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COMBINING ELEMENTS FROM DISTINCT FINITE FIELDS IN MIXED FACTORIALS

By B. L. Raktoe

University of Guelph, Ontario

1. Summary. This paper extends and unifies the use of Galois or finite fields in mixed factorials utilizing concepts of ring and ideal theory. A method of combining elements from k distinct prime fields is presented and its use indicated in formulating properties of treatment combinations and effects in a mixed factorial.

2. Introduction. White and Hultquist (1965) have given a technique to combine elements from distinct finite fields. Their method consists essentially in defining addition and multiplication of elements from two distinct finite fields by mapping these into a finite commutative ring containing subrings isomorphic to each of the fields in question. It was indicated that their method could be generalized but no general approach was presented in their paper. The aim of this paper is:

1. To provide a new and equivalent theoretical basis for the results obtained by White and Hultquist (1965).
2. To generalize the technique of combining elements from two fields to k finite fields.
3. To illustrate the use of the technique under point 2 in deriving properties of the general mixed factorial.

3. Combining elements from distinct prime fields. Let $p_1, p_2, \cdots, p_k$ be k distinct primes and $GF(p_1), GF(p_2), \cdots, GF(p_k)$ the corresponding Galois fields, i.e. each $GF(p_j)$ consists of the residue classes of integers modulo the prime p_j. Also let $p = \prod_{j=1}^{k} p_j$ and $R(p)$ be the ring of residue classes of integers modulo p and $I(w)$ be an ideal generated by an arbitrary element w of $R(p)$. We are now ready to prove the following:

Lemma 3.1. *The elements of the form* $a_j = \prod_{i\neq j}^{k} p_i - p_j = c_j - p_j$, (*where* $c_j = \prod_{i\neq j}^{k} p_i$), *in the ring* $R(p)$ *are prime to the number* p *and hence* a_j^{-1} *exists in* $R(p)$ *for* $j = 1, 2, \cdots, k$. *In other words the* a_j*'s belong to the multiplicative group of non-zero divisors in* $R(p)$.

Proof. By inspection a_j and p have no common factor, hence a_j is prime to p, which implies the existence of a_j^{-1} in $R(p)$ for $j = 1, 2, \cdots, k$.

Lemma 3.2. *The elements of the form* $b_j = c_j \cdot a_j^{-1} = 1 + p_j \cdot a_j^{-1}$, $j = 1, 2, \cdots, k$, *in* $R(p)$ *are idempotent.*

Proof. We must show that $b_j^2 = b_j + r \cdot p$, i.e., $b_j^2 - b_j = r \cdot p$, where r is an element of $R(p)$. Now, $b_j^2 - b_j = b_j \cdot (b_j - 1) = b_j \cdot (p_j \cdot a_j^{-1}) = (1 + p_j \cdot a_j^{-1}) \cdot p_j \cdot a_j^{-1} = (a_j + p_j) \cdot a_j^{-1} \cdot p_j \cdot a_j^{-1} = c_j \cdot p_j \cdot a_j^{-2} = a_j^{-2} \cdot p = r \cdot p$, where $r = a_j^{-2}$ exists in $R(p)$ by Lemma 3.1.

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Lemma 3.3. *The product* $b_j \cdot b_{j^*} = 0$ *in the ring* $R(p)$ *if* $j \neq j^*$, j *and* j^* *taking on the values* $1, 2, \cdots, k$.

Proof. Without loss of generality let $p_j = p_1$ and $p_{j^*} = p_2$, since they can be brought in those positions, then: $b_1 \cdot b_2 = c_1 \cdot a_1^{-1} \cdot c_2 \cdot a_2^{-1} = c_1 \cdot a_1^{-1} \cdot (p_1 \cdot \prod_{i=3}^{k} p_i) \cdot a_2^{-1} = p_1 \cdot c_1 \cdot r = r \cdot p = 0$, where $r = a_1^{-1} \cdot a_2^{-1} \cdot \prod_{i=3}^{k} p_i$.

Lemma 3.4. *The element* b_j *of Lemma* 3.2 *generates the ideal* $I(b_j)$ *in the ring* $R(p)$, *which annihilates the ideal* $I(b_{j^*})$ *if* $j \neq j^*$, j *and* j^* *taking on the values* $1, 2, \cdots, k$.

Proof. An element of the ideal $I(b_j)$ in $R(p)$ is of the form $r \cdot b_j$, where $r \in R(p)$; similarly any element of $I(b_{j^*})$ is of the form $r^* \cdot b_{j^*}$, $r^* \in R(p)$. Hence the product $(r \cdot b_j) \cdot (r^* \cdot b_{j^*}) = r \cdot r^* \cdot b_j \cdot b_{j^*} = 0$, by Lemma 3.3.

Lemma 3.5. *The ideal generated by the element* p_j *in* $R(p)$ *annihilates the ideal* $I(b_j)$, $j = 1, 2, \cdots, k$.

Proof. The proof follows directly from the fact that $p_j \cdot b_j = p_j \cdot c_j \cdot a_j^{-1} = p \cdot a_j^{-1} = 0$ in $R(p)$.

Lemma 3.6. *The multiplicative identity element of the ideal* $I(b_j)$ *of Lemma* 3.4 is b_j, $j = 1, 2, \cdots, k$.

Proof. Let $d = r \cdot b_j$ be an arbitrary element of $I(b_j)$, where $r \in R(p)$, then $b_j \cdot d = b_j \cdot r \cdot b_j = r \cdot b_j^2 = r \cdot b_j = d$, by Lemma 3.2.

Lemma 3.7. *The multiplicative identity element* 1 *of the ring* $R(p)$ *is the sum of the multiplicative identities of the* $I(b_j)$*'s,* i.e. $1 = \sum_{j=1}^{k} b_j$.

Proof. First of all we have

$$\prod_{j=1}^{k} a_j = (-1)^{k+1} \cdot \sum_{j=1}^{k} c_j^2 \qquad \text{and}$$

$$\sum_{j=1}^{k} [p_j \cdot (\prod_{i \neq j}^{k} a_i)] = (-1)^{k} \cdot (k-1) \cdot \sum_{j=1}^{k} c_j^2.$$

Hence

$$\begin{aligned} \sum_{j=1}^{k} b_j &= \sum_{j=1}^{k} (1 + a_j^{-1} \cdot p_j) = k + \sum_{j=1}^{k} a_j^{-1} \cdot p_j \\ &= k + [\sum_{j=1}^{k} p_j \cdot (\prod_{i \neq j}^{k} a_i)] \cdot (\prod_{j=1}^{k} a_j^{-1}) \\ &= k + [(-1)^{k} \cdot (k-1) \cdot \sum_{j=1}^{k} c_j^2] \cdot [(-1)^{k+1} \cdot \sum_{j=1}^{k} c_j^2]^{-1} \\ &= k + (k-1) \cdot (-1) = k - k + 1 = 1. \end{aligned}$$

Q.E.D.

Theorem 3.1. *The ring* $R(p)$ *is the direct sum of the ideals* $I(b_j)$*'s,* i.e. $R(p) = \sum_{j=1}^{k} \oplus I(b_j)$.

Proof. By Lemmas 3.2, 3.3 and 3.7 we have respectively $b_j^2 = b_j$, $b_j \cdot b_{j^*} = 0$, $j \neq j^*$, and $1 = \sum_{j=1}^{k} b_j$. Hence it follows immediately from van der Waerden [(1950), page 147], that the theorem is true.

Lemma 3.8. *The field* $GF(p_j)$ *is isomorphic to the ideal* $I(b_j)$, $j = 1, 2, \cdots, k$.

Proof. For $x \in GF(p_j)$ and $y \in I(b_j)$ define the mapping $\sigma : GF(p_j) \to I(b_j)$ by $\sigma(x) = b_j \cdot x$. Obviously σ is 1:1 and onto and for x_1 and $x_2 \in GF(p_j)$ and y_1 and

$y_2 \,\varepsilon\, I(b_j)$ we have

$$\begin{aligned}\sigma(x_1 + x_2) &= b_j\cdot(x_1 + x_2) = b_j\cdot x_1 + b_j\cdot x_2 = \sigma(x_1) + \sigma(x_2)\\ &= y_1 + y_2\,;\\ \sigma(x_1\cdot x_2) &= b_j\cdot(x_1\cdot x_2) = b_j^2\cdot(x_1\cdot x_2) = (b_j\cdot x_1)\cdot(b_j\cdot x_2)\\ &= \sigma(x_1)\cdot\sigma(x_2) = y_1\cdot y_2\,.\end{aligned}$$

Hence σ is an isomorphism and $GF(p_j)$ is isomorphic to $I(b_j)$.

DEFINITION 3.1. Define addition and multiplication of elements from distinct Galois fields $x \,\varepsilon\, GF(p_j)$ and $x^* \,\varepsilon\, GF(p_j)$ by the rules

$$\begin{aligned}x + x^* &= \sigma(x) + \sigma(x^*),\\ x\cdot x^* &= \sigma(x)\cdot\sigma(x^*).\end{aligned}$$

DEFINITION 3.2. If $r \,\varepsilon\, R(p)$ and $x \,\varepsilon\, GF(p_1)$ then we define the addition and multiplication of x and r by

$$x + r = \sigma(x) + r, \qquad x\cdot r = \sigma(x)\cdot r.$$

THEOREM 3.2. *The ring $R(p)$ is the direct sum of the $GF(p_j)$'s*, i.e. $R(p) = \sum_{j=1}^{k} \oplus\, GF(p_j)$.

PROOF. The proof follows directly from Theorem 3.1, Lemma 3.8 and Definitions 3.1 and 3.2.

4. Applications to mixed factorials. Consider the mixed factorial $2^2 \times 3 \times 5$, i.e. two factors at two levels, one factor at 3 levels, one factor at five levels. In this example, three finite fields are involved, namely $GF(2)$, $GF(3)$ and $GF(5)$. The b_j's of Lemma 3.2 are calculated as follows:

$$\begin{aligned}b_1 &= 3\cdot 5\;(3\cdot 5 - 2)^{-1} = 15\cdot(13)^{-1} = 15\cdot 7 = 105 = 15 \text{ (mod. 30)},\\ b_2 &= 2\cdot 5\;(2\cdot 5 - 3)^{-1} = 10\cdot(7)^{-1} = 10\cdot 13 = 130 = 10 \text{ (mod. 30)},\\ b_3 &= 2\cdot 3\;(2\cdot 3 - 5)^{-1} = 6\cdot(1)^{-1} = 6\cdot 1 = 6 \text{ (mod. 30)}.\end{aligned}$$

The ideals generated by the b_j's and the mapping of the $GF(p_j)$'s into these ideals are found using Lemma 3.8.

$$\begin{array}{cccc} GF(2) & & I(15) \\ \left.\begin{matrix}0\\1\end{matrix}\right\} & \xrightarrow{\sigma} & \left\{\begin{matrix}0\\15\end{matrix}\right. \end{array} \qquad \begin{array}{ccc} GF(3) & & I(10) \\ \left.\begin{matrix}0\\1\\2\end{matrix}\right\} & \xrightarrow{\sigma} & \left\{\begin{matrix}0\\10\\20\end{matrix}\right. \end{array} \qquad \begin{array}{ccc} GF(5) & & I(6) \\ \left.\begin{matrix}0\\1\\2\\3\\4\end{matrix}\right\} & \xrightarrow{\sigma} & \left\{\begin{matrix}0\\6\\12\\18\\24\end{matrix}\right. \end{array}$$

Using Theorem 3.2 we see that the ring $R\ (30) = GF\ (2) \oplus GF\ (3) \oplus GF\ (5)$, i. e.

$$\begin{array}{ccccccccccccc} GF(2) & + & GF(3) & + & GF(5) & = I(15) & + & I(10) & + & I(6) & = & R(30) \\ 0 & + & 0 & + & 0 & = \sigma(0) & + & \sigma(0) & + & \sigma(0) & = & 0 \\ 1 & + & 1 & + & 1 & = \sigma(1) & + & \sigma(1) & + & \sigma(1) & = & 1 \\ 0 & + & 2 & + & 2 & = \sigma(0) & + & \sigma(2) & + & \sigma(2) & = & 2 \end{array}$$

etc. etc.

The classical way of writing out the set T of treatment combinations in any mixed $\prod_{j=1}^{k} p_j^{m_j}$ factorial is using $(\sum_{j=1}^{k} m_j)$-tuples $z = (z_{11}, z_{12}, \cdots, z_{1m_1}, z_{21}, z_{22}, \cdots, z_{2m_2}, z_{k1}, z_{k2}, \cdots, z_{km_k})$, $z_{ji} \varepsilon GF(p_j)$, $j = 1, 2, \cdots, k$, $i = 1, 2, \cdots, m_j$. Using Lemma 3.8 and Theorem 3.2 we can equivalently work with a new set of treatment combinations consisting of $(\sum_{j=1}^{k} m_j)$-tuples of the form $y = (y_{11}, y_{12}, \cdots, y_{1m_1}, y_{21}, y_{22}, \cdots, y_{2m_2}, \cdots, y_{k1}, y_{k2}, \cdots, y_{km_k})$, where $y_{ji} \varepsilon I(b_j)$, $j = 1, 2, \cdots, k$, $i = 1, 2, \cdots, m_j$. Thus in our $2^2 \times 3 \times 5$ mixed factorial the two equivalent representations are

$T = \{z = (z_{11},$	$z_{12},$	$z_{21},$	$z_{31})\}$	$N = \{y = (y_{11},$	$y_{12},$	$y_{21},$	$y_{31})\}$
0	0	0	0	0	0	0	0
0	1	0	0	0	15	0	0
1	0	0	0	15	0	0	0
1	1	0	0	15	15	0	0
0	0	1	0	0	0	10	0
0	1	1	0	0	15	10	0
	etc.				etc.		

The following lemma relates to the algebraic structure of N.

LEMMA 4.1. *The set N of $(\sum_{j=1}^{k} m_j)$-tuples $y = (y_{11}\ y_{12}, \cdots, y_{1m_1}, y_{21}, y_{22}, \cdots, y_{2m_2}, \cdots, y_{k1}, y_{k2}, \cdots, y_{km_k})$ with $y_{ji} \varepsilon I(b_j)$, $j = 1, 2, \cdots, k$, $i = 1, 2, \cdots, m_j$, form a submodule N of the module G consisting of $(\sum_{j=1}^{k} m_j)$-tuples over the ring $R(p)$.*

This lemma enables us to carry out operations with the new treatment combinations within the submodule N or the larger module G. After having performed the desired operations, the results can always be brought back in the context of the classical treatment set T.

The usual way of writing out the effects in a mixed factorial is done utilizing a subset of $(\sum_{j=1}^{k} m_j)$-tuples from T. An effect in classical notation is of the form

$$A_1^{z_{11}^*} A_2^{z_{12}^*} \cdots A_{m_1}^{z_{1m_1}^*} B_1^{z_{21}^*} B_2^{z_{22}^*} \cdots B_{m_2}^{z_{2m_2}^*} \cdots K_1^{z_{k}^*} \cdot K_2^{z_{k2}^*} \cdots K_{m_k}^{z_{km_k}^*},$$

where the $(\sum_{j=1}^{k} m_j)$-tuple $z^* \neq 0$ is an element of T with the understanding that z^* represents the class $\{(\rho_1 \cdot z_{11}^*, \rho_1 \cdot z_{12}^*, \cdots, \rho_1 \cdot z_{1m_1}^*, \rho_2 \cdot z_{21}^*, \rho_2 \cdot z_{22}^*, \cdots, \rho_2 \cdot z_{2m_2}^*, \cdots, \rho_k \cdot z_{k1}^*, \rho_k \cdot z_{k2}^*, \cdots, \rho_k \cdot z_{km_k}^*)\}$, ρ_j being a non-zero mark of $GF(p_j)$.

Thus in our $2^2 \times 3 \times 5$ factorial the effect $A_1A_2B = A_1^1A_2^1B^1C^0$ represents the class $\{A_1^1A_2^1B^1C^0, A_1^1A_2^1B^2C^0\} = \{A_1A_2B, A_1A_2B^2\}$.

Now denote the set of superscripts of the classical effects by the letter E and denote the subset in N corresponding to E by the set F, then clearly F is a new and equivalent representation of the effects in the mixed factorial. An element in F now represents the class $\{\rho \cdot y^*\} = \{\rho \cdot y_{11}^*, \rho \cdot y_{12}^*, \cdots, \rho \cdot y_{1m_1}^*, \rho \cdot y_{21}^*, \rho \cdot y_{22}^*, \cdots, \rho \cdot y_{2m_2}^*, \cdots, \rho \cdot y_{k1}^*, \rho \cdot y_{k2}^*, \cdots, \rho \cdot y_{km_k}^*)\}$, where ρ is an element of the multiplicative group of $\phi(p)$ non-zero divisors in $R(p)$, ϕ being Euler's function. This last statement follows directly from the following lemma.

LEMMA 4.2. *The ideal $I(b_j)$ is mapped into itself when all its elements are multiplied by a fixed element of the multiplicative group of non-zero divisors. (In other words, the multiplication produces an automorphism of $I(b_j)$).*

In the mixed factorial $2^2 \times 3 \times 5$ we now have the following explicit correspondence between the effects in classical notation (i.e. the set E) and the new notation given by F.

$$
\begin{array}{rclcrcl}
E & = & \{z^* = (z_{11}^*, z_{12}^*, z_{21}^*, z_{31}^*) & \qquad & F & = & \{y^* = (y_{11}^*, y_{12}^*, y_{21}^*, y_{31}^*)\} \\
A_1 & = & A_1^1 A_2^0 B^0 C^0 & & A_1^{15} & = & A_1^{15} A_2^0 B^0 C^0 \\
A_2 & = & A_1^0 A_2^1 B^0 C^0 & & A_2^{15} & = & A_1^0 A_2^{15} B^0 C^0 \\
A_1 A_2 & = & A_1^1 A_2^1 B^0 C^0 & & A_1^{15} A_2^{15} & = & A_1^{15} A_2^{15} B^0 C^0 \\
B & = & A_1^0 A_2^0 B^1 C^0 & & B^{10} & = & A_1^0 A_2^0 B^{10} C^0 \\
A_1 B & = & A_1^1 A_2^0 B^1 C^0 & & A_1^{15} B^{10} & = & A_1^{15} A_2^0 B^{10} C^0 \\
A_2 B & = & A_1^0 A_2^1 B^1 C^0 & & A_2^{15} B^{10} & = & A_1^0 A_2^{15} B^{10} C^0 \\
& & \text{etc.} & & & & \text{etc.}
\end{array}
$$

Also implicit here is the fact that for example the effect $A_1^{15} A_2^{15} B^{10}$ in F represents the class $\{A_1^{15} A_2^{15} B^{10} C^0, A_1^{15} A_2^{15} B^{20}\}$. All operations with the elements of F can now be carried out conveniently within the treatment module N or within the larger module G and if desirable one can always go back to the classical notation after obtaining the results. An operation of fundamental importance in our example is the concept of generalized interaction of two effects in F. This is given by the product:

$$
\begin{aligned}
&A_1^{y_{11}^*} A_2^{y_{12}^*} B^{y_{21}^*} C^{y_{31}^*} \times A_1^{y_{11}^{*\prime}} A_2^{y_{12}^{*\prime}} B^{y_{21}^{*\prime}} C^{y_{31}^{*\prime}} \\
&\qquad = \{A_1^{\lambda_1 \cdot y_{11}^* + \lambda_2 \cdot y_{11}^{*\prime}} A_2^{\lambda_1 \cdot y_{12}^* + \lambda_2 \cdot y_{12}^{*\prime}} B^{\lambda_1 \cdot y_{21}^* + \lambda_2 \cdot y_{21}^{*\prime}} C^{\lambda_1 \cdot y_{31}^* + \lambda_2 \cdot y_{31}^{*\prime}}\}
\end{aligned}
$$

where λ_1 and λ_2 are elements of the multiplicative group of non-zero divisors in $R(30)$, i.e. λ_1 and λ_2 take on the values 1, 7, 11, 13, 17, 19, 23, 29. As an example, consider the generalized interaction

$$
\begin{aligned}
A_1^{15} C^6 \times A_1^{15} A_2^{15} B^{10} &= \{A_2^{15} B^{10} C^6,\ A_2^{15} B^{20} C^6,\ A_2^{15} B^{10} C^{12},\ A_2^{15} B^{20} C^{12},\ A_2^{15} B^{10} C^{18}. \\
&\qquad A_2^{15} B^{20} C^{18},\ A_2^{15} B^{10} C^{24},\ A_2^{15} B^{20} C^{24}\} \\
&= \{A_2^{\rho \cdot 15} B^{\rho \cdot 10} C^{\rho \cdot 6}\} = A_2^{15} B^{10} C^6,
\end{aligned}
$$

since $A_2^{15} B^{10} C^6$ represents the class $\{A_2^{\rho \cdot 15} B^{\rho \cdot 10} C^{\rho \cdot 6}\}$.

Before we extend the usual definition of the level of an effect of the symmetrical factorial [e.g. see Federer (1955)] to the mixed factorial, consider the set of ideals $\{I(b_1), I(b_2), \cdots, I(b_k)\}$ and all possible direct sums of subsets of this. We know (e.g. see van der Waerden (1950), that these direct sums are ideals in $R(p)$. A direct sum of a subset of r ideals $I(b_{i_1}) \oplus I(b_{i_2}) \oplus \cdots \oplus I(b_{i_r})$ is generated by the greatest common divisor of the elements $b_{i_1}, b_{i_2}, \cdots, b_{i_r}$. Some examples for the $2^2 \times 3 \times 5$ factorial are the following:

$$\begin{aligned} I(15) \oplus I(15) &= I(15) = \{0, 15\}, \\ I(15) \oplus I(10) &= I(5) = \{0, 5, 10, \cdots, 25\}, \\ I(15) \oplus I(15) \oplus I(10) \oplus I(6) &= I(1) = R(p) \\ &= \{0, 1, 2, \cdots, 29\}, \end{aligned}$$

etc.

In the mixed factorial $2^2 \times 3 \times 5$ the number of levels of *factors* A_1 and A_2 is two, each level being an element of the ideal $I(15)$. Similarly the factors B and C have three and five levels, the levels being defined as the elements of the ideals $I(10)$ and $I(6)$ respectively. A product of factors has its levels defined in the direct sum of the corresponding ideals and the number of its levels is equal to the order of the direct sum ideal. For example the number of levels of the product $A_1 \times B$ is equal to six, each level being an element of $I(15) + I(10) = I(5)$.

The ith level of an *effect* in F consists of a set of treatment combinations in N defined by the module equation:

$$\begin{aligned} &(A_1^{y^*_{11}} A_2^{y^*_{12}} \cdots A_{m_1}^{y^*_{1m_1}} B_1^{y^*_{21}} B_2^{y^*_{22}} \cdots B_{m_2}^{y^*_{2m_2}} \cdots K_1^{y^*_{k1}} K_2^{y^*_{k2}} \cdots K_{m_k}^{y^*_{km_k}})_i \\ &\quad = \{y_{11}, y_{12}, \cdots, y_{1m_1}, y_{21}, y_{22}, \cdots, y_{2m_2}, \cdots, y_{k1}, y_{k2}, \cdots, y_{km_k}): \\ &\qquad y_{11} \cdot \overset{*}{y}_{11} + y_{12} \cdot \overset{*}{y}_{12} + \cdots + y_{1m_k} \cdot \overset{*}{y}_{1m_1} + y_{21} \cdot \overset{*}{y}_{12} + y_{22} \cdot \overset{*}{y}_{22} + \cdots + y_{2m_2} \cdot \overset{*}{y}_{2m_2} \\ &\qquad + \cdots + y_{k1} \cdot \overset{*}{y}_{k1} + y_{k2} \cdot \overset{*}{y}_{k2} + \cdots + y_{km_k} \cdot \overset{*}{y}_{km_k} = i\} \end{aligned}$$

where i is an element of the direct sum of the ideals corresponding to the factors with non-zero $\overset{*}{y}_{ji}$'s, $j = 1, 2, \cdots, k, i = 1, 2, \cdots, m_j$.

For example the 10 treatment combinations comprising the 5th level of the

$$\begin{aligned} &\text{effect } A_1^{15} B^{10} = (A_1^{15} B^{10})_5 = \{(y_{11}, y_{12}, y_{21}, y_{31}): 15 y_{11} + 10 y_{21} = 5\} \\ &\quad = \{(15\ 0\ 20\ 0), (15\ 15\ 20\ 0), (15\ 0\ 20\ 6), (15\ 15\ 20\ 6), (15\ 0\ 20\ 12), \\ &\qquad (15\ 15\ 20\ 12), (15\ 0\ 20\ 18), (15\ 15\ 20\ 18), (15\ 0\ 20\ 24), (15\ 15\ 20\ 24)\}. \end{aligned}$$

In classical notation this set corresponds to $\{(1\ 0\ 2\ 0), (1\ 1\ 2\ 0), (1\ 0\ 2\ 1), (1\ 1\ 2\ 1), (1\ 0\ 2\ 2), (1\ 1\ 2\ 2), (1\ 0\ 2\ 3), (1\ 1\ 2\ 3), (1\ 0\ 2\ 4), (1\ 1\ 2\ 4)\}$.

In the same way levels of other effects can be found and rewritten in classical notation.

5. Discussion. All the results in this paper are a consequence of Dedekinds theorem [mentioned in van der Waerden (1950), page 149], which states that *"Every* *commutative ring without radical satisfying the minimal condition is a direct sum of commutative fields, which mutually annihilate one another.*" The principal contribution of this paper was the identification of the b_j's, which led to the mutually annihilating finite fields. The ideas developed in Section 4 are straight forward extensions of concepts encountered in symmetrical factorials. Hence the theory of mixed factorials has been unified with the theory of symmetrical factorials, this last one being now a special case.

Finally, confounding plans and fractional replicates of the mixed factorial and also "mixed" lattices can now be easily constructed utilizing the results of the paper.

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