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A SYSTEM OF MARKOV CHAINS WITH RANDOM LIFE TIMES[1]

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0. Introduction. The purpose of this paper is to investigate the limiting properties of random variables associated with a system of random processes. The system is described as follows. At each discrete integer time $n \geqq 0, M_n$ particles enter a denumerable set of states Λ at a given state denoted by $(0,0)$. Assume $\{M_n, n \in I\}$ to be a sequence of independent Poisson variables with common mean λ. (Here and throughout, I denotes the set of nonnegative integers.) Moreover, at each integer time $n \geqq 1$, each particle already in the system may undergo a transition independently of the other particles and independently of $\{M_n, n \in I\}$. A particle which entered the system at time $k \leqq n$, moves according to the probability law of $Z(n-k)$ where $\{Z(n), n \in I\}$ is a random process described below.

1. Preliminaries. Let $\{X(n), n \in I\}$ be an irreducible aperiodic Markov chain having state space Γ, taken to be the nonnegative integers, and having stationary transition probabilities $P(x,y)$. Let $P_n(x,y)$ denote the n-step transition probabilities and $P_n(x,B) = \sum_{y \in B} P_n(x,y)$ for sets $B \subseteq \Gamma$. Let $\{Y(n), n \in I\}$ be a random process with state space $\{0,1\}$, independent of $\{X(n), n \in I\}$. Let $p(n) = P[Y(n) = 0]$, $p = \{p(n), n \in I\}$, and assume $Y(n) = 1$ implies $Y(n+1) = 1$ for each $n \in I$. Thus $p(n) \geqq p(n+1)$ and $\pi \equiv \lim_{n \to \infty} p(n)$ exists. Define $Z(n) = (X(n), Y(n))$. The process $\{Z(n), n \in I\}$ has state space $\Lambda = \{(x,y) : x \in \Gamma, y = 0 \text{ or } 1\}$. The independence assumption of the introduction means that the sequence $\{M_n, n \in I\}$ is independent of the processes $\{X(n), n \in I\}$ and $\{Y(n), n \in I\}$. One can think of the transition of a particle in its y coordinate from state 0 to state 1 as the death of this particle. Accordingly, transitions of the process $\{Z(n), n \in I\}$ through states of the form $(x,0), x \in \Gamma$, can be thought of as the transitions of a particle according to the law of the Markov chain while the particle is still alive. Two special cases of the $Y(n)$ process are of interest. If $\pi = 1$ no deaths occur and $Z(n)$ is Markov with transition probabilities $P(x,y)$. If for some $n_0 \in I, n_0 > 0, p(n) = 1$ if $n \leqq n_0$ and $p(n) = 0$ for $n > n_0$ the particles have fixed life times. In this case it will be seen that the system of live particles attains a macroscopic equilibrium. See Section 2 for details.

In what follows, $B \subset \Gamma$ is assumed finite and, to avoid trivialities, not to include state 0. Let V_B^r denote the time of rth visit to B by $X(n)$ and $N_k(B)$ the occupation time of B by $X(n)$ to time k. Formally,

$$V_B^1 = \min\{n : X(n) \in B\}, \qquad V_B^r = \min\{n : n > V_B^{r-1}, X(n) \in B\}$$

where if for some integer $r > 0$, $X(n) \notin B$ for all $n > V_B^{r-1}$, take $V_B^r = \infty$. Further $N_k(B) = \sum_{j=1}^k \delta_B(X_j)$ where $\delta_B(x) = 1$ (or 0) if $x \in B$ (or $x \notin B$). Let $N(B) = \lim_{k \to \infty} N_k(B)$ whether finite or infinite. Probabilities for the random variables V_B^r,

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$N_k(B)$, and $N(B)$ are understood to be conditional given $X(0) = 0$. That is, $P[N(B) = k]$ means $P[N(B) = k \mid X(0) = 0]$.

The main results of the investigations of these systems are given in the theorems below. Let $B^* = \{(x, 0) : x \in B\}$. Let $M_n(B^*; r)$ denote the number of particles which have visited B^* exactly r times by time n.

THEOREM 1.

$$P[\lim_{n\to\infty} M_n(B^*; r)/n = \lambda\{h(B, r, p) - h(B, r+1, p)\} \leqq \lambda] = 1,$$

where $h(B,r,p) = \lambda \sum_{k=1}^{\infty} p(k)P[V_B{}^r = k]$. *In particular, if* $\pi = 1$,

$$P[\lim_{n\to\infty} M_n(B^*; r)/n = \lambda P[N(B) = r]] = 1.$$

THEOREM 2.

(a) $$P[\lim_{n\to\infty} \textstyle\sum_{r=1}^{n} M_n(B^*; r)/n = \lambda h(B, 1, p)] = 1.$$

(b) *Let* $\gamma_n = \sum_{r=1}^{n} EM_n(B^*; r)$. *Unless there exists an integer* k_0 *such that* $p(k) = 0$ *for* $k > k_0$ *and* $P[V_B{}^1 \leqq k_0] = 0$,

$$\lim_{n\to\infty} P[\{\textstyle\sum_{r=1}^{n} M_n(B^*; r) - \gamma_n\}\gamma_n{}^{-\frac{1}{2}} < \alpha] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\alpha} e^{-z^2/2}\, dz.$$

(Note that under the condition given no live particle can ever hit B^* and $M_n(B^*; r) = 0$ for every integer $r \geqq 1$.)

THEOREM 3. *Let* $A_n(B^*)$ *denote the number of particles in* B^* *at time* n, *and* $S_n(B^*) = \sum_{k=1}^{n} A_k(B^*)$.

(a) *If either* (i) $X(n)$ *is transient or* (ii) $X(n)$ *is persistent*, $\lim_{n\to\infty} EA_n(B^*) < \infty$ *and for each* $x \in B$, $\sum_{k=1}^{n} p(k) \sum_{j=1}^{k} P_j(0, x) \leqq \alpha n^{1-\delta}$ *for some* $\alpha > 0$, $0 < \delta < 1$, *then*

$$P[\lim_{n\to\infty} n^{-1} S_n(B^*) = \lambda \textstyle\sum_{k=1}^{\infty} p(k) P_k(0, B)] = 1.$$

(b) $$P[\lim_{n\to\infty} n^{-1} \textstyle\sum_{m=1}^{n} m^{-1} A_m(B^*) = \lambda\pi u(B)] = 1$$

where $u(x) = \lim_n P_n(x, x)$ *and* $u(B) = \sum_{x\in B} u(x)$.

(c) *Suppose* $X(n)$ *is persistent and* $\lim_{n\to\infty} EA_n(x^*) = \infty$. *Let* $f(n) = nE^{-1}S_n(x^*)$ *and assume for some integer* $v > 0$, $\sum_{n=1}^{\infty} f(n^v) < \infty$ *and* $\lim_{n\to\infty} f(n^v)/f([n+1]^v) = 1$. *Then* $P[\lim_{n\to\infty} S_n(x^*)/ES_n(x^*) = 1] = 1$. *Furthermore, if the hypotheses hold for each* x^* *in a finite set* B^*, $P[\lim_{n\to\infty} S_n(B^*)/ES_n(B^*) = 1] = 1$.

THEOREM 4. *If for some* $\delta > 0$

(i) $\lim_{n\to\infty} n^{\delta} p(n) = \infty$ *and*

(ii) *there exists an* $\alpha > 0$ *such that* $k^{1+\delta}/EN_k{}^2(B) < \alpha$ *for* $k \geqq k_0$ *where*

$$k_0 = \min\{j \mid P_j(0, B) > 0\}$$

then

$$\lim_{n\to\infty} P[\{S_n(B^*) - ES_n(B^*)\}\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}} < \beta] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\beta} e^{-z^2/2}\, dz.$$

In an investigation of a similar type Derman [2] and Port [4] consider a system of transient Markov chains. At time 0, M_x particles are placed in each state x of the state space of a transient Markov chain P. The M_x are assumed independent Poisson with means $\mu(x)$, where $\{\mu(x), x \in I\}$ is a stationary measure assumed to exist for P. At each time $n \geqq 1$ all particles in the system are allowed to move, each independently of the others and each according to P. No further particles are added to the system nor do deaths occur. The system is found to be in macroscopic equilibrium. The techniques used by Port are applicable in the study of the system of the paper. The results obtained here are without reference to the dual chain nor is the Markov chain assumed necessarily transient. In Port [5], he has extended his previous work to general discrete time Markov processes.

2. Distributional results. The probability distributions of several random variables associated with this system are established as a consequence of the following lemma. The proof is one used by both Derman [2] and Port [4].

LEMMA. *Let* $\{M_n, n \in I\}$ *denote a sequence of* i.i.d. *Poisson random variables,* $EM_n = \lambda$. *Let* $\{Y_{jk}(\alpha), \alpha \in I, k \in I, 0 \leqq j \leqq M_k\}$ *denote a sequence of random variables, independent in j and k for fixed* α *and independent of the sequence* $\{M_n, n \in I\}$. *Further assume*

(i) $p_k(\alpha) = P[Y_{jk}(\alpha) = 1], 1 - p_k(\alpha) = P[Y_{jk}(\alpha) = 0]$ *for* $0 \leqq j \leqq M_k$;

(ii) $\sum_{\alpha=0}^{\infty} Y_{jk}(\alpha) = 0$ *or* 1 a.e. *for each pair* (j, k).

Let $S_n(\alpha) = \sum_{k=1}^{n} \sum_{j=1}^{M_k} Y_{jk}(\alpha)$. *Then for each integer* $n \geqq 1$, $\{S_n(\alpha), \alpha \in I\}$ *is a sequence of mutually independent Poisson random variables with* $ES_n(\alpha) = \lambda \sum_{k=0}^{n} p_k(\alpha)$.

PROOF. It suffices to calculate the characteristic function for the vector $(S_n(\alpha_1), \cdots, S_n(\alpha_m))$ for an arbitrary finite sequence $\{\alpha_j \in I, 1 \leqq j \leqq m\}$. Note that (i) and (ii) imply at most one $Y_{jk}(\alpha)$ can equal one for each pair (j, k). Using this observation and that (i) and (ii) imply $\sum_{\alpha=0}^{\infty} p_k(\alpha) \leqq 1$ for each k one finds the joint characteristic function of the vector $\mathbf{Y}_{jk} = (Y_{jk}(\alpha_1), \cdots, Y_{jk}(\alpha_m))$ is given by

$$\phi(t_1, \cdots, t_m) = [p_k(\alpha_1)(e^{it_1} - 1) + \cdots + p_k(\alpha_m)(e^{it_m} - 1) + 1].$$

Since the M_k are Poisson variables with parameter λ, it follows that the characteristic function of $\sum_{j=1}^{M_k} \mathbf{Y}_{jk}$ is given by $\exp \lambda\{p_k(\alpha_1)(e^{it_1} - 1) + \cdots + p_k(\alpha_m)(e^{it_m} - 1)\}$ and finally that the desired characteristic function is $\exp \lambda\{(e^{it_1} - 1) \sum_{k=0}^{n} p_k(\alpha_1) + \cdots + (e^{it_m} - 1) \sum_{k=0}^{n} p_k(\alpha_m)\}$, which is a product of Poisson characteristic functions as required.

Consequences of this lemma pertinent to this paper are:

(2.1) For each $n \in I$, $\{A_n(x^*), x^* \in \Gamma^*\}$ is a sequence of mutually independent Poisson random variables with $EA_n(x^*) = \lambda \sum_{k=0}^{n} p(k) P_k(0, x)$, where $\Gamma^* = \{(x, 0) \mid x \in \Gamma\}$ and $x^* = (x, 0)$.

(2.2) Let $I_n(B^*; r)$, $n \in I$, $r \in I$ denote the number of particles in B^* for the rth time at time n. Then for fixed B^* and $r \in I$, $\{I_n(B^*; r), n \in I\}$ is a sequence of mutually independent Poisson random variables with $EI_n(B^*; r) = \lambda \sum_{k=1}^{n} p(k) P[V_B{}^r = k]$.

(2.3) For each fixed B^* and $n \in I$, $\{M_n(B^*;r), r \in I\}$ is a sequence of mutually independent Poisson random variables with

$$EM_n(B^*;r) = \sum_{j=1}^{n} \sum_{k=1}^{j} p(k)\{P[V_B^r = k] - P[V_B^{r+1} = k]\}.$$

In (2.1) and (2.2) the expressions for the expected values given in terms of quantities already defined follow in a natural way from the lemma. This is not so in (2.3). However, $EM_n(B^*;r)$ can be calculated from (2.2) and the observation $M_n(B^*;r) = \sum_{j=1}^{n} I_j(B^*;r) - \sum_{j=1}^{n} I_j(B^*;r+1)$.

To prove (2.1) let X_{jk} denote the state at time n of the jth particle to enter the system at time k. Let $Y_{jk}(x^*) = 1$ (or 0) if $X_{jk} = x^*$ (or $X_{ij} \neq x^*$). The sequence $Y_{jk}(x^*)$ satisfies the hypotheses of the lemma with $P[Y_{jk}(x^*) = 1] = p(n-k)P_{n-k}(0,x)$. Conclusions (2.2) and (2.3) follow similarly.

REMARK. If for some integer $d > 0, p(k) = 0$ whenever $k > d$, the lifetime of a particle cannot exceed d with probability one. Thus for each $n > d$, $\{A_n(x^*), x^* \in \Gamma^*\}$ is a sequence of mutually independent Poisson random variables with $EA_n(x^*) = \lambda \sum_{k=0}^{d} p(k)P_k(0,x)$. Thus, d time units after the system began, it has attained a state of macroscopic equilibrium of the sort exhibited by the system investigated by Derman and by Port. Note if $p(k) > 0$ for all k that since $\{X(n), n \in I\}$ is irreducible, the sequence $\sum_{k=0}^{m} p(k)P_k(0,x)$ is nondecreasing and tends to $\sum_{k=0}^{\infty} p(k)P_k(0,x)$. This limit is not attained and therefore the system does not attain equilibrium unless p has only finitely many non-zero terms.

3. Proofs of Theorems 1 and 2. The random variable $M_n(B^*;r)$ is the number of particles to visit B^* exactly r times by time n. Thus $\sum_{r=1}^{n} M_n(B^*;r)$ is the number of particles to visit B^* at least once by time n and $\sum_{r=1}^{n} M_n(B^*;r)/n$ is the average number of particles to visit B^* for the first time per unit of time. Some limiting properties of these random variables are established in this section.

PROOF OF THEOREM 1. Note first that

$$M_n(B^*;r) = \sum_{j=1}^{n} I_j(B^*;r) - \sum_{j=1}^{n} I_j(B^*;r+1).$$

From (2.2) observe that each sum on the right is a sum of independent random variables and moreover that $\lim_{n\to\infty} \sum_{j=1}^{n} \operatorname{Var} I_j(B^*;s)/j^2 < \infty$, for $s = r, r+1$.

Thus the strong law of large numbers is used to conclude that

$$\lim_{n\to\infty} M_n(B^*;r)/n = \lim_{n\to\infty} \lambda\{h_n(B,r,p) - h_n(B,r+1,p)\}/n \quad \text{a.e.}$$

where $h_n(B,r,p) \equiv \sum_{k=1}^{n}(n-k+1)p(k)P[V_B^r = k]$. Since $\sum_{k=1}^{\infty} P[V_B^r = k] \leqq 1$ and $p(k) \leqq 1$ it follows that $\lim_{n\to\infty} h_n(B,r,p)/n = h(B,r,p)$. Thus $\lim_{n\to\infty} M_n(B^*;r)/n = \lambda\{h(B,r,p) - h(B,r+1,p)\} \leqq \lambda\{p(1)P[V_B^r < \infty] - \pi P[V_B^{r+1} < \infty]\} \leqq \lambda$, a.e. Finally observe that if $\pi = 1$, then $p(k) = 1$ for $k = 1, 2, \cdots$ and $\lambda\{h(B,r,p) - h(B,r+1,p)\} = \lambda P[N(B) = r]$.

PROOF OF THEOREM 2. For part (a) note

$$\sum_{r=1}^{n} M_n(B^*;r) = \sum_{r=1}^{n} \sum_{j=1}^{n} \{I_j(B^*;r) - I_j(B^*;r+1)\}.$$

Interchange the order of summation and observe that $I_j(B^*;r) = 0$ when $j < r$. Thus $\lim_{n\to\infty}\sum_{r=1}^{n} M_n(B^*;r)/n = \lim_{n\to\infty}\sum_{j=1}^{n} I_j(B^*;1)/n$. As in Theorem 1, the strong law of large numbers applies and with probability one this limit is equal to $\lim_{n\to\infty}\lambda\sum_{j=1}^{n}\sum_{k=1}^{j} p(k)P[V_B{}^1 = k]/n$. Since $\sum_{n=1}^{\infty} P[V_B{}^1 = k] \leqq 1$ and $p(k) \leqq 1$, this last limit equals $\lambda h(B,1,p)$.

(b) From (2.3) one concludes $\sum_{r=1}^{n} M_n(B^*;r)$ is a Poisson random variable. It is known that the distributions of a sequence of normalized Poisson random variables with means tending to infinity tend to the standard normal distribution. Thus to prove part (b) one need only show $\lim_{n\to\infty}\gamma_n = \infty$. Note $\gamma_n = \sum_{r=1}^{n} EM_n(B^*;r) = \sum_{j=1}^{n} EI_j(B^*;1) = \lambda\sum_{j=1}^{n}\sum_{k=1}^{j} p(k)P[V_B{}^1 = k]$. Thus unless $\sum_{k=1}^{\infty} p(k)P[V_B{}^1 = k] = 0$ which can happen only under the condition given, $\lim_{n\to\infty}\gamma_n = \infty$ as required.

REMARK. One can interpret $\lim_{n\to\infty}\sum_{r=1}^{n} M_n(B^*;r)/n$ as the rate at which particles hit B^* for the first time. If $X(n)$ is persistent and $\pi = 1$, this limit equals

$$\lambda\sum_{k=1}^{\infty} P[V_B{}^r = k] = \lambda P[V_B{}^r < \infty] = \lambda,$$

the rate at which new particles enter the system. In all other cases this rate is, not surprisingly, less than λ.

4. Proof of Theorem 3. For $X(n)$ persistent two cases for the strong law of large numbers for the random variables $S_n(x^*)$ are pertinent. The case where $\lim_{n\to\infty} A_n(x^*) < \infty$ and consequently $\lim_{n\to\infty} n^{-1}S_n(x^*) < \infty$ is considered in part (a). The case where $\lim_{n\to\infty} A_n(x^*)$ is infinite and consequently where

$$\lim_{n\to\infty} n^{-1}ES_n(x^*)$$

is infinite is considered in part (c).

(a) From (2.1) it is known that $\operatorname{Var} A_n(x^*) = \lambda\sum_{k=1}^{n} p(k)P_k(0,x)$. For a fixed state $x^* \in B^*$, under either hypothesis (i) or (ii) the sequence $\{\operatorname{Var} A_n(x^*), n \in I\}$ is bounded. A law of large numbers for dependent random variables is applicable. See Parzen ([3] page 420). For $P[\lim_{n\to\infty} S_n(x^*)/n = \lim_{n\to\infty} ES_n(x^*)/n] = 1$ to hold, it is sufficient that there exist positive constants a and γ such that $|C(n)| \leqq a/n^\gamma$, where $C(n) \equiv \operatorname{Cov}(A_n(x^*), S_n(x^*)/n)$. Finding an expression for $C(n)$ is a tedious exercise in calculating moments and can be accomplished with the use of the following observations. First note

$$E(A_m(x^*)A_n(x^*)) = E(\{\sum_{k=0}^{m-1}\sum_{j=0}^{M_k} X_{jk}\}\{\sum_{v=0}^{n-1}\sum_{u=0}^{M_v} Y_{u,v}\})$$

where X_{jk} (or Y_{jk}) equals one if the jth particle to enter the system at time k is in state x^* at time m (or n) and equals zero otherwise. Now use the properties that X_{jk} and $Y_{u,v}$ are independent unless $j = u$ and $k = v$, and that $\{M_k, k \in I\}$ is a sequence of independent Poisson, λ, random variables independent of X_{jk} and $Y_{u,\iota}$ to find

$$E(A_m(x^*)A_n(x^*)) = \lambda\sum_{k=0}^{m-1} p(n-k)P_{m-k}(0,x)P_{n-m}(x,x) + EA_m(x^*)EA_n(x^*).$$

Thus

$$nC(n) = \lambda \sum_{m=1}^{n} \sum_{k=0}^{m-1} p(n-k)P_{m-k}(0, x)P_{n-m}(x, x)$$

$$= \lambda \sum_{j=1}^{n-1} P_j(0, x) \sum_{k=1}^{n-j} P_k(x, x)p(k+j) \tag{4.1}$$

$$= \lambda \sum_{k=2}^{n} p(k) \sum_{j=1}^{k-1} P_j(0, x)P_{k-j}(x, x). \tag{4.2}$$

From (4.1) and under hypothesis (i), $nC(n) \leqq \lambda\sum_{j=1}^{\infty} P_j(0, x)\sum_{k=1}^{\infty} P_k(x, x) < \infty$ since $X(n)$ is transient. Under hypothesis (ii) use (4.2) to note that $nC(n) \leqq an^{-\delta}$. Hence under either hypothesis $|C(n)|$ decreases sufficiently quickly.

Note $ES_n(x^*) = \lambda\sum_{k=1}^{n}\sum_{j=1}^{k} p(j)P_j(0, x)/n$. Under either hypothesis (i) or (ii) the series $\sum_{k=1}^{\infty} p(k)P_k(0, x)$ is convergent. Hence it follows that $\lim_{n\to\infty} ES_n(x^*)/n = \lambda\sum_{k=1}^{\infty} p(k)P_k(0, x)$. Finally since $A_n(B^*) = \sum_{x^*\in B^*} A_n(x^*)$ and $P_k(0, B) = \sum_{x\in B} P_k(0, x)$ part (a) is proved.

REMARK. Note that if $\pi = 1$, $ES(B^*) = EN(B)$.

(b) It is easily seen that for $x^* \in \Gamma^*$, the sequence $\{\operatorname{Var} A_m(x^*)/m, m = 1, 2, \cdots\}$ is bounded. The same law of large numbers used in part (a) applies here. It must be shown that there exist positive constants a and γ such that $|C^*(n)| \leqq a/n^{\gamma}$ where $nC^*(n) = \operatorname{Cov}(A_n(x^*)/n, \sum_{m=1}^{n} A_m(x^*)/m)$. Using the calculation of the proof in part (a) to note $n^2C^*(n) = \lambda\sum_{m=1}^{n}\sum_{k=0}^{m-1} p(n-k)P_{m-k}(0, x)P_{n-m}(x, x)/m \leqq \lambda n$. Hence $|C^*(n)| \leqq \lambda/n$ and

$$P\left[\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{A_m(x^*)}{m} = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{EA_m(x^*)}{m}\right] = 1.$$

From (2.1) note $E[\sum_{m=1}^{n} A_m(x^*)/m] = \lambda\sum_{m=1}^{n}\sum_{k=1}^{m} p(k)P_k(0, x)/m$. Further $\lim_{k\to\infty} p(k)P_k(0, x) = \pi u(x)$. Thus $\lim_{m\to\infty} Q_m(x) \equiv \lim_{m\to\infty} \sum_{k=1}^{m} p(k)P_k(0, x)/m = \pi u(x)$. A repetition of the argument yields $\lim_{n\to\infty} \lambda\sum_{m=1}^{n} Q_m(x)/n = \lambda\pi u(x)$. Finally, since $A_n(B^*) = \sum_{x^*\in B^*} A_n(x^*)$ and $u(B) = \sum_{x\in B} u(x)$, part (b) is proved.

Before proceeding with the proof of part (c), note that

$$\operatorname{Var} S_n(x^*) = \sum_{k=1}^{n} \operatorname{Var} A_k(x^*) + 2 \operatorname{Cov} \sum_{j=2}^{n} \sum_{i=1}^{j-1} A_i(x^*)A_j(x^*)$$

$$= \lambda ES_n(x^*) + 2\lambda \sum_{j=2}^{n} \sum_{k=2}^{j-1} p(k) \sum_{i=1}^{k-1} P_i(0, x)P_{k-i}(x, x).$$

The last equality is found using (4.2). Since $P_i(x, x) \leqq 1$ and $p(k) \leqq p(i)$ for $i \leqq k$ one obtains $\operatorname{Var} S_n(x^*) \leqq \lambda ES_n(x^*) + 2\lambda\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{i=1}^{k} p(i)P_i(0, x) \leqq \lambda ES_n(x^*) + 2\lambda nES_n(x^*) \leqq 3\lambda nES_n(x^*)$. From this it is easy to see that

$$\lim_{n\to\infty} P\left[\left|\frac{S_n(x^*)}{ES_n(x^*)} - 1\right| > \varepsilon\right] = 0$$

since

$$P\left[\left|\frac{S_n(x^*)}{ES_n(x^*)} - 1\right| > \varepsilon\right] \leqq \frac{\operatorname{Var} S_n(x^*)}{\varepsilon^2 E^2 S_n(x^*)} \leqq \frac{3\lambda n}{\varepsilon^2 ES_n(x^*)},$$

and since the hypothesis $\lim_{n\to\infty} A_n(x^*) = \infty$ implies $\lim_{n\to\infty} nE^{-1}S_n(x^*) = 0$. Now for the proof of part (c).

(c) Let ν and f be as in the hypothesis. Use the Tchebychev inequality as above to obtain

$$P\left[\left|\frac{S_{n^\nu}(x^*)}{ES_{n^\nu}(x^*)}-1\right|>\varepsilon\right]<cf(n^\nu)$$

where c is the constant factor. Since by hypothesis $\sum_{n=1}^\infty f(n^\nu)<\infty$, one has, using the Borel–Cantelli Lemma that $\lim_{n\to\infty} S_{n^\nu}(x^*)E^{-1}S_{n^\nu}(x^*)=1$ a.e. The theorem is proved if it is shown that

$$P\left[\lim_{n\to\infty}\max_{(n-1)^\nu+1\leqq k\leqq n^\nu}\left|\frac{S_{n^\nu}(x^*)}{ES_{n^\nu}(x^*)}-\frac{S_k(x^*)}{ES_k(x^*)}\right|=0\right]=1. \tag{4.3}$$

Define

$$U_n=\max_{(n-1)^\nu+1\leqq k\leqq n^\nu}\left|\frac{S_{n^\nu}(x^*)-S_k(x^*)}{ES_{n^\nu}(x^*)}\right|,\quad V_n=\max_{(n-1)^\nu+1\leqq k\leqq n^\nu}\left|\frac{S_k(x^*)}{ES_{n^\nu}(x^*)}-\frac{S_k(x^*)}{ES_k(x^*)}\right|.$$

It will be shown that $\lim_{n\to\infty}U_n=0$ a.e. and $\lim_{n\to\infty}V_n=0$ a.e. Since the $S_n(x^*)$ are increasing random variables in n, $U_n\leqq\{S_{n^\nu}(x^*)-S_{(n-1)^\nu}(x^*)\}E^{-1}S_{n^\nu}(x^*)$. Note that from above $\lim_{n\to\infty}S_{n^\nu}(x^*)E^{-1}S_{n^\nu}(x^*)=1$ a.e. It is also easily seen using the hypothesis $\lim_{n\to\infty}f(n^\nu)/f([n-1]^\nu)=1$ that $\lim_{n\to\infty}S_{(n-1)^\nu}(x^*)E^{-1}S_{n^\nu}(x^*)=1$ a.e. Thus $\lim_{n\to\infty}U_n=0$ a.e. Since $ES_n(x^*)$ and $S_n(x^*)$ are increasing in n, $V_n\leqq S_{n^\nu}(x^*)\{E^{-1}S_{(n-1)^\nu}(x^*)-E^{-1}S_{n^\nu}(x^*)\}$. Similarly as in the case for U_n, $\lim_{n\to\infty}V_n=0$ a.e. Hence (4.3) holds. Thus $\lim_{n\to\infty}S_n(x^*)E^{-1}S_n(x^*)=1$ a.e.

Finally if the hypotheses hold for each $x^*\in B^*$

$$S_n(B^*)E^{-1}S_n(B^*)=E^{-1}S_n(B^*)\textstyle\sum_{x^*\in B^*}\{S_n(x^*)E^{-1}S_n(x^*)\}ES_n(x^*).$$

Since B^* is finite, $ES_n(B^*)=\sum_{x^*\in B^*}ES_n(x^*)$, and since each term in parentheses in the sum above tends to one a.e., it follows that $\lim_{n\to\infty}S_n(B^*)E^{-1}S_n(B^*)=1$ a.e.

COROLLARY. *Let $X(n)$ be persistent non-null and let $\pi>0$. The conclusion of part* (c) *of the theorem holds.*

PROOF. Under these hypotheses for $\varepsilon>0$, there exists a $K>0$ such that $k>K$ implies

$$\left|k^{-1}\textstyle\sum_{j=1}^k p(j)P_j(0,x)-u(x)\pi\right|<\varepsilon.$$

Thus

$$\lim_{n\to\infty}n^{-2}ES_n(x^*)=\lim_{n\to\infty}n^{-2}\textstyle\sum_{k=K+1}^n\sum_{j=1}^k p(j)P_j(0,x).$$

However,

$$n^{-2}\textstyle\sum_{k=K+1}^n k(u(x)\pi-\varepsilon)\leqq n^{-2}\sum_{k=K+1}^n\sum_{j=1}^k p(j)P_j(0,x)\leqq n^{-2}\sum_{k=K+1}^n k(u(x)\pi+\varepsilon).$$

Thus taking limits as $n\to\infty$, it is clear that $\lim_{n\to\infty}n^{-2}ES_n(x^*)=u(x)\pi/2$. Thus $f(n)$ is asymptotic to cn^{-1} and the hypotheses of the theorem holds for $\nu=2$.

5. Proof of Theorem 4.

LEMMA. *If* $\lim n\{\mathrm{Var}\,S_n(B^*)\}^{-\frac{1}{2}}=0$, *then*

$$\lim_{n\to\infty}P[\{S_n(B^*)-ES_n(B^*)\}\{\mathrm{Var}\,S_n(B^*)\}^{-\frac{1}{2}}<\beta]=(2\pi)^{-\frac{1}{2}}\textstyle\int_{-\infty}^{\beta}e^{-z^2/2}\,dz.$$

PROOF. Observe that $S_n(B^*) = \sum_{r=1}^{n} rM_n(B^*;r)$. The random variable

$$\{rM_n(B^*;r) - rEM_n(B^*;r)\}\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}}$$

is of Poisson type and is thus infinitely divisible. Using the Kolmogorov canonical form, the logarithm of its characteristic function is given by

$$\log \phi_{r,n}(t) = \int_{-\infty}^{\infty}(e^{itx} - 1 - itx)x^{-2}\,dG_{r,n}(x)$$

where

$$\begin{aligned} G_{r,n}(x) &= 0 && \text{for } x \leqq r\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}}, \\ &= \{r^2 EM_n(B^*;\, r)\}\{\operatorname{Var} S_n(B^*)\}^{-1} && \text{otherwise.} \end{aligned}$$

Thus the logarithm of the characteristic function of

$$\{S_n(B^*) - ES_n(B^*)\}\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}}$$

is given by $\log \phi_n(t) = \int_{-\infty}^{\infty}(e^{itx} - 1 - itx)\,dG_n(x)$ where

$$\begin{aligned} G_n(x) &= 0 && \text{for } x \leqq \{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}}, \\ &= \{\textstyle\sum_{r=1}^{k} r^2 M_n(B^*;\, r)\}\{\operatorname{Var} S_n(B^*)\}^{-1} && \text{for } k\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}} < x \leqq \\ & && (k+1)\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}} \\ & && \text{where } 1 \leqq k \leqq n-1, \\ &= 1 && \text{for } x > n\{\operatorname{Var} S_n(B^*)\}^{-\frac{1}{2}}. \end{aligned}$$

Thus using the hypothesis, $\lim_{n\to\infty} \log \phi_n(t) = -t^2/2$, and the lemma is proved.

To prove Theorem 4, first use (2.3) to note that

$$\begin{aligned} \operatorname{Var} S_n(B^*) &= \lambda \textstyle\sum_{r=1}^{n} r^2 \sum_{j=1}^{n} \sum_{k=1}^{j} p(k)\{P[V_B{}^r = k] - P[V_B{}^{r+1} = k]\} \\ &\geqq \lambda p(n) \textstyle\sum_{r=1}^{n} r^2 \sum_{j=1}^{n} P[N_j(B) = r] = \lambda p(n) \sum_{j=1}^{n} EN_j{}^2(B). \end{aligned}$$

The inequality above follows by first bounding $p(k)$ below by $p(n)$ and then observing $P[V_B{}^r \leqq j] - P[V_B{}^{r+1} \leqq j] = P[N_j(B) = r]$. Using hypothesis (ii) of the theorem one finds

$$n^{-2} \operatorname{Var} S_n(B^*) \geqq n^{-2}\lambda\alpha^{-1} p(n) \textstyle\sum_{j=k_0+1}^{n} j^{1+\delta} \geqq \lambda\alpha^{-1}(2+\delta)^{-1} p(n)(n^{\delta} - k_0{}^{2+\delta}/n).$$

Taking limits as n tends to infinity and using (i) in the hypothesis of the theorem one finds the hypothesis of the lemma is satisfied and hence the theorem is proved.

COROLLARY. *Let $X(n)$ be a persistent non-null Markov chain and let*

$$\lim_{n\to\infty} np(n) = \infty.$$

Then the conclusion of the theorem holds.

PROOF. It is known from the strong law of large numbers for Markov chains that $\lim_{n\to\infty} n^{-1}N_n(B) = u(B)$ a.e. (See Chung [1], page 93.) Therefore, since also

$\lim_{n\to\infty} n^{-1}N_n(B) = u(B)$ in probability and $n^{-1}N_n(B) \leqq 1$ with probability 1, it follows that $\lim_{n\to\infty} n^{-2}EN_n{}^2(B) = u^2(B)$. Hence the hypotheses of the theorem are satisfied for $\delta = 1$.

6. Possible applications. In investigating this system no serious attempt has been made to consider applications. It would appear, however, that models of this type can be used to simulate the flow of objects within a set of states. One example of an operations research nature is given tentatively. The movement of rental automobiles within the set Γ of agencies of a national firm if unregulated is Markovian in nature. The removal from service of an auto corresponds to the death of a particle. Since eventual removal is certain, $\pi = 0$. Times are picked as appropriate, (say the beginning of the nth week corresponds to time n.) Then in this example $A_n(x^*)$ denotes the number of automobiles in the territory of agency x at the beginning of the nth week the system is under observation.

According to Theorem 3a, the average number of autos at agency x tends to $\lambda\sum_{k=0}^{\infty} p(k)P_k(0,x)$ with probability one, and by the remark of Section 2 the distribution of $A_n(x^*)$ tends to that of a Poisson random variable with parameter $\lambda\sum_{k=0}^{\infty} p(k)P_k(0,x)$. This last observation suggests a question in control theory. Specifically the natural macroscopic equilibrium of the system may not be a desirable one for the operators of the system. Considering the costs involved, how can one optimally regulate the system by the redistribution of particles to maintain a desirable distribution within the system?

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