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ONE-WAY EXPECTED UTILITY WITH FINITE CONSEQUENCE SPACES

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1. Introduction. Throughout this paper I shall assume that X is a nonempty finite set and that $\mathscr{P}$ is the set of probability distributions on X. The purpose of the paper is to examine critically conditions for a binary relation $\succ$ (preference) on $\mathscr{P}$ that imply

PROPOSITION 1. *There is a real-valued function u on X such that, for all $P, Q \in \mathscr{P}$,*

$$P \succ Q \Rightarrow E(u, P) > E(u, Q). \tag{1}$$

In (1), $E(u, P) = \sum u(x)P(x)$, the expected value of u under P.

The expected-utility representation in (1) can be thought of as unidimensional one-way ($\Rightarrow$) expected utility, in contrast to the unidimensional two-way ($\Leftrightarrow$) representation

$$P \succ Q \Leftrightarrow E(u, P) > E(u, Q) \tag{2}$$

that is implied by the von Neumann–Morgenstern axioms [8]. Our interest in (1) stems primarily from the fact that (1), unlike (2), does not imply that the relation (not $P \succ Q$, not $Q \succ P$) is transitive. For further comments on this point see Aumann [2] and Fishburn [4].

The one-way representation (1) has been studied previously by Aumann [2], [3] and Kannai [5]. Both remark on the difficulties encountered when X is allowed to be infinite, and Kannai investigates multidimensional expected-utility in this case. Although Aumann's formulation differs slightly from mine, there is no difficulty in translating his formulation into the one used here. In particular, the following theorem is similar to Aumann's Theorem A [2], and its proofs in ([4] Chapter 9) and in this paper are similar to Aumann's proof. [$\alpha P+(1-\alpha)R$ is the convex linear combination of P and R so that, for all $A \subseteq X$, $(\alpha P+(1-\alpha)R)(A) = \alpha P(A)+(1-\alpha)R(A)$. Recall that X is assumed to be finite.]

THEOREM 1. *Proposition* 1 *is true if the following four conditions hold throughout* $\mathscr{P}$:

A1. $\succ$ *is transitive.*

A2. $\alpha \in (0, 1)$ *and* $P \succ Q \Rightarrow \alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)R$.

A3. $\alpha \in (0, 1)$ *and* $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)R \Rightarrow P \succ Q$.

A4. $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$ *for all* $\alpha \in (0, 1] \Rightarrow$ *not* $S \succ R$.

To quote Aumann ([2] page 451), A2 "asserts that a preference is not changed by 'dilution'," and A3 says "that if we have a diluted preference, then the correspond-

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ing undiluted preference also holds." The final condition is an Archimedean condition that may be stated alternatively thus: if $S \succ R$ then there is some $\alpha \in (0, 1]$ such that not $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$. When $S \succ R$, we would expect $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$ to be false when α is sufficiently near to zero.

Of the four conditions in Theorem 1, only A4 is implied by (1). In the course of the paper we shall develop a necessary and sufficient condition for Proposition 1. However, this condition (Section 4) defies simple interpretation. Nevertheless, it illustrates the structure of (1) and can serve as a basis for proving theorems such as Theorem 1. In fact we shall prove Theorem 1 in this way although the proof is somewhat incidental to an argument that the sufficient conditions of Theorem 1 probably cannot be improved upon (by not assuming all of A1, A2 and A3 for example) without an uncompensating loss in simplicity of interpretation. Put differently, our contention is that there is no set of sufficient conditions for Proposition 1 that is more elegant than the set provided by Aumann.

Prior to a defense of this contention we shall investigate the special case where $\succ$ is finite (or "finitely generated" as explained later). Although A1, A2 and irreflexivity are sufficient for Proposition 1 in this case (Theorem 4 and Corollary 2), we shall focus our attention on another condition (B1, Section 3) that is necessary and sufficient for (1). Although condition B1 presents some interpretive problems it has the advantage that it applies as well to any nonempty subset of $\mathscr{P}$ as to $\mathscr{P}$ itself. (This is not true for example of A2 where if $P \succ Q$ then a whole host of mixtures is taken to be relevant for preference comparisons.) For example, it is immaterial under B1 whether the set of distributions on X that are of real concern to the decision maker is closed under convex combinations. However, to avoid unnecessary notation I shall use $\mathscr{P}$ in connection with B1.

The proofs of later theorems for Proposition 1 will be based on some theory for convex cones that we shall now review.

2. Convex cones. A nonempty subset C of n-dimensional Euclidean space E^n is a *convex cone* if and only if $x, y \in C$ and $\alpha, \beta > 0 \Rightarrow \alpha x+\beta y \in C$. The origin 0 of E^n need not be in a convex cone. $-C = \{-x: x \in C\}$. $\bar{C}$ is the closure of C.

The convex cone *generated* by nonempty $D \subseteq E^n$ is

$$C = \{\textstyle\sum \lambda_i x^i : \lambda_i > 0 \quad \text{and} \quad x^i \in D \quad \text{for} \quad \text{all} \quad i\}$$

in which all sums are nonempty and finite. C is *finitely generated* if it is generated by some finite subset of itself. $\bar{C} = C \cup \{0\}$ when C is finitely generated.

With $w, x \in E^n$, $w \cdot x = \sum_{i=1}^n w_i x_i$. The following theorem is proved in Aumann [2] and Fishburn [4]. The corollary follows easily.

THEOREM 2. *Suppose that C is a nonempty convex cone in E^n. Then there is a $w \in E^n$ such that*

$$w \cdot x > 0 \quad \textit{for all} \quad x \in C \tag{3}$$

if and only if $\bar{C} \cap (-C) = \varnothing$.

COROLLARY 1. *If C is finitely generated then* (3) *holds for some $w \in E^n$ if and only if* $0 \notin C$.

This corollary is used in the next section. In Section 4 we shall use Theorem 2 along with the following characterization of $\bar{C} \cap (-C) = \varnothing$.

THEOREM 3. *Suppose that C is a nonempty convex cone in E^n. Then $\bar{C} \cap (-C) = \varnothing$ if and only if, for all $x, y \in E^n$,*

$$(4) \qquad \alpha x + (1-\alpha)y \in C \quad \textit{for all} \quad \alpha \in (0, 1] \Rightarrow -y \notin C.$$

Clearly, (3) implies (4), and (4) implies that $0 \notin C$. Assume that (4) holds and that x is in the interior of $C \subseteq E^n$. Then, for any $y \in \bar{C}$ it can be shown (see, e.g., [6] page 110) that $\alpha x + (1-\alpha)y \in C$ for all $\alpha \in (0, 1]$. Hence, by (4), $-y \notin C$, and therefore (4) implies $\bar{C} \cap (-C) = \varnothing$. Suppose next that (4) holds and that C has an empty interior. It can then be shown with little difficulty that there is an $i \in \{1, \cdots, n\}$ such that x_i is uniquely determined by the other $n-1$ components of x for every $x \in C$. With $i = n$ let $C' = \{(x_1, \cdots, x_{n-1}) : x \in C\}$. C' is a convex cone in E^{n-1} and the copy of (4) holds for C'. Therefore, if C' has an interior point then $\bar{C}' \cap (-C') = \varnothing$ and hence $\bar{C} \cap (-C) = \varnothing$. If C' has no interior point, further reduction leads to a C'' with an interior point, from which $\bar{C} \cap (-C) = \varnothing$ follows.

3. Independence axioms. This section focuses on three related independence axioms. Although our main concern will be with the first of these, the other two are of interest both in themselves and in their relationships to the first.

B1. *$[m \in \{1, 2, \cdots\}$ and $\alpha_j > 0$ and $P^j \succ Q^j$ for $j = 1, \cdots, m$ and $\sum \alpha_j = 1] \Rightarrow \sum \alpha_j P^j \neq \sum \alpha_j Q^j$.*

B2. *$[m \in \{1, 2, \cdots\}$ and $\alpha_j > 0$ and $P^j \succ Q^j$ for $j = 1, \cdots, m$ and $\sum \alpha_j = 1] \Rightarrow \sum \alpha_j P^j \succ \sum \alpha_j Q^j$.*

B3. *$[m \in \{2, 3, \cdots\}$ and $\alpha_j > 0$ for $j = 1, \cdots, m$ and $\sum \alpha_j = 1$ and $P^j \succ Q^j$ for $j = 1, \cdots, m-1$ and $\sum_1^m \alpha_j P^j = \sum_1^m \alpha_j Q^j] \Rightarrow Q^m \succ P^m$.*

Of these three, only B1 is necessary for Proposition 1. Moreover, B1 is the only axiom that does not posit the relevance of preference comparisons that are not supposed in its hypotheses. It may be noted that if $Q^m \succ P^m$ in B3 is changed to not $P^m \succ Q^m$, then the modified B3 is equivalent to B1.

Several interesting relationships among these axioms and A1, A2 and A3 are noted in the following theorem (proofs omitted).

THEOREM 4. B3 $\Leftrightarrow$ A1 *and* A2 *and* A3. A1 *and* A2 $\Rightarrow$ B2. B2 *and* $\succ$ *is irreflexive* $\Rightarrow$ B1. B1 $\Rightarrow$ $\succ$ *is irreflexive and asymmetric.*

Since B3 is equivalent to A1 and A2 and A3 (a fact that we shall use in the next section), the remainder of this section concentrates on B1 and B2. Interpretive aspects of these axioms will be examined later in the section.

If X has only one element then B1 implies that $\succ = \varnothing$. In general, $(\succ = \varnothing) \Rightarrow$ Proposition 1.

Suppose then that $X = \{x_1, \cdots, x_n\}$ and that $\succ \neq \varnothing$. Let $D(\succ) = \{(P(x_1) - Q(x_1), \cdots, P(x_n) - Q(x_n)): P \succ Q\}$ and let $C(\succ)$ be the convex cone in E^n generated by $D(\succ)$. Then B1 $\Rightarrow 0 \notin C(\succ)$, for if $0 \in C(\succ)$ then there are $p^1, \cdots, p^m$ in $D(\succ)$ and positive $\alpha_1, \cdots, \alpha_m$ such that $0 = \sum \alpha_j p^j$ with $P^j \succ Q^j$ for each j where $p^j = ((P^j(x_1) - Q^j(x_1), \cdots, (P^j(x_n) - Q^j(x_n))$.

Hence, if $C(\succ)$ is finitely generated, Corollary 1 in the presence of B1 yields a $w \in E^n$ with $w \cdot p > 0$ for all $p \in C(\succ)$, and this implies (1) with $u(x_i) = w_i$ for $i = 1, \cdots, n$. This proves

THEOREM 5. *Suppose that $\succ = \varnothing$ or else that $\succ \neq \varnothing$ and $C(\succ)$ is finitely generated. Then Proposition* 1 *is true if and only if* B1 *holds.*

COROLLARY 2. *Suppose that $\succ = \varnothing$ or else that $\succ \neq \varnothing$ and $C(\succ)$ is finitely generated. Then Proposition* 1 *is true if* B2 *holds and $\succ$ is irreflexive.*

One other implication of B1 seems noteworthy. Suppose that B1 holds and $C = C(\succ)$ is not finitely generated. Then it is possible to have $\bar{C} \cap (-C) \neq \varnothing$ which, by Theorem 2, implies that Proposition 1 is false. By applying a theorem similar to Theorem 2 [to get $w^1 \in E^n$ such that $p \cdot w^1 \geqq 0$ for all $p \in C$ and $p \cdot w^1 > 0$ for some $p \in C$] and reducing dimensionality in successive steps [in the first to handle $\{p: p \in C \text{ and } p \cdot w^1 = 0\}$] it is possible to develop a finite-dimensional lexicographic replacement for (1). Omitting details, B1 implies that there is a positive integer N and real-valued functions $u_1, \cdots, u_N$ on X such that, for all $P, Q \in \mathscr{P}$, $P \succ Q \Rightarrow (E(u_1, P), \cdots, E(u_N, P)) >_L (E(u_1, Q), \cdots, E(u_N, Q))$. (Recall that $(a_1, \cdots, a_N) >_L (b_1, \cdots, b_N)$ is defined by $a_1 > b_1$ or $(a_1 = b_1$ and $a_2 > b_2)$ or $\cdots$ or $(a_i = b_i$ for all $i < N$ and $a_N > b_N)$.)

Turning now to interpretive aspects, consider B2. A typical defense of **B2** proceeds by interpreting $\sum \alpha_j P^j$ as a two-stage gamble. One of the P^j is chosen in the first stage according to the "probabilities" $\alpha_1, \cdots, \alpha_m$. This P^j then determines an $x \in X$ according to its probabilities. If $P^j \succ Q^j$ for $j = 1, \cdots, m$ and if $\sum \alpha_j P_j$ and $\sum \alpha_j Q^j$ are considered relevant then it seems reasonable to have $\sum \alpha_j P^j \succ \sum \alpha_j Q^j$.

There are, however, several arguments that counter the conclusion of B2. As already noted, the decision maker may be completely unconcerned about the mixtures $\sum \alpha_j P^j$ and $\sum \alpha_j Q^j$ even though it might be argued that he can consider such mixtures at little if any extra expense. A second argument against $\sum \alpha_j P^j \succ \sum \alpha_j Q^j$ notes that, even though the decision maker prefers P^j to Q^j for each j, the α_j (especially with m large) may dilute the P^j and Q^j in the mixed forms to such an extent that the individual may not care which mixture he chooses if he has a choice between them. A third argument against B2 follows an example suggested by Allais [1] that is expanded on by Savage ([7] pages 101–103). In a slightly modified form this goes as follows. We consider four gambles for monetary prizes:

$Q(\$500{,}000) = 1.0$

$P(\$10) = .01$, $P(\$500{,}000) = .89$, $P(\$2{,}500{,}000) = .10$

$S(\$0) = .90$, $S(\$2{,}500{,}000) = .10$

$R(\$10) = .89$, $R(\$500{,}000) = .11$.

Experience indicates that $Q \succ P$ and $S \succ R$ for many people. B2 would then require $\frac{1}{2}Q+\frac{1}{2}S \succ \frac{1}{2}P+\frac{1}{2}R$. But an examination of these mixtures shows the following probabilities:

$\frac{1}{2}Q+\frac{1}{2}S$: .45 for \$0; .50 for \$500,000; .05 for \$2,500,000

$\frac{1}{2}P+\frac{1}{2}R$: .45 for \$10; .50 for \$500,000; .05 for \$2,500,000

which make it likely that $\frac{1}{2}P+\frac{1}{2}R \succ \frac{1}{2}Q+\frac{1}{2}S$. (For further comments on this see Savage [7].)

Of the three arguments against B2 the first seems to challenge B1 indirectly and the last seems to challenge it directly. (Change \$10 to \$0 to get $\frac{1}{2}Q+\frac{1}{2}S = \frac{1}{2}P+\frac{1}{2}R$.) On the other hand, the usual sure-thing defense of B2 does not apply directly to B1 unless we grant the relevance of the mixtures $\sum\alpha_j P^j$ and $\sum\alpha_j Q^j$ for preference comparison. Since part of the purpose of B1 is to avoid the assumption of such relevance, it appears that we pay the cost of losing an "obvious" justification for the gain of a more general theory.

4. General analysis. Turning now to the general case of Proposition 1 with X finite, we shall argue that any significant weakening of the sufficient conditions in Theorem 1 (say to necessary and sufficient conditions) can be made only with great loss in interpretability of the new conditions compared to those of Theorem 1. We shall let $|X| = n$ and write $P \in \mathscr{P}$ in its vector form $(P(x_1), \cdots, P(x_n))$ in E^n. $C = C(\succ)$ is as defined in the preceding section.

Our analysis will be based on the following consequence of Theorem 2 and Theorem 3.

LEMMA 1. *Proposition* 1 *is true if and only if*

$$\alpha(P-Q)+(1-\alpha)(R-S) \in C \quad \textit{for all} \quad \alpha \in (0, 1] \Rightarrow S-R \notin C. \tag{5}$$

The necessity of (5) follows easily from (1) and the definition of C. For sufficiency we show that if (4) fails then (5) fails. Hence suppose that

$$-q \in C \quad \text{and} \quad \alpha p+(1-\alpha)q \in C \quad \text{for all} \quad \alpha \in (0, 1]. \tag{6}$$

By the definition of C we have $p = \sum\lambda_i(P^i-Q^i)$ and $q = \sum\sigma_j(R^j-S^j)$ with $\lambda_i, \sigma_j > 0$ and $P^i \succ Q^i$ and $S^j \succ R^j$ for all i and j. Let $\lambda = \sum\lambda_i$, $\sigma = \sum\sigma_i$, $P = \sum(\lambda_i/\lambda)P^i$, $Q = \sum(\lambda_i/\lambda)Q^i$, $R = \sum(\sigma_j/\sigma)R^j$ and $S = \sum(\sigma_j/\sigma)S^j$. Clearly $S-R \in C$. Hence to show a violation of (5) we need to show that

$$\alpha(P-Q)+(1-\alpha)(R-S) \in C$$

when $\alpha \in (0, 1]$. We proceed from the latter part of (6):

$$\alpha\lambda(P-Q)+(1-\alpha)\sigma(R-S) \in C, \qquad (\alpha > 0).$$

If $\sigma \geqq \lambda$ then

$$(1/\sigma)[\alpha\lambda(P-Q)+(1-\alpha)\sigma(R-S)]+\alpha[(\sigma-\lambda)/\sigma](P-Q) \in C,$$

or $\alpha(P-Q)+(1-\alpha)(R-S) \in C$. If $\lambda > \sigma$ then with $\delta \in (0, 1]$ and $k > 0$

$$\alpha\lambda(P-Q)+(1-\alpha)\sigma(R-S)+k[\delta\lambda(P-Q)+(1-\delta)\sigma(R-S)] \in C.$$

Taking δ close to zero and $k = \alpha(1-\alpha)(\lambda-\sigma)/[\alpha\sigma(1-\delta)-\delta\lambda(1-\alpha)]$, the preceding expression equals a positive number times $\alpha(P-Q)+(1-\alpha)(R-S)$ so that the latter is in C. This completes the proof.

Now Lemma 1 gives a necessary and sufficient condition for Proposition 1, but (even when it is written with appropriate expressions involving $\succ$ as replacements for C) there does not seem to be any simple intuitive interpretation for the condition. Nevertheless, Lemma 1 is of some interest since it presents a structural representation of Proposition 1 and can be used to establish sufficient conditions for Proposition 1, such as those in Theorem 1.

Since some Archimedean axiom is required for Proposition 1 and I know of no axiom other than A4 (and a similar axiom in Aumann [2]) that is both necessary for (1) and has a relatively straightforward interpretation, let us consider (5) in the presence of

A4. $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$ *for all* $\alpha \in (0, 1] \Rightarrow$ *not* $S \succ R$.

Suppose (5) fails and $P-Q \in C \Rightarrow P \succ Q$. Then A4 fails. Hence A4 and $P-Q \in C \Rightarrow P \succ Q$ imply (5). Even though $P-Q \in C \Rightarrow P \succ Q$ is not necessary for (5), I know of no weakening of this condition that implies (5) in the presence of A4 and has an easily understood interpretation or justification. That $P-Q \in C \Rightarrow P \succ Q$ has such an interpretation follows from the fact that it holds if and only if A1, A2 and A3 hold. This follows from Theorem 4 and the easily proved

LEMMA 2. B3 $\Leftrightarrow (P-Q \in C \Rightarrow P \succ Q)$.

This analysis, besides completing a proof of Theorem 1, suggests that there is little room for weakening its conditions while maintaining simplicity of interpretation. This should of course be regarded as a tentative conclusion and a challenge to the reader.

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REFERENCES

[1] ALLAIS, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école Americaine. *Econometrica* **21** 503–546.
[2] AUMANN, R. J. (1962). Utility theory without the completeness axiom. *Econometrica* **30** 445–462.
[3] AUMANN, R. J. (1964). Utility theory without the completeness axiom: a correction. *Econometrica* **32** 210–212.
[4] FISHBURN, P. C. (1970). *Utility Theory for Decision Making*. Wiley, New York.
[5] KANNAI, Y. (1963). Existence of a utility in infinite dimensional partially ordered spaces. *Israel J. Math.* **1** 229–234.
[6] KELLEY, J. L., NAMIOKA, I., et al. (1963). *Linear Topological Spaces*. Van Nostrand, Princeton.
[7] SAVAGE, L. J. (1954). *The Foundations of Statistics*. Wiley, New York.
[8] VON NEUMANN, J. and MORGENSTERN, O. (1947). *Theory of Games and Economic Behavior*, 2nd ed. Princeton Univ. Press.