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# ON THE REDUCTION OF ASSOCIATE CLASSES FOR CERTAIN PBIB DESIGNS

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Kronecker product and reduced designs were defined by M. N. Vartak (1955). The main purpose of this paper is to give necessary and sufficient conditions for partially balanced incomplete block (PBIB) designs with $l$ associate classes constructed by the Kronecker product of some balanced incomplete block (BIB) designs to be reducible to PBIB designs with $l_1$ associate classes for a positive integer $l_1$ which is less than $l$.

**1. Introduction and summary.** The Kronecker product of designs and reduced designs were defined by Vartak (1955), but the association schemes connecting these designs were not considered explicitly. When an arrangement with the parameters of a PBIB design is given, it is important to determine the association scheme matching its design in relation to the problem showing the uniqueness of the association scheme. This paper gives necessary and sufficient conditions (NASCs) for PBIB designs with $l$ associate classes constructed by the Kronecker product of some BIB designs to be reducible to PBIB designs with $l_1$ associate classes for a positive integer $l_1$ satisfying $l_1 < l$. The problem considered here will be of both theoretical and practical importance with regard to constructing and analyzing certain PBIB designs.

Vartak's approach and the approach given in this paper differ; in his approach the numbers of blocks containing a pair of treatments and the second kind parameters (i.e., $p^i_{jk}$) of the PBIB design $N$ are used, while in the approach considered in this paper, the numbers of blocks containing a pair of treatments and the latent roots of the matrix $NN'$ are used.

In Section 2, some results concerned with the association scheme are introduced. In Section 3, NASCs for a PBIB design with at most three associate classes constructed by the Kronecker product of two BIB designs to be reducible to a PBIB design with only two distinct associate classes are given. In Section 4, NASCs for a PBIB design with at most seven associate classes constructed by the Kronecker product of three BIB designs to be reducible to a PBIB design with only three distinct associate classes are given. In Section 5, NASCs for a PBIB design with at most $2^m - 1$ associate classes constructed by the Kronecker product of $m$ BIB designs to be reducible to a PBIB design with only $m$ distinct associate classes are given. In Section 6, NASCs for a PBIB design with at most three associate classes constructed by a sum of the Kronecker products of two sets of BIB designs to be reducible to a PBIB design with only two distinct associate classes are given. In Section 7 a complementary remark is described. Finally some illustrations of a few of the theorems are given in Section 8.

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**2. Association schemes and some preliminary results.** An association scheme is defined for a set of $v$ treatments. Let the association matrices $A_0, A_1, \cdots, A_m$ be

$$A_i = ||a_{\alpha i}^{\beta}|| , \qquad \alpha, \beta = 1, 2, \cdots, v ; \quad i = 0, 1, \cdots, m ,$$

where

$$\begin{aligned} a_{\alpha i}^{\beta} &= 1 , \qquad \text{if } \alpha\text{th and } \beta\text{th treatments are } i\text{th associates} , \\ &= 0 , \qquad \text{otherwise} . \end{aligned}$$

Then it is known (cf. [1]) that the linear closure $\mathfrak{A}$ of matrices $A_0, A_1, \cdots, A_m$ over the real field is an $(m+1)$-dimensional commutative algebra containing a unit, which is called the association algebra $\mathfrak{A} = [A_0, A_1, \cdots, A_m]$ of treatments. If the mutually orthogonal idempotents of $\mathfrak{A}$ are given by $A_0^{\#}, A_1^{\#}, \cdots, A_m^{\#}$, then $\mathfrak{A}$ may also be expressed by indicating its ideal basis as $\mathfrak{A} = [A_0^{\#}, A_1^{\#}, \cdots, A_m^{\#}]$.

It should be remarked that an association scheme can be defined and characterized independently of treatment-block incidence of the design [12]. The axioms of an association scheme have been derived from describing the relation among treatments in terms of the structure of treatment-block incidence of the design, in particular, the numbers $\lambda_i$. This paper is based on this thinking (cf. [2]).

When the parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$ of a PBIB design are not all different, the $m$ associate classes of the PBIB design based on a certain association scheme may not be all distinct. The theorems in this paper give some criteria to determine whether a PBIB design with at most $m$ associate classes is reducible to a PBIB design with $m$ or fewer distinct associate classes when its parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$ are not all different. The number $\lambda_i$, which is the number of blocks containing a pair of treatments which are $i$th associates, may be called the coincidence number of the pair.

The incidence matrix $N = ||n_{\alpha a}||$ of an $m$-associate class PBIB design with parameters $v, n_i, p_{jk}^i, b, r, k$ and $\lambda_i$ is defined by

$$\begin{aligned} n_{\alpha a} &= 1 , \qquad \text{if } \alpha\text{th treatment occurs in } a\text{th block} , \\ &= 0 , \qquad \text{otherwise} . \end{aligned}$$

The following lemma useful to this paper is given [11]:

Lemma A. *$NN'$ belongs to the association algebra $\mathfrak{A}$ and can be expressed as*

$$NN' = \textstyle\sum_{j=0}^{m} \lambda_j A_j = \sum_{i=0}^{m} \rho_i A_i^{\#} ,$$

*where the last member of the expression is the spectral expansion of $NN'$ in $\mathfrak{A}$. The densities*

$$\rho_i = \textstyle\sum_{j=0}^{m} \lambda_j z_{ij} , \qquad i = 0, 1, \cdots, m ,$$

*where $z_{ij}$ are latent roots of $\mathscr{P}_j = ||p_{ij}^k||$ $(k = 0, 1, \cdots, m)$ and $N'$ is the transpose of the matrix $N$, are the latent roots of $NN'$. In particular,*

$$\rho_0 = rk = \textstyle\sum_{i=0}^{m} n_i \lambda_i .$$

*The $\rho_i$ satisfy the inequalities*

$$0 \leqq \rho_i \leqq rk , \qquad i = 0, 1, \cdots, m .$$

*The multiplicity of $\rho_i$ is the trace $(A_i^{\#})$.*

Lemma A shows that for a PBIB design $N$ with $m$ associate classes, $NN'$ has at most $m+1$ coincidence numbers $\lambda_i$ and distinct latent roots $\rho_i$. This fact is used to derive the necessary conditions in the subsequent theorems.

Finally, since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix may be denoted by the same symbol throughout this paper.

**3. Kronecker product design of two BIB designs.** Let $N_i = ||n_{\alpha\beta}^{(i)}||$ be BIB designs with parameters $v_i$, $b_i$, $r_i$, $k_i$ and $\lambda_i$ $(i = 1, 2)$ and consider the Kronecker product of these designs in the form $N = N_1 \otimes N_2 = ||n_{\alpha\beta}^{(1)} N_2||$.

It is clear that there exist orthogonal matrices $H_i$ which make $N_i N_i'$ diagonal, such as

$$H_i N_i N_i' H_i' = D_i = \operatorname{diag}\{r_i k_i, r_i - \lambda_i, \cdots, r_i - \lambda_i\}, \qquad i = 1, 2, \tag{3.1}$$

where $\operatorname{diag}\{r_i k_i, r_i - \lambda_i, \cdots, r_i - \lambda_i\}$ denotes a diagonal matrix whose diagonal elements are the elements $r_i k_i$ and $r_i - \lambda_i$, and $r_i - \lambda_i$ appears $v_i - 1$ times. Hence we obtain

$$\begin{aligned}(H_1 \otimes H_2)&NN'(H_1 \otimes H_2)' \\ &= D_1 \otimes D_2 = \operatorname{diag}\{r_1k_1r_2k_2, r_1k_1(r_2 - \lambda_2), \cdots, r_1k_1(r_2 - \lambda_2), (r_1 - \lambda_1)r_2k_2, \cdots, \\ &\qquad (r_1 - \lambda_1)r_2k_2, (r_1 - \lambda_1)(r_2 - \lambda_2), \cdots, (r_1 - \lambda_1)(r_2 - \lambda_2)\},\end{aligned}$$

that is, $r_1k_1r_2k_2$, $r_1k_1(r_2 - \lambda_2)$, $(r_1 - \lambda_1)r_2k_2$ and $(r_1 - \lambda_1)(r_2 - \lambda_2)$ are nonzero latent roots of $NN'$ with multiplicities $1$, $v_2 - 1$, $v_1 - 1$ and $(v_1 - 1)(v_2 - 1)$, respectively. In this new design $N$, all the coincidence numbers of a pair of treatments are clearly $r_1r_2$, $r_1\lambda_2$, $r_2\lambda_1$ and $\lambda_1\lambda_2$.

The above statement and the proof of Theorem 4.2 of Vartak [9] imply that $N = N_1 \otimes N_2$ is a PBIB design with at most three associate classes. Among $v_1v_2$ treatments in design $N$, an association scheme, i.e., rectangular association scheme ($F_2$ association scheme [12]), of three associate classes can be naturally defined (cf. [10]). Hence the parameters of design $N$ are as follows:

$$v^* = v_1v_2, \qquad b^* = b_1b_2, \qquad r^* = r_1r_2, \qquad k^* = k_1k_2,$$
$$\lambda_0^* = r^*, \qquad \lambda_1^* = r_1\lambda_2, \qquad \lambda_2^* = r_2\lambda_1, \qquad \lambda_3^* = \lambda_1\lambda_2,$$
$$n_1^* = v_2 - 1, \qquad n_2^* = v_1 - 1, \qquad n_3^* = (v_1 - 1)(v_2 - 1),$$

$$\begin{aligned}
&\mathscr{P}_1 = ||p_{i1}^k|| = \left\|\begin{matrix} v_2 - 2, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & v_2 - 1, & v_2 - 2 \end{matrix}\right\|, \\
&\mathscr{P}_2 = ||p_{i2}^k|| = \left\|\begin{matrix} 0, & 0, & 1 \\ 0, & v_1 - 2, & 0 \\ v_1 - 1, & 0, & v_1 - 2 \end{matrix}\right\|, \\
&\mathscr{P}_3 = ||p_{i3}^k|| = \left\|\begin{matrix} 0 & v_2 - 1, & v_2 - 2 \\ v_1 - 1, & 0 & v_1 - 2 \\ (v_1 - 1)(v_2 - 2), & (v_1 - 2)(v_2 - 1), & (v_1 - 2)(v_2 - 2) \end{matrix}\right\|,
\end{aligned} \tag{3.2}$$

where $i, k = 1, 2, 3$.

Now if the Kronecker product design $N = N_1 \otimes N_2$ is a PBIB design with only two associate classes, then from $\lambda_i < r_i$ $(i = 1, 2)$ and Lemma A the following relations must hold:

$$r_1 k_1(r_2 - \lambda_2) = r_2 k_2(r_1 - \lambda_1)\,, \qquad r_1 \lambda_2 = r_2 \lambda_1\,. \tag{3.3}$$

If condition (3.3) is satisfied, then from the relations among the parameters of the BIB designs, i.e., $\lambda_i(v_i - 1) = r_i(k_i - 1)$, $i = 1, 2$, we can obtain

$$v_1 = v_2 = v\,, \quad \text{say}\,; \qquad k_1 = k_2 = k\,, \quad \text{say} \tag{3.4}$$

and vice versa.

We shall show that if a PBIB design $N = N_1 \otimes N_2$ with at most three associate classes satisfies condition (3.4), then the Kronecker product design $N = N_1 \otimes N_2$ is reducible to a PBIB design with only two distinct associate classes.

When a PBIB design $N$ with the rectangular association scheme satisfies condition (3.4), we can naturally define the $L_2$ association scheme among $v_1 v_2 = v^2$ treatments as follows:

$v^* = v_1 v_2 = v^2$ treatments of design $N$ can be written in a $v \times v$ square such that any two treatments in the same row or the same column are first associates, whereas any two treatments not in the same row and not in the same column are second associates.

Then we can easily verify from condition (3.4) that the association defined above satisfies three conditions of the association scheme with two associate classes. Thus in comparison with (3.2) the parameters of this reduced design are as follows:

$$v^* = v^2\,, \qquad b^* = b_1 b_2\,, \qquad r^* = r_1 r_2\,, \qquad k^* = k^2\,,$$
$$\lambda_0^* = r^*\,, \qquad \lambda_1^* = r_1 \lambda_2 = r_2 \lambda_1\,, \qquad \lambda_2^* = \lambda_1 \lambda_2\,,$$
$$n_1^* = 2(v - 1)\,, \qquad n_2^* = (v - 1)^2\,,$$
$$\mathscr{P}_1 = ||p_{i1}^k|| = \begin{Vmatrix} v - 2\,, & 2 \\ v - 1\,, & 2(v - 2) \end{Vmatrix},$$
$$\mathscr{P}_2 = ||p_{i2}^k|| = \begin{Vmatrix} v - 1\,, & 2(v - 2) \\ (v - 1)(v - 2)\,, & (v - 2)^2 \end{Vmatrix},$$

where $i, k = 1, 2$.

Therefore the following theorem is obtained:

THEOREM 1. *Given the* BIB *designs $N_i$ with parameters $v_i$, $b_i$, $r_i$, $k_i$ and $\lambda_i$ $(i = 1, 2)$. Then necessary and sufficient conditions for the Kronecker product* PBIB *design $N = N_1 \otimes N_2$, which has at most three associate classes having the rectangular association scheme, to be reducible to a* PBIB *design with only two distinct associate classes having the $L_2$ association scheme are*

$$v_1 = v_2\,, \qquad k_1 = k_2\,.$$

Note that this is the least reducible number of associate classes. Theorem 1 is an alternative derivation of Corollary 4.2.2 of Vartak [9].

**4. Kronecker product design of three BIB designs.** Let $N_i$ be BIB designs with parameters $v_i$, $b_i$, $r_i$, $k_i$ and $\lambda_i$ $(i = 1, 2, 3)$ and consider the Kronecker product of these designs in the form $N = N_1 \otimes N_2 \otimes N_3$.

Since relation (3.1) remains valid for $i = 1$, 2 and 3, we obtain

$$\begin{aligned}(H_1 \otimes H_2 \otimes H_3)NN'(H_1 \otimes H_2 \otimes H_3)'\\ = D_1 \otimes D_2 \otimes D_3 = \operatorname{diag}\{&r_1 k_1 r_2 k_2 r_3 k_3, r_1 k_1 r_2 k_2(r_3 - \lambda_3), \cdots,\\ &r_1 k_1 r_2 k_2(r_3 - \lambda_3), r_1 k_1 r_3 k_3(r_2 - \lambda_2), \cdots,\\ &r_1 k_1 r_3 k_3(r_2 - \lambda_2), r_2 k_2 r_3 k_3(r_1 - \lambda_1), \cdots,\\ &r_2 k_2 r_3 k_3(r_1 - \lambda_1), r_1 k_1(r_2 - \lambda_2)(r_3 - \lambda_3), \cdots,\\ &r_1 k_1(r_2 - \lambda_2)(r_3 - \lambda_3), r_2 k_2(r_1 - \lambda_1)(r_3 - \lambda_3), \cdots,\\ &r_2 k_2(r_1 - \lambda_1)(r_3 - \lambda_3), r_3 k_3(r_1 - \lambda_1)(r_2 - \lambda_2), \cdots,\\ &r_3 k_3(r_1 - \lambda_1)(r_2 - \lambda_2), (r_1 - \lambda_1)(r_2 - \lambda_2)(r_3 - \lambda_3), \cdots,\\ &(r_1 - \lambda_1)(r_2 - \lambda_2)(r_3 - \lambda_3)\},\end{aligned}$$

that is, $r_1 k_1 r_2 k_2 r_3 k_3$, $r_1 k_1 r_2 k_2(r_3 - \lambda_3)$, $r_1 k_1 r_3 k_3(r_2 - \lambda_2)$, $r_2 k_2 r_3 k_3(r_1 - \lambda_1)$, $r_1 k_1(r_2 - \lambda_2)(r_3 - \lambda_3)$, $r_2 k_2(r_1 - \lambda_1)(r_3 - \lambda_3)$, $r_3 k_3(r_1 - \lambda_1)(r_2 - \lambda_2)$ and $(r_1 - \lambda_1)(r_2 - \lambda_2)(r_3 - \lambda_3)$ are nonzero latent roots of $NN'$ with multiplicities 1, $v_3 - 1$, $v_2 - 1$, $v_1 - 1$, $(v_2 - 1)(v_3 - 1)$, $(v_1 - 1)(v_3 - 1)$, $(v_1 - 1)(v_2 - 1)$ and $(v_1 - 1)(v_2 - 1)(v_3 - 1)$, respectively. It is not difficult to see from the structure of $N = N_1 \otimes N_2 \otimes N_3$ that all the coincidence numbers of a pair of treatments are $r_1 r_2 r_3$, $\lambda_1 r_2 r_3$, $\lambda_2 r_1 r_3$, $\lambda_3 r_1 r_2$, $r_1 \lambda_2 \lambda_3$, $r_2 \lambda_1 \lambda_3$, $r_3 \lambda_1 \lambda_2$ and $\lambda_1 \lambda_2 \lambda_3$.

The above statement and the proof of Theorem 4.2 of Vartak [9] imply that $N = N_1 \otimes N_2 \otimes N_3$ is a PBIB design with at most seven associate classes. Among $v_1 v_2 v_3$ treatments in design $N = N_1 \otimes N_2 \otimes N_3$, an association scheme of seven associate classes is a special case of those defined in the next Section 5.

Now if the Kronecker product design $N = N_1 \otimes N_2 \otimes N_3$ is a PBIB design with only three distinct associate classes, then from $\lambda_i < r_i$ $(i = 1, 2, 3)$ and Lemma A, by investigating those among the latent roots of $NN'$ and among all the coincidence numbers which may be equal to each other, the following relations must hold:

$$\begin{aligned} r_1 k_1(r_2 - \lambda_2) &= r_2 k_2(r_1 - \lambda_1), & r_1 \lambda_2 &= r_2 \lambda_1,\\ r_2 k_2(r_3 - \lambda_3) &= r_3 k_3(r_2 - \lambda_2), & r_2 \lambda_3 &= r_3 \lambda_2,\\ r_1 k_1(r_3 - \lambda_3) &= r_3 k_3(r_1 - \lambda_1), & r_1 \lambda_3 &= r_3 \lambda_1. \end{aligned} \tag{4.1}$$

It should be noted that condition (4.1) contains condition (3.3). Hence we can easily show that condition (4.1) is equivalent to the following relation:

$$v_1 = v_2 = v_3 = v, \quad \text{say}; \qquad k_1 = k_2 = k_3 = k, \quad \text{say}. \tag{4.2}$$

We shall show that if a PBIB design $N = N_1 \otimes N_2 \otimes N_3$ with at most seven associate classes satisfies condition (4.2), then the Kronecker product design $N = N_1 \otimes N_2 \otimes N_3$ is reducible to a PBIB design with only three distinct associate classes.

When a PBIB design $N = N_1 \otimes N_2 \otimes N_3$ satisfies condition (4.2), we can naturally define an association scheme, i.e., cubic association scheme [6], of three associate classes for $v^* = v^3$ treatments as follows:

Geometrically, when $v^* = v_1 v_2 v_3 = v^3$ treatments are arranged in a cube of side $v$, any two treatments lying on the same axis are first associates, those lying on the same plane are second associates and the rest are third associates.

Then we can easily verify from condition (4.2) that the association defined above satisfies three conditions of the association scheme with three associate classes. Thus the parameters of this reduced design are as follows:

$$v^* = v^3\,, \qquad b^* = b_1 b_2 b_3\,, \qquad r^* = r_1 r_2 r_3\,, \qquad k^* = k^3\,,$$
$$\lambda_0^* = r^*\,, \qquad \lambda_1^* = \lambda_1 r_2 r_3 = \lambda_2 r_1 r_3 = \lambda_3 r_1 r_2\,,$$
$$\lambda_2^* = r_1 \lambda_2 \lambda_3 = r_2 \lambda_1 \lambda_3 = r_3 \lambda_1 \lambda_2\,, \qquad \lambda_3^* = \lambda_1 \lambda_2 \lambda_3\,,$$
$$n_1^* = 3(v-1)\,, \qquad n_2^* = 3(v-1)^2\,, \qquad n_3^* = (v-1)^3\,,$$
$$\mathscr{P}_1 = ||p_{i1}^k|| = \begin{Vmatrix} v-2\,, & 2\,, & 0 \\ 2(v-1)\,, & 2(v-2)\,, & 3 \\ 0\,, & v-1\,, & 3(v-2) \end{Vmatrix},$$
$$\mathscr{P}_2 = ||p_{i2}^k|| = \begin{Vmatrix} 2(v-1)\,, & 2(v-2)\,, & 3 \\ 2(v-1)(v-2)\,, & 2(v-1)+(v-2)^2\,, & 6(v-2) \\ (v-1)^2\,, & 2(v-1)(v-2)\,, & 3(v-2)^2 \end{Vmatrix},$$
$$\mathscr{P}_3 = ||p_{i3}^k|| = \begin{Vmatrix} 0\,, & v-1 & 3(v-2) \\ (v-1)^2\,, & 2(v-1)(v-2)\,, & 3(v-2)^2 \\ (v-1)^2(v-2)\,, & (v-1)(v-2)^2\,, & (v-2)^3 \end{Vmatrix},$$

where $i, k = 1, 2, 3$. In this case, all distinct latent roots of $NN'$ are

$$\begin{aligned}
\rho_0^* &= r^* k^* && \text{with multiplicity } 1\,, \\
\rho_1^* &= r_1 r_2 (r_3 - \lambda_3) k^2 && \text{with multiplicity } 3(v-1)\,, \\
&= r_1 r_3 (r_2 - \lambda_2) k^2\,, \\
&= r_2 r_3 (r_1 - \lambda_1) k^2\,, \\
&= (r^* - \lambda_1^*) k^2\,, \\
\rho_2^* &= r_1 (r_2 - \lambda_2)(r_3 - \lambda_3) k && \text{with multiplicity } 3(v-1)^2\,, \\
&= r_2 (r_1 - \lambda_1)(r_3 - \lambda_3) k\,, \\
&= r_3 (r_1 - \lambda_1)(r_2 - \lambda_2) k\,, \\
&= (r^* - 2\lambda_1^* + \lambda_2^*) k\,, \\
\rho_3^* &= (r_1 - \lambda_1)(r_2 - \lambda_2)(r_3 - \lambda_3) && \text{with multiplicity } (v-1)^3\,, \\
&= r^* - 3\lambda_1^* + 3\lambda_2^* - \lambda_3^*\,.
\end{aligned}$$

Therefore the following theorem is obtained:

THEOREM 2. *Given the* BIB *designs* $N_i$ *with parameters* $v_i$, $b_i$, $r_i$, $k_i$ *and* $\lambda_i$ ($i = 1, 2, 3$). *Then necessary and sufficient conditions for the Kronecker product* PBIB *design* $N = N_1 \otimes N_2 \otimes N_3$, *which has at most seven associate classes, to be reducible* *to a* PBIB *design with only three distinct associate classes having the cubic association scheme are*

$$v_1 = v_2 = v_3\,, \qquad k_1 = k_2 = k_3\,.$$

Note that this is the least reducible number of associate classes. Theorem 2 also gives a method, which is somewhat different from that of Raghavarao and Chandrasekhararao [6], of constructing a PBIB design with the cubic association scheme.

**5. Kronecker product design of $m$ BIB designs.** Let $N_i$ be BIB designs with parameters $v_i$, $b_i$, $r_i$, $k_i$ and $\lambda_i$ $(i = 1, 2, \cdots, m)$ and consider the Kronecker product of these designs in the form $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$.

Then the following lemma is given (cf. [3]):

LEMMA 5.1. *The design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ is a* PBIB *design with at most $2^m - 1$ associate classes having the $F_m$ type association scheme.*

In this case, the $F_m$ type association scheme of $2^m - 1$ associate classes for $v^* = v_1\ v_2 \cdots v_m$ treatments is naturally defined as follows:

Suppose that there are $v^* = v_1\ v_2 \cdots v_m$ treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ indexed by $m$-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ where $\alpha_i = 1, 2, \cdots, v_i$ and $i = 1, 2, \cdots, m$. Any two treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\phi(\beta_1, \beta_2, \cdots, \beta_m)$ are $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)$th associates if $[\varepsilon(\alpha_1 - \beta_1), \varepsilon(\alpha_2 - \beta_2), \cdots, \varepsilon(\alpha_m - \beta_m)] = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)$, where $\varepsilon(x)$ is a function of $x$ which assumes either the value zero or one according as $x$ is zero or not. Each treatment is the $(0, \cdots, 0)$th associate of itself.

Designs with the above association scheme have been investigated by Hinkelmann [3], Hinkelmann and Kempthorne [4] and Yamamoto, Fujii and Hamada [12].

LEMMA 5.2. *Given the* BIB *designs $N_i$ with parameters $v$, $b_i$, $r_i$, $k$ and $\lambda_i$ $(i = 1, 2, \cdots, m)$. Then design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ is a* PBIB *design with $m$ distinct associate classes having the following parameters based on the hypercubic association scheme*:

$$\begin{aligned} &v^* = v^m\,, \qquad b^* = \textstyle\prod_{i=1}^m b_i\,, \qquad r^* = \prod_{i=1}^m r_i\,, \qquad k^* = k^m\,, \\ &\lambda_0^* = r^*\,, \qquad \lambda_1^* = \textstyle\prod_{i \in \{1,2,\cdots,m\}-\{j\}} r_i \lambda_j\,, \\ &\lambda_2^* = \textstyle\prod_{i \in \{1,2,\cdots,m\}-\{j,l\}} r_i \lambda_j \lambda_l, \cdots, \lambda_{m-1}^* = r_i \prod_{j \in \{1,2,\cdots,m\}-\{i\}} \lambda_j\,, \\ &\lambda_m^* = \textstyle\prod_{i=1}^m \lambda_i\,, \end{aligned} \tag{5.1}$$

*where the notation $i \in \{1, 2, \cdots, m\} - \{j, l\}$ means that suffix $i$ extends over all the integers excluding pairs $j$, $l$ $(j \neq l) = 1, 2, \cdots, m$.*

PROOF. It is sufficient to show that if the PBIB design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ in Lemma 5.1 satisfies the condition

$$v_1 = v_2 = \cdots = v_m = v\,, \quad \text{say}\,; \qquad k_1 = k_2 = \cdots = k_m = k\,, \quad \text{say}\,,$$

then design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ can lead to a PBIB design with only $m$ distinct associate classes having the hypercubic association scheme. In this case an association scheme, i.e., hypercubic association scheme [5], [12], of $m$ associate classes for $v^* = v^m$ treatments can be naturally defined as follows:

Suppose that there are $v^* = v^m$ treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ indexed by $m$-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ where $\alpha_i = 1, 2, \cdots, v$ and $i = 1, 2, \cdots, m$. Any two treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\phi(\beta_1, \beta_2, \cdots, \beta_m)$ are $i$th associates if and only if $\sum_{k=1}^{m} \varepsilon(\alpha_k - \beta_k) = i$. Each treatment is the 0th associate of itself.

Then it is clear from the structure of $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$ that the parameters of this PBIB design are given by (5.1).

Lemma 5.2 also gives a method, which is somewhat different from that of Kusumoto [5], of constructing a PBIB design with the hypercubic association scheme.

Thus from Lemmas 5.1 and 5.2 and the method of proof of Theorems 1 and 2, the following theorem can be established:

THEOREM 3. *Given the* BIB *designs* $N_i$ *with parameters* $v_i$, $b_i$, $r_i$, $k_i$ *and* $\lambda_i$ $(i = 1, 2, \cdots, m)$. *Then necessary and sufficient conditions for the Kronecker product* PBIB *design* $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m$, *which has at most* $2^m - 1$ *associate classes having the* $F_m$ *type association scheme, to be reducible to a* PBIB *design with only* $m$ *distinct associate classes having the hypercubic association scheme are*

$$v_1 = v_2 = \cdots = v_m , \qquad k_1 = k_2 = \cdots = k_m .$$

Note that this is the least reducible number of associate classes. Theorems 1 and 2 described in the preceding sections are obviously special cases of Theorem 3.

**6. Kronecker product design of another type.** Let $N_i$ be BIB designs with parameters $v_i$, $b_i$, $r_i$, $k_i$ and $\lambda_i$ $(i = 1, 2)$ and consider the Kronecker product of these designs in the form $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$, where $N_i^*$ is a complementary BIB design with parameters $v_i^* = v_i$, $b_i^* = b_i$, $r_i^* = b_i - r_i$, $k_i^* = v_i - k_i$ and $\lambda_i^* = b_i - 2r_i + \lambda_i$ of a BIB design $N_i$ $(i = 1, 2)$.

From the properties of a BIB design we have

$$\begin{aligned} N_i N_i' &= (r_i - \lambda_i) I_{v_i} + \lambda_i G_{v_i} , \\ N_i^* N_i^{*\prime} &= (r_i - \lambda_i) I_{v_i} + \lambda_i^* G_{v_i} , \\ N_i N_i^{*\prime} &= (\lambda_i - r_i) I_{v_i} + (r_i - \lambda_i) G_{v_i} , \end{aligned} \tag{6.1}$$

where $G_{v_i}$ is a $(v_i \times v_i)$ matrix whose elements are all unity and $i = 1, 2$. Noting that the matrices (6.1) are symmetrical and mutually commutative, we can find orthogonal matrices $H_i$ which make all $N_i N_i'$, $N_i^* N_i^{*\prime}$, $N_i N_i^{*\prime}$ diagonal simultaneously, such as

$$\begin{aligned} H_i N_i N_i' H_i' &= D_i = \operatorname{diag}\{r_i k_i, r_i - \lambda_i, \cdots, r_i - \lambda_i\} , \\ H_i N_i^* N_i^{*\prime} H_i' &= D_i^* = \operatorname{diag}\{(b_i - r_i)(v_i - k_i), r_i - \lambda_i, \cdots, r_i - \lambda_i\} , \\ H_i N_i N_i^{*\prime} H_i' &= \tilde{D}_i = \operatorname{diag}\{(v_i - 1)(r_i - \lambda_i), \lambda_i - r_i, \cdots, \lambda_i - r_i\} , \end{aligned} \tag{6.2}$$

where $r_i - \lambda_i$ and $\lambda_i - r_i$ appear $v_i - 1$ times respectively and $i = 1, 2$. Hence we obtain

$$(H_1 \otimes H_2) N N' (H_1 \otimes H_2)' = D_1 \otimes D_2 + D_1^* \otimes D_2^* + 2\tilde{D}_1 \otimes \tilde{D}_2 . \tag{6.3}$$

Further from the structure of $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$, the parameters of design $N$ are obviously given by

$$\begin{aligned}
&v' = v_1 v_2 , \qquad b' = b_1 b_2 , \\
&r' = r_1 r_2 + (b_1 - r_1)(b_2 - r_2) , \\
&k' = k_1 k_2 + (v_1 - k_1)(v_2 - k_2) , \\
&\lambda_0' = r' , \qquad \lambda_1' = r_1 \lambda_2 + (b_1 - r_1)(b_2 - 2r_2 + \lambda_2) , \\
&\lambda_2' = r_2 \lambda_1 + (b_2 - r_2)(b_1 - 2r_1 + \lambda_1) , \\
&\lambda_3' = \lambda_1 \lambda_2 + (b_1 - 2r_1 + \lambda_1)(b_2 - 2r_2 + \lambda_2) + 2(r_1 - \lambda_1)(r_2 - \lambda_2) .
\end{aligned}$$

Then from (6.2) and (6.3) the latent roots of $NN'$ are as follows:

$$\begin{aligned}
&\rho_0 = r'k' && \text{with multiplicity } 1 , \\
&\rho_1 = (r_2 - \lambda_2)(4r_1 k_1 + b_1 v_1 - 4b_1 k_1) && \text{with multiplicity } v_2 - 1 , \\
&\rho_2 = (r_1 - \lambda_1)(4r_2 k_2 + b_2 v_2 - 4b_2 k_2) && \text{with multiplicity } v_1 - 1 , \\
&\rho_3 = 4(r_1 - \lambda_1)(r_2 - \lambda_2) && \text{with multiplicity } (v_1 - 1)(v_2 - 1) .
\end{aligned}$$

The above statement and the proof of Theorem 4.2 of Vartak [9] imply that $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ is a PBIB design with at most three associate classes. In like manner to Section 3, among $v_1 v_2$ treatments in design $N$ a rectangular association scheme can be defined. But it was shown by Sillitto [8] and Shrikhande [7] that if the parameters of the original BIB designs $N_i$ ($i = 1, 2$) satisfy condition $b_i = 4(r_i - \lambda_i)$, then $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ is a BIB design with parameters $\lambda' = \lambda_1' = \lambda_2' = \lambda_3' = r' - b'/4$. In this case the latent roots are $\rho_0 = r'k'$ with multiplicity 1 and $\rho_1 = \rho_2 = \rho_3 = r' - \lambda'$ with multiplicity $v' - 1$.

Here we consider the derivation of necessary and sufficient conditions for a PBIB design $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ with at most three associate classes to be reducible to a PBIB design with only two distinct associate classes. The following relations can be obtained:

$$\begin{aligned}
&\lambda_1' = \lambda_3' \quad \text{if and only if} \quad b_2 = 4(r_2 - \lambda_2) , \\
&\lambda_2' = \lambda_3' \quad \text{if and only if} \quad b_1 = 4(r_1 - \lambda_1) , \\
&\lambda_1' = \lambda_2' \quad \text{if and only if} \quad b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1) , \\
&\rho_1 = \rho_3 \quad \text{if and only if} \quad b_1 = 4(r_1 - \lambda_1) , \\
&\rho_2 = \rho_3 \quad \text{if and only if} \quad b_2 = 4(r_2 - \lambda_2) , \\
&\rho_1 = \rho_2 \quad \text{if and only if} \quad (r_1 - \lambda_1)(4r_2 k_2 + b_2 v_2 - 4b_2 k_2) \\
&\qquad\qquad = (r_2 - \lambda_2)(4r_1 k_1 + b_1 v_1 - 4b_1 k_1) .
\end{aligned} \tag{6.4}$$

Hence

$$\left.\begin{aligned} \lambda_1' = \lambda_2' \neq \lambda_3' , \\ \rho_1 = \rho_2 \neq \rho_3 \end{aligned}\right\} \text{ if and only if } \begin{cases} b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1) , \\ b_i \neq 4(r_i - \lambda_i) , \qquad i = 1, 2 , \\ (r_1 - \lambda_1)(4r_2 k_2 + b_2 v_2 - 4b_2 k_2) \\ \quad = (r_2 - \lambda_2)(4r_1 k_1 + b_1 v_1 - 4b_1 k_1) . \end{cases} \tag{6.5}$$

Further it should be noted, from the relations among the parameters of the BIB designs $N_i$, that condition (6.5) is equivalent to the following condition:

$$(6.6) \qquad v_1 = v_2 = v\,, \quad \text{say}\,; \qquad b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1)\,,$$
$$b_i \neq 4(r_i - \lambda_i)\,, \quad i = 1, 2\,.$$

Now it is clear from Lemma A and (6.4) that if a PBIB design $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ with at most three associate classes is a PBIB design with only two distinct associate classes, then we must have (6.5), i.e., (6.6). Conversely if the parameters of a PBIB design $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ satisfy condition (6.6), i.e., (6.5), then the design can obviously lead to a PBIB design with only two distinct associate classes. For under condition (6.6), in like manner to Section 3, among $v_1 v_2 = v^2$ treatments in design $N$, an $L_2$ association scheme can be defined.

Thus the following theorem is obtained:

THEOREM 4. *Given the* BIB *designs $N_i$ with parameters $v_i$, $b_i$, $r_i$, $k_i$ and $\lambda_i$ ($i = 1, 2$). Then necessary and sufficient conditions for a* PBIB *design, which has at most three associate classes having the rectangular association scheme and which is constructed by the Kronecker product $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$, to be reducible to a* PBIB *design with only two distinct associate classes having the $L_2$ association scheme are*

$$v_1 = v_2\,, \qquad b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1)\,, \qquad b_i \neq 4(r_i - \lambda_i)\,, \qquad i = 1, 2\,.$$

A generalization of this type is easily given and omitted here. It should be noted that a comparison of Theorems 1 and 4 can be made from a structural point of view of the design $N$.

**7. Complementary remark.** Under the conditions obtained by equalizing all those among the latent roots of $NN'$ and among all the coincidence numbers which may be equal to each other, a PBIB design with at most $2^m - 1$ associate classes is reducible to a PBIB design with only $m$ distinct associate classes. But under the conditions obtained by equalizing suitably those in part which may be equal to each other, a PBIB design with at most $2^m - 1$ associate classes may be reducible to a PBIB design with $m_1$ associate classes for a positive integer $m_1$ satisfying $m < m_1 \leqq 2^m - 1$. It is not generally easy to determine the association scheme matching this design with $m_1$ associate classes.

For example, we consider the case of Section 4, i.e., $m = 3$. If in condition (4.1) we assume the conditions to hold in any two rows, then the same result as Theorem 2 is obtained, because the conditions in the remaining row are necessarily derived from the conditions in the two chosen rows. Next we assume only the conditions in any one row to hold in (4.1). Without loss of generality we can assume that

$$r_1 k_1(r_2 - \lambda_2) = r_2 k_2(r_1 - \lambda_1)\,, \qquad r_1 \lambda_2 = r_2 \lambda_1\,.$$

From Section 3, these are equivalent to the condition

$$(7.1) \qquad v_1 = v_2 = v\,, \quad \text{say}\,; \qquad k_1 = k_2 = k\,, \quad \text{say}\,.$$

For the remaining conditions we further assume that

$$(7.2)\qquad \begin{aligned} &r_1 k_1(r_3 - \lambda_3) \neq r_3 k_3(r_1 - \lambda_1)\,, \qquad r_1\lambda_3 \neq r_3\lambda_1\,, \\ &r_2 k_2(r_3 - \lambda_3) \neq r_3 k_3(r_2 - \lambda_2)\,, \qquad r_2\lambda_3 \neq r_3\lambda_2\,, \\ &r_1 k_1 r_2 k_2(r_3 - \lambda_3) \neq r_3 k_3(r_1 - \lambda_1)(r_2 - \lambda_2)\,, \qquad r_1 r_2 \lambda_3 \neq \lambda_1\lambda_2 r_3\,. \end{aligned}$$

Under conditions (7.1) and (7.2) all distinct nonzero latent roots of $NN'$ are six among eight latent roots, i.e., $r_1k_1r_2k_2r_3k_3$, $r_1k_1r_2k_2(r_3-\lambda_3)$, $r_1k_1r_3k_3(r_2-\lambda_2) = r_2k_2r_3k_3(r_1-\lambda_1)$, $r_1k_1(r_2-\lambda_2)(r_3-\lambda_3) = r_2k_2(r_1-\lambda_1)(r_3-\lambda_3)$, $r_3k_3(r_1-\lambda_1)(r_2-\lambda_2)$ and $(r_1-\lambda_1)(r_2-\lambda_2)(r_3-\lambda_3)$. Further all distinct coincidence numbers of a pair of treatments are also six among eight coincidence numbers, i.e., $r_1r_2r_3$, $r_1r_2\lambda_3$, $r_1\lambda_2r_3 = \lambda_1r_2r_3$, $r_1\lambda_2\lambda_3 = \lambda_1r_2\lambda_3$, $\lambda_1\lambda_2r_3$ and $\lambda_1\lambda_2\lambda_3$.

Thus under conditions (7.1) and (7.2) we may obtain a PBIB design with $m_1$ associate classes ($m_1 = 5, 6, 7$) having the parameters:

$$\begin{gathered} v^* = v^2v_3\,, \qquad b^* = b_1b_2b_3\,, \qquad r^* = r_1r_2r_3\,, \qquad k^* = k^2k_3\,, \\ \lambda_0^* = r^*\,, \qquad \lambda_1^* = r_1r_2\lambda_3\,, \qquad \lambda_2^* = r_1\lambda_2r_3 = \lambda_1r_2r_3\,, \\ \lambda_3^* = r_1\lambda_2\lambda_3 = \lambda_1r_2\lambda_3\,, \qquad \lambda_4^* = \lambda_1\lambda_2r_3\,, \qquad \lambda_5^* = \lambda_1\lambda_2\lambda_3\,. \end{gathered}$$

Then what association schemes are introduced among these $v^2v_3$ treatments? For $m_1 = 7$ we shall be able to introduce still the $F_3$ type association scheme. For $m_1 = 5$ we get a PBIB design with only five distinct associate classes. However, there are not well-known association schemes aside from the $F_3$ type association scheme for $m_1 = 5$ and 6. This is one of the unsolved problems concerned with the classification of association schemes.

**8. Some illustrations.** Examples for a special case, i.e., $m = 2$, of Lemmas 5.1 and 5.2 are given by Examples 5.1 and 5.2 of Vartak [9]. Here examples of a comparison of Theorems 1 and 4 are given.

Let $N_1$ and $N_2$ be BIB designs with the following parameters

$$v_1 = b_1 = 3\,, \qquad r_1 = k_1 = 2\,, \qquad \lambda_1 = 1$$

and

$$v_2 = 5\,, \qquad b_2 = 10\,, \qquad r_2 = 4\,, \qquad k_2 = 2\,, \qquad \lambda_2 = 1\,,$$

respectively.

From Section 3 the parameters of a PBIB design $N^{(1)} = N_1 \otimes N_2$ with three associate classes are given by

$$\begin{gathered} v^{(1)} = 15\,, \qquad b^{(1)} = 30\,, \qquad r^{(1)} = 8\,, \qquad k^{(1)} = 4\,, \\ \lambda_1^{(1)} = 2\,, \qquad \lambda_2^{(1)} = 4\,, \qquad \lambda_3^{(1)} = 1\,, \\ n_1^{(1)} = 4\,, \qquad n_2^{(1)} = 2\,, \qquad n_3^{(1)} = 8\,, \end{gathered}$$

$$\mathscr{P}_1^{(1)} = \begin{Vmatrix} 3, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 4, & 3 \end{Vmatrix}, \qquad \mathscr{P}_2^{(1)} = \begin{Vmatrix} 0, & 0, & 1 \\ 0, & 1, & 0 \\ 2, & 0, & 3 \end{Vmatrix}, \qquad \mathscr{P}_3^{(1)} = \begin{Vmatrix} 0, & 4, & 3 \\ 2, & 0, & 1 \\ 6, & 4, & 3 \end{Vmatrix}.$$

The associate classes and blocks of design $N^{(1)}$ are given by Vartak [9].

Now from Section 6 the parameters of a PBIB design $N^{(2)} = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ with three associate classes are given by

$$v^{(2)} = 15\,, \qquad b^{(2)} = 30\,, \qquad r^{(2)} = 14\,, \qquad k^{(2)} = 7\,,$$
$$\lambda_1^{(2)} = 5\,, \qquad \lambda_2^{(2)} = 4\,, \qquad \lambda_3^{(2)} = 7\,.$$

Since the same rectangular association scheme is introduced in both designs $N^{(1)}$ and $N^{(2)}$, the parameters of the association scheme of designs $N^{(1)}$ and $N^{(2)}$ are all the same. Hence it should be noted that there may be more than one PBIB design based on the same association scheme.

Let the treatments of $N^{(2)}$ be designated by integers $1, 2, \cdots, 15$. Then the blocks of design $N^{(2)}$ are as follows:

| | | |
|---|---|---|
| (1, 2, 6, 7, 13, 14, 15) , | (1, 3, 6, 8, 12, 14, 15) , | (1, 4, 6, 9, 12, 13, 15) , |
| (1, 5, 6, 10, 12, 13, 14) , | (2, 3, 7, 8, 11, 14, 15) , | (2, 4, 7, 9, 11, 13, 15) , |
| (2, 5, 7, 10, 11, 13, 14) , | (3, 4, 8, 9, 11, 12, 15) , | (3, 5, 8, 10, 11, 12, 14) , |
| (4, 5, 9, 10, 11, 12, 13) , | (1, 2, 8, 9, 10, 11, 12) , | (1, 3, 7, 9, 10, 11, 13) , |
| (1, 4, 7, 8, 10, 11, 14) , | (1, 5, 7, 8, 9, 11, 15) , | (2, 3, 6, 9, 10, 12, 13) , |
| (2, 4, 6, 8, 10, 12, 14) , | (2, 5, 6, 8, 9, 12, 15) , | (3, 4, 6, 7, 10, 13, 14) , |
| (3, 5, 6, 7, 9, 13, 15) , | (4, 5, 6, 7, 8, 14, 15) , | (3, 4, 5, 6, 7, 11, 12) , |
| (2, 4, 5, 6, 8, 11, 13) , | (2, 3, 5, 6, 9, 11, 14) , | (2, 3, 4, 6, 10, 11, 15) , |
| (1, 4, 5, 7, 8, 12, 13) , | (1, 3, 5, 7, 9, 12, 14) , | (1, 3, 4, 7, 10, 12, 15) , |
| (1, 2, 5, 8, 9, 13, 14) , | (1, 2, 4, 8, 10, 13, 15) , | (1, 2, 3, 9, 10, 14, 15) . |

These blocks are simply constructed by the Kronecker product of

$$N_1 = \left\|\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}\right\| \qquad \text{and} \qquad N_2 = \left\|\begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}\right\|$$

in an incidence matrix form.

Further, let us take $N_1 = N_2$ to be BIB designs with parameters $v = b = 3$, $r = k = 2$ and $\lambda = 1$. Then since the parameters of these designs satisfy the conditions of Theorems 1 and 4, the parameters of two PBIB designs $N^{(1)} = N_1 \otimes N_2$ and $N^{(2)} = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ with two associate classes are

$$v^{(1)} = b^{(1)} = 9\,, \qquad r^{(1)} = k^{(1)} = 4\,, \qquad \lambda_1^{(1)} = 2\,, \qquad \lambda_2^{(1)} = 1$$

and

$$v^{(2)} = b^{(2)} = 9\,, \qquad r^{(2)} = k^{(2)} = 5\,, \qquad \lambda_1^{(2)} = 2\,, \qquad \lambda_2^{(2)} = 3$$

and

$$n_1^{(1)} = n_1^{(2)} = 4\,, \qquad n_2^{(1)} = n_2^{(2)} = 4\,,$$
$$\mathscr{P}_1^{(1)} = \mathscr{P}_1^{(2)} = \left\|\begin{matrix} 1\,, & 2 \\ 2\,, & 2 \end{matrix}\right\|, \qquad \mathscr{P}_2^{(1)} = \mathscr{P}_2^{(2)} = \left\|\begin{matrix} 2\,, & 2 \\ 2\,, & 1 \end{matrix}\right\|,$$

respectively.

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