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CONVERGENCE RATES FOR THE LAW OF THE ITERATED LOGARITHM

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Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables and S_n denote the partial sums $\sum_{k=1}^{n} X_k$. For such sequences of random variables, with mean zero and variance one, the Law of the Iterated Logarithm states that $P[\limsup S_n(2n \lg\lg n)^{-\frac{1}{2}} = 1] = 1$. In this paper probabilities where this law is applicable are considered and appropriate convergence rates are determined.

According to Levy's terminology, for a given sequence of independent random variables $\{X_n\}$ a monotonic sequence $\{\varphi_n\}$ is said to be in the lower class $\mathfrak{L}$ if $P[S_n > n^{\frac{1}{2}}\varphi_n$ infinitely often$] = 1$. Otherwise the above probability is zero and the sequence is said to be in the upper class $\mathfrak{U}$. In 1946 Feller [6] characterized the upper and lower classes for independent identically distributed random variables with $EX = 0$, $EX^2 = 1$, and $EX^2 \lg\lg |X| < \infty$. Namely, $\{\varphi_n\}$ is in the upper (lower) class if the series $\sum \varphi_n e^{-\varphi_n^2/2} n^{-1}$ converges (diverges).

In this paper, the initial results are directed toward obtaining a convergence rate for $P[\sup_{k \geqq n} |S_k[(2 + \epsilon)k \lg\lg k]^{-\frac{1}{2}}| > 1]$ for independent identically distributed random variables satisfying the above moment conditions. In Theorem 3 it is shown that under a somewhat stronger moment condition Feller's criterion $(\sum \varphi_n e^{-\varphi_n^2/2} n^{-1} < \infty)$ is equivalent to the convergence of $\sum \varphi_n^2 n^{-1} P[|S_n| > n^{\frac{1}{2}}\varphi_n]$. Finally, random variables with $EX = 0$ and $EX^2 = 1$ are considered and it is shown that a weaker criterion than Feller's is sufficient to guarantee a convergence rate for $P[|S_n| > n^{\frac{1}{2}}\varphi_n]$. Thus there are monotonic sequences $\{\varphi_n\}$ such that $P[S_n > n^{\frac{1}{2}}\varphi_n$ infinitely often$] = 1$ and yet the series $\sum n^{-1} P[|S_n| > n^{\frac{1}{2}}\varphi_n]$ converges. To obtain some of the preceding, extensive use has been made of results and techniques developed in [8] where Friedman, Katz, and Koopmans applied the convergence rate concept to the Central Limit Theorem.

In this paper $\{X_n\}$ denotes sequences of independent identically distributed random variables with common distribution function F; $\Phi(x)$ represents the standard normal distribution function, and $[x]$ will stand for the largest integer less than or equal to x. Also

$$\begin{aligned} \lg x &= \log_e x, & x > 1 \\ &= 0 & \text{otherwise.} \end{aligned}$$

Results. Initially, it is easily observed that in [8] the following somewhat stronger result is actually proven.

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THEOREM (Friedman, Katz, Koopmans). *If* $EX_1 = 0$, $EX_1^2 = 1$ *the series*

$$\sum_{n=1}^{\infty} n^{-1} \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x\sigma_n^{-1})|$$

converges, where

$$\sigma_n^2 = \int_{|x|<n^{\frac{1}{2}}} x^2\, dF - (\int_{|x|<n^{\frac{1}{2}}} x\, dF)^2.$$

More generally, if in addition it is assumed that $E|X_1|^{2+\delta} < \infty$, *for* $0 < \delta < 1$, *then the series*

$$\sum_{n=1}^{\infty} n^{(\delta/2)-1} \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x\sigma_n^{-1})|$$

converges.

Lemma 1 allows the second theorem of [8] to be stated in a form more suited to application in the following.

LEMMA 1. *Let* X *be a random variable with* $EX = 0$ *and* $EX^2 = 1$, *and let*

$$\sigma_n^2 = \int_{|x|<n^{\frac{1}{2}}} x^2\, dF - (\int_{|x|<n^{\frac{1}{2}}} x\, dF)^2.$$

Then

$$\sum_{n=1}^{\infty} (1 - \sigma_n^2)n^{-1} < \infty \Leftrightarrow EX^2 \lg |X| < \infty.$$

PROOF. Notice

$$1 - \sigma_n^2 = \int_{|x| \geqq n^{\frac{1}{2}}} x^2\, dF + (\int_{|x| \geqq n^{\frac{1}{2}}} x\, dF)^2$$

so that

$$\int_{|x| \geqq n^{\frac{1}{2}}} x^2\, dF \leqq 1 - \sigma_n^2 \leqq 2 \int_{|x| \geqq n^{\frac{1}{2}}} x^2\, dF.$$

Thus it suffices to show

$$\sum n^{-1} \int_{|x| \geqq n^{\frac{1}{2}}} x^2\, dF < \infty \Leftrightarrow EX^2 \lg |X| < \infty$$

which is easily done by standard techniques.

THEOREM (Friedman, Katz, Koopmans). *If* $EX_1 = 0$ *and* $EX_1^2 = 1$ *then* $EX_1^2 \lg |X_1| < \infty$ *is equivalent to the convergence of*

$$\sum_{n=1}^{\infty} n^{-1} \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x)|.$$

In [9], which has appeared since the initial preparation of this paper, Heyde proves the above theorem. It is interesting to note the existence of these two radically different methods of proof for this result. The following results, Theorem 1 and Proposition 1, are directed toward the proof of Theorem 2, but each is of some intrinsic interest. The first of these is motivated by and proven essentially as the corresponding result of [8].

THEOREM 1. $EX_1 = 0$, $EX_1^2 = 1$, *and* $EX_1^2 \lg \lg |X_1| < \infty$ *imply that*

$$\sum_{n=3}^{\infty} n^{-1} \lg \lg n \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x\sigma_n^{-1})| < \infty,$$

where

$$\sigma_n^2 = \int_{|x|<n^{\frac{1}{2}}} x^2\, dF - (\int_{|x|<n^{\frac{1}{2}}} x\, dF)^2.$$

PROPOSITION 1. *Let* $EX_1 = 0$ *and* $EX_1^2 = 1$. *If*

$$\sum_{n=3}^\infty n^{-1} \lg\lg n P\{|S_n| > [(2+\epsilon)n \lg\lg n]^{\frac{1}{2}}\}$$

converges for all $\epsilon > 0$ *then the same is true for*

$$\sum_{n=3}^\infty (n \lg n)^{-1} P[\sup_{k \geqq n} |S_k[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}} > 1].$$

PROOF. Fix $\epsilon > 0$ and choose $\alpha > 1$ such that $(2+\epsilon)/\alpha^6 = 2 + \delta > 2$:

$$\sum_{n=3}^\infty (n \lg n)^{-1} P[\sup_{k \geqq n} |S_k[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}| > 1]$$

$$\leqq c_1 \sum_{i=3}^\infty (\lg[\alpha^i])^{-1} P[\sup_{k \geqq \alpha^i} |S_k[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}| > 1]$$

$$\leqq c_2 \sum_{i=3}^\infty i^{-1} \sum_{j=i}^\infty P[\max_{\alpha^j \leqq k < \alpha^{j+1}} |S_k[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}| > 1]$$

$$\leqq c_2 \sum_{i=3}^\infty i^{-1} \sum_{j=i}^\infty P[\max_{\alpha^j \leqq k < \alpha^{j+1}} |[S_k - \mu(S_k - S_{[\alpha^{j+1}]})][(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}|$$

$$> 1 - \max_{\alpha^j \leqq k < \alpha^{j+1}} |\mu(S_k - S_{[\alpha^{j+1}]})[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}|]$$

$$(1) \quad \leqq c_3 \sum_{i=3}^\infty i^{-1} \sum_{j=i}^\infty$$

$$\cdot P\{\max_{\alpha^j \leqq k < \alpha^{j+1}} |[S_k - \mu(S_k - S_{[\alpha^{j+1}]})][(2+\epsilon)\alpha^{-1} k \lg\lg k]^{-\frac{1}{2}}| > 1\}$$

$$\leqq c_3 \sum_{i=3}^\infty i^{-1} \sum_{j=i}^\infty P\{\max_{\alpha^j \leqq k < \alpha^{j+1}} |S_k - \mu(S_k - S_{[\alpha^{j+1}]})|$$

$$> [(2+\epsilon)\alpha^{-1}\alpha^j \lg\lg \alpha^j]^{\frac{1}{2}}\}$$

$$(2) \quad \leqq 2c_3 \sum_{i=3}^\infty i^{-1} \sum_{j=i}^\infty P\{|S_{[\alpha^{j+1}]}| > [(2+\epsilon)\alpha^{-1}\alpha^j \lg\lg \alpha^j]\}$$

$$(3) \quad \leqq c_4 \sum_{i=3}^\infty i^{-1} \sum_{j=i}^\infty P\{|S_{[\alpha^{j+1}]}| > [(2+\epsilon)\alpha^{-3}\alpha^{j+1} \lg\lg \alpha^{j+1}]^{\frac{1}{2}}\}$$

$$= c_4 \sum_{j=3}^\infty P\{|S_{[\alpha^{j+1}]}| > [(2+\epsilon)\alpha^{-3}\alpha^{j+1} \lg\lg \alpha^{j+1}]^{\frac{1}{2}}\} \sum_{i=3}^j i^{-1}$$

$$\leqq c_5 \sum_{j=3}^\infty \lg\lg \alpha^j P\{|S_{[\alpha^j]}| > [(2+\epsilon)\alpha^{-3}\alpha^j \lg\lg \alpha^j]^{\frac{1}{2}}\}$$

$$(4) \quad \leqq c_6 \sum_{j=3}^\infty \alpha^{-j} \lg\lg \alpha^{-j} \sum_{n=[\alpha^j]}^{[\alpha^{j+1}]-1} P\{|S_n| > [(2+\epsilon)\alpha^{-6} n \lg\lg n]^{\frac{1}{2}}\}$$

$$\leqq c_7 \sum_{n=3}^\infty n^{-1} \lg\lg n \, P\{|S_n| > [(2+\delta)n \lg\lg n]^{\frac{1}{2}}\}$$

$$< \infty.$$

(1) follows from

$$\begin{aligned} |\mu(S_k - S_{[\alpha^{j+1}]})| &= |\mu(S_k - S_{[\alpha^{j+1}]}) - E(S_k - S_{[\alpha^{j+1}]})| \\ &\leqq \{2\sigma^2(S_k - S_{[\alpha^{j+1}]})\}^{\frac{1}{2}} \\ &\leqq \{2\sigma^2(S_{[\alpha^j]} - S_{[\alpha^{j+1}]})\}^{\frac{1}{2}} \\ &= [2\sigma^2(X_1)\alpha^j(\alpha - 1)]^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\max |\mu(S_k - S_{[\alpha^{j+1}]})| [(2+\epsilon)\alpha^j \lg\lg \alpha^j]^{-\frac{1}{2}}$$

$$\leqq [2\sigma^2(X_1)(\alpha - 1)/(2+\epsilon) \lg\lg \alpha^j]^{\frac{1}{2}} \to 0$$

and (1) holds for $\alpha > 1$.

(2) obtains by Levy's inequalities.
(3) follows from

$$\alpha^{j+1} \lg\lg \alpha^{j+1}(\alpha^j \lg\lg \alpha^j)^{-1} < \alpha^2 \quad \text{for all} \quad j \geqq j_0 .$$

(4) can be shown using Levy's inequalities. Let $\alpha^j \leqq n \leqq \alpha^{j+1}$. Then

$$\begin{aligned} P\{|S_{[\alpha^j]}| > [(2+\epsilon)\alpha^{-3}\alpha^j \lg\lg \alpha^j]^{\frac{1}{2}}\} \\ &\leqq P[\max_{\alpha^j \leqq k \leqq n} |S_k - \mu(S_k - S_n)| > [(2+\epsilon)\alpha^{-3}\alpha^j \lg\lg \alpha^j]^{\frac{1}{2}} \\ &\quad - \max_{\alpha^j \leqq k \leqq n} |\mu(S_k - S_n)|] \\ &\leqq P\{\max_{\alpha^j \leqq k \leqq n} |S_k - \mu(S_k - S_n)| > [(2+\epsilon)\alpha^{-4}\alpha^j \lg\lg \alpha^j]^{\frac{1}{2}}\} \\ &\leqq 2P\{|S_n| > [(2+\epsilon)\alpha^{-4}\alpha^j \lg\lg \alpha^j]^{\frac{1}{2}}\} \\ &\leqq 2P\{|S_n| > [(2+\epsilon)\alpha^{-6}\alpha^{j+1} \lg\lg \alpha^{j+1}]^{\frac{1}{2}}\} \\ &\leqq 2P\{|S_n| > [(2+\epsilon)\alpha^{-6} n \lg\lg n]^{\frac{1}{2}}\}. \end{aligned}$$

With this result the proof of Theorem 2 is straightforward. It should be noted that this conclusion has been demonstrated earlier in [1] under somewhat stronger hypotheses. Various considerations seem to indicate the assumptions made here cannot be improved significantly. In comparing this theorem with others for convergence rates, it seems the proof is extremely laborious. This is related to the dependence of the results on the actual magnitude of the variance. For example, it is for this reason that the ordinary application of symmetrization fails.

THEOREM 2. $EX_1 = 0$, $EX_1^2 = 1$, *and* $EX_1^2 \lg\lg |X_1| < \infty$ *imply that*

$$\sum_{n=3}^{\infty} (n \lg n)^{-1} P[\sup_{k \geqq n} |S_k[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}| > 1] < \infty .$$

PROOF. The random variables satisfy the hypotheses of Theorem 1, thus

$$\sum_{n=3}^{\infty} n^{-1} \lg\lg n \sup_{x \in \mathfrak{R}} |P[S_n n^{-\frac{1}{2}} x] - \Phi(x\sigma_n^{-1})| < \infty .$$

Specifically, taking $[(2+\epsilon)\lg\lg n]^{\frac{1}{2}}$ for x, one obtains the convergence of

$$\sum_{n=3}^{\infty} n^{-1} \lg\lg n |P\{|S_n| > [(2+\epsilon)n \lg\lg n]^{\frac{1}{2}}\} - 2\Phi(-[(2+\epsilon)\lg\lg n]^{\frac{1}{2}}\sigma_n^{-1})|.$$

But one obtains by the approximation of Φ in the tail

$$\begin{aligned} \sum_{n=3}^{\infty} n^{-1} \lg\lg n \Phi(-[(2+\epsilon)\lg\lg n]^{\frac{1}{2}}\sigma_n^{-1}) \\ &\leqq c \sum_{n=3}^{\infty} \sigma_n \lg\lg n e^{(-(2+\epsilon)\lg\lg n (2\sigma_n^2)^{-1})} n^{-1} [(2+\epsilon)\lg\lg n]^{-\frac{1}{2}} \end{aligned}$$

which converges since $\sigma_n^2 \to 1$.

Thus, since this series converges absolutely, it must be that

$$\sum_{n=3}^{\infty} n^{-1} \lg\lg n P\{|S_n| > [(2+\epsilon)n \lg\lg n]^{\frac{1}{2}}\} < \infty$$

and the hypothesis of Proposition 1 is satisfied. Hence

$$\sum_{n=3}^{\infty} (n \lg n)^{-1} P[\sup_{k \geqq n} |S_k[(2+\epsilon)k \lg\lg k]^{-\frac{1}{2}}| > 1] < \infty$$

for all $\epsilon > 0$.

In Theorem 3 an attempt has been made to determine convergence rates for arbitrary functions from the upper class according to Feller's criterion. It was found, however, that a more restrictive hypothesis was needed on the random varibles. Again, a similar conclusion was drawn in [1] under stronger moment conditions.

THEOREM 3. *Let* $EX_1 = 0$, $EX_1^2 = 1$, $EX_1^2 \lg |X_1| \lg \lg |X_1| < \infty$ *and* φ *be a nonnegative increasing function on* $[1, \infty)$. *Then the series*

$$\sum_{n=1}^{\infty} \varphi^2(n) n^{-1} P[|S_n| > n^{\frac{1}{2}}\varphi(n)]$$

converges, if and only if,

$$\int_1^{\infty} \varphi(t) e^{-\varphi^2(t)/2}\, dt t^{-1} < \infty.$$

PROOF. From [4], p. 70, one obtains

$$\begin{aligned}\varphi^2(n)n^{-1}|P[|S_n| > n^{\frac{1}{2}}\varphi(n)] - 2(1 - \Phi(\varphi(n)))| \\ \leqq n^{-1} \sup_{x \geqq 0} (1 + x^2)|P[|S_n n^{-\frac{1}{2}}| > x] - 2(1 - \Phi(x))| \\ \leqq 2n^{-1}(2/\pi)^{\frac{1}{2}}(a^2 + 1)a^{-1}e^{-a^2/2} + 5a^2\Delta(n))\end{aligned}$$

where $\pm a$ are continuity points of $P[S_n/n^{\frac{1}{2}} < x]$ and $\Delta(n) = \sup_x |P[S_n/n^{\frac{1}{2}} < x] - \Phi(x)|$. Now ϵ can be chosen arbitrarily small and positive such that $\pm a_n = \pm[(2 + \epsilon) \lg \lg n]^{\frac{1}{2}}$ are continuity points for each n, and one obtains

$$\begin{aligned}\varphi^2(n)n^{-1}|P[|S_n| > n^{\frac{1}{2}}\varphi(n)] - 2(1 - \Phi(\varphi(n)))| \\ \leqq c[\{(2 + \epsilon) \lg \lg n + 1\}\{n[(2 + \epsilon) \lg \lg n]^{\frac{1}{2}}(\lg n)^{1+(\epsilon/2)}\}^{-1} \\ + (2 + \epsilon) \lg \lg n n^{-1}\Delta(n)].\end{aligned}$$

The series formed from the first terms above converges, and now one must consider

$$\begin{aligned}\sum_{n=3}^{\infty} n^{-1} \lg \lg n \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x)| \\ \leqq \sum_{n=3}^{\infty} n^{-1} \lg \lg n \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x\sigma_n^{-1})| \\ + \sum_{n=3}^{\infty} n^{-1} \lg \lg n \sup_x |\Phi(x\sigma_n^{-1}) - \Phi(x)|.\end{aligned}$$

Here

$$\sigma_n^2 = \int_{|x| \leqq n^{\frac{1}{2}}} x^2\, dF - \left(\int_{|x| \leqq n^{\frac{1}{2}}} x\, dF\right)^2.$$

Now the hypotheses of Theorem 1 are satisfied, and hence the first series converges. As in [8], the convergence of the second is equivalent to

$$\sum_{n=3}^{\infty} n^{-1} \lg \lg n (1 - \sigma_n^2) < \infty.$$

This, in turn, can be shown to be equivalent to the existence of $EX_1^2 \lg |X_1| \cdot \lg \lg |X_1|$. Thus

$$\sum_{n=1}^{\infty} \varphi^2(n)n^{-1}|P[|S_n| > n^{\frac{1}{2}}\varphi(n)] - 2(1 - \Phi(\varphi(n))| < \infty.$$

Now consider

$$\sum_{n=1}^{\infty} \varphi^2(n)n^{-1}(1 - \Phi(\varphi(n))).$$

By [5], p. 166, this converges or diverges with

$$\int_1^\infty \varphi(t)t^{-1}e^{-\varphi^2(t)/2}\,dt$$

and the theorem is proven.

The final result considers a weakening of Feller's criterion for sequences of the upper class. A convergence rate result is obtained for random variables with only two moments satisfying this criterion, and thus applies for certain sequences from the lower class. It is then possible to exhibit sequences $\{\varphi_n\}$ such that $P[|S_n| > \varphi_n n^{\frac{1}{2}}$ infinitely often$] = 1$ and yet the prescribed convergence rate holds.

THEOREM 4. *Let $\varphi(t)$ be a positive increasing function on $[1, \infty)$. Then (1) and (2) are equivalent:*

$$\int_1^\infty e^{-\varphi^2(t)/2}[t\varphi(t)]^{-1}\,dt < \infty, \tag{1}$$

$$\sum_{n=1}^\infty n^{-1}P[|S_n| > n^{\frac{1}{2}}\varphi(n)] < \infty, \tag{2}$$

for all sequences $\{X_n\}$ with $EX_1 = 0$ and $EX_1^2 = 1$.

PROOF. Clearly (1) and (2) diverge if $\varphi(n)$ is bounded. Thus without loss of generality assume $\varphi(n) \uparrow \infty$.

(1) $\Rightarrow$ (2). Let σ_n^2 denote the variance of the random variables truncated at $n^{\frac{1}{2}}$. Then

$$\sum_{n=1}^\infty n^{-1}P[|S_n| > n^{\frac{1}{2}}\varphi(n)] \leqq 2\{\sum_{n=1}^\infty n^{-1}(\sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x\sigma_n^{-1})|) + \sum_{n=1}^\infty n^{-1}\Phi(-\varphi(n)\sigma_n)^{-1}\}.$$

Now by Theorem 1 of [8], the first series above converges. Considering the latter the approximation for the tail can again be used:

$$\sum_{n=1}^\infty n^{-1}\Phi(-\varphi(n)\sigma_n^{-1}) \leqq c\sum_{n=1}^\infty \sigma_n e^{-\varphi^2(n)/(2\sigma n^2)}[n\varphi(n)]^{-1} \leqq c_1\int_1^\infty e^{-\varphi^2(t)/2}[t\varphi(t)]^{-1}\,dt.$$

Thus, if the integral is finite, the convergence of

$$\sum_{n=1}^\infty n^{-1}P[|S_n| > n^{\frac{1}{2}}\varphi_n]$$

is demonstrated.

For (2) $\Rightarrow$ (1) let each X_n have the standard normal distribution. Then

$$P[|S_n| > n^{\frac{1}{2}}\varphi(n)] = 2(1 - \Phi(\varphi(n)) \approx 2(2\pi)^{-\frac{1}{2}}e^{-\varphi^2(n)/2}[\varphi(n)]^{-1}$$

and

$$\infty > \sum_{n=1}^\infty n^{-1}P[|S_n| > n^{\frac{1}{2}}\varphi(n)] \geqq c\int_1^\infty e^{-\varphi^2(t)/2}[t\varphi(t)]^{-1}\,dt.$$

Examples of sequences from the lower class $\mathfrak{L}$ which satisfy the hypothesis $\int_1^\infty e^{-\varphi^2(t)/2}[t\varphi(t)]^{-1}\,dt < \infty$, thus the prescribed convergence rate holds, are provided by

$$\varphi^2(n) = 2 \lg\lg n + \delta \lg\lg\lg n$$

for $1 < \delta \leqq 3$.

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