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ESTIMATION FOR DISTRIBUTIONS WITH MONOTONE FAILURE RATE

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1. Introduction and summary. In this paper, we shall investigate a problem analogous to the one treated in Prakasa Rao [6]. We shall now suppose that there is a sample of n independent observations from a distribution F with monotone failure rate r and the problem is to obtain a maximum likelihood estimator (MLE) of r and its asymptotic distribution.

Grenander [4] and Marshall and Proschan [5] have obtained the MLE of r and the latter showed that these estimators are consistent. We obtain the asymptotic distribution of this estimator. The estimation problem is reduced at first to that of a stochastic process and the asymptotic distribution of MLE is obtained by means of theorems on convergence of distributions of stochastic processes.

Methods used in this paper are similar to those in Prakasa Rao [6] and therefore, proofs are given only at places where they differ from the proofs of that paper. We shall consider the case of distributions with increasing failure rate (IFR) in detail. Results in the case of distributions with decreasing failure rate (DFR) are analogous to those of IFR and we shall state the main result in Section 7.

Sections 2 and 3 deal with the definition and properties of distributions with monotone failure rate r. Some results related to the asymptotic properties of the MLE of r are given in Section 4. The problem is reduced to that of a stochastic process in Section 5. The asymptotic distribution of the MLE is obtained in Section 6.

2. Some definitions and properties of distributions with monotone failure rate. Let F be a distribution function with density f. The failure rate r of F is defined by

$$r(x) = \frac{f(x)}{1-F(x)} \tag{2.1}$$

for x such that $F(x) < 1$. Let $R(x) = -\log(1-F(x))$. It is easily seen that R is convex on the support of F if and only if r is nondecreasing and that R is concave on the support of F if and only if r is nonincreasing. We say that F is an IFR or DFR distribution according as r is nondecreasing or nonincreasing. Properties of distributions with monotone failure rate are discussed in Barlow, Marshall and Proschan [1].

3. MLE for IFR. Suppose that F is an IFR with failure rate r. Further suppose that the support of F is the interval $[0, \infty)$. Let X_i, $1 \leqq i \leqq n$ be the order statistics of a random sample of size n from the population with distribution F. Let $\mathscr{F}$ denote

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the class of all IFR distributions. It is not possible to obtain the MLE for r directly by maximizing the likelihood

$$L(F) = \prod_1^n f(X_i)$$

since $f(X_n)$ can be chosen arbitrarily large. Therefore, we consider a subfamily $\mathscr{F}^M$ of $\mathscr{F}$ consisting of distributions F with $r(x) \leqq M$ for all x, obtaining

$$\sup_{F \in \mathscr{F}^M} \prod_1^n f(X_i) \leqq M^n.$$

There is a unique distribution $\hat{F}_n{}^M \in \mathscr{F}^M$ for which the above supremum is attained. The failure rate $\hat{r}_n{}^M$ of $\hat{F}_n{}^M$ converges to a failure rate $\hat{r}_n$ as M approaches ∞ for $x < X_n$. For $x \geqq X_n$, $\hat{r}_n{}^M = M$ for all M and therefore $\hat{r}_n{}^M \to \infty$ as $M \to \infty$. This estimator $\hat{r}_n$, which is infinite for $x \geqq X_n$, is called the MLE of r. Let $X_0 \equiv 0$.

From the results of Grenander [4] or as an application of the results of Van Eeden [8], the estimator $\hat{r}_n$ can be derived and it is given by

$$\begin{aligned} \hat{r}_n(x) &= 0 && \text{for} \quad x < X_1, \\ &= \hat{r}_n(X_i) && \text{for} \quad X_i \leqq x < X_{i+1}, 1 \leqq i \leqq n-1, \\ &= \infty && \text{for} \quad x \geqq X_n, \end{aligned}$$

where

$$\hat{r}_n(X_i) = \min_{v \geqq i+1} \max_{u \leqq i} \{[v-u][\textstyle\sum_{j=u}^{v-1} (n-j)(X_{j+1}-X_j)]^{-1}\}.$$

This estimator has been shown to be strongly consistent at all x at which r is continuous by Marshall and Proschan [5]. This estimator can also be written in the form

$$\hat{r}_n(X_i) = \inf_{v \geqq x} \sup_{u < x} \frac{F_n(v) - F_n(u)}{\int_u^v [1 - F_n(y)]\, dy} \tag{3.1}$$

where F_n is the empirical distribution function. In fact

$$[\hat{r}_n(X_i)]^{-1} = \sup_{v \geqq (i+1)/n} \inf_{u \leqq i/n} \frac{\Phi_n(v) - \Phi_n(u)}{v-u} \tag{3.2}$$

where

$$\Phi_n(j/n) = \int_0^{X_j} [1 - F_n(x)]\, dx. \tag{3.3}$$

Define

$$\begin{aligned} \Phi_n(\eta) &= 0, && 0 \leqq \eta < 1/n, \\ &= \Phi_n(j/n), && j/n \leqq \eta < (j+1)/n, \qquad 1 \leqq j \leqq n-1 \\ &= \Phi_n(1), && \eta = 1. \end{aligned} \tag{3.4}$$

Let $\hat{\Phi}_n$ be the concave majorant of Φ_n. It follows that $[\hat{r}_n(x)]^{-1}$ is the left-hand derivative of the concave majorant $\hat{\Phi}_n$ at $F_n(x)$ from (3.2).

4. Some results related to the asymptotic properties of the MLE. Let ξ be such that $0 < F(\xi) < 1$. Let $\hat{r}_n(\xi)$ denote the MLE of r at ξ and let $[r^*_{n,c}(\xi)]^{-1}$ denote the left-hand derivative of the concave majorant of Φ_n at $F_n(\xi)$ when the argument of Φ_n is restricted to the interval $[F(\xi)-cn^{-\frac{1}{3}}, F(\xi)+cn^{-\frac{1}{3}}]$. We shall assume that the failure rate r is differentiable at ξ with $r(\xi) > 0$ and $r'(\xi)$ different from zero. The following lemma can be proved by methods analogous to those used in Prakasa Rao [6]. We shall make use of Theorem 4.6 (proved later in this section) in the proof of the lemma. We shall indicate only those steps in the proof which will guide the reader in constructing a proof of this lemma.

LEMMA 4.1. *There is a function $\lambda(c)$ such that*

(i) $\limsup_{n\to\infty} P[r^*_{n,c}(\xi) \neq \hat{r}_n(\xi)] \leqq \lambda(c)$ *and*
(ii) $\lambda(c) \to 0$ *as* $c \to \infty$.

Let $F(\xi) = \eta$, $A = cr'(\xi)[f(\xi)r^2(\xi)]^{-1}$ and $g(\xi) = [r(\xi)]^{-1}$, and define

$$\begin{aligned} I_n(\xi) &= n[\Phi_n(\eta+cn^{-\frac{1}{3}}) - cn^{-\frac{1}{3}}\{g(\xi) - An^{-\frac{1}{3}}\} - \Phi_n(\eta)] \\ &= n[\Phi_n(\eta+cn^{-\frac{1}{3}}) - \Phi_n(\eta)] - cn^{\frac{2}{3}}g(\xi) + Acn^{\frac{1}{3}}. \end{aligned}$$

Now by Theorem 4.6, it follows that

$$\begin{aligned} I_n(\xi) &= n^{\frac{1}{3}}W_n(c)g(\xi) - \tfrac{1}{2}c^2n^{\frac{1}{3}}r'(\xi)[f(\xi)r^2(\xi)]^{-1} + cn^{\frac{2}{3}}g(\xi) \\ &\quad + O_p(n^{\frac{1}{6}}) - cn^{\frac{2}{3}}g(\xi) + Acn^{\frac{1}{3}}, \\ &= n^{\frac{1}{3}}[W_n(c)g(\xi) + \tfrac{1}{2}c^2r'(\xi)\{f(\xi)r^2(\xi)\}^{-1} + O_p(n^{-\frac{1}{6}})], \\ &= n^{\frac{1}{3}}[W_n(c)g(\xi) + \tfrac{1}{2}c^2B(\eta) + O_p(n^{-\frac{1}{6}})], \end{aligned}$$

where $B(\eta) = r'(\xi)[f(\xi)r^2(\xi)]^{-1}$. Notice that $B(\eta) > 0$. Let $B_n = \frac{1}{2}c^2B(\eta) + O_p(n^{-\frac{1}{6}})$ and $V_n = c^{-1}W_n(c)$. Then we obtain that

$$I_n(\xi) = n^{\frac{1}{3}}[cg(\xi)V_n + B_n].$$

Clearly $E[V_n] = 0$ and $\mathrm{Var}[V_n] = 1$.

It is now easy to see with these definitions of I_n, V_n and B_n, one stage of the proof of Lemma 4.1 can be completed along the same lines as in Lemma 4.1 of Prakasa Rao [6]. At a later stage in the proof in [6], we have introduced a new distribution. In an analogous way, we introduce an exponential distribution function $\tilde{F}$ with failure rate

$$\begin{aligned} r(x) &= 0 \qquad && x \leqq 0 \\ &= r(\xi) && x > 0. \end{aligned}$$

Let Y be distributed as $\tilde{F}$. Then $Y_i = -[r(\xi)]^{-1}\log(1-\tilde{F}(X_i))$ form a set of order statistics from the exponential distribution and $(n-i)(Y_{i+1} - Y_i)$ are independent identically distributed exponential random variables with mean $[r(\xi)]^{-1}$. It is not hard to see that $Y_{i+1}-Y_i \leqq X_{i+1}-X_i$ for all $i \leqq k(n)$ where $k(n)$ is chosen so that $X_{k(n)+1} \leqq \xi < X_{k(n)+2}$. In view of this fact, it is easily seen that an inequality similar to (4.16) of Prakasa Rao [6] holds good, since for $\zeta \leqq \eta$,

$$\Phi_n(\eta)-\Phi_n(\zeta) = \textstyle\sum_{j=[\zeta n]}^{[\eta n]-1} (n-j)(X_{j+1}-X_j).$$

Furthermore, we can introduce a Poisson process as in Prakasa Rao [6] since the inter-arrival times in a Poisson process are independent random variables with the same exponential distribution, and $\Phi_n(\eta)-\Phi_n(\zeta)$ is the sum of independent random variables with the same exponential distribution when $\tilde{F}$ is the underlying distribution. The rest of the proof can be completed in a way similar to that in Prakasa Rao [6]. With these remarks, we omit the complete proof of Lemma 4.1.

As a consequence of our assumptions, it follows that there exist constants γ, α and K such that

$$f(x) \geqq \gamma, \qquad r(x) \geqq \alpha \tag{4.1}$$

and

$$|f'(x)| \leqq K\,|f(x)| \tag{4.2}$$

for x in a neighborhood of ξ. Let $F_n(\xi) = \eta_n$ and $F(\xi) = \eta$. It is well known that $\eta_n-\eta = O_p(n^{-\frac{1}{2}})$. Let

$$U_j = F(X_{j+1})-F(X_j)-E[F(X_{j+1})-F(X_j)\,|\,X_j] \tag{4.3}$$

where X_i, $1 \leqq i \leqq n$ are the order statistics and $E[Y|X]$ denotes the conditional expectation of Y given X. It is easy to see that

$$U_j = F(X_{j+1})-F(X_j)-\lambda(X_j) \tag{4.4}$$

where

$$\lambda(X_j) = (1-F(X_j))/(n-j+1). \tag{4.5}$$

We shall obtain the necessary asymptotic expansions in a series of lemmas which will be combined at the end of the section to give the final result. We mention here that the approximations which are of the order O_p are all satisfied uniformly for δ in $[-c, c]$. Let

$$a = [\eta n], \qquad b = [\eta n+\delta n^{\frac{2}{3}}]. \tag{4.6}$$

LEMMA 4.2.

$$n\left[\Phi_n\left(\frac{b}{n}\right)-\Phi_n\left(\frac{a}{n}\right)\right] = \sum_{j=a}^{b-1}(n-j)\frac{\lambda(X_j)+U_j}{f(X_j)}+O_p(n^{-\frac{1}{3}}).$$

PROOF. It is easy to see from (4.4) by Taylor's theorem, that

$$X_{j+1}-X_j = \frac{\lambda(X_j)+U_j}{f(X_j)}-\tfrac{1}{2}[\lambda(X_j)+U_j]^2\frac{f'(\zeta_j)}{f^3(\zeta_j)} \tag{4.7}$$

where $\zeta_j = F^{-1}(\theta_j)$ and θ_j lies between $F(X_j)$ and $F(X_{j+1})$. By definition of Φ_n,

$$(4.8)\quad n\left[\Phi_n\left(\frac{b}{n}\right)-\Phi_n\left(\frac{a}{n}\right)\right]$$
$$= \sum_{j=a}^{b-1}(n-j)(X_{j+1}-X_j)$$
$$= \sum_{j=a}^{b-1}\left\{\frac{\lambda(X_j)+U_j}{f(X_j)}\right\}(n-j)-\tfrac{1}{2}\sum_{j=a}^{b-1}(n-j)\frac{f'(\zeta_j)}{f^3(\zeta_j)}(\lambda(X_j)+U_j)^2.$$

Since X_j, for $a \leqq j \leqq b-1$, lies in a neighborhood of ξ for large n and since $\zeta_j = F^{-1}(\theta_j)$ where $F(X_j) \leqq \theta_j \leqq F(X_{j+1})$, it follows that ζ_j is in a neighborhood of ξ for $a \leqq j \leqq b-1$ for large n. Hence we obtain from (4.1) and (4.2) that $f(\zeta_j) \geqq \gamma$ and $|f'(\zeta_j)| \leqq K|f(\zeta_j)|$. It now follows that for sufficiently large n,

$$E\sum_{j=a}^{b-1}\frac{f'(\zeta_j)}{f^3(\zeta_j)}(n-j)\{\lambda(X_j)+U_j\}^2 \leqq K\gamma^{-2}\sum_{j=a}^{b-1}(n-j)E[\lambda(X_j)+U_j]^2$$
$$(4.9)\quad = K\gamma^{-2}\sum_{j=a}^{b-1}(n-j)E[F(X_{j+1})-F(X_j)]^2$$
$$= 2K\gamma^{-2}(n+1)^{-1}(n+2)^{-1}\sum_{j=a}^{b-1}(n-j)$$
$$= O(n^{-\frac{1}{3}}).$$

Equations (4.8) and (4.9) together prove the lemma.

LEMMA 4.3.

$$n[\Phi_n(b/n)-\Phi_n(a/n)] = \textstyle\sum_{j=a}^{b-1}(n-j)U_jf^{-1}(X_j)+\sum_{j=a}^{b-1}r^{-1}(X_j)+O_p(n^{-\frac{1}{3}}).$$

PROOF. By Lemma 4.2, we have

$$n[\Phi_n(b/n)-\Phi_n(a/n)]$$
$$= \textstyle\sum_{j=a}^{b-1}(n-j)f^{-1}(X_j)[\lambda(X_j)+U_j]+O_p(n^{-\frac{1}{3}})$$
$$(4.10)\quad = \textstyle\sum_{j=a}^{b-1}(n-j)U_jf^{-1}(X_j)+\sum_{j=a}^{b-1}(n-j)(n-j+1)^{-1}r^{-1}(X_j)+O_p(n^{-\frac{1}{3}})$$
$$= \textstyle\sum_{j=a}^{b-1}(n-j)U_jf^{-1}(X_j)+\sum_{j=a}^{b-1}r^{-1}(X_j)$$
$$-\textstyle\sum_{j=a}^{b-1}(n-j+1)^{-1}r^{-1}(X_j)+O_p(n^{-\frac{1}{3}}).$$

It is easy to see that for large n,

$$(4.11)\quad E\left|\textstyle\sum_{j=a}^{b-1}(n-j+1)^{-1}r^{-1}(X_j)\right| \leqq \alpha\textstyle\sum_{j=a}^{b-1}(n-j+1)^{-1} = O(n^{-\frac{1}{3}}).$$

This observation together with (4.10) proves the lemma.

LEMMA 4.4.

$$\sum_{j=a}^{b-1} r^{-1}(X_j) = (b-a)r^{-1}(Z(\eta)) - \tfrac{1}{2}\delta^2 n^{\frac{1}{3}} r'(Z(\eta))r^{-2}(Z(\eta))f^{-1}(Z(\eta)) + O_p(n^{\frac{1}{6}})$$

where $Z(t) = F^{-1}(t)$ *for* $0 \leqq t \leqq 1$.

PROOF. Let $Z_j = Z(j/n)$. It follows by the Kolmogorov–Smirnov Theorem and the fact that f is bounded away from zero, that

$$\sup_{a \leqq j \leqq b} |X_j - Z_j| = O_p(n^{-\frac{1}{2}}). \tag{4.12}$$

Since $r^{-1}(X_j) - r^{-1}(Z_j) = -r'(\zeta_j)r^{-2}(\zeta_j)(X_j - Z_j)$ for some ζ_j between X_j and Z_j, we have

$$\sum_{j=a}^{b-1} r^{-1}(X_j) - \sum_{j=a}^{b-1} r^{-1}(Z_j) = -\sum_{j=a}^{b-1} r'(\zeta_j)r^{-2}(\zeta_j)(X_j - Z_j). \tag{4.13}$$

The term on the right-hand side is $O_p(n^{\frac{1}{6}})$ by (4.12) since $r'(\cdot)r^{-2}(\cdot)$ is bounded in a neighborhood of ξ. Again, applying Taylor's Theorem, we get that

$$r^{-1}(Z_j) = r^{-1}(Z(\eta)) - (n^{-1}j - \eta)r'(Z(\eta))r^{-2}(Z(\eta)f^{-1}(Z(\eta)) + (n^{-1}j - \eta)o(1) \tag{4.14}$$

which implies that

$$\sum_{j=a}^{b-1} r^{-1}(Z_j) = (b-a)r^{-1}(Z(\eta)) - \tfrac{1}{2}\delta^2 n^{\frac{1}{3}} r'[Z(\eta)]r^{-2}[Z(\eta)]f^{-1}[Z(\eta)] + o(n^{\frac{1}{3}}). \tag{4.15}$$

Equations (4.13) and (4.15) together prove this lemma.

LEMMA 4.5.

$$\sum_{j=a}^{b-1} (n-j)U_j f^{-1}(X_j) = f^{-1}[Z(\eta)] \sum_{j=a}^{b-1} (n-j)U_j + O_p(n^{\frac{1}{6}}).$$

PROOF. By the Kolmogorov–Smirnov Theorem, it can be shown as before that

$$f^{-1}(X_j) = f^{-1}(Z_j) + O_p(n^{-\frac{1}{2}}) \tag{4.16}$$

uniformly in j. Hence

$$\sum_{j=a}^{b-1}(n-j)U_j f^{-1}(X_j) = \sum_{j=a}^{b-1}(n-j)U_j f^{-1}(Z_j) + \sum_{j=a}^{b-1} (n-j)U_j O_p(n^{-\frac{1}{2}}).$$

But

$$E|(n-j)U_j O_p(n^{-\frac{1}{2}})|^2 \leqq E[(n-j)U_j]^2 E[O_p(n^{-\frac{1}{2}})]^2 = O(n^{-1}) \tag{4.17}$$

uniformly in j. (4.16) and (4.17) together imply that

$$\sum_{j=a}^{b-1} (n-j)U_j f^{-1}(X_j) = \sum_{j=a}^{b-1} (n-j)U_j f^{-1}(Z_j) + O_p(n^{\frac{1}{6}}). \tag{4.18}$$

Since $f^{-1}(Z_j) = f^{-1}(Z(\eta)) + (n^{-1}j - \eta)[-f'(Z(\eta))f^{-2}(Z(\eta)) + o(1)]$, it follows that

$$\sum_{j=a}^{b-1} (n-j)U_j f^{-1}(Z_j) = f^{-1}(Z(\eta)) \sum_{j=a}^{b-1} (n-j)U_j + M_n \tag{4.19}$$

where

$$M_n = \sum_{j=a}^{b-1} (n-j)(n^{-1}j - \eta)[-f'(Z(\eta))f^{-2}(Z(\eta)) + o(1)]U_j. \tag{4.20}$$

Clearly $E(M_n) = 0$ and $\operatorname{Var}(M_n) \leqq K^2\gamma^{-2}\sum_{j=a}^{b-1}\{(n-j)^2(n^{-1}j - \eta)^2 \operatorname{Var}(U_j)\} = O(1)$, since $\operatorname{Var}(U_j) = O(n^{-2})$ uniformly in j. Therefore, we have $M_n = O_p(1)$. Now (4.18), (4.19) and (4.20) together show that

$$\begin{aligned}\sum_{j=a}^{b-1}(n-j)U_j f^{-1}(X_j) &= f^{-1}(Z(\eta))\sum_{j=a}^{b-1}(n-j)U_j + o_p(n^{\frac{1}{6}}) + O_p(1)\\ &= f^{-1}(Z(\eta))\sum_{j=a}^{b-1}(n-j)U_j + O_p(n^{\frac{1}{6}})\end{aligned}$$

which proves the lemma.

We have the following theorem from Lemmas 4.3, 4.4 and 4.5.

THEOREM 4.6. *Let ξ be such that $0 < F(\xi) < 1$ and let $F(\xi) = \eta$. Suppose that $-c \leqq \delta \leqq c$. Let $a = [\eta n]$ and $b = [\eta n + \delta n^{\frac{2}{3}}]$. Then*

$$n\left[\Phi_n\left(\frac{b}{n}\right) - \Phi_n\left(\frac{a}{n}\right)\right] = \frac{n^{\frac{1}{3}}W_n(\delta)}{r[Z(\eta)]} - \frac{\delta^2 n^{\frac{1}{3}} r'[Z(\eta)]}{2f[Z(\eta)]r^2[Z(\eta)]} + \frac{\delta n^{\frac{2}{3}}}{r[Z(\eta)]} + O_p(n^{\frac{1}{6}})$$

where $W_n(\delta) = n^{\frac{2}{3}}(n-a)^{-1}\sum_{j=a}^{b-1}(n-j+1)U_j$.

PROOF. By Lemmas 4.3, 4.4 and 4.5, we have the following result.

$$n\left[\Phi_n\left(\frac{b}{n}\right) - \Phi_n\left(\frac{a}{n}\right)\right] = \frac{1}{f[Z(\eta)]}\sum_{j=a}^{b-1}(n-j)U_j + \frac{\delta n^{\frac{2}{3}}}{r[Z(\eta)]} - \tfrac{1}{2}[\delta^2 n^{\frac{1}{3}} B(\eta)] + O_p(n^{\frac{1}{6}}) \tag{4.21}$$

where

$$B(\eta) = f^{-1}(Z(\eta) r'(Z(\eta)) r^{-2}(Z(\eta))). \tag{4.22}$$

Therefore

$$n\left[\Phi_n\left(\frac{b}{n}\right) - \Phi_n\left(\frac{a}{n}\right)\right] = \frac{n(1-\eta)}{f|Z(\eta)|}\sum_{j=a}^{b-1}\frac{n-j}{n-a}U_j + T_n + \frac{\delta n^{\frac{2}{3}}}{r[Z(\eta)]} - \tfrac{1}{2}[\delta^2 n^{\frac{1}{3}} B(\eta)] + O_p(n^{\frac{1}{6}}) \tag{4.23}$$

where $T_n = (\eta n - a)(n-a)^{-1} f^{-1}(Z(\eta))\sum_{j=a}^{b-1}(n-j)U_j$. Obviously $E|T_n| = O(n^{-\frac{1}{3}})$ since $b-a = O(n^{\frac{2}{3}})$ and $E|U_j| = O(n^{-1})$ uniformly in j. We obtain now from (4.23), that

$$n\left[\Phi_n\left(\frac{b}{n}\right) - \Phi_n\left(\frac{a}{n}\right)\right] = \frac{n}{r[Z(\eta)]}\sum_{j=a}^{b-1}\frac{n-j+1}{n-a}U_j - R_n + \frac{\delta n^{\frac{2}{3}}}{r[Z(\eta)]} - \tfrac{1}{2}[\delta^2 n^{\frac{1}{3}} B(\eta)] + O_p(n^{\frac{1}{6}}) \tag{4.24}$$

where $R_n = n(n-a)^{-1} r^{-1}(Z(\eta))\sum_{j=a}^{b-1} U_j$. Clearly $E|R_n| = O(n^{-\frac{1}{3}})$. Hence

$$n\left[\Phi_n\left(\frac{b}{n}\right) - \Phi_n\left(\frac{a}{n}\right)\right] = \frac{n^{\frac{1}{3}}}{r[Z(\eta)]}W_n(\delta) - \tfrac{1}{2}[\delta^2 n^{\frac{1}{3}} B(\eta)] + \frac{\delta n^{\frac{2}{3}}}{r[Z(\eta)]} + O_p(n^{\frac{1}{6}})$$

where $B(\eta)$ is given by (4.22) and

$$W_n(\delta) = n^{\frac{2}{3}}(n-a)^{-1}\sum_{j=a}^{b-1}(n-j+1)U_j. \tag{4.25}$$

REMARK. Note that $B(\eta) > 0$ since r is nondecreasing and $r'(\xi) \neq 0$.

5. Reduction to a problem in stochastic processes. In this section, we shall reduce the problem of calculating the asymptotic distribution of the slope of the concave majorant of $\Phi_n(y)$ over $[F(\xi)-cn^{-\frac{1}{3}}, F(\xi)+cn^{-\frac{1}{3}}]$ at $y = F_n(\xi)$ to the corresponding problem of the Wiener process over $[-c, c]$ after suitable normalization. We shall assume that the failure rate r is differentiable at ξ with $r'(\xi) \neq 0$. Furthermore, we assume that ξ is such that $0 < F(\xi) < 1$, $r(\xi) > 0$ as before.

Let $F(\xi) = \eta$, $F_n(\xi) = \eta_n$, $a = [\eta n]$ and $b = [\eta n + \delta n^{\frac{2}{3}}]$ as before where F_n is the empirical distribution function and $-c \leqq \delta \leqq c$. It follows from Theorem 4.6, that

$$n^{\frac{2}{3}}\left[\Phi_n\left(\frac{b}{n}\right)-\Phi_n\left(\frac{a}{n}\right)\right] = \frac{W_n(\delta)}{r[Z(\eta)]}-\tfrac{1}{2}\delta^2 B(\eta)+\frac{\delta n^{\frac{1}{3}}}{r[Z(\eta)]}+O_p(n^{-\frac{1}{6}}). \tag{5.1}$$

Let $\alpha_n(\eta)+\delta\beta_n(\eta)$ denote the tangent to the concave majorant of

$$\frac{W_n(\delta)}{r[Z(\eta)]}-\tfrac{1}{2}\delta^2 B(\eta)+O_p(n^{-\frac{1}{6}})$$

at $\delta_n = (\eta_n-\eta)n^{\frac{1}{3}}$. We note that

$$\beta_n(\eta) = n^{\frac{1}{3}}[r_{n,c}^{*-1}(Z(\eta))-r^{-1}(Z(\eta))] \tag{5.2}$$

where $r_{n,c}^{*-1}(Z(\eta))$ denotes the left-hand derivative of the concave majorant of $\Phi_n(y)$ over $[\eta-cn^{-\frac{1}{3}}, \eta+cn^{-\frac{1}{3}}]$ at $y = \eta_n$. Let

$$\delta = \lambda\zeta; \qquad \lambda = 2^{\frac{2}{3}}[r[Z(\eta)]B(\eta)]^{-\frac{2}{3}}; \qquad V_n(\zeta) = \lambda^{-\frac{1}{2}}W_n(\delta). \tag{5.3}$$

It is easy to see that

$$\begin{aligned}\frac{W_n(\delta)}{r[Z(\eta)]}&-\tfrac{1}{2}\delta^2 B(\eta)-\alpha_n(\eta)-\delta\beta_n(\eta)+O_p(n^{-\frac{1}{6}}) \\ &= \lambda^{\frac{1}{2}}r^{-1}[Z(\eta)]\left[V_n(\zeta)-\left(\zeta+\frac{\beta_n(\eta)}{\lambda B(\eta)}\right)^2-\left(\frac{2\alpha_n(\eta)}{\lambda^2 B(\eta)}-\frac{\beta_n^{\,2}(\eta)}{\lambda^2 B^2(\eta)}\right)\right]+O_p(n^{-\frac{1}{6}}).\end{aligned} \tag{5.4}$$

We note that $2\beta_n(\eta)\lambda^{-1}B^{-1}(\eta)$ is the slope of the concave majorant at $\zeta = \zeta_n = \lambda^{-1}(\eta_n-\eta)n^{\frac{1}{3}}$ of the process

$$X_n(\zeta) = V_n(\zeta)-\zeta^2+O_p(n^{-\frac{1}{6}}) \tag{5.5}$$

on $[-q, q]$ where $q = c\lambda^{-1}$. Note that $\zeta_n = O_p(n^{-\frac{1}{6}})$ and $2\lambda^{-1}B^{-1}(\eta)$ is the scale factor which multiplies the slope $\beta_n(\eta)$ by applying the transformations in (5.3).

Let $C[a, b]$ denote the space of continuous functions on $[a, b]$ and $D[a, b]$ denote the space of functions on $[a, b]$ with at most discontinuities of first kind only. Let us induce convergence in $D[a, b]$ by Skorokhod's J_1-topology. It is well known that $C[a, b]$ with a supremum norm topology is a closed subset of $D[a, b]$ with J_1-topology. We say that a sequence of stochastic processes X_n with trajectories in $D[a, b]$ a.s. converges in distribution to another process X with trajectories in $D[a, b]$ a.s. if the measures ν_n induced by X_n on $D[a, b]$ converge weakly to the measure ν induced by X on $D[a, b]$.

A stochastic process W on $[-q,q]$ is said to be a Wiener process if it is a Gaussian process with stationary independent increments with (i) $W(0)=0$ (ii) $E[W(t)]=0$ for $|t| \leqq q$ and (iii) $\operatorname{Var}[W(t)]=|t|$. It is well known that the trajectories of W are in $C[-q,q]$ a.s. and hence, in particular, in $D[-q,q]$ a.s. Proof of this fact can be found in Doob [3].

It is obvious that the trajectories of V_n are in $D[-q,q]$. Let μ_n and μ be the measures induced by V_n and W respectively on $D[-q,q]$. Our aim is to prove that μ_n converges to μ weakly. Some lemmas will be proved which lead us to the required result.

LEMMA 5.1. *For any δ in $[-c,c]$, $W_n(\delta)$ is asymptotically normal with mean* 0 *and variance* $|\delta|$.

PROOF. Let Δ_i, $1 \leqq i \leqq n+1$ be $(n+1)$ independent random variables each with the exponential distribution with density function

$$\begin{aligned} g(x) &= n\,e^{-nx} \qquad x \geqq 0, \\ &= 0 \qquad\quad\;\; x<0. \end{aligned}$$

Let $D_n=\Delta_1+\cdots+\Delta_{n+1}$ and $G_i=D_n^{-1}[\Delta_1+\cdots+\Delta_i]$, $1 \leqq i \leqq n$. Then G_i, $1 \leqq i \leqq n$ form order statistics of a sample of size n from the uniform distribution on $[0,1]$. By definition,

$$\begin{aligned} W_n(\delta) &= n^{\frac{2}{3}} \sum_{j=a}^{b-1} \frac{n-j+1}{n-a} U_j \\ &= n^{\frac{2}{3}} \sum_{j=a}^{b-1} \frac{n-j+1}{n-a}\left[F(X_{j+1})-F(X_j)-\frac{1-F(X_j)}{n-j+1}\right]. \end{aligned} \tag{5.6}$$

It is easy to see from our earlier observation that $W_n(\delta)$ has the same distribution as

$$\begin{aligned} W_n^*(\delta) &= n^{\frac{2}{3}} \sum_{j=a}^{b-1} \frac{n-j+1}{n-a}\left[G_{j+1}-G_j-\frac{1-G_j}{n-j+1}\right] \\ &= \frac{n^{\frac{2}{3}}}{D_n(n-a)} \sum_{j=a}^{b-1} (n-j+1)\left[\left(\Delta_{j+1}-\frac{1}{n}\right)-\left(\frac{\Delta_{j+1}+\cdots+\Delta_{n+1}}{n-j+1}-\frac{1}{n}\right)\right]. \end{aligned} \tag{5.7}$$

Let

$$A_n(\delta) = \frac{n^{\frac{2}{3}}}{n-a} \sum_{j=a}^{b-1} (n-j+1)\left(\Delta_{j+1}-\frac{1}{n}\right), \tag{5.8}$$

and

$$B_n(\delta) = \frac{n^{\frac{2}{3}}}{n-a} \sum_{j=a}^{b-1} \left(\Delta_{j+1}+\cdots+\Delta_{n+1}-\frac{n-j+1}{n}\right). \tag{5.9}$$

Clearly $E(B_n(\delta))=0$ and

$$\begin{aligned} \operatorname{Var}(B_n(\delta)) &= n^{\frac{4}{3}}(n-a)^{-2} \operatorname{Var}\left[\textstyle\sum_{j=a}^{b-1} (\Delta_{j+1}+\cdots+\Delta_{n+1})\right] \\ &= n^{\frac{4}{3}}(n-a)^{-2} \operatorname{Var}\left[\textstyle\sum_{j=a}^{b-1} (j-a)\Delta_j + (b-a)\sum_{j=b}^{n+1} \Delta_j\right] \\ &= o(1) \end{aligned}$$

since Δ_j are independent, $\mathrm{Var}(\Delta_j) = n^{-2}$ and $b-a = O(n^{\frac{2}{3}})$. Hence $B_n(\delta) \to 0$ in probability as $n \to \infty$. Furthermore, $D_n \to 1$ in probability. Since $W_n^*(\delta) = D_n^{-1} [A_n(\delta) - B_n(\delta)]$, it is enough to prove that $A_n(\delta)$ is asymptotically normal wtih mean 0 and variance $|\delta|$ in order to complete the proof of the lemma, by Slutsky's Theorem (See Cramér [2]). Let $\psi_n(t)$ denote the characteristic function of $A_n(\delta)$. It is easy to show after some computations that

$$\begin{aligned}\log \psi_n(t) &= -t^2 n^{-\frac{4}{3}} [2(n-a)]^{-2} \textstyle\sum_{j=a}^{b-1} (n-j+1)^2 + o(1) \\ &= -\tfrac{1}{2} t^2 |\delta| + o(1).\end{aligned}$$

Hence by the continuity theorem for characteristic functions, it follows that $A_n(\delta)$ is asymptotically normal with mean 0 and variance $|\delta|$, which proves the lemma.

LEMMA 5.2. *For any ζ in $[-q, q]$, $V_n(\zeta)$ is asymptotically normal with mean* 0 *and variance* $|\zeta|$.

PROOF. This follows immediately from the previous lemma since $V_n(\zeta) = \lambda^{-\frac{1}{2}} W_n(\delta)$.

REMARK. In a similar way, it can be shown that for any collection ζ_i, $1 \leqq i \leqq K$, $|\zeta_i| \leqq q$, the joint distribution of $[V_n(\zeta_1), \cdots, V_n(\zeta_K)]$ converges to the multivariate normal distribution with mean 0 and variance-covariance matrix $(\delta(\zeta_i, \zeta_j) \min(|\zeta_i|, |\zeta_j|))$, where

$$\begin{aligned}\delta(a,b) &= 1 \qquad \text{if } a, b \text{ are of the same sign} \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Define $V_n^*(\zeta) = \lambda^{-\frac{1}{2}} W_n^*(\delta)$ for $\zeta \in [-q, q]$. Clearly V_n^* has the same distribution as V_n. The next lemma shows that the family of processes $\{D_n V_n^*\}$ on $[-q, q]$ (D_n is as defined in Lemma 5.1) satisfies an equi-continuity condition. In other words, the class of measures induced by the processes $\{D_n V_n^*\}$ on $D[-q, q]$ forms a tight family. See Sethuraman [7] for the definition of tightness of family of probability measures.

LEMMA 5.3. *For any ζ_1, ζ_2 in $[-q, q]$,*

$$E|D_n V_n^*(\zeta_1) - D_n V_n^*(\zeta_2)|^4 \leqq C|\zeta_1 - \zeta_2|^2 + |\zeta_1 - \zeta_2| o(1)$$

where C is a constant independent of n, ζ_1 and ζ_2.

PROOF. Define $A_n(\delta)$ and $B_n(\delta)$ as in Lemma 5.1. Consider

$$\begin{aligned}(5.10) \qquad & E|D_n V_n^*(\zeta_1) - D_n V_n^*(\zeta_2)|^4 \\ &= \lambda^{-2} E|D_n W_n^*(\delta_1) - D_n W_n^*(\delta_2)|^4 \\ &= \lambda^{-2} E|\{A_n(\delta_1) - A_n(\delta_2)\} - \{B_n(\delta_1) - B_n(\delta_2)\}|^4 \\ &\leqq 8\lambda^{-2}\{E|A_n(\delta_1) - A_n(\delta_2)|^4 + E|B_n(\delta_1) - B_n(\delta_2)|^4\}\end{aligned}$$

by the elementary inequality $E|X+Y|^4 \leqq 8[E|X|^4 + E|Y|^4]$.

Let $b_1 = [\eta n + \delta_1 n^{\frac{2}{3}}]$ and $b_2 = [\eta n + \delta_2 n^{\frac{2}{3}}]$. Let us first compute

$$(5.11)\quad E\,|A_n(\delta_1) - A_n(\delta_2)|^4 = n^{-\frac{4}{3}}(1-an^{-1})^{-4}E\,|\sum_{j=b_1}^{b_2-1}(n-j+1)(\Delta_{j+1}-n^{-1})|^4.$$

Since Δ_j are independent and identically distributed with $E(\Delta_j) = n^{-1}$, $\operatorname{Var}(\Delta_j) = n^{-2}$ and $E(\Delta_j - n^{-1})^4 = 9n^{-4}$, it follows from (5.11), that

$$\begin{aligned}(5.12)\quad E\,|A_n(\delta_1) - A_n(\delta_2)|^4 &= n^{-\frac{4}{3}}(1-an^{-1})^{-4}\{9\sum_{j=b_1}^{b_2-1}(n-j+1)^4 n^{-4} \\ &\quad + 6n^{-4}\sum_{i=b_1}^{b_2-1}\sum_{i\neq j, j=b_1}^{b_2-1}(n-i+1)^2(n-j+1)^2\} \\ &\leqq 9n^{-16/3}(1-an^{-1})^{-4}[\sum_{j=b_1}^{b_2-1}(n-j+1)^2]^2 \\ &\leqq 9n^{-16/3}(1-an^{-1})^{-4}(b_2-b_1)^2 n^4 \leqq C_1\,|\delta_2-\delta_1|^2\end{aligned}$$

for some constant C_1 independent of n, δ_1 and δ_2. Now consider

$$\begin{aligned}(5.13)\quad & E\,|B_n(\delta_1) - B_n(\delta_2)|^4 \\ &= n^{-\frac{4}{3}}(1-an^{-1})^{-4}E\,|\sum_{j=b_1}^{b_2-1}\{(\Delta_{j+1}-n^{-1}) + \cdots + (\Delta_{n+1}-n^{-1})\}|^4 \\ &= n^{-\frac{4}{3}}(1-an^{-1})^{-4}E\,|\sum_{j=1}^{b_2-b_1} j(\Delta_{b_1+j}-n^{-1}) \\ &\qquad\qquad + (b_2-b_1)\sum_{j=b_2+1}^{n+1}(\Delta_j - n^{-1})|^4 \\ &\leqq 8n^{-\frac{4}{3}}(1-an^{-1})^{-4}[E\,|\sum_{j=1}^{b_2-b_1} j(\Delta_{b_1+j}-n^{-1})|^4 \\ &\qquad\qquad + (b_2-b_1)^4 E\,|\sum_{j=b_2+1}^{n+1}(\Delta_j-n^{-1})|^4] \\ &\leqq 72n^{-16/3}(1-an^{-1})^{-4}[\{\sum_{j=1}^{b_2-b_1} j^2\}^2 + (b_2-b_1)^4(n-b_2)^2] \\ &\leqq 72n^{-16/3}(1-an^{-1})^{-4}[(b_2-b_1)^2 n^4 + (b_2-b_1)^4 n^2] \\ &\leqq C_2[|\delta_2-\delta_1|^2 + 8\,|\delta_2-\delta_1|\,n^{-\frac{2}{3}}c^3]\end{aligned}$$

for some constant C_2 since $|\delta_2 - \delta_1| \leqq 2c$. Combining (5.12), (5.13) and (5.10), we get that there exists a constant C independent of n, ζ_1 and ζ_2 such that

$$E\,|D_n V_n^*(\zeta_1) - D_n V_n^*(\zeta_2)|^4 \leqq C\,|\zeta_2-\zeta_1|^2 + |\zeta_2-\zeta_1|\,o(1).$$

REMARK. Since D_n converges to 1 in probability, $D_n V_n^*(\zeta)$ is asymptotically normal with mean 0 and variance $|\zeta|$. Similarly $[D_n V_n^*(\zeta_1), \cdots, D_n V_n^*(\zeta_K)]$ is asymptotically multivariate normal with mean 0 and variance-covariance matrix

$$(\delta(\zeta_i, \zeta_j)\min(|\zeta_i|, |\zeta_j|)).$$

We notice that the process $D_n V_n^*$ is a process with independent increments since $D_n V_n^*$ can be represented in terms of sums of independent exponentials from (5.8) and (5.9).

We shall now state a theorem connected with convergence of distributions of stochastic processes with independent increments in $D[a,b]$.

THEOREM 5.4. *Let $\{X_n\}$ be a sequence of stochastic processes with independent increments in $D[a,b]$ and X be another process in $D[a,b]$ such that* (i) *for any t_i, $1 \leqq i \leqq k$, the joint distribution of $[X_n(t_1), \cdots, X_n(t_k)]$ converges weakly to the joint* *distribution of* $[X(t_1), \cdots, X(t_k)]$, *and* (ii) *there exist constants* $\tau > 0$, $C > 0$ *independent of n such that for every* t_1, t_2 *in* $[a, b]$,

$$E\left|X_n(t_1)-X_n(t_2)\right|^{\tau} \leqq C\left|t_1-t_2\right|^2+\left|t_1-t_2\right| o(1).$$

Let ν_n *and* ν *be the measures induced by* X_n *and* X *respectively on* $D[a,b]$. *Then* ν_n *converges to* ν *weakly.*

PROOF. From the condition (ii) of the hypothesis of the theorem, it follows that there exists a constant A independent of n, t_1 and t_2, such that $E|X_n(t_1)-X_n(t_2)|^{\tau} \leqq A|t_1-t_2|$. Therefore, for any $\lambda > 0$,

$$P\{|X_n(t_1)-X_n(t_2)| > \lambda\} \leqq A\lambda^{-\tau}|t_2-t_1| \leqq A\delta\lambda^{-\tau}$$

for all t_1, t_2 in $[a,b]$ such that $|t_2-t_1| \leqq \delta$. Let $\psi(\delta,\lambda) = A\,\delta\lambda^{-\tau}$. Notice that $\psi(\delta,\lambda) \to 0$ as $\delta \to 0$. Now from the remarks on page 140 of Sethuraman [7], it follows that the sequence of measures ν_n converges weakly to ν.

As a consequence of Lemma 5.3, and the remarks made following the lemma, it follows that the sequence of processes $D_n V_n{}^*$ converges in distribution to the Wiener process W on $[-q,q]$. Since D_n converges to 1 in probability, it follows by an extension of Slutsky's theorem for processes that $V_n{}^*$ converges in distribution to W. In other words, V_n converges in distribution to W since the finite dimensional distributions of V_n and $V_n{}^*$ are the same. Clearly, $\zeta^2+O_p(n^{-\frac{1}{6}})$ converges to ζ^2 uniformly in ζ since $|\zeta| \leqq q$ and O_p is uniform for $|\zeta| \leqq q$. Hence, by a simple extension of Slutsky's Theorem (Cramér [2]) for processes, we obtain the following theorem.

THEOREM 5.5. *The sequence of processes* X_n *on* $[-q,q]$ *given by* $X_n(\zeta) = V_n(\zeta)-\zeta^2+O_p(n^{-\frac{1}{6}})$ *converges in distribution to the process* X *cn* $[-q,q]$ *given by* $X(\zeta) = W(\zeta)-\zeta^2$ *where* W *is the Wiener process on* $[-q,q]$.

6. Asymptotic distribution of the MLE for IFR. Define

$$\psi(t) = \tfrac{1}{2}U_x(t^2,t)U_x(t^2,-t) \tag{6.1}$$

where $u(x,z) = P[W(t) > t^2$ for some $t > z \mid W(z) = x]$ is a solution of the heat equation $\frac{1}{2}U_{xx} = -U_z$ subject to the boundary conditions (i) $u(x,z) = 1$ for $x \geqq z^2$ and (ii) $u(x,z) \to 0$ as $x \to -\infty$. Here U_x denotes the partial derivative of $u(x,z)$ with respect to x.

In view of Theorem 5.5, the following final result can be obtained by methods analogous to those in Section 6 of Prakasa Rao [6].

THEOREM 6.1. *Let F be an* IFR *with failure rate r. Let* ξ *be such that* $0 < F(\xi) < 1$. *Further suppose that r is differentiable at* ξ *with non-zero derivative and* $r(\xi) > 0$. *Let* $\hat{r}_n(\xi)$ *denote the* MLE *of* $r(\xi)$ *based on n independent observations. Then the asymptotic distribution of*

$$n^{\frac{1}{3}}\left\{\frac{r'(\xi)r^{-4}(\xi)}{2f(\xi)}\right\}^{-\frac{1}{3}}\left\{\frac{1}{\hat{r}_n(\xi)}-\frac{1}{r(\xi)}\right\}$$

has density $\frac{1}{2}\psi(\frac{1}{2}x)$ *where* ψ *is given by* (6.1).

7. Asymptotic distribution of MLE for DFR. In this section, we shall state the main result for DFR distributions. Proofs are analogous to those in the IFR case.

THEOREM 7.1. *Let F be a* DFR *with failure rate r. Let ξ be such that* $0 < F(\xi) < 1$. *Further suppose that r is differentiable at ξ with non-zero derivative and* $r(\xi) > 0$. *Let $\hat{r}_n(\xi)$ denote the* MLE *of $r(\xi)$ based on n independent observations. Then the asymptotic distribution of*

$$n^{\frac{1}{3}}\left\{\frac{-r'(\xi)r^{-4}(\xi)}{2f(\xi)}\right\}^{-\frac{1}{3}}\left\{\frac{1}{\hat{r}_n(\xi)}-\frac{1}{r(\xi)}\right\}$$

has density $\frac{1}{2}\psi(\frac{1}{2}x)$ where ψ is given by (6.1).

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