$, the equations (6) are soluble if and only if there exists a solution to the reduced system of equations

$$\mathbf{W}\mathbf{v}_\beta = \mathbf{n}_\beta. \tag{7}$$

Let j be any element of G_{ux}, $x = 1, \cdots, z_u - 1$. Then there exists some element

$g \varepsilon G_{u,x-1}$ for which $n_{ij}n_{ig} > 0$ for some i, $i = 1, \cdots, a$. When the statistical model is the two-way classification with fixed main effects α_i, $i = 1, \cdots, a$, and β_j, $j = 1, \cdots, b$, we have that $E\{\bar{y}_{ij\cdot} - \bar{y}_{ig\cdot}\} = \beta_j - \beta_g$ and, consequently, that $\beta_j - \beta_g$ is linearly estimable. This implies that $\beta_j - \beta_f$ is linearly estimable where f is the single element of G_{u0}. Thus, $\beta_k - \beta_h$ is estimable where k and h are any two elements of G_u, and the number of linearly-independent vectors $\mathbf{n}_\beta$ for which a solution to (7) exists is at least $\sum_{j=1}^{u^*} (m_j - 1)$. Since these linearly-independent vectors can all be expressed as linear combinations of the columns of $\mathbf{W}$, the rank of $\mathbf{W}$ is at least $\sum_{j=1}^{u^*} (m_j - 1)$.

It is clear from the statement of the algorithm that, for any choice of i, j, and k, if $n_{ij} > 0$, $n_{ik} > 0$, and $j \varepsilon G_u$, then $k \varepsilon G_u$. Thus, taking $\mathbf{w}_i$, $i = 1, \cdots, b$, to be the ith row of W,

$$\sum_{j \varepsilon G_u} \mathbf{w}_j = \mathbf{0}, \qquad u = 1, \cdots, u^*,$$

and consequently the rank of $\mathbf{W}$ is at most $\sum_{j=1}^{u^*} (m_j - 1)$. Q.E.D.

REMARK 2. *An $m \times m$ matrix formed by deleting from* $\mathbf{W}$ *rows $j_1, \cdots, j_{u^*}$ and columns $j_1, \cdots, j_{u^*}$, where j_k, $k = 1, \cdots, u^*$, is any element of G_k, has full rank.*

PROOF. The proof follows from Remark 1 and the last paragraph of its proof. Q.E.D.

REMARK 3. *If $p = a + m$, then $d_{33} \equiv n_{\cdot\cdot} - \theta_1 - \theta_6 = t_3 = 0$; and, if $p > a + m$, then $d_{33} > 0$.*

PROOF. From least squares theory, we have that $R_\gamma \geqq R_{\alpha\beta}$. Clearly, $R_\gamma = R_{\alpha\beta}$ if and only if there exists a solution to the normal equations for the two-way classification with interaction when the estimates of the interaction effects are all taken equal to zero; that is, if and only if there exists a solution for $\hat{\mathbf{\delta}}$ (where $\hat{\mathbf{\delta}}$ is defined as before) in the equations

$$\mathbf{S}^*\hat{\mathbf{\delta}} = \mathbf{y}^* \tag{8}$$

in which; with $p_{ij} = 1$ if $n_{ij} > 0$ and $= 0$ otherwise, $p_0 = 0$, and $p_f = \sum_{g=1}^{b} p_{fg}$, $f = 1, \cdots, a$; the elements y_h^* of the $p \times 1$ vector $\mathbf{y}^*$ and the elements s_{hk}^* of the $p \times (a + b + 1)$ matrix $\mathbf{S}^*$ are given by

$$y_h^* = n_{ux}\bar{y}_{ux\cdot}, \qquad s_{h1}^* = s_{h,u+1}^* = s_{h,a+x+1}^* = n_{ux},$$

and $s_{hk}^* = 0$ for k not equal to 1, $u + 1$, or $a + x + 1$, where, for a given value of h, u is taken to be the smallest value of u and x is taken to be the smallest (for that u) value of x such that $h = \sum_{f=0}^{u-1} p_f + \sum_{g=1}^{x} p_{ug}$.

Those columns $\mathbf{s}_{a+k+1}^*$, $k = 1, \cdots, b$, of $\mathbf{S}^*$ such that $k \varepsilon G_u$ sum to the same vector as those columns $\mathbf{s}_{i+1}^*$, $i = 1, \cdots, a$, for which $n_{ik} > 0$ for some $k \varepsilon G_u$. Also, $\sum_{i=1}^{a} \mathbf{s}_{i+1}^* = \mathbf{s}_1^*$. Consequently, the rank of $\mathbf{S}^*$ is at most $a + m$.

Each of the rows of the matrix $\mathbf{S}$ introduced earlier is a linear combination of the rows of $\mathbf{S}^*$. Therefore, the rank of $\mathbf{S}^*$ is greater than or equal to the rank of $\mathbf{S}$. But $\mathbf{S}$ has rank $a + m$ since, in the system of equations (5), some $b - m + 1$

of the unknowns can be prescribed arbitrarily and the remainder are then uniquely determined.

Thus, the rank of $\mathbf{S}^*$ is $a + m$.

If $p = a + m$, the equations (8) are soluble for all vectors $\mathbf{y}^*$. However, if $p > a + m$ and $\sigma_\gamma^2 > 0$, a solution for $\hat{\hat{\boldsymbol{\delta}}}$ in the equations (8) does not exist for certain sets of $\mathbf{y}^*$ vectors having nonzero probability, and, for those vectors, $R_\gamma > R_{\alpha\beta}$. Thus, if $p > a + m$, $d_{33} > 0$, and, if $p = a + m$, $d_{33} = t_3 = 0$. Q.E.D.

The proof of the result specified by line 4 is typical of the proofs of the results specified by the lines of Table 4 and set forth in Theorem 2. This result is: If conditions 1, 2, 4, and 6 are satisfied and 5 is unsatisfied, then $\mathbf{D}$ has rank 3, $t_3 = 0$, $\mathbf{d}_3 = \mathbf{0}$, $\mathbf{d}^1 + \mathbf{d}^2 = \mathbf{d}^3$, and Method-3 estimates exist for $\sigma_\alpha^2 + \sigma_\gamma^2$, $\sigma_\beta^2 + \sigma_\gamma^2$, and σ_ϵ^2. That $t_3 = 0$, $\mathbf{d}_3$, $= \mathbf{0}$, and $\mathbf{d}^1 + \mathbf{d}^2 = \mathbf{d}^3$ when the conditions are as specified follows from Remark 3. Taking $\mathbf{D}^*$ to be the 3×3 triangular matrix formed by deleting the 3rd row and the 3rd column of $\mathbf{D}$, we find by use of Lemma 2 that when the conditions are as specified,

$$\det \mathbf{D}^* = (n_{..} - \theta_1)(n_{..} - \theta_2)(n_{..} - p) > 0$$

and consequently $\mathbf{D}$ has rank 3. An appropriate solution $\tilde{\boldsymbol{\delta}}^*$ to the system of equations (3) is obtained by taking $\tilde{\sigma}_3^2 = 0$ and

$$\begin{Vmatrix} \tilde{\sigma}_1^2 \\ \tilde{\sigma}_2^2 \\ \tilde{\sigma}_4^2 \end{Vmatrix} = \mathbf{D}^{*-1} \begin{Vmatrix} t_1 \\ t_2 \\ t_4 \end{Vmatrix}.$$

Since

$$E \begin{Vmatrix} t_1 \\ t_2 \\ t_4 \end{Vmatrix} = \mathbf{D}^* \begin{Vmatrix} \sigma_\alpha^2 + \sigma_\gamma^2 \\ \sigma_\beta^2 + \sigma_\gamma^2 \\ \sigma_\epsilon^2 \end{Vmatrix},$$

$E\{\tilde{\boldsymbol{\delta}}^{*\prime}\} = (\sigma_\alpha^2 + \sigma_\gamma^2, \sigma_\beta^2 + \sigma_\gamma^2, 0, \sigma_\epsilon^2)$, and the result specified by line 4 of Table 4 is proved.

Proofs for the results specified by the other lines of Table 4 can also be constructed by using the above approach.

5. An example. As an example of a sequence $\{n_{ij}\}(i = 1, \cdots, a; j = 1, \cdots, b)$ for which Method-1 estimates of certain linear functions of the variance components exist but Method-3 estimates do not, take $a = 5$ and $b = 3$ and let the 5×3 matrix of subclass numbers be given by

$$\begin{vmatrix} 5 & 0 & 0 \\ 7 & 3 & 0 \\ 9 & 0 & 7 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{vmatrix}.$$

We have $m = 2$ and $p = a + m = 7$. By reference to Tables 3 and 4, we find that Method-1 estimates exist for σ_α^2, σ_β^2, and σ_γ^2, but Method-3 estimates do not exist for these three components.

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