The Annals of Mathematical Statistics
1969, Vol. 40, No. 6, 2216–2226

ABSTRACTS OF PAPERS

(Abstract of a paper presented at the Central Regional meeting, Iowa City, Iowa, April 23–25, 1969. Additional abstracts appeared in earlier issues.)

45. A sequential approach to classification. M. S. SRIVASTAVA, University of Toronto. (Invited)

Let π_i, $i = 0, 1, 2$, denote the three p-variate normal populations with means μ_i and unknown covariance matrix Σ. It is known that $\mu_0 = \mu_i$ for exactly one $i \varepsilon (1, 2)$. On the basis of samples from each population, the problem is to find for which i this is true when the two errors of misclassification are fixed and specified. Since no fixed-sample procedure will meet our requirements, a class of sequential procedures is proposed. It is shown that the average sample size, $EN \leqq C + n_0$, where $[C]$ is the sample size required when Σ is known; $[C]$ is the smallest integer $\geqq C$, and n_0 is a fixed integer $\geqq \frac{1}{3}(p + 4)$. When $\mu_1 - \mu_2$ is known, it is also shown that the cost of not knowing the covariance matrix is some finite number of k observations than prescribed by the sequential rule; k is a positive number which depends only on the specified errors of misclassification, and is independent of μ_1, μ_2, and Σ. (Received 18 August 1969.)

(Abstract of a paper presented at the Western Regional meeting, Monterey, California, May 7–9, 1969. Additional abstracts appeared in earlier issues.)

20. Confidence regions based on transformation models. J. A. HARTIGAN, Yale University. (Invited)

Assume an observation X, a parameter θ, and an error model $E = e(X, \theta)$ where E is a random variable invariant in distribution under the transformations T in a group G. A real valued ordering function ρ is assumed with $P[\rho(E) = \rho(TE)] = 0$ unless T is the identity i. Define $\phi_T(E) = 1$ if $\rho(E) \leqq \rho(TE)$, and $\phi_T(E) = 0$ if $\rho(E) > \rho(TE)$. Then the variables $(\phi_T, T \varepsilon G, T \neq i)$ are *splitting variables*, i.e., the function $\sum \phi_T$ takes the values 0, 1, 2, $\cdots$, N (where N is the number of non-identity transformations in G) each with probability $1/(N + 1)$. For each k, the region $\{\theta \mid \sum \phi_T \leqq k\}$ is a confidence region for θ of size $(k + 1)/(N + 1)$. In order to reduce computational expenditures, a subgroup of the transformation group G, or a randomly selected subset, may be used. (Received 11 August 1969.)

(Abstracts of papers presented at the Annual meeting, New York, New York, August 19–22, 1969. Additional abstracts appeared in earlier issues.)

111. On joint distribution of several nonparametric test statistics (preliminary report). S. G. MOHANTY and C. I. PETROS, McMaster University.

Consider two independent random samples from continuous populations with distribution functions F and G respectively. In order to test $F = G$ against $F \neq G$ or $F > G$ several nonparametric tests such as Kolmogorov–Smirnov test, median test, rank test and Haga test are well known. In this paper, the authors derive the joint distributions of several of these statistics and discuss tests based on the joint distributions. (Received 14 July 1969.)

112. On fixed-width confidence bounds for regression parameters II. P. K. BANERJEE and M. S. SRIVASTAVA, University of Toronto.

Consider $\{y_n\}$ $(n = 1, 2, \cdots)$, a sequence of independent observations with $y_i = \beta' \mathbf{x}^{(i)} + \epsilon_1$, β' an unknown $1 \times p$ row vector, $\mathbf{x}^{(i)}$ a known p-vector and ϵ_i a random observation obeying an unknown distribution function F with mean zero and finite, but unknown, variance σ^2. We wish to find a region R in p-dimensional space such that $P(\beta \varepsilon R) = \alpha$ and such that the maximum diameter of $R \leqq 2d$. Srivastava ([1] (1967) University of Toronto) obtained two confidence regions, an ellipsoid and a spheroid, under weaker condition than Srivastava ([2] (1967) *J. Roy. Statist. Soc. Ser. B.*) and Albert (1966); Gleser (1969) has obtained an extension of Anscombe's Theorem for the regression case, using Srivastava's [1] condition. In this paper, another set of conditions is given. An example is given to show that the condition in [1] does not imply these. Bounds for $E(N)$ are also obtained. Under certain conditions the cost of not knowing the variances has been shown to be a finite number $k \geqq 0$ of observations. (Received 15 July 1969.)

113. Elementary characterization of discrete distributions. Z. GOVINDARAJULU and R. T. LESLIE, University of Kentucky.

If the conditional distribution of a set of discrete random variables for each specified value of the sum of the variables is given, then the distributions of the summands are characterized, provided certain conditions are realized. Further, there is a discussion of the minimum number of the determinations of the conditional distribution that are required in order to characterize the distributions of the summands. Several examples are also provided. (Received 22 July 1969.)

114. Applications of weak convergence of stochastic processes to statistics. PETER J. BICKEL, University of California at Berkeley. (Invited)

Various applications of weak convergence of stochastic processes to the derivation of asymptotic theorems in statistics will be discussed. These will include the following. (1) Applications of weak convergence of the Quantile Process to (a) Asymptotic theory of linear combinations of order statistics (results of Hájek, Bickel, Shorack) and (b) The Galton Test and estimate (Bickel and Hodges). (2) Applications of weak convergence of the two sample empirical process (Pyke and Shorack) to (a) Chernoff-Savage statistics and (b) Two sample problems in higher dimensions. (3) Applications of weak convergence of the empirical process for finite populations (Rosen, Bickel) to two sample problems in higher dimensions. (4) Weak convergence of the likelihood function in a neighborhood of the true parameter value (LeCam and Hoeffding) applied to asymptotic normality of maximum likelihood estimates (Huber). (Received 22 July 1969.)

(An abstract of a paper to be presented at the Central Regional meeting, Dallas, Texas, April 8–10, 1969. Additional abstracts will appear in future issues.)

1. Useful bounds in packing problem. BRUCE McK. JOHNSON, UWE KOEHN and BODH RAJ GULATI, University of Connecticut and Eastern Connecticut State College.

Let $m_t(r, s)$ denote the maximum number of points in finite projective space $PG(r-1, s)$ of $(r-1)$-dimensions based on the Galois field $GF(s)$, where s is prime or power of a prime, so that no t of the chosen points are linearly dependent. Bose has shown (*Sankhyā*, **8**) that $m_t(r, s)$ also symbolizes the maximum number of factors that can be accommodated in a symmetrical factorial design in which each factor operates at $s = p^n$ levels, blocks are of size s^r and no main effect or t-factor ($t > 1$) or lower order interaction is confounded. We have established that $m_t(t + r, s) = t + r + 1$ for $t \geqq s(r + 1)$, from which an earlier result, $m_t(t, s) = t + 1$ for $s \leqq t$, as shown by Bush [*Ann. Math. Statist.* **23** (1952), 424–434] follows immediately. We have further shown that $m_t(t + r, s) \leqq t + r + s$ for $sr \leqq t \leqq s(r + 1) - 1$, $s > 2$ and $r \geqq 2$. For $r = 2$, $s = 3$, the upper bound is achieved while for $r > 2$ and $s = 3$, the bound is improved by unity. (Received 12 August 1969.)

(An abstract of a paper to be presented at the Eastern Regional meeting, Chapel Hill, North Carolina, May 5–7, 1969. Additional abstracts will appear in future issues.)

2. On maximum number of two-level factors in symmetrical factorial designs (preliminary report). BODH RAJ GULATI and E. G. KOUNIAS, Eastern Connecticut State College and McGill University.

It is well known that the maximum number of factors in a symmetrical factorial design in which each factor is at s levels, blocks are of size s^r, and no t-factor or lower order interaction is confounded, is given by the maximum number of points in finite projective space $PG(r - 1, s)$ of $(r - 1)$-dimensions based on Galois field GF(s), where s is prime or power of a prime, so that no t of these points are linearly dependent. Some results are known for $t = 3$ but very little is known for $t \geqq 4$. In this paper, we have established the following results: (i) $m_t(t + r, 2) = t + r + 2$ for $f_1(r) \leqq t \leqq 2r + 1$ where $r \geqq 2$ and $f_1(r) = (r + 2) + [(r + 2)/3]$ if $r = 0$ or $1 \bmod 3$, $f_1(r) = (r + 1) + [(r + 2)/3]$ if $r = 2 \bmod 3$; (ii) $m_t(t + r, 2) = t + r + 3$ for $f_2(r) \leqq t \leqq f_1(r) - 1$, where $f_2(r) = (r + 3) + [(r + 3)/7]$ if $r = 0, 1$ or $4 \bmod 7$, $f_2(r) = (r + 2) + [(r + 3)/7]$ if $r = 0, 2$ or $5 \bmod 7$, $f_2(r) = (r + 1) + [(r + 3)/7]$ otherwise; (iii) $m_t(t + r, 2) = t + r + 4$ for $f_3(r) \leqq t \leqq f_2(r) - 1$ where $f_3(r) = (r + 4) + [(r + 4)/15]$ for $r = 4, 8, 10$ or $11 \bmod 15$, $f_3(r) = (r + 3) + [(r + 4)/15]$ for $r = 1, 3, 5, 7, 9$ or $12 \bmod 15$, $f_3(r) = (r + 2) + [(r + 4)/15]$ for $r = 0, 2, 6$, or $13 \bmod 15$, $f_3(r) = (r + 1) + [(r + 4)/15]$ otherwise; and (iv) $m_t(t + r, 2) \geqq t + r + 5$ for $t \leqq f_3(r) - 1$. These results improve some of the previously reported results. [*Ann. Math. Statist.* **40** (1969), 1880.] For other values of t and r, bounds are being investigated. (Received 11 August, 1969.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. A law of large numbers for sampling finite populations with different inclusion probabilities for different individuals. V. P. GODAMBE, University of Waterloo.

Let a finite population consist of N individuals $i = 1, \cdots, N$ and $\mathbf{x}_N = (x_1, \cdots, x_i, \cdots, x_N)$, x_i being the real variate value associated with i, $i = 1, \cdots, N$. Further let n_N be the (fixed) size of the sample drawn from the population with the sampling design (Definition 2.3, Godambe and Joshi (1965) *Ann. Math. Statist.*) d_N, giving, for the individual i, $i = 1, \cdots, N$, the *inclusion probability* π_i such that $\inf \pi_i = \pi > 0$. If, given d_N, $\bar{e}$ denotes the Horvitz-Thompson estimator for the population mean $\bar{x}_N = \sum_1^N x_i/N$, using the Tschebyscheff's inequality we have, $\Pr(|\bar{e} - \bar{x}_N| < k \mid \mathbf{x}_N, d_N) \geqq 1 - V(\bar{e} \mid \mathbf{x}_N, d_N)/k^2 \cdots$ (I) for all $\mathbf{x}_N \varepsilon R^N$, the Euclidean N-space and all $k > 0$, V being the corresponding variance. Now let ζ^N be a (prior) distribution on R^N satisfying the conditions: (i) ζ^N is a product measure, (ii) $\int x_i \, d\zeta^N = c\pi_i$ $i = 1, \cdots, N$, c being any constant and (iii) for $i = 1, \cdots, N$, $\int x_i^2 \, d\zeta^N -$

$c^2\pi_i^2 = \sigma_i^2$, $\sup \sigma_i^2 = \sigma^2 < \infty$. Integrating both sides of the inequality (I) with respect to ζ^N (and using equation (50) of Godambe and Joshi (1965), *Ann. Math. Statist.*) we get $\Pr(|\bar{e} - \bar{x}_N| < k \mid \zeta^N, d_N) \geqq 1 - \sum_1^N \sigma_i^2(1 - \pi_i)/(\pi_i N^2 k^2) \cdots$ (II). Further since $\sup \sigma_i^2 = \sigma^2$ and $\inf \pi_i = \pi$ from (II) we get $\Pr(|\bar{e} - \bar{x}_N| < k \mid \zeta^N, d_N) \geqq 1 - \sigma^2(1 - \pi)/(\pi N k^2) \cdots$ (III) for all $k > 0$. A version of a law of large numbers is obtained by using the fact that in (III), $\sigma^2(1 - \pi)/(\pi N k^2) \to 0$ as $N \to \infty$. Of course the sample size $n_N = \sum_1^N \pi_i > \pi N \to \infty$, but $1 > n_N/N > \pi$. The probabilistic convergence $|\bar{e} - \bar{x}_N| \to 0$ is obviously with respect to any (prior) distribution ζ^∞ satisfying the conditions (i), (ii) and (iii) above and a sequence of sampling designs $\cdots, d_N, \cdots$ with a *fixed* sequence of inclusion probabilities $\cdots, \pi_i, \cdots$ having $\inf \pi_i = \pi > 0$. (Received 14 July 1969.)

2. On ϵ-comparison of experiments. ERIK NIKOLAI TORGERSEN, University of California at Berkeley.

This paper treats ϵ-comparison with arbitrary (but fixed) parameter sets. Most of the results are proved within a slightly generalized set up which permits families of finite measures (not necessarily probability measures). We show that ϵ-deficiency may be decided by a minimax risk criterion. Using an average risk criterion we obtain an "extension by repetitions in the parameter set" result, which yields criteria for ϵ-deficiency for k-decision problems in terms of operational characteristics and—under regularity conditions—an alternative proof of a Markov kernel criterion for ϵ-deficiency given by LeCam in 1964. Extending a result proved by C. Boll in his 1955 thesis (see also LeCam (1964)), we show that it suffices to consider invariant Markov kernels, provided the experiments are invariant. Measures of information based upon deficiencies and weighted deficiencies are given and deficiencies are expressed by weighted deficiencies. We show that weighted deficiencies w.r.t. the maximum information experiment yields—essentially—the "complete invariant" treated by N. Morse and R. Sacksteder in 1966. Conditions for two experiments being at maximal distance from each other are given where the distance is either the distance Δ defined by LeCam in 1964 or the analogous distance for testing problems. (Received 15 July 1969.)

3. An optimal property of the difference estimator. V. P. GODAMBE, University of Waterloo.

Consider a finite population of N individuals i, $i = 1, \cdots, N$, the corresponding (real) variate values being denoted by x_i, $i = 1, \cdots, N$. The population vector $\mathbf{x} = (x_1, \cdots, x_N)$. We restrict to simple random sampling without replacement with fixed number of draws, say n. If corresponding to the individuals i, $(i = 1, \cdots, N)$, y_i are the *given* values of the auxiliary variate and if s denotes the set of sampled individuals i the difference estimator of the population total $\sum_1^N x_i$ is given by $\bar{e} = N\bar{x}_s + N(\bar{Y} - \bar{y}_s)$ where $n\bar{x}_s = \sum_{i\varepsilon s} x_i$, $n\bar{y}_s = \sum_{i\varepsilon s} y_i$ and $N\bar{Y} = \sum_1^N y_i$. Let e be any unbiased estimator of the population total $\sum_1^N x_i$ and $V(e, \mathbf{x})$ be the variance of e given the population vector $\mathbf{x}$. Then it is shown that $\int_{R_N} V(\bar{e}, \mathbf{x})\, d\zeta \leqq \int_{R_N} V(e, \mathbf{x})\, d\zeta$, for *any* (prior) distribution ζ on the Euclidean N-Space R_N satisfying the conditions (i) ζ is a product measure and (ii) $\int_{R_N} x_i\, d\zeta = \alpha + y_i$, $i = 1, \cdots, N$, α being an arbitrarily fixed number. (Received 18 July 1969.)

4. Optimum allocation of quantiles in disjoint intervals for the BLUES of the parameters of the exponential distribution when the sample is censored in the middle. A. K. MD. EHSANES SALEH, Carleton University.

This paper extends the results on the author's work ((1966) *Ann. Math. Statist.*) on estimation of the parameters of the exponential distribution based on K (fixed suitably chosen quantiles from censored samples). Here we consider the sample to be censored in the middle under the Type II censoring scheme. The optimum allocation of K quantiles in the disjoint intervals $(0, \alpha)$ and $[\beta, 1)$ along with the coefficients of the BLUE's have been determined for the proportion of censoring $\beta - \alpha$, $\alpha = .40(.05).90$ and $\beta = .50(.05)1$. (Received 22 July 1969.)

5. Estimation of the mean after a preliminary test on regression coefficient (preliminary report). M. AHSANULLAH and A. K. MD. EHSANES SALEH, Food and Drug Directorate, Tunney's Pasture and McGill University.

In this paper we consider estimation of the parameter β_0 in the linear model. $Y_j = \beta_0 + \beta_1 X_j + \epsilon_j$ $j = 1, 2, \cdots n$. (Where Y_j are normally distributed variables with fixed variance σ^2) after a preliminary test on the regression coefficient β_1. We define the estimate of β_0 as $\hat{\beta} = \bar{Y}$ if the hypothesis $\beta_1 = 0$ is accepted; $\hat{\beta} = \bar{Y} - \hat{\beta}_1 \bar{X}$ if the hypothesis $\beta_1 = 0$ is rejected; where $\hat{\beta}_1$ is the usual estimate of β_1. The test is based on the usual Student's t-statistic with $n - 2$ degrees of freedom. Various numerical results on the bias and efficiency of the estimate have been obtained for recommendation on its use. The above result reduces to the two-sample pooled mean (On pooling means when variance is unknown. *J. Amer. Statist. Assoc.* **63** (1968)) when we define $x_j = 0$ for $j = 1, 2 \cdots n_1$; $x_j = 1$ for $j = n_1 + 1, \cdots n$. (Received 23 July 1969.)

6. Generalization of the Hâjek-Rênyi inequality. ESTRATIOS KOUNIAS, McGill University.

In 1955 J. Hâjek and A. Rényi established an inequality for independent random variables which generalized an inequality of Kolmogorov. Here we generalize that inequality for martingale difference sequences and Hâjek-Rényi's inequality is a particular case of ours. The main result is summarized in the following: THEOREM. *Let* $X_1, X_2, \cdots, X_n$ *be a sequence of random variables such that* $E|X_i|^r < \infty$ *for* $i = 1, 2, \cdots, n$ *and* $1 < r \leqq 2$ *and* $E(X_i \mid X_1, \cdots, X_{i-1}) = 0$, $i = 2, 3, \cdots, n$; $EX_1 = 0$. If $C_1, C_2, \cdots, C_n$ *is a non-increasing sequence of positive constants, then for any positive integers* m, n *with* $m < n$ *and arbitrary* $h > 0$, $P(\max_{m \leqq k \leqq n} C_k|X_1 + \cdots + X_n| \geqq h) \leqq C(C_m^r \sum_{i=1}^{m} E|X_1|^r + \sum_{k=m+1}^{n} C_k^r E|X_n|^r)$ *where* $1 \leqq C = f(r) \leqq 2^{2-r} < 2$, *with* $f(2) = 1$. For the proof we use a previous paper by the author and a tightened form of Minkovski inequality due to Esseen and Von Bahr. (Received 23 July 1969.)

7. Subordination of infinite-dimensional stationary stochastic processes. V. MANDREKAR and H. SALEHI, Michigan State University.

The notion of subordination for weakly stationary processes was introduced by A. N. Kolmogorov in 1941. For a pair of univariate processes which are stationarily correlated he established analytic characterization of this notion. The notion of subordination is extended to the infinite-dimensional stationary processes studied by R. Payen [*Ann. Inst. Henri Poincaré* (1967) **3** 323–396] and an analytic characterization of subordination in terms of spectral and cross-spectral distributions is given. These constitute a natural extension of Kolmogorov's result to the infinite-dimensional case. Such processes arise in practice in the study of stochastic differential equations. (See, e.g., Vakhaniya, N. N. *Theor. Probability Appl.* (1967) 666–667). (Received 28 July 1969.)

8. On using an incorrect value of ρ in group divisible two-associate PBIB designs. ROBERT CLÉROUX, University of Montreal.

In this paper we examine the problem of recovery of inter-block information in incomplete block designs. The ratio ρ of the inter-block variance to the intra-block variance which plays a key role is usually unknown and estimated from the data. We study the loss of information on treatment estimation and treatment contrast estimation due to using the estimate value of ρ in place of ρ. It is seen that for the group divisible two-associate partically balanced incomplete block designs this loss is generally small. A similar study has been made for the balanced incomplete block designs by V. Siskind, *Biometrika,* (1968) **55** 254. (Received 28 July 1969.)

9. Corrections to spectral estimates for the effect of tapering. DAVID R. BRILLINGER, London School of Economics.

J. W. Tukey, (*Spectral Analysis of Time Series* ed. B. Harris. Wiley, New York, 41–42), has suggested the tapering of observed time series prior to the estimation of spectra. The procedure has the following form; a function $h(\alpha) = 0$ for $\alpha < 0$, $\alpha > 1$, is given. If $\mathbf{Y}(t)$ is a vector-valued stationary series whose values are available for $0 \leqq t \leqq T$, then one forms the series $\mathbf{X}(t) = h(t/T)\mathbf{Y}(t)$ and estimates the spectra of this tapered series in the usual manner. Suppose $\mathbf{f}_{XX}^{(T)}(\lambda)$ is an estimate of the spectral density matrix of the $\mathbf{X}(t)$. This paper demonstrates that under regularity conditions $\mathbf{f}_{YY}^{(T)}(\lambda) = [\int h(\alpha)^2\, d\alpha]^{-1}\mathbf{f}_{XX}^{(T)}(\lambda)$ is an asymptotically unbiased estimate of the spectral density matrix of the $\mathbf{Y}(t)$. It is also shown that the asymptotic covariances of the entries of $\mathbf{f}_{YY}^{(T)}(\lambda)$ are $[\int h(\alpha)^4\, d\alpha][\int h(\alpha)^2\, d\alpha]^{-2}$ times the asymptotic covariances that would have been obtained if the series had not been tapered. For a cosine bell tapering of the first and last 10 per cent of a series $[\int h(\alpha)^2\, d\alpha]^{-1} = 1.143$ and $[\int h(\alpha)^4\, d\alpha][\int h(\alpha)^2\, d\alpha]^{-2} = 1.116$. (Received 28 July 1969.)

10. Nonisomorphic solutions of pseudo–(3, 5, 2) and pseudo–(3, 6, 3) graphs (preliminary report). S. S. SHRIKHANDE and VASANTI N. BHAT, University of Bombay.

R. C. Bose [Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* **13** (1963)] has defined the concept of partial geometries, their induced graphs, and the pseudo-geometric graphs. There are precisely two nonisomorphic solutions of a (13, 26, 6, 3, 1) balanced incomplete block design. The existence of a latin square of order 5 having an orthogonal mate as well as that of one without an orthogonal mate is known. We define the concept of an ascendent graph. Utilizing this and the results on descedent graphs as given by R. C. Bose and S. S. Shrikhande [Graphs in which each pair of vertices is adjacent to the same number of other vertices, sub. *Studia Sci. Math. Hungar.*] we construct several mutually nonisomorphic solutions of pseudo–(3, 5, 2) and pseudo–(3, 6, 3) graphs. The concept of Seidel–equivalence has also been used in this connection. [Bhagwandas and Shrikhande, Seidel-equivalence of Strongly Regular Graphs, *Sankhyā, Ser. A*, **30** (1968)]. Some interesting analytical results have been obtained. For example, it has been proved that in a pseudo–(3, k, 2) graph if the existence of $k - 2$ parallel cliques of size k is known then the existence of one more clique of size k implies that the graph is geometrisable. (Received 28 July 1969.)

11. Asymptotically most powerful rank tests. M. S. SRIVASTAVA, University of Toronto.

It is well known that the one-sample, or c-sample ($c \geqq 2$) problems are special cases of the general linear regression model $Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_p x_{pi} + \epsilon_i$, where we wish to test the hypothesis $H : \beta_1 = \beta_2 = \cdots = \beta_q = 0$, $q \leqq p$. The object of this paper is to propose a class of asymptotically most powerful rank score tests for the hypothesis H, and to develop the theory for asymptotic normality under the hypothesis and the alternative (tending to the hypothesis at a suitable rate) on the lines of Hájek, who considered the case $p = 2$, $q = 1$, and $x_{2i} = 1$, $i = 1, 2, \cdots, n$, without any loss of generality we are taking the scale parameter as one throughout the investigation. Hájek's (*Ann. Math. Statist.* (1962)) conditions are, however, too restrictive and it has been shown that the polynomial regression problems do not meet his requirements for the asymptotic normality to hold; these conditions have been relaxed in this paper, and the results have been obtained under weaker conditions than Hájek (*op. cit.*). The limiting distribution of the test statistic is shown to be central chi-square with q degrees of freedom under H and non-central chi-square under a sequence of alternatives tending to the hypothesis at a suitable rate. The Pitman efficiency of the proposed tests, relative to the classical F-test, is proved to be the same as the efficiency of the corresponding rank score tests relative to the t-test in the two sample problem. The c-sample problem has been considered as a special case of the regression problem. (Received 28 July 1969.)

12. Asymptotically most powerful rank tests for censored data. M. S. SRIVASTAVA, University of Toronto.

It is well known that one-sample or c-sample problems are special cases of the general linear regression model $Y_i = \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + \epsilon_i$, where we wish to test the hypothesis $H: \beta_1 = \cdots = \beta_q = 0$, $q < k$. This problem has been considered by Hájek (*Ann. Math. Statist.* (1962) 697, and in the above abstract) and a class of asymptotically most powerful rank score tests has been proposed. In this paper, the above problem of testing H has been considered for the *censored* data, i.e., when only the first r ordered observations are available. A class of rank score tests has been proposed. It has been shown that the proposed test is superior to those proposed by Gastwirth (*Ann. Math. Statist.* (1965)), and Basu (*Ann. Math. Statist.* (1967, 1968)); no large sample comparison with Rao, Savage and Sobel (*Ann. Math. Statist.* (1960)) statistic is possible since its asymptotic distribution is not known. The c-sample problem, as a special case of the regression model, has been considered. In this case, however, the design matrix X_r becomes a random variable. (Received 28 July 1969.)

13. On a class of non-parametric tests for independence-bivariate case. M. S. SRIVASTAVA, University of Toronto.

This paper is concerned with the problem of testing the independence of two random variables X, Y on the basis of a random sample (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n). The joint distribution function H is assumed to be absolutely continuous, and $H(x, y) = F_\rho(x - \rho u(y))G(y)$, where F_ρ denotes the conditional distribution function of X given Y, and $G(y)$ the marginal distribution of Y; $u(y)$ is a function of y. In the case of a standard bivariate normal (means 0 and variances 1) with correlation ρ, $u(y) = -g'(y)/g(y) = y$. Without assuming the knowledge of F_ρ and G, we wish to test the hypothesis that $\rho = 0$ in the above model. An advantage of the above model is that it includes both sided alternatives also. A class of rank score tests is considered and its distribution under contiguous alternatives is obtained. The rank score tests are locally most powerful only if $u(y) = -g'(y)/g(y)$. In this case the normal score test is at least as efficient as the parametric correlation coefficient r_n–test. (Received 28 July 1969.)

14. On a class of non-parametric tests for multivariate regression parameters. M. S. SRIVASTAVA, University of Toronto.

Srivastava (*Ann. Math. Statist.* (1968) 697) proposed a class of rank score tests for testing the hypothesis that $\beta_1 = \cdots = \beta_p = 0$ in the linear regression model $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_p x_{pi} + \epsilon_i$ under weaker conditions than Hájek (*Ann. Math. Statist.* (1962)). In this note, under the same weak conditions, a class of rank score tests is proposed for testing $\boldsymbol{\beta} \equiv (\boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_p) = 0$ in the multivariate linear regression model $\mathbf{y}^{(i)} = \boldsymbol{\beta}_1 x_{1i} + \cdots + \boldsymbol{\beta}_p x_{pi} +$

$\epsilon^{(i)}$, where β_i's are k-vectors. The limiting distribution of the test statistic is shown to be central χ^2_{pk} under H and non-central χ^2_{pk} under a sequence of alternatives tending to the hypothesis at a suitable rate. (Received 31 July 1969).

15. Optimal asymptotic tests of composite hypotheses for stochastic processes (preliminary report). B. R. BHAT and S. R. KULKARNI, Karnatak University.

Let $\{X_n\}$ be a stochastic process. Suppose the conditional densities of X_i given $X_1 \cdots X_{i-1}$ depend on parameters ξ and $\boldsymbol{\theta} = (\theta_1, \cdots \theta_r)$. We proceed to test the hypothesis $H:\xi = \xi_0 = 0$, $\boldsymbol{\theta}$ unspecified, as in Neyman(*Cramér Volume* (1959)). Let u_i be a function of $x_1, x_2, \cdots x_i$ and $\boldsymbol{\theta}$ with $E(u_i \mid x_1 \cdots x_{i-1}) = 0$ and $E(u_i^2 \mid x_1 \cdots x_{i-1}) = 1$ under H, and satisfy some regularity conditions. It is proved that an asymptotic test of size α is given by the rejection region $[(n^{-\frac{1}{2}}) \sum u_i > \nu(\alpha)]$ where $\int_{\nu(\alpha)}^{\infty} e^{-\frac{1}{2}t^2}\, dt = \alpha(2\pi)^{\frac{1}{2}}$. If the conditional density functions $p(x_i \mid x_1 \cdots x_{i-1}, \xi, \boldsymbol{\theta})$ satisfy some regularity conditions and $E(u_i\varphi_{\theta_j i}(x_1 \cdots x_{i-1})$ is zero for all $j = 1, 2, \cdots r$ for all i then we may substitute for $\boldsymbol{\theta}$ in the test criterion its locally root n consistent estimate, where $\varphi_{\theta_j i} = \partial \log p(x_i \mid x_1 \cdots x_i, \xi_0, \boldsymbol{\theta})/\partial\theta_j$ $(j = 1, 2, \cdots r)$. It is shown that asymptotically optimal test in a class $C_s(\alpha)$ of tests is given by the rejection region $[(n^{-\frac{1}{2}}) \sum(g_i/E^{\frac{1}{2}}(g_i^2 \mid x_1 \cdots x_{i-1})) > \nu(\alpha)]$ where $g_i = \varphi^{s}_{\xi i} - b_{1i}\varphi_{\theta_1 i} - \cdots - b_{ri}\varphi_{\theta_r i}$, $\varphi_{\xi i} = \partial \log p(x_i \mid x_1 \cdots x_{i-1}, \xi_i, \boldsymbol{\theta})/\partial\xi \mid_{\xi=0}$, and b_{ji}'s are partial regression coefficients of $\varphi_{\xi i}$ on $\varphi_{\theta_j i}$'s based on conditional distribution of x_i given $x_1 \cdots x_{i-1}$. (Received 8 August 1969.)

16. Random rank statistics and confidence intervals. HIRA LAL KOUL, Michigan State University.

In a simple linear regression model $Y = \theta x + z$, a bounded length confidence interval using Wilcoxon type signed rank statistics is constructed. It is shown that this interval asymptotically (as length of the interval goes to zero) attains the preassigned coverage probability $1 - 2\alpha$. Method of proof uses asymptotic linearity in θ of rank statistics based on random number of observations. While proving this fact, we also have that certain weighted empirical cumulatives of random number of observations are relatively compact. (Received 8 August 1969.)

17. Some sequential procedures for a multivariate slippage problem. M. S. SRIVASTAVA and V. S. TANEJA, University of Toronto and New Mexico State University.

Consider $k + 1$ p-variate normal populations Π_i with means μ_i and common covariance matrix Σ; the population Π_0 is called control or standard population. The problem is to decide that none of the k experimental categories is better than the control or decide that a certain category is better using some distance function. We consider two distance functions $\mu_i'\mu_i$ and $\mu_i'\Sigma^{-1}\mu_i$. Sequential procedures on the lines of Chow and Robbins (*Ann. Math. Statist.* (1965)) and Paulson (*Ann. Math. Statist.* (1962)) are obtained. (Received 8 August 1969.)

18. A class of sequential selection procedures based on ranks. M. S. SRIVASTAVA, University of Toronto.

Let X_{is} denote the sth observation from category Π_i $(i = 1, 2, \cdots, k, s = 1, 2, \cdots)$. Let $P\{X_{is} \leqq x\} = F(x - \theta_i)$, where F is unknown but $\varepsilon\mathfrak{F}$ (satisfying certain regularity conditions). In this paper a class of sequential procedures based on ranks has been proposed for selecting the population with the largest θ. Except for some regularity conditions, no assumption regarding the form of the distribution is made as contrast to the so called "nonparametric" selection procedures based on ranks proposed by Lehmann (*Math. Ann.* (1963) 268–275), where actually one requires the exact knowledge of the density function to carry out the ranking. The procedure can be described as follows. Let $\psi(u, f_0) = -[f_0'(F_0^{-1}(u))/f_0(F_0^{-1}(u))]$, $0 < u < 1$; F_0 is known and $\varepsilon\mathfrak{F}$. Let C_n be a consistent estimator of $\int_0^1 \psi(u, f_0)\psi(u, f)\, du$; F is unknown but $\varepsilon\mathfrak{F}$. Let $R_{ij}^{(m)}$ be the rank of X_{ij} and $\psi_m = \psi(j/m + 1)$, $(j-1)/m < u \leqq j/m$, $m = kn$. Let $\{a_n\}$ be a sequence of numbers such that $\lim_{n\to\infty} a_n = a$ and $P\{Z \leqq -a\} = \alpha/(k-1)$, where Z is $N(0, 1)$. The observations are taken sequentially according to the stopping rule N = smallest integer $n \geqq 1$ such than $n \geqq a_n^2\gamma^2/d^2C_n^2$ where $\gamma^2 = \int_0^1 \psi^2(u)\, du$. When sampling is stopped at $N = n$, select the population with the largest $\sum_{j=1}^n \psi_m(R_{ij}^{(m)}/m + 1, f_0)$ as the best population. (Received 8 August 1969.)

19. On unbiased estimation of density functions. C. P. Quesenberry and Allan H. Seheult, North Carolina State University.

Let $X^{(n)} = (X_1, X_2, \cdots, X_n)$ be n independent observations on a random variable X with unknown distribution P which is assumed to be a member of a family of probability measures $\mathcal{P}$, dominated by a σ-finite measure μ on a measurable space $(\mathfrak{X}, \mathfrak{A})$; and denote by $\mathfrak{p}$ the family of densities (with respect to μ) corresponding to $\mathcal{P}$. M. Rosenblatt *Ann. Math. Statist.* **27** (1956) 832–837) proved that if $\mathcal{P}$ is the family of Lebesgue-continuous probability measures on the real line then there exists no unbiased estimator for an element $f \varepsilon \mathfrak{p}$. His result follows from the fact that the empirical distribution function, based on $X^{(n)}$, is not absolutely continuous with respect to Lebesgue measure. We generalize this result in the following: Theorem. *An unbiased estimator of f exists if and only if there exists an unbiased estimator $\hat{P}$, of P, that is absolutely continuous with respect to the original dominating measure, μ. Moreover, when such a $\hat{P}$ exists, the corresponding unbiased estimator of f is given by the Radon-Nikodym derivative, $d\hat{P}/d\mu$.* If there exists a complete and sufficient statistic for $\mathcal{P}$, the theorem can be rephrased in terms of unique minimum variance unbiased estimators of density functions. Several of the well-known families of distributions are considered to illustrate the theorem. (Received 12 August 1969.)

20. Asymptotically most powerful rank tests for regression parameters in MANOVA. M. S. Srivastava, University of Toronto.

Srivastava (*Ann. Math. Statist.* **40** 697), proposed a class of rank score tests for testing the hypothesis that $\beta_1 = \cdots \beta_p = 0$ in the linear regression model $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_p x_{pi} + \epsilon_i$ under weaker conditions than Hájek (*Ann. Math. Statist.* (1962)). In this paper, under the same weak conditions, a class of rank score tests is proposed for testing $\boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_q = 0$ in the multivariate linear regression model $\mathbf{y}_i = \boldsymbol{\beta}_1 x_{1i} + \cdots + \boldsymbol{\beta}_p x_{pi} + \boldsymbol{\epsilon}_i$, $q \leqq p$, where $\boldsymbol{\beta}_i$'s are k-vectors. The limiting distribution of the test statistic is shown to be central χ_{qk}^2 under H and non-central χ_{qk}^2 under a sequence of alternatives tending to the hypothesis at a suitable rate. (Received 18 August 1969.)

21. On ellipsoidal confidence region of maximum fixed-width for the multivariate normal mean and the cost of not knowing the covariance matrix. M. S. Srivastava and R. P. Bhargava, University of Toronto.

Srivastava ([1], *J. Roy. Statist. Soc. Ser. B*, (1967)) proposed a spherical confidence region of fixed diameter $2d$ for the mean vector θ of any p-variate distribution with finite but unknown covariance matrix Σ. It was shown there that (i) the specified coverage probability α is attained in the limit as $d \to 0$, and (ii) the sequential sample size $N(d)$ has the property that $EN(d)/C \to 1$ as $d \to 0$ where $[C]$ is the fixed sample size required when Σ is known. Later Srivastava and Bhargava (*Ann. Math. Statist.* Abstract 31, October (1969)), showed that for the normal distribution the coverage probability α can be attained, independently of d, Θ and Σ, by taking a fixed number of k (depending on α) additional observations than prescribed by the sequential procedure [1]. It was also shown that $E(N) \leqq C + O(1)$. In this paper a class of ellipsoidal confidence regions of maximum (random) width $\leqq 2d$ at a prescribed confidence level α is proposed for the normal case. The cost of not knowing the covariance matrix is only a finite number k of additional observations than prescribed by the sequential rule [1]; k depends on α and the sequence $\{a_n\}$, but does not depend on θ, Σ and d. Naturally these k observations are used only in improving the estimate of the mean. It may be noted that the ellipsoidal confidence region is contained in the spheroid [1] confidence region, both having the property that the width of the maximum diameter is $\leqq 2d$. However, if one desires to have control on all the diameters, a modified sequential procedure is proposed. (Received 18 August 1969.)

22. On the cost of not knowing the variance when making a maximum fixed-width confidence interval for the normal mean using t-statistic. M. S. Srivastava and R. P. Bhargava, University of Toronto.

Using t-statisic, it is shown that the mean of a normal distribution with unknown variance σ^2 may be estimated to lie within an interval of random width less than or equal to $2d$ (d specified) at a prescribed confidence level α. A class of sequential procedures which overcome the ignorance about σ^2 with no more than a finite number of observations is given. That is, the expected sample size exceeds the fixed sample size one would use if σ^2 were known by a finite amount, the difference depending on the confidence level α but not depending on the values of the mean μ, the variance σ^2 and the maximum width $2d$. An upper bound of the expected value of N is given. (Received 18 August 1969.)

23. Collisions of stable processes (preliminary report). Richard Gisselquist, Dartmouth College.

Let $\{x_i(t): i = \cdots, -1, 0, 1, \cdots\}$ be a collection of one-dimensional stable processes of order $\gamma \varepsilon (1, 2]$ with the property that the starting positions $\cdots < x_{-1}(0) < x_0(0) = 0 < x_1(0) < \cdots$ form a Poisson system with rate one. T. Harris, *J. Appl. Probability* **2** (1965) 323–338, has shown how to define a collection of "collision processes" $\{y_i(t)\}$ from the $\{x_i(t)\}$ by means of idealized elastic collisions. An idea due to Frank Spitzer is then used to prove that for large A the processes $y_0(At)/A^{(2\gamma)^{-1}}$ approach the Gaussian process with mean zero and covariance $r(t, s) = c_\gamma(t)^{\gamma^{-1}} + s^{\gamma^{-1}} - |t - s|^{\gamma^{-1}})$ in the sense of convergence of finite dimensional distributions. (Received 3 September 1969.)

24. On the method of likelihood-modulation and non-central distributions. Franz Streit, University of Toronto.

The method of likelihood-modulation is a technique for deriving from a given density function a class of related density functions. It is shown that a very simple derivation of the non-central χ^2–, F–, and t–distribution from the corresponding central distributions can be given using the following version of this procedure. Let X_μ be random variables with density functions $f(x:\mu)$ and let T be a statistic with density functions $g(t:\mu)$ for $T(X_\mu)$. For any structural model (even models with a non-unitary group) with class of error densities $\{f(x:\mu)\}$, reference variable D and normalizing constants $k_\mu(D)$ for the error probability distributions the following is true. If $T(X)$ depends on X only in terms of $D(X)$, if $T(X)$ is sufficient with respect to the class of marginal distribution $\{h(D(X):\mu)\}$ and $k_{\mu_0}(D) \neq 0$ then $g(t:\mu) = [k_\mu(D)]^{-1}k_{\mu_0}(D)g(t:\mu_0)$. [See for instance D. A. S. Fraser, *The Structure of Inference*. (1968).] This derivation of the non-central distributions offers several advantages. It is solely based on the basic invariance-properties of the statistics and thus easily remembered. In fact the rotation invariance of the non-central χ^2-variable $C_{\mathbf{u}}^{(n)}$ $(= |\mathbf{X}|^2$ where $\mathbf{X} = (x_1, \cdots, x_n)$ and x_i is $N(\mu_i, 1))$ and the invariance with respect to suitable scaling-operations is all that is needed in the proofs. Furthermore the procedure is geometrically intuitive and may be applied for the derivation of many other non-central distributions. (Received 3 September 1969.)

25. On the supercritical Galton-Watson process with immigration. E. Seneta, Australian National University.

Let $\{X_n\}$ be a Galton-Watson process initiated by a single ancestor $(X_0 = 1)$, whose offspring distribution is $\{f_j\}$ where $f_j \neq 1, j \geqq 0$ and $1 < m \equiv \sum jf_j < \infty$, and which is subject at each generation to an independent immigration component with distribution $\{b_j\}$, where $b_0 < 1$. Write $V_n = X_n/c_n$, where $\{c_n\}$ is the sequence of generalized norming constants constructed for the ordinary supercritical Galton-Watson process by the author (*Ann. Math. Statist.*, **39** (1968) 2098–2102). Then if $\sum b_j \log j < \infty$, the sequence $\{V_n\}$ converges almost surely to a non-degenerate random variable V having a continuous distribution on $(0, \infty)$. If the logarithmic moment condition fails, $V_n \to \infty$ almost surely. The almost sure behaviour, from which the rest follows along the lines of the author's earlier work, stems from the fact that the process $\{-\exp(-V_n)\}$ is a submartingale. (Received 8 September 1969.)

26. The application of invariance to unbiased estimation. Carl Morris and Morris L. Eaton, The Rand Corporation and University of Chicago.

The following main theorem is proved. Let X be a random variable taking values in $\mathfrak{X}$ and $\mathcal{P} = \{P_\theta: \theta \in \Theta\}$ be a family of measures on $\mathfrak{X}$. Let G be a group of transformations on $\mathfrak{X}$ which preserves $\mathcal{P}$, and define the group operation by $(g_2 \circ g_1)(x) = g_2(g_1(x))$. Let $\bar{G}$, the induced group acting on Θ, act transitively on Θ. Assume that a complete sufficient statistic $T: \mathfrak{X} \to G$ can be defined, denoting its value at x by T_x. Suppose $Z(x) \equiv T_x^{-1}(x)$ (T_x^{-1} is the G-inverse), satisfies $Z(g(x)) = Z(x)$ for all $g \in G, x \in \mathfrak{X}$. Let $\varphi(\theta)$ be any real-valued function of θ with unbiased estimate $f(x)$. Then a minimum variance unbiased estimate (MVUE) of $\varphi(\theta)$ is $f^*(T_x) \equiv E_Z f(T_x(Z))$ where $T_x(Z)$ is the value of the group element T_x at Z. Z is stochastically independent of T_x and E_Z denotes expectation over the marginal distribution of Z which is independent of θ. The above provides a method for finding the MVUE of estimable functions in many standard problems. Examples given to illustrate the use of the theorem include the estimation of functions of the parameters of a multivariate normal distribution. (Received 8 September 1969.)

27. Nonparametric density estimation. Edward J. Wegman, University of North Carolina.

This paper is an expository review of density estimation including several early parametric methods, kernel estimates, orthogonal estimates and maximum likelihood estimates. Also included is a Monte Carlo study of competing estimates as well as several graphs for visual comparisons. (Received 16 September 1969.)