The Annals of Applied Probability
2001, Vol. 11, No. 1, 121–181

SPATIAL STRUCTURE IN LOW DIMENSIONS FOR DIFFUSION LIMITED TWO-PARTICLE REACTIONS

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Consider the system of particles on $\mathbb{Z}^d$ where particles are of two types, A and B, and execute simple random walks in continuous time. Particles do not interact with their own type, but when a type A particle meets a type B particle, both disappear. Initially, particles are assumed to be distributed according to homogeneous Poisson random fields, with equal intensities for the two types. This system serves as a model for the chemical reaction $A + B \to inert$. In Bramson and Lebowitz [7], the densities of the two types of particles were shown to decay asymptotically like $1/t^{d/4}$ for $d < 4$ and $1/t$ for $d \geq 4$, as $t \to \infty$. This change in behavior from low to high dimensions corresponds to a change in spatial structure. In $d < 4$, particle types segregate, with only one type present locally. After suitable rescaling, the process converges to a limit, with density given by a Gaussian process. In $d > 4$, both particle types are, at large times, present locally in concentrations not depending on the type, location or realization. In $d = 4$, both particle types are present locally, but with varying concentrations. Here, we analyze this behavior in $d < 4$; the behavior for $d \geq 4$ will be handled in a future work by the authors.

CONTENTS

1. Introduction. Consider a system of particles of two types on $\mathbb{Z}^d$, A and B, which execute simple random walks in continuous time at rate d. That is, the motion of different particles is independent, and a particle at site x will jump to a given one of its $2d$ nearest neighbors at rate $1/2$. Particles are

Received May 1999; revised February 2000.
[1]Supported in part by NSF Grant DMS-96-26196.
[2]Supported in part by NSF Grant DMR-95-23266, DIMACS and supporting agencies, NSF Contract STC-91-19999 and the New Jersey Commission on Science and Technology.
AMS 2000 *subject classification*. 60K35.
Key words and phrases. Diffusion limited reaction, annihilating random walks, asymptotic densities, spatial structure.

assumed not to interact with their own type; multiple A particles or multiple B particles can occupy a given site. However, when a particle meets a particle of the opposite type, both disappear. (When a particle simultaneously meets more than one particle of the opposite type, it will cause only one of these particles to disappear.)

We assume that particles are initially distributed according to independent homogeneous Poisson random fields, with intensity λ for each type of particle. That is, the probability of there being j_1 type A particles and j_2 type B particles at a given site x is $e^{-2\lambda}\lambda^{j_1+j_2}/j_1!\,j_2!$. If there are initially both A and B particles at a site x, they immediately cancel each other out as much as possible. We denote by $\xi_t^A(x)$ and $\xi_t^B(x)$ the number of A particles, respectively, the number of B particles at site x, and by $\xi_t^A(E)$ and $\xi_t^B(E)$ the number of such particles in a finite set $E \subset \mathbb{Z}^d$. Also, set $\xi_t^{\#}(E) = \xi_t^A(E) + \xi_t^B(E)$, for the total number of particles in E. We associate with each A particle the value -1 and with each B particle the value 1, and denote by $\xi_t(x)$ the signed number of particles at $x \in \mathbb{Z}^d$, that is, $\xi_t(x) = \xi_t^B(x) - \xi_t^A(x)$. Similarly, $\xi_t(E) = \xi_t^B(E) - \xi_t^A(E)$. We denote by ξ_t, $\xi_t \in (\mathbb{Z}_+^2)^{\mathbb{Z}^d}$, the random state of the system at time t, where $\mathbb{Z}_+$ designates the nonnegative integers; the first coordinate at each site corresponds to the number of A particles there, and the second coordinate to the number of B particles. We write ξ_{0-} for the initial state before A and B particles originally at the same site have annihilated one another.

The above two-particle annihilating random walk can serve as a model for the irreversible chemical reaction $A + B \to \mathit{inert}$, where both particle types are mobile. A and B can also represent matter and antimatter. There has been substantial interest in this model in the physics literature over the last two decades following papers by Ovchinnikov and Zeldovich [17], Toussaint and Wilczek [19], and Kang and Redner [11]; see [7] for a more complete set of references, and [14] and [15] for more recent work.

Let $\rho_A(t)$ and $\rho_B(t)$ denote the densities of A and B particles at the origin, that is,

$$
\begin{aligned}
\rho_A(t) &= E[\#A \text{ particles at 0 at time } t],\\
\rho_B(t) &= E[\#B \text{ particles at 0 at time } t].
\end{aligned}
\tag{1.1}
$$

(In the paper, E will be used for both expectations and sets.) Since ξ_{0-} is translation invariant, its densities do not depend on the site x. The difference $\rho_B(t) - \rho_A(t)$ remains constant for all t, because particles annihilate in pairs. Since $\rho_A(0-) = \rho_B(0-) = \lambda$, one has $\rho_A(t) = \rho_B(t)$ for all t, which we will denote by $\rho(t)$. In [7], it was shown that

$$
\begin{aligned}
c_d\lambda^{1/2}/t^{d/4} \le \rho(t) &\le c'_d\lambda^{1/2}/t^{d/4} && \text{for } d < 4,\\
c_4(\lambda^{1/2} \vee 1)/t \le \rho(t) &\le c'_4(\lambda^{1/2} \vee 1)/t && \text{for } d = 4,\\
c_d/t \le \rho(t) &\le c'_d/t && \text{for } d > 4,
\end{aligned}
\tag{1.2}
$$

for large t and appropriate positive constants c_d and c'_d; here, $a \vee b = \max(a, b)$. (Bounds were also derived when the initial densities are unequal.) The asymptotic power laws in (1.2) were previously obtained, in [19] and [11], using heuristic arguments.

The asymptotics of $\rho(t)$ in (1.2) are tied to the spatial structure of ξ_t, which also depends on d. The slow rate of decay for $d < 4$ corresponds to the presence locally, at large times, of only one type of particle, typically. This behavior is a consequence of the random fluctuations in the numbers of A and B particles locally in the initial state, and the tendency for particles of the local minority type to be annihilated before they can be replenished by the arrival of particles from outside the region. In particular, the random walks executed by these particles are diffusive, which imposes limitations on the rate of mixing of particles. For $d > 4$, A and B particles remain sufficiently mixed so that the behavior is different, and mean field reasoning gives the correct asymptotics. Namely, assuming that $d\rho_A(t)/dt$ is proportional to $-\rho_A(t)\rho_B(t) = -(\rho_A(t))^2$, then $\rho(t) = \rho_A(t)$ will decay like a multiple of $1/t$, which is the right answer. In this latter setting, the limiting density does not depend on the initial densities. The dimension $d = 4$ is a hybrid of the previous two cases, with both mechanisms playing a role.

It is the purpose of this paper to analyze the spatial structure of ξ_t in $d < 4$. The behavior of ξ_t in $d \geq 4$ will be covered in [9]. Our main results are Theorems 1 and 2. Theorem 1 gives the macroscopic limiting behavior of the process. It says, in essence, that ξ_t, under diffusive scaling, converges to a limit which is the convolution of white noise with the normal kernel. Regions where this convolution is positive correspond to regions where only B particles are present, with negative regions corresponding to the presence of A particles.

By white noise, we mean the stochastic process Φ whose domain is the set of finite rectangular solids $D \subset \mathbb{R}^d$, with sides parallel to the coordinate axes, such that any linear combination $\sum_{j=1}^n a_j \Phi(D_j)$ is normally distributed with mean 0, and, for any D_1 and D_2,

$$E[\Phi(D_1)\Phi(D_2)] = |D_1 \cap D_2|. \tag{1.3}$$

Loosely speaking, $\langle \Phi(x)\Phi(y)\rangle = \delta(x - y)$ for $x, y \in \mathbb{R}^d$, that is, $\Phi(x)$ is a Gaussian field with a δ-function covariance, in physics terminology. [One can alternatively define Φ as the linear functional on the Schwartz space of rapidly decreasing functions $f\colon \mathbb{R}^d \to \mathbb{R}$, where $\Phi(f)$ is normally distributed with mean 0, and $E[\Phi(f_1)\Phi(f_2)] = \int_{\mathbb{R}^d} f_1(x) f_2(x)\,dx$.] These rules specify a generalized Gaussian random field on $\mathbb{R}^d$. We will assume that $\Phi(D)$ is, for each realization, continuous in the coordinates of D; a version of the process exists for which this holds. White noise is closely connected with Brownian sheet, and the above definition is motivated by this relationship. (More detail will be given in Section 8.)

We will write $N_t(\cdot)$ for the density of a normal random variable with mean 0 and covariance matrix tI, that is,

$$N_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t} \quad \text{for } x \in \mathbb{R}^d, \tag{1.4}$$

where $|\cdot|$ is the Euclidean norm. We write $N_t(E) = \int_E N_t(x)\,dx$ for measurable $E \subset \mathbb{R}^d$. Let $\Phi * N_t$ denote the convolution of Φ with N_t, that is,

$$(\Phi * N_t)(D) = \int_{\mathbb{R}^d} \Phi(D - x) N_t(x)\,dx, \tag{1.5}$$

where $D + y$ designates D translated by y. Since $\Phi * N_t$ is the average of translates of a generalized Gaussian random field, $\Phi * N_t$ is also Gaussian. Because of the smoothness of N_t, $\Phi * N_t$ will have a density $(\Phi * N_t)(x)$; it is Gaussian with variance $(4\pi t)^{-d/2}$ at each point. We will write $(\Phi * N_t)^-(D)$ and $(\Phi * N_t)^+(D)$ for the integrals of the negative and positive parts of $(\Phi * N_t)(x)$ over D.

To state Theorem 1, we need to normalize ξ_t. The notation ${}^T\hat{\xi}_t$ (respectively, ${}^T\hat{\xi}_t^A$ and ${}^T\hat{\xi}_t^B$) will denote ξ_t (respectively, ξ_t^A and ξ_t^B) after scaling time by T, space by $T^{1/2}$ in each direction, and the weight of individual particles by $T^{d/4}$. That is,

$${}^T\hat{\xi}_t(E) = \xi_{Tt}(T^{1/2}E)/T^{d/4}, \tag{1.6}$$

where $E \subset \mathbb{Z}^d_{T^{1/2}}$, which is $\mathbb{Z}^d$ scaled by $T^{1/2}$ in each direction. The factor $T^{d/4}$ is mandated by the first line of (1.2). For $E \subset \mathbb{R}^d$, we set ${}^T\hat{\xi}_t(E) = {}^T\hat{\xi}_t(E \cap \mathbb{Z}^d_{T^{1/2}})$.

Theorem 1 gives the limiting macroscopic behavior of ξ_t. It states that $({}^T\hat{\xi}_t^A(D), {}^T\hat{\xi}_t^B(D))$ converges weakly, on $\mathbb{R}^2$, to $(2\lambda)^{1/2}((\Phi * N_t)^-(D), (\Phi * N_t)^+(D))$, where λ is the initial density for the A and B particles. Here and elsewhere in the paper, unless stated otherwise, rectangular solids D will be of the form $\prod_{j=1}^d (y_j, x_j]$. They will be called "rectangles" for short.

THEOREM 1. *Let Φ and N_t be defined as above, with $t > 0$, and let $D \subset \mathbb{R}^d$ be any finite rectangle. Then, for $d < 4$,*

$$\left({}^T\hat{\xi}_t^A(D), {}^T\hat{\xi}_t^B(D)\right) \Rightarrow \left((2\lambda)^{1/2}(\Phi * N_t)^-(D), (2\lambda)^{1/2}(\Phi * N_t)^+(D)\right) \tag{1.7}$$

as $T \to \infty$.

A more general version of Theorem 1, Theorem 4, is demonstrated in Section 8. There, it is shown that ${}^T\hat{\xi}_t^A(D)$ and ${}^T\hat{\xi}_t^B(D)$ are uniformly well approximated by $(2\lambda)^{1/2}(\Phi * N_t)^-(D)$ and $(2\lambda)^{1/2}(\Phi * N_t)^+(D)$ over all $t \in [1/M, M]$, $M > 1$, and all D in a fixed cube. The rectangles D in both theorems can easily be generalized, although one needs the cardinality of the collection of sets employed in Theorem 4 not to be too large, in order to avoid the piling up of small probability events where either ${}^T\hat{\xi}_t^A(E)$ or ${}^T\hat{\xi}_t^B(E)$, $E \subset \mathbb{R}^d$, is badly behaved.

In the course of demonstrating Theorem 4, one obtains estimates that give the asymptotic behavior of $\rho(t)$. It is shown at the end of Section 6 that

$$\lim_{t\to\infty} t^{d/4}\rho(t) = (\lambda/\pi)^{1/2}(4\pi)^{-d/4} \tag{1.8}$$

when $d < 4$. This strengthens the first line of (1.2). These limits were given in [19]. We also note it follows immediately from (1.7) that

$$^{T}\hat{\xi}_t(D) \Rightarrow (2\lambda)^{1/2}(\Phi * N_t)(D) \quad \text{as } T \to \infty. \tag{1.9}$$

Conversely, (1.7) will follow from (1.9), if one also knows that the particle types segregate.

In order to understand the spatial structure of ξ_t, one also needs to know its behavior on the microscopic scale. By (1.8), the correct spatial scaling will be $t^{1/4}$ in each direction, and so we set

$$\check{\xi}_t(E) = \xi_t(t^{1/4}E), \tag{1.10}$$

with $E \subset \mathbb{Z}^d_{T^{1/4}}$, and $\check{\xi}^A_t$ and $\check{\xi}^B_t$ being defined analogously. One can also guess at the limiting spatial structure of $\check{\xi}_t$. Particles will, at large times, only be annihilated occasionally. This allows particles the time to mix locally, without interaction. They should therefore be independently distributed locally, as $t \to \infty$. This produces a Poisson random field, after conditioning on the intensity at $x = 0$, $(2\lambda)^{1/2}(\Phi * N_1)(0)$, with the type of particle present depending on the sign of $(\Phi * N_1)(0)$. The random variable $(\Phi * N_1)(0)$ is normally distributed, with mean 0 and variance $(4\pi)^{-d/2}$.

With this behavior in mind, we denote by $\mathscr{P}_c$ the Poisson random field of A particles with intensity c^- if $c \le 0$, and the Poisson random field of B particles with intensity c^+ if $c > 0$. We interpret $\mathscr{P}_c$ as a vector, with the first coordinate corresponding to A particles and the second coordinate to B particles. Also, for F a probability distribution function on $\mathbb{R}$, set

$$\mathscr{P}_F = \int \mathscr{P}_c \, dF(c), \tag{1.11}$$

that is, $\mathscr{P}_F$ is the convex combination of homogeneous Poisson random fields with intensities weighted according to F.

In Theorem 2, $\Rightarrow$ denotes weak convergence with respect to the Borel measures on $\mathbb{R}^d$ having finite mass on all compact subsets. [The space of measures is assumed to be equipped with the topology of vague convergence on $\mathbb{R}^d$, that is, integration is against $f \in C^+_c(\mathbb{R}^d)$, where $C^+_c(\mathbb{R}^d)$ is the set of nonnegative continuous functions on $\mathbb{R}^d$ with compact support.]

THEOREM 2. *For $d < 4$,*

$$(\check{\xi}^A_t, \check{\xi}^B_t) \Rightarrow \mathscr{P}_F \quad \textit{as } t \to \infty, \tag{1.12}$$

where F is the distribution of a normal random variable with mean 0 *and variance* $2\lambda(4\pi)^{-d/2}$.

In this paper, we will demonstrate Theorem 2 and Theorem 4, the more general version of Theorem 1 mentioned earlier. Versions of these results were summarized in [8]. An outline of the main steps leading to these results will be given in the next section.

Certain features of the asymptotic behavior of the model considered here are shared by two simpler systems, coalescing random walk and annihilating random walk. Both cases consist of particles on $\mathbb{Z}^d$, of a single type, which execute independent simple random walks. In the first case, when two particles meet, they coalesce into a single particle, whereas, in the second case, they annihilate one another. The two models can be interpreted in terms of the chemical reactions $A + A \to A$ and $A + A \to \textit{inert}$, respectively. For both models, it is natural to assume that all sites are initially occupied.

The asymptotic behavior of both models is known. For the coalescing random walk, the density is asymptotically $1/\sqrt{\pi t}$ in $d = 1$, $(\log t)/\pi t$ in $d = 2$, and $1/\gamma_d t$, for appropriate γ_d, in $d \geq 3$ [6]. The asymptotic density of annihilating random walk is, in each case, one half as great [2]. Scaling, so as to compensate for the decrease in density, produces analogs of Theorem 2. For $d \geq 2$, the limiting measure is again Poisson [2], but, in $d = 1$, it is not [1].

Recent work [12] considers a generalization of the above coalescing random walk. There, coalescence is not automatic when two particles meet, and occurs with a probability that depends on the number of particles present at a site. Results are obtained for $d \geq 6$.

2. Summaries of the proofs of Theorems 1 and 2. In this section, we summarize the proofs of Theorems 1 and 2. We present the main steps, providing motivation in each case. Proofs of the individual steps are given in the remaining sections.

Rather than directly show Theorem 1, our approach will be to first show Theorem 3, which is given below. This result is a more concrete analog of Theorem 1, which compares ξ_t with $\xi_0 * N_t$ along individual sample paths, instead of showing weak convergence of ${}^T\hat{\xi}_t$ to $(2\lambda)^{1/2}(\Phi * N_t)$. The result also shows that the distribution of particles for ξ_t, at large times, is essentially deterministic if ξ_0 is known. Error bounds for the corresponding estimates, in (2.2) and (2.3), are given in terms of powers of T; one has $T^{d/4-1/9{,}000}$ inside of $P(\cdot)$, and $T^{-1/9{,}000}$ on the right side of the inequalities. (The exact values of the small constants are not important, but show that convergence occurs at least at a polynomial rate.)

Here and later on in the paper, $\mathscr{D}_R$ will denote the set of all rectangles contained in $D_R = \prod_{j=1}^{d}(-R/2, R/2]$, the semiclosed cube of length R centered at the origin. [Recall that rectangles are always assumed to be of the form $\prod_{j=1}^{d}(y_j, x_j]$.] Since the particles in ξ_t reside on $\mathbb{Z}^d$, we will implicitly interpret such rectangles as subsets of $\mathbb{Z}^d$, when there is no risk of ambiguity; $|D|$ will denote the number of sites in $D \cap \mathbb{Z}^d$. Since ξ_0 is discrete, the convolution $\xi_0 * N_t$ will be defined by summing over $\mathbb{Z}^d$, that is,

$$(\xi_0 * N_t)(x) = \sum_{y \in \mathbb{Z}^d} \xi_0(x - y) N_t(y) \quad \text{for } x \in \mathbb{Z}^d. \tag{2.1}$$

This contrasts with the convolution in (1.5), where one integrates over $\mathbb{R}^d$. Throughout the paper, the initial density λ of A and B particles will be considered to be fixed, with $\lambda > 0$. As always, $f(x)^+ = f(x) \vee 0$ and $f(x)^- = -f(x) \vee 0$.

THEOREM 3. *For $d < 4$ and $M > 1$,*

$$P\Big(\sup_{t\in[T/M,\,MT]}\ \sup_{D\in\mathscr{D}_{MT^{1/2}}} \Big|\xi_t^A(D) - \sum_{x\in D}(\xi_0 * N_t)(x)^-\Big| \ge T^{d/4-1/9{,}000}\Big) \le T^{-1/9{,}000} \tag{2.2}$$

and

$$P\Big(\sup_{t\in[T/M,\,MT]}\ \sup_{D\in\mathscr{D}_{MT^{1/2}}} \Big|\xi_t^B(D) - \sum_{x\in D}(\xi_0 * N_t)(x)^+\Big| \ge T^{d/4-1/9{,}000}\Big) \le T^{-1/9{,}000} \tag{2.3}$$

hold for sufficiently large T.

In Section 8, we will derive Theorem 4, and hence Theorem 1, from Theorem 3. The basic procedure will be to show that ξ_0, when scaled as in (1.6), converges weakly to white noise Φ, and then to use Theorem 3 and the continuity of $*$ to obtain (1.7).

In order to demonstrate Theorem 3, we first demonstrate the following analog for $\xi_t(D)$, with $t \in [T/M, T]$ and $D \in \mathscr{D}_{T^{1/2}}$. (Rescaling T will allow us to extend $[T/M, T]$ to $[T/M, MT]$, and $\mathscr{D}_{T^{1/2}}$ to $\mathscr{D}_{MT^{1/2}}$, when convenient later on.)

PROPOSITION 2.1. *For $d < 4$ and $M > 1$,*

$$P\Big(\sup_{t\in[T/M,\,T]}\ \sup_{D\in\mathscr{D}_{T^{1/2}}} |\xi_t(D) - (\xi_0 * N_t)(D)| \ge T^{d/4-1/80}\Big) \le \exp\{-T^{1/42}\} \tag{2.4}$$

holds for sufficiently large T.

This bound is considerably weaker than those in (2.2) and (2.3), in that it only measures the imbalance between the numbers of A and B particles locally, rather than their absolute numbers. (The exponential bound on the right side of the inequality is, of course, stronger.) Proposition 2.1 will be shown in Section 5.

In order to derive (2.2) and (2.3), one also needs to know that, locally, the number of particles of the "minority type" is negligible. For this, it will be sufficient to show that the expected number of such particles is small at specific times that are not too far apart. One will then be able to fill in the behavior at intermediate times and apply Markov's inequality to the expectation. For these purposes, we will employ Proposition 2.2. Together with Proposition 2.1, it will be used to demonstrate Theorem 3, in Section 7. Throughout the paper, we will employ the notation

$$\xi_t^m(E) = \xi_t^A(E) \wedge \xi_t^B(E) \tag{2.5}$$

for the number of particles of the minority type in E, where $E \subset \mathbb{Z}^d$. Here and later on, we use $C_1, C_2, \dots$ for positive constants whose exact values do not concern us.

PROPOSITION 2.2. *Let $d < 4$, $M > 1$, and choose R_T so that $R_T = \delta_1(T)T^{1/2}$, where $\delta_1(T) \geq T^{-d/48}$ and $\delta_1(T) \to 0$ as $T \to \infty$. For sufficiently large T, there exist K and $t_1 < t_2 < \cdots < t_K$, with $t_k - t_{k-1} \leq \delta_1(T)T, [t_1, t_K] \supset [T/M, T], [t_1, t_{K-1}] \subset [T/2M, T]$, and C_1 (depending on λ and M), so that*

$$E[\xi_{t_k}^m(D_{R_T})] \leq C_1\delta_1(T)T^{-d/4}(R_T)^d \quad \text{for } k = 1, \dots, K. \tag{2.6}$$

The cube D_{R_T} contains approximately $(R_T)^d$ sites, and so, by (1.2), will contain of order of magnitude $T^{-d/4}(R_T)^d$ particles. Inequality (2.6) implies that $\xi_{t_k}^m(D_{R_T})$ is, on the average, much smaller than this, for R_T chosen as above.

Proposition 2.2 will follow from machinery introduced in [7]. The basic idea is that, if $E[\xi_t^m(D_{R_T})]$ is large for too long a stretch of time, enough annihilation will occur to contradict the bounds on $\rho(t)$ in (1.2). The bound (2.6) will also enable us to derive precise asymptotics on $\rho(t)$, for $d < 4$, in Section 6. These are an improvement of the upper and lower bounds on $\rho(t)$ in (1.2).

The proof of Proposition 2.1 employs two main results. To state these, we need to introduce some additional notation. Let $K_t(x)$ denote the probability that a simple rate-d continuous time random walk in $\mathbb{Z}^d$, starting at the origin, is at x at time t. Denote by ${}_s\eta_t, s > 0$, the stochastic process in t that is identical to ξ_t up until time s, and for which, starting at time s, the existing particles continue to execute independent simple random walks as before, but without annihilation. We let η_t denote the process of independent random walks with initial state $\eta_0 = \xi_{0-}$, the initial configuration of ξ_t before A and B particles at the same site have annihilated one another. We also set ${}_0\eta_t = \eta_t$. The processes ξ_t and ${}_s\eta_t, s \geq 0$, can all be constructed on the same probability space, so that they are all adapted to the same family of increasing σ-algebras $\mathcal{F}_t, t \geq 0$. To do this, we specify an arbitrary ranking of all of the particles initially in the system, with the rule that when more than two particles of opposite types meet, the highest ranked A and B particles are the ones which are annihilated. Then, $\mathcal{F}_t$ is defined to be the σ-algebra generated by the labeled random walks corresponding to η_r, for $r \leq t$. Later on, we will also employ the σ-algebras $\mathcal{F}_t^\xi \subset \mathcal{F}_t$, where $\mathcal{F}_t^\xi$ is generated by ξ_r, for $r \leq t$.

It is easy to see that for any finite set $E \subset \mathbb{Z}^d$ and $s \leq t$,

$$E[\xi_t(E)|\mathcal{F}_s] = E[{}_s\eta_t(E)|\mathcal{F}_s] = (\xi_s * K_{t-s})(E), \tag{2.7}$$

where $*$ is defined as in (2.1). [In (2.7), the outer E stands for expectation, whereas the inner E is a subset of $\mathbb{Z}^d$.] The following proposition says that, for large t, $(\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)$ is a good approximation of $\xi_t(E)$. The reasons for this are basically that (1) there are few enough particles locally at time $t^{1/4}$, and therefore little enough randomness, so that, up to an error which is of smaller order than $|E|/t^{d/4}$, ${}_{t^{1/4}}\eta_t(E)$ can be replaced by its conditional expectation, and (2) the annihilation of pairs of A and B particles over $[t^{1/4}, t]$ reduces this randomness still further, and so $\xi_t(E)$ can also be replaced by the same conditional expectation. Using (2.7), one can then substitute $(\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)$ for this conditional expectation. Since, for our applications, $\varepsilon|E|$ will not be much less than $t^{d/2}$ and ε will not be too small, the bound on the right side of (2.8) will be quite small. This result is demonstrated in Section 3. Here and later on, we use the abbreviation $v_t(\varepsilon) = \varepsilon \wedge t^{3/16}$.

PROPOSITION 2.3. *For $d < 4$ and sufficiently large t,*

$$(2.8)\qquad \begin{aligned} &P\big(|\xi_t(E) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)| \geq \varepsilon|E|t^{-d/4}\big) \\ &\qquad \leq 6\exp\big\{-((\varepsilon v_t(\varepsilon)|E|t^{-9d/20}) \wedge t^{1/8})\big\} \end{aligned}$$

holds for all ε and $E \subset \mathbb{Z}^d$, with $|E| \leq t^d$.

The other estimate needed for Proposition 2.1 is a comparison of $\xi_0 * N_t$ with $\xi_{t^{1/4}} * K_{t-t^{1/4}}$. These two quantities will typically be close since particles do not wander far by time $t^{1/4}$, and since $K_{t-t^{1/4}}$ can be approximated by N_t, by using an appropriate version of the local central limit theorem. The desired result, Proposition 2.4, is demonstrated in Section 4.

PROPOSITION 2.4. *For any d, let t be sufficiently large and $s \leq t^{1/4}$. Then, for all $\varepsilon \geq 0$,*

$$(2.9)\qquad P\big(\big|(\xi_0 * N_t)(0) - (\xi_s * K_{t-s})(0)\big| \geq \varepsilon t^{-d/4}\big) \leq 4\exp\big\{-(\varepsilon^2 \wedge 1)t^{1/4}\big\}.$$

Note that since ξ_0 is translation invariant, the analog of (2.9) holds at all $x \in \mathbb{Z}^d$. Corresponding bounds therefore hold for finite $E \subset \mathbb{Z}^d$, when factors of $|E|$ are inserted for both inequalities.

Propositions 2.3 and 2.4 are employed in Section 5 to show Proposition 2.1. After combining the two results, we still need to show that the bounds hold simultaneously over all $t \in [T/M, T]$ and $D \in \mathscr{D}_{T^{1/2}}$. Since the probabilities of the exceptional sets in (2.8) and (2.9) will be exponentially small, they can be summed over a fine lattice of elements in $[T/M, T] \times \mathscr{D}_{T^{1/2}}$, while maintaining such bounds. One can then "fill in" the events corresponding to the values between the lattice points, to produce the desired uniformity over $[T/M, T] \times \mathscr{D}_{T^{1/2}}$, as in (2.4) of Proposition 2.1.

Retracing the steps taken so far in this section, we have just discussed Propositions 2.3 and 2.4, which are the main steps in showing Proposition 2.1. As we discussed earlier, Proposition 2.1, together with Proposition 2.2, is employed to derive Theorem 3. By rescaling the process ξ_t in Theorem 3, one then obtains Theorem 1.

We still need to discuss Theorem 2. The additional work required to demonstrate the theorem from the previous results is done in Section 9. The basic reasoning is that, over intervals of time ending in t that are short relative to t, relatively little annihilation occurs, because of the smooth decrease in the density in (1.8). For the space scale of interest to us for $\check{\xi}_t$, namely $t^{1/4}$, this is long enough for the local particles (typically of only one type) to thoroughly mix. Such a mixed state will, for a typical realization, be nearly Poisson for large t. Its intensity near 0 will be given by $t^{d/4}|(\xi_0 * N_t)(0)|$. Laplace transforms are employed to carry out the proof.

As mentioned earlier, the behavior of ξ_t, for $d \geq 4$, will be handled in the future paper [9]. The behavior will be different than that considered here, for $d < 4$, since both types of particles will coexist locally. This leads to a different rate of decay for $\rho(t)$, which is given in (1.2). There are certain similarities, though, and the analogs of the results from Sections 3–5, for $d < 4$, will also be stated for $d \geq 4$ at the end of their respective sections. They will be applied in [9].

3. Approximation of $\boldsymbol{\xi_t}$ by earlier conditional expectations. In this section, we demonstrate Proposition 2.3, which states that for finite sets $E \subset \mathbb{Z}^d$ in $d < 4$, $\xi_t(E)$ is approximated by $(\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)$ with high probability. As outlined in Section 2, we do this by analyzing ${}_{t^{1/4}}\eta_t$, where, we recall, ${}_s\eta_r$, $r \in [0, t]$, is the process that evolves like ξ_r up until time s, but where, over $(s, t]$, the annihilation of particles is quenched. The main goal of this section will be to show the following analog of Proposition 2.3, with ${}_{t^{1/4}}\eta_t$ substituted for ξ_t.

PROPOSITION 3.1. *For $d < 4$ and sufficiently large t,*

$$\begin{aligned} P(|{}_{t^{1/4}}\eta_t(E) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)| \geq \varepsilon |E| t^{-d/4}) \\ \leq 3\exp\{-((\varepsilon v_t(\varepsilon)|E| t^{-9d/20}) \wedge t^{1/8})\} \end{aligned} \tag{3.1}$$

holds for all ε and $E \subset \mathbb{Z}^d$, with $|E| \leq t^d$.

(The exponent $1/4$ in the subscript $t^{1/4}$ is not crucial here; other choices would require a modification of the term $t^{9d/20}$.)

It is not difficult to deduce Proposition 2.3 from Proposition 3.1. The main step is given by the following lemma.

LEMMA 3.1. *For all d and $s \leq t$,*

$$\begin{aligned} P({}_s\eta_t(E) - \xi_t(E) \geq 0 \mid \mathscr{F}_t^{\xi}) \geq 1/2, \\ P({}_s\eta_t(E) - \xi_t(E) \leq 0 \mid \mathscr{F}_t^{\xi}) \geq 1/2, \end{aligned} \tag{3.2}$$

hold a.s. for all $E \subset \mathbb{Z}^d$, with $|E| < \infty$.

PROOF. Both parts of (3.2) follow from the symmetric behavior of A and B particles. Two particles of types A and B, which meet at some $\tau \in (s, t]$ under ξ_r, continue to evolve as independent simple random walks, $Y^A(r)$ and $Y^B(r)$, on $[\tau, t]$, under ${}_s\eta_t$. The difference of indicator functions $1(Y^B(r) \in E) - 1(Y^A(r) \in E)$ is symmetric, and is independent of all other such pairs of random walks, when conditioned on $\mathscr{F}_t^{\xi}$. The sum of all such differences equals ${}_s\eta_t(E) - \xi_t(E)$, and will again be symmetric when conditioned on $\mathscr{F}_t^{\xi}$. This implies (3.2). □

Using Lemma 3.1, Proposition 2.3 follows immediately from Proposition 3.1. Setting $s = t^{1/4}$, one sees that at least half of the time when the exceptional event in (2.8) holds, the same is true for the event in (3.1). So, the upper bound on the probability on the event in (3.1) implies that in (2.8).

We now turn our attention to demonstrating Proposition 3.1. Most of the work required for the proposition is to show that $(|\xi_{t^{1/4}}| * K_{t-t^{1/4}})(E)$ is typically not too large. One then uses this bound in conjunction with a large deviation estimate. The desired bound on this convolution is given by the following result.

PROPOSITION 3.2. *Let $d < 4$. For given $\delta > 0$, suppose that s is sufficiently large, and that $t \geq s^4$. Then,*

$$P((|\xi_s| * K_{t-s})(x) \geq 2s^{-(d/4-\delta)}) \leq e^{-s^{2/3}} \tag{3.3}$$

for all x.

We first demonstrate Proposition 3.2. This will require Lemmas 3.2–3.4, which are given below. We will then show how Proposition 3.1 follows from Proposition 3.2.

We will employ moment generating functions to show (3.3). Rather than analyzing $|\xi_s| * K_{t-s}$ directly, we will look at $|{}^x\xi_s| * K_{t-s}$, for $x \in \mathbb{Z}^d$. The process ${}^x\xi_s$ denotes the analog of ξ_s, but where the initial state is restricted to $\mathscr{A}_s(x)$; that is, for $|E| < \infty$, ${}^x\xi_0(E) = \xi_0(E \cap \mathscr{A}_s(x))$. The set $\mathscr{A}_s(x)$ is defined by

$$\mathscr{A}_s(x) = \{y \in \mathbb{Z}^d\colon |y - x|_\infty \leq s/3\}, \tag{3.4}$$

where $|\cdot|_\infty$ denotes the sup norm. We construct ${}^x\xi_s$ on the same space as ξ_s, by assigning the same random walk paths to corresponding particles as was done before (2.7). Let $\mathscr{C}_s$ denote the set of all $x \in \mathbb{Z}^d$ for which each coordinate is a multiple of $\lfloor s \rfloor$, the integer part of s. For the first part of the argument, we will restrict x to $\mathscr{C}_s$. There, we will use the independence of the processes ${}^{x_1}\xi_r, {}^{x_2}\xi_r, \ldots$, for $x_j \in \mathscr{C}_s$, $j = 1, 2, \ldots$, and $r \in [0, t]$. Then, when we analyze $|\xi_s| * K_{t-s}$, we will also consider translates of $\mathscr{C}_s$.

Lemma 3.2 states that, for any x, $\xi_s(x)$ and ${}^x\xi_s(x)$ are close in expectation. This is not surprising. The initial states ξ_0 and ${}^x\xi_0$ only differ at sites further than $s/3$ from x. By time s, the probability will be small that this difference will have worked its way to x.

LEMMA 3.2. *For all d and sufficiently large s,*

$$E[|\xi_s(x) - {}^x\xi_s(x)|] \le e^{-C_2 s} \tag{3.5}$$

for all x and appropriate $C_2 > 0$.

Together with (1.2), (3.5) gives the following bound on $E[|{}^x\xi_s(x)|]$. Lemma 3.2 will also be used in (3.19).

COROLLARY 3.1. *For $d < 4$ and sufficiently large s,*

$$E[|{}^x\xi_s(x)|] \le C_3/s^{d/4} \tag{3.6}$$

for all x and appropriate C_3.

The corollary will be used in the proof of Lemma 3.3.

PROOF OF LEMMA 3.2. We consider the "discrepancy" ${}^x\mathscr{G}_r$ between ξ_r and ${}^x\xi_r$, at each time r. This set is defined as those A and B particles, from either ξ_r or ${}^x\xi_r$, which still exist by time r for one process but not for the other. [The initial discrepancy consists of particles of ξ_0 lying outside $\mathscr{A}_s(x)$.] In order for $\xi_r(y) \neq {}^x\xi_r(y)$, ${}^x\mathscr{G}_r$ must contain a particle at y. One can check that the particles in ${}^x\mathscr{G}_r$ execute independent random walks except when A and B particles from the same process meet one another, and at least one of them is in ${}^x\mathscr{G}_{r-}$. If both are in ${}^x\mathscr{G}_{r-}$, the particles annihilate one another, and both disappear from ${}^x\mathscr{G}_r$. If only one is in ${}^x\mathscr{G}_{r-}$, then, upon annihilation, this particle is replaced in the discrepancy by a particle of the opposite type (e.g., B instead of A) at the same site, which belongs to the opposite process. So, ${}^x\mathscr{G}_r$ is dominated by a set of random walks, whose initial positions are given by the initial positions of the A and B particles outside $\mathscr{A}_s(x)$.

The distance between x and $\mathscr{A}_s(x)^c$ is $s/3$. The initial positions of A and B particles for ξ_r are given by Poisson random fields with intensity λ. It is therefore not difficult to show, using moment generating functions, that, for large s, the expected number of random walks at x at time s, which were originally in $\mathscr{A}_s(x)^c$, is bounded above by

$$\lambda \sum_{z=\lfloor s/3\rfloor+1} z^{d-1} e^{-C_4 z} \le \lambda e^{-C_4 s/4}$$

for appropriate $C_4 > 0$ (see, e.g., the proof of Lemma 7.3). Since this dominates $E[|\xi_s(x) - {}^x\xi_s(x)|]$, (3.5) follows. $\square$

In order to derive (3.3) of Proposition 3.2, we will need the following bound on the moment generating function of $|{}^x\xi_s(x)|$.

LEMMA 3.3. *Let $d < 4$, and fix $\delta > 0$. Suppose that $\theta > 0$ is bounded, s is sufficiently large, and $m \in \mathbb{Z}^+$ is chosen so that $m! \ge s^{d/4}$. Then,*

$$E[e^{\theta|{}^x\xi_s(x)|}] \le 1 + C_5(e^{\theta m} - 1)/s^{d/4-\delta} \tag{3.7}$$

for all x and appropriate C_5.

PROOF. For given $m \in \mathbb{Z}^+$ and $\theta > 0$, it is easy to check that

$$E[e^{\theta|^x\xi_s(x)|}] \le 1 + (e^{\theta m} - 1)P(^x\xi_s(x) \ne 0) + E[e^{\theta|^x\xi_s(x)|} - 1; |^x\xi_s(x)| > m]. \tag{3.8}$$

By (3.6), the second term on the right is bounded above by

$$C_4(e^{\theta m} - 1)/s^{d/4}. \tag{3.9}$$

Denote by $\eta_s^{\#}(x)$ the total number of particles at x for the process η_s. To handle the third term, we note that it is at most

$$E[e^{\theta\eta_s^{\#}(x)} - 1; \eta_s^{\#}(x) > m]. \tag{3.10}$$

Since $\eta_s^{\#}(x)$ is Poisson with mean 2λ, (3.10) is less than or equal to

$$e^{-2\lambda} \sum_{j>m} (e^{\theta j} - 1)(2\lambda)^j/j!.$$

For given $\delta > 0$, bounded θ, $m! \ge s^{d/4}$ and large enough s, this is less than or equal to

$$(2\lambda)^m(e^{\theta m} - 1)/m! \le (e^{\theta m} - 1)/s^{d/4-\delta}. \tag{3.11}$$

This provides an upper bound on the third term. Together, (3.8)–(3.11) give the bound in (3.7). □

Using Lemma 3.3, we derive Lemma 3.4. It is the analog of Proposition 3.2, but with $|^y\xi_s(y)|$ in place of $|\xi_s(y)|$, and the sum being over $y \in \mathscr{C}_s$ rather than $y \in \mathbb{Z}^d$. In the proof, we will use the local central limit bound,

$$\limsup_{t\to\infty}\{t^{d/2} \sup(K_{t-s}(x): x \in \mathbb{Z}^d,\ s \le t/2)\} < \infty, \tag{3.12}$$

as well as

$$\limsup_{t\to\infty}\left\{s^d \sup\left(\sum_{y\in\mathscr{C}_s} K_{t-s}(x - y): x \in \mathbb{Z}^d,\ s \le t^{1/2}\right)\right\} < \infty, \tag{3.13}$$

which hold for all d. These bounds follow from (4.5) and (4.8), respectively, together with simple bounds on the normal kernel N_{t-s}. In Section 4, we will go into greater detail on such bounds.

LEMMA 3.4. *Let $d < 4$. For given $\delta > 0$, suppose that s is sufficiently large, and that $t \ge s^2$. Then,*

$$P\left(\sum_{y\in\mathscr{C}_s} |^y\xi_s(y)|K_{t-s}(x - y) \ge s^{-(5d/4-\delta)}\right) \le \exp\{-t^{d/2}/s^{5d/4}\} \tag{3.14}$$

for all x.

PROOF. Set $\theta = \theta_y = t^{d/2} K_{t-s}(x-y)/m$ in (3.7), where m is the smallest integer satisfying $m! \geq s^{d/4}$. By (3.12), $m\theta_y$ is bounded for large s and $t \geq 2s$. So, by Lemma 3.3, one has, for given $\delta' > 0$,

$$\begin{aligned} &E[\exp\{t^{d/2}|{}^y\xi_s(y)|K_{t-s}(x-y)/m\}] \\ &\quad \leq 1 + C_5(\exp\{t^{d/2}K_{t-s}(x-y)\}-1)/s^{d/4-\delta'} \\ &\quad \leq 1 + C_6 t^{d/2}K_{t-s}(x-y)/s^{d/4-\delta'} \\ &\quad \leq \exp\{C_6 t^{d/2}K_{t-s}(x-y)/s^{d/4-\delta'}\} \end{aligned} \tag{3.15}$$

for all x and y, and appropriate C_6. One also has $s^{\delta'} \geq m$ for large s, and m chosen as above. So, by (3.15),

$$\begin{aligned} &E[\exp\{t^{d/2}|{}^y\xi_s(y)|K_{t-s}(x-y)/s^{\delta'}\}] \\ &\quad \leq \exp\{C_6 t^{d/2}K_{t-s}(x-y)/s^{d/4-\delta'}\}. \end{aligned} \tag{3.16}$$

Now, for $y_j \in \mathscr{C}_s$, $j = 1, 2, \ldots$, the processes ${}^{y_1}\xi_r, {}^{y_2}\xi_r, \ldots$ are independent, as was mentioned below (3.4). Setting $r = s$, it therefore follows from (3.16) that

$$\begin{aligned} &E\left[\exp\left\{t^{d/2}\sum_{y\in\mathscr{C}_s}|{}^y\xi_s(y)|K_{t-s}(x-y)/s^{\delta'}\right\}\right] \\ &\quad \leq \exp\left\{(C_6 t^{d/2}/s^{d/4-\delta'})\sum_{y\in\mathscr{C}_s}K_{t-s}(x-y)\right\}. \end{aligned}$$

By (3.13), this is less than or equal to

$$\exp\{C_7 t^{d/2}/s^{5d/4-\delta'}\}$$

for large s with $t \geq s^2$, and appropriate C_7. So, by Chebyshev's inequality,

$$\begin{aligned} &P\left(\sum_{y\in\mathscr{C}_s}|{}^y\xi_s(y)|K_{t-s}(x-y) \geq s^{-(5d/4-\delta)}\right) \\ &\quad \leq \exp\{(C_7 s^{\delta'} - s^{\delta-\delta'})t^{d/2}/s^{5d/4}\}. \end{aligned} \tag{3.17}$$

If we set $\delta' = \delta/3$, this is at most $\exp\{-t^{d/2}/s^{5d/4}\}$ for large s, which implies (3.14). $\square$

Using Lemmas 3.2 and 3.4, we demonstrate Proposition 3.2.

PROOF OF PROPOSITION 3.2. The bound (3.14) holds independently of x. Consequently, it also holds if one instead sums over translates $\mathscr{C}_s + z$ of $\mathscr{C}_s$, with $z \in \mathbb{Z}^d$. By summing over all such $\lfloor s \rfloor^d$ translates, one obtains, for given $\delta > 0$,

$$P\left(\sum_{y\in\mathbb{Z}^d}|{}^y\xi_s(y)|K_{t-s}(x-y) \geq s^{-(d/4-\delta)}\right) \leq s^d \exp\{-t^{d/2}/s^{5d/4}\} \tag{3.18}$$

for all x, and for large enough s with $t \geq s^2$. On the other hand, since $\sum_{y\in\mathbb{Z}^d} K_{t-s}(x-y) = 1$, it follows from Lemma 3.2 and Chebyshev's inequality that for large s,

$$P\left(\sum_{y\in\mathbb{Z}^d} (|\xi_s(y)| - |^y\xi_s(y)|)K_{t-s}(x-y) \geq s^{-(d/4-\delta)}\right) \leq s^{d/4-\delta}e^{-C_2 s} \tag{3.19}$$

for all x. Together, (3.18) and (3.19) imply that for large s and $t \geq s^4$,

$$P\left(\sum_{y\in\mathbb{Z}^d} |\xi_s(y)|K_{t-s}(x-y) \geq 2s^{-(d/4-\delta)}\right) \leq e^{-s^{2/3}}.$$

This is equivalent to (3.3), and completes the proof of Proposition 3.2. □

We now proceed to prove Proposition 3.1. This will complete our proof of Proposition 2.3. The argument consists of applying moment generating functions to ${}_{t^{1/4}}\eta_t$ conditioned on $\mathscr{F}_{t^{1/4}}$. Since the conditioned process ${}_{t^{1/4}}\eta_r$ evolves according to independent simple random walks over $(t^{1/4}, t]$, and since ${}_{t^{1/4}}\eta_{t^{1/4}} = \xi_{t^{1/4}}$ is known, the computations are fairly explicit. Proposition 3.2 supplies the main technical estimate needed to bound the right side of (3.23).

PROOF OF PROPOSITION 3.1. We abbreviate the left side of (3.1) by setting

$$\bar{\eta} = {}_{t^{1/4}}\eta_t(E) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(E). \tag{3.20}$$

We also set $\overline{K}(x) = K_{t-t^{1/4}}(E-x)$.

We proceed to estimate the moment generating function of $\bar{\eta}$, conditioned on $\mathscr{F}_{t^{1/4}}$. Since over $[t^{1/4}, t]$, ${}_{t^{1/4}}\eta_r$ evolves according to independent simple random walks, one can, for given θ, write

$$\begin{aligned} E[e^{\theta\bar{\eta}}|\mathscr{F}_{t^{1/4}}] = \prod_{x\in\mathbb{Z}^d} &\left[\overline{K}(x)e^{-\theta(1-\overline{K}(x))} + (1-\overline{K}(x))e^{\theta\overline{K}(x)}\right]^{\xi^A_{t^{1/4}}(x)} \\ &\times \left[\overline{K}(x)e^{\theta(1-\overline{K}(x))} + (1-\overline{K}(x))e^{-\theta\overline{K}(x)}\right]^{\xi^B_{t^{1/4}}(x)}. \end{aligned} \tag{3.21}$$

For small θ, this is less than or equal to

$$\begin{aligned} &\prod_x [1+\theta^2\overline{K}(x)(1-\overline{K}(x))]^{|\xi_{t^{1/4}}(x)|} \\ &\quad \leq \exp\left\{\theta^2 \sum_x |\xi_{t^{1/4}}(x)|\overline{K}(x)\right\} \\ &\quad = \exp\left\{\theta^2(|\xi_{t^{1/4}}| * K_{t-t^{1/4}})(E)\right\}. \end{aligned} \tag{3.22}$$

It follows from (3.21), (3.22) and Chebyshev's inequality (applied to both $\theta > 0$ and $\theta < 0$) that, for $\gamma > 0$,

$$P(|\bar{\eta}| \geq \gamma \,|\, \mathscr{F}_{t^{1/4}}) \leq 2\exp\{\theta^2(|\xi_{t^{1/4}}| * K_{t-t^{1/4}})(E) - \gamma\theta\}. \tag{3.23}$$

Let G denote the set where

$$(|\xi_{t^{1/4}}| * K_{t-t^{1/4}})(E) < 2|E|t^{-(d-4\delta)/16}$$

for given $\delta > 0$. By Proposition 3.2, applied to each $x \in E$ with $s = t^{1/4}$,

$$P(G^c) \le |E| e^{-t^{1/6}} \le e^{-t^{1/8}} \tag{3.24}$$

for sufficiently large t and $|E| \le t^d$. (Of course, any power of t suffices.)

On the other hand, on G, the right side of (3.23) is at most

$$2\exp\{(2\theta^2|E|t^{-(d-4\delta)/16}) - \gamma\theta\}, \tag{3.25}$$

which provides an upper bound on $P(|\bar{\eta}| \ge \gamma \,|\, \mathscr{F}_{t^{1/4}})$, for small θ. For a given $\varepsilon > 0$, let

$$\gamma = \varepsilon|E|t^{-d/4}, \qquad \theta = \tfrac{1}{4}t^{-\delta/4}((\varepsilon t^{-3d/16}) \wedge 1). \tag{3.26}$$

Setting $v_t(\varepsilon) = \varepsilon \wedge t^{3d/16}$, one can write $\theta = \frac{1}{4}t^{-(3d+4\delta)/16}v_t(\varepsilon)$. Substitution of γ and θ into (3.23) and (3.25), with $\delta < d/20$, implies that on G,

$$\begin{aligned} P\big(|\bar{\eta}| \ge \varepsilon|E|t^{-d/4} \,|\, \mathscr{F}_{t^{1/4}}\big) &\le 2\exp\left\{-\frac{\varepsilon v_t(\varepsilon)|E|}{8t^{(7d+4\delta)/16}}\right\} \\ &\le 2\exp\big\{-\varepsilon v_t(\varepsilon)|E|t^{-9d/20}\big\} \end{aligned} \tag{3.27}$$

for large t. It follows from (3.24) and (3.27) that, for large t,

$$P\big(|\bar{\eta}| \ge \varepsilon|E|t^{-d/4}\big) \le 3\exp\big\{-((\varepsilon v_t(\varepsilon)|E|t^{-9d/20}) \wedge t^{1/8})\big\},$$

which implies (3.1). $\square$

As mentioned at the end of Section 2, we will want to employ the higher dimensional analogs of results, such as Proposition 2.3, in [9]. The modifications required for Proposition 2.3 are straightforward to make. The restriction to $d < 4$ was needed for Corollary 3.1, which employed (1.2). If one instead employs the corresponding bound for $d \ge 4$, one obtains

$$E[|{}^x\xi_s(x)|] \le C_3/s \tag{3.28}$$

for large s and all x, in place of (3.6). The bound in (3.6) was used in (3.9); replacing the term $s^{d/4}$ by s there gives the corresponding bound for $d \ge 4$. This leads to the analog of Lemma 3.3, with s replacing $s^{d/4}$ in both places, and to the analog of Lemma 3.4, with s^{d+1} replacing $s^{5d/4}$ in both places. This last change requires us to replace the bound $s^{d/4-\delta}$ with $s^{1-\delta}$ in Proposition 3.2. In the proof of Proposition 3.1, one replaces $t^{(d-4\delta)/16}$ with $t^{(1-\delta)/4}$ in the bound defining G. This leads to the upper bound, for large t,

$$P(|\bar{\eta}| \ge \varepsilon|E|t^{-1}|\mathscr{F}_{t^{1/4}}) \le 2\exp\{-\varepsilon v_t(\varepsilon)|E|t^{-9/5}\} \tag{3.29}$$

corresponding to (3.27), where now $v_t(\varepsilon) = \varepsilon \wedge t^{3/4}$, and to the analogous bounds corresponding to (3.1). Application of Lemma 3.1, as before, then produces the following analog of Proposition 2.3 for $d \ge 4$.

PROPOSITION 3.3. *For $d \geq 4$ and sufficiently large t,*

$$P(|\xi_t(E) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)| \geq \varepsilon |E| t^{-1}) \\ \leq 6\exp\{-(\varepsilon v_t(\varepsilon)|E|t^{-9/5}) \wedge t^{1/8})\} \tag{3.30}$$

holds for all ε and $E \subset \mathbb{Z}^d$, with $|E| \leq t^d$.

4. Approximation of $\xi_s * K_{t-s}$ by $\xi_0 * N_t$. In this section, we demonstrate Proposition 2.4, which states that $(\xi_s * K_{t-s})(0)$ is approximated by $(\xi_0 * N_t)(0)$, with high probability, when $s \leq t^{1/4}$. In order to demonstrate the proposition, it is enough to verify the following two results.

PROPOSITION 4.1. *For any d, let t be sufficiently large and $s \leq t^{1/4}$. Then, for all $\varepsilon \in [0, 1]$,*

$$P\big(\big|(\xi_s * N_t)(0) - (\xi_s * K_{t-s})(0)\big| \geq \varepsilon t^{-d/4}\big) \leq 2\exp\{-4\varepsilon^2 t^{1/2}\}. \tag{4.1}$$

PROPOSITION 4.2. *For any d, let t be sufficiently large and $s \leq t^{1/4}$. Then, for all $\varepsilon \in [0, 1]$,*

$$P\big(\big|(\xi_0 * N_t)(0) - (\xi_s * N_t)(0)\big| \geq \varepsilon t^{-d/4}\big) \leq 2\exp\{-4\varepsilon^2 t^{1/4}\}. \tag{4.2}$$

For these propositions, we will need estimates on K_{t-s}, which follow from a standard local central limit theorem. Related bounds are also used in Sections 3 and 9. We employ the references [3] and [18] for these purposes.

Assume that $d = 1$. By [18], Theorem 16, page 207,

$$(1 + (x/t^{1/2})^3)(N_t(x) - K_t(x)) = o(t^{-1}) \quad \text{as } t \to \infty \tag{4.3}$$

holds uniformly in $x \in \mathbb{Z}$. [Because K_t is symmetric, its cumulant of order 3 will be 0, which gives the simplified form in (4.3).] One can also employ [3], Theorem 22.1, page 231. (Both results are stated for discrete times, although the derivation for continuous t is, of course, the same.)

Assume now that d is arbitrary. Since the evolution of K_t in different coordinates is independent, one can apply (4.3) to conclude that

$$(1 + (|x|/t^{1/2})^{3d})(N_t(x) - K_t(x)) = o(t^{-(d+1)/2}). \tag{4.4}$$

The next two bounds follow quickly from (4.4):

$$\sup_{x \in \mathbb{Z}^d} |N_t(x) - K_t(x)| \leq C_8 t^{-(d+1)/2} \tag{4.5}$$

and

$$\sum_{x \in \mathbb{Z}^d} |N_t(x) - K_t(x)| \leq C_8 t^{-1/2} \tag{4.6}$$

for appropriate C_8 and large t. From these bounds, one also obtains that

$$\sum_{x \in \mathbb{Z}^d} (N_t(x) - K_t(x))^2 \leq C_9 t^{-d/2-1} \tag{4.7}$$

for appropriate C_9.

Suppose that $\mathbb{R}^d$ is partitioned into sets E_j, $j = 1, 2, \dots$, such that each E_j contains a cube of length M with $M \le C_{10}t^{1/2}$, for a given C_{10}. Again using (4.4), one can generalize (4.6) so that, for $x_j \in E_j$,

$$\sum_j \left|N_t(x_j) - K_t(x_j)\right| \le C_{11} M^{-d} t^{-1/2} \tag{4.8}$$

holds for some C_{11} independently of the choice of x_j and the partition $\{E_j\}$. Using this and simple estimates on $\sum_j N_t(x_j)$, it is not difficult to derive (3.13), which was employed in the proof of Lemma 3.4.

We will also need some basic estimates on N_t. Set $g(s, r) = N_{t-s}(x)$, where $r = |x|$ and $x \in \mathbb{R}^d$. It is easy to check that, for appropriate C_{12},

$$\left|\frac{\partial g}{\partial s}(s, r)\right| \le C_{12} t^{-(d/2+1)} \left(1 + \frac{r^2}{t}\right) e^{-r^2/4t} \tag{4.9}$$

for all $s \le t/2$ and x. So, for all $x \in \mathbb{Z}^d$,

$$\left|N_t(x) - N_{t-s}(x)\right| \le C_{12} s t^{-(d/2+1)} \left(1 + \frac{r^2}{t}\right) e^{-r^2/4t}. \tag{4.10}$$

With a little work, it follows from this that

$$\sum_{x \in \mathbb{Z}^d} (N_t(x) - N_{t-s}(x))^2 \le C_{13} s^2 t^{-(d/2+2)} \tag{4.11}$$

for appropriate C_{13}.

One can also check that

$$\left|\frac{\partial g}{\partial r}(0, r)\right| \le C_{14} t^{-(d/2+1)} r e^{-r^2/2t}. \tag{4.12}$$

For $|x - x'| \le M \le t^{1/2}$, one can use this to show

$$|N_t(x) - N_t(x')| \le C_{15} t^{-(d/2+1)} (|x| + M) M e^{-|x|^2/4t}, \tag{4.13}$$

for appropriate C_{15}. With a little work, one can then show that, for appropriate C_{16} and any $y \in \mathbb{R}^d$,

$$\sum_{x \in \mathbb{Z}^d} \max_{x'} \left\{(N_t(x - y) - N_t(x' - y))^2 \colon |x - x'| \le M\right\} \le C_{16} t^{-(d/2+1)} M^2. \tag{4.14}$$

In order to show Proposition 4.1, we first show its analog, where ξ_s is replaced by ξ_0. In Proposition 4.3 and all following results in this section, all dimensions d are allowed.

PROPOSITION 4.3. *Let t be sufficiently large and $s \le t^{1/2}$. Then, for all $\varepsilon \in [0, 1]$,*

$$P\left(\left|(\xi_0 * N_t)(0) - (\xi_0 * K_{t-s})(0)\right| \ge \varepsilon t^{-d/4}\right) \le \exp\{-4\varepsilon^2 t^{1/2}\}. \tag{4.15}$$

PROOF. To obtain (4.15), we compute an upper bound on the corresponding moment generating function, and then apply Chebyshev's inequality. First, recall that $\xi_0(x)$, at each site x, is the difference of two Poisson random variables, each with intensity λ. Since these random variables are independent at different sites, one has that, for given θ,

$$\begin{aligned}
&E\big[\exp\{\theta((\xi_0 * N_t)(0) - (\xi_0 * K_{t-s})(0))\}\big]\\
&\quad = \prod_{x\in\mathbb{Z}^d} E\big[\exp\{\theta R(x)\xi_0(x)\}\big] \qquad (4.16)\\
&\quad = \exp\left\{\lambda \sum_{x\in\mathbb{Z}^d}(\exp\{\theta R(x)\} + \exp\{-\theta R(x)\} - 2)\right\},
\end{aligned}$$

where $R(x) = N_t(-x) - K_{t-s}(-x)$.

Together, (4.5) and (4.10) imply that, for appropriate C_{17},

$$|R(x)| \le C_{17} t^{-(d+1)/2}$$

for large t and all x, since $s \le t^{1/2}$. So, for $|\theta| \le C_{18} t^{(d+1)/2}$ and appropriate C_{18}, (4.16) is at most

$$\exp\left\{2\lambda\theta^2 \sum_x (R(x))^2\right\}.$$

By (4.7) and (4.11), this is less than or equal to

$$\exp\{C_{19}\lambda\theta^2 t^{-(d/2+1)}\}, \qquad (4.17)$$

for appropriate C_{19}. Combining the inequalities from (4.16) through (4.17), it follows that

$$E\big[\exp\{\theta((\xi_0 * N_t)(0) - (\xi_0 * K_{t-s})(0))\}\big] \le \exp\big\{C_{19}\lambda\theta^2 t^{-(d/2+1)}\big\}.$$

Applying Chebyshev's inequality for both $\theta > 0$ and $\theta < 0$, one obtains

$$\begin{aligned}
&P\big(\big|(\xi_0 * N_t)(0) - (\xi_0 * K_{t-s})(0)\big| \ge \varepsilon t^{-d/4}\big)\\
&\quad \le 2\exp\big\{C_{19}\lambda\theta^2 t^{-(d/2+1)} - |\theta|\varepsilon t^{-d/4}\big\}.
\end{aligned}$$

Setting $|\theta| = 5\varepsilon t^{(d+2)/4}$, one has, for large t, the upper bound $\exp\{-4\varepsilon^2 t^{1/2}\}$, which implies (4.15). □

In order to obtain Proposition 4.1 from Proposition 4.3, we need to replace ξ_0 by ξ_s. The following lemma will enable us to do that.

LEMMA 4.1. *Let $f(\cdot)$ be any nonrandom function. For all s,*

$$P\left(\sum_{x\in\mathbb{Z}^d} f(x)(\eta_s(x) - \xi_s(x)) \ge 0\, t \mid t\, \mathscr{F}_s^{\xi}\right) \ge 1/2. \qquad (4.18)$$

The statement in (4.18) is similar to that in (3.2) of Lemma 3.1. The reasoning that is required is analogous, with the point being that exchanging the random walk motions of A and B particles in η_t, after annihilation occurs in the corresponding process ξ_t, does not change the law of $\sum_x f(x)(\eta_s(x)-\xi_s(x))$, conditioned on $\mathscr{F}_s^{\xi}$.

By first setting $f(x) = N_t(-x) - K_{t-s}(-x)$ and then $f(x) = K_{t-s}(-x) - N_t(-x)$, one obtains the following corrollary of Lemma 4.1.

COROLLARY 4.1. *For all $s \le t$ and ε,*

$$P\big(\big|(\xi_s * N_t)(0) - (\xi_s * K_{t-s})(0)\big| \ge \varepsilon t^{-d/4}\big) \\ \le 2P\big(\big|(\eta_s * N_t)(0) - (\eta_s * K_{t-s})(0)\big| \ge \varepsilon t^{-d/4}\big). \tag{4.19}$$

The distribution of $\eta_s(x)$, $x \in \mathbb{Z}^d$, for each s, is given by the difference of two Poisson random fields, each with intensity λ. So, the distribution of $\eta_s(x)$ is constant over s, and one may substitute η_0 for η_s on the right side of (4.19). This, in turn, may be replaced by ξ_0, since the joint distributions of $\xi_0(x)$ and $\eta_0(x)$, over $x \in \mathbb{Z}^d$, are the same. Consequently, one obtains the following result.

COROLLARY 4.2. *For all $s \le t$ and ε,*

$$P\big(\big|(\xi_s * N_t)(0) - (\xi_s * K_{t-s})(0)\big| \ge \varepsilon t^{-d/4}\big) \\ \le 2P(|(\xi_0 * N_t)(0) - (\xi_0 * K_{t-s})(0)| \ge \varepsilon t^{-d/4}). \tag{4.20}$$

Proposition 4.1 is an immediate consequence of Proposition 4.3 and Corollary 4.2.

We now turn our attention to showing Proposition 4.2. Our first step is to replace ξ_0 by η_0 and ξ_s by η_s in (4.2). For this, we apply Lemma 4.1 again, this time with $f(x) = N_t(-x)$. Since $\xi_0(x) = \eta_0(x)$ for all x, and $\xi_0 \in \mathscr{F}_s^{\xi}$, we obtain the following result.

COROLLARY 4.3. *For all $s \le t$ and ε,*

$$P\big(\big|(\xi_0 * N_t)(0) - (\xi_s * N_t)(0)\big| \ge \varepsilon t^{-d/4}\big) \\ \le 2P(|(\eta_0 * N_t)(0) - (\eta_s * N_t)(0)| \ge \varepsilon t^{-d/4}). \tag{4.21}$$

On account of Corollary 4.3, in order to show Proposition 4.2, it suffices to demonstrate the following variant.

PROPOSITION 4.4. *Let t be sufficiently large and $s \le t^{1/4}$. Then, for all $\varepsilon \in [0, 1]$,*

$$P\big(\big|(\eta_0 * N_t)(0) - (\eta_s * N_t)(0)\big| \ge \varepsilon t^{-d/4}\big) \le \exp\{-4\varepsilon^2 t^{1/4}\}. \tag{4.22}$$

In order to demonstrate (4.22), it is more convenient to instead focus on the motion of the individual particles corresponding to η_s, which are undergoing rate-d simple random walks on $\mathbb{Z}^d$. We will show, in effect, that for $s \le t^{1/4}$, only a negligible number of particles will have moved far enough by time s to alter $N_t(\cdot)$ by more than a negligible amount from its initial value. We will employ the following notation. Label the positions at time s of the $\eta_0^{\#}(x) = \eta_0^A(x) + \eta_0^B(x)$ particles initially at x by $X_s(x, j)$, $j = 1, \dots, \eta_0^{\#}(x)$, where the ordering is chosen independently of the type of particle; $\eta_0^{\#}(x)$, $x \in \mathbb{Z}^d$, are independent mean-2λ Poisson random variables. Set

$$\mathscr{I} = \{(x, j): 1 \le j \le \eta_0^{\#}(x)\},$$

and let $\operatorname{sgn}(x, j) = 1$ whenever the corresponding particle is a B particle, and $\operatorname{sgn}(x, j) = -1$ whenever it is an A particle. Also, for $(x, j) \in \mathscr{I}$, set

$$Y(x, j) = \operatorname{sgn}(x, j)(N_t(x) - N_t(X_s(x, j))).$$

[Since s and t are thought of as being fixed here, they are suppressed in $Y(x, j)$.] Using the above notation, we can rewrite (4.22) as

$$P\left(\left|\sum_{(x,j)\in\mathscr{I}} Y(x, j)\right| \ge \varepsilon t^{-d/4}\right) \le \exp\{-4\varepsilon^2 t^{1/4}\}. \tag{4.23}$$

We break the demonstration of (4.23) into two steps. For the first step, Lemma 4.2, we set

$$Y(x) = W(N_t(x) - N_t(X_s(x))),$$

where $X_s(x)$ is a rate-d simple random walk on $\mathbb{Z}^d$ starting at x, and W is an independent random variable taking values 1 and -1 with equal probability. We introduce the quantities

$$\begin{aligned} \psi_1(x) &= \max_{x'}\{(N_t(x) - N_t(x'))^2: |x - x'| \le t^{1/4}\}, \\ \psi_2(x) &= N_t(0) \sum_{|y|>t^{1/4}} K_s(y)(N_t(x) + N_t(x - y)), \end{aligned} \tag{4.24}$$

with $\psi(x) = \psi_1(x) + \psi_2(x)$.

LEMMA 4.2. *Let t be sufficiently large. Then, for all s, x and $|\theta| \le C_{20} t^{d/2}$,*

$$E\left[e^{\theta Y(x)}\right] \le \exp\{C_{21}\theta^2\psi(x)\} \tag{4.25}$$

for appropriate C_{21} (depending on C_{20}).

PROOF. Note that $Y(x)$ is symmetric, and that $|Y(x)| \le t^{-d/2}$ for all s, t and x. So, for $|\theta| \le C_{20} t^{d/2}$,

$$\begin{aligned} E\left[e^{\theta Y(x)}\right] &= E\left[\sum_{k=0}^{\infty} (\theta Y(x))^{2k}/(2k)!\right] \le 1 + C_{21}\theta^2 E[(Y(x))^2] \\ &\le \exp\{C_{21}\theta^2 E[(Y(x))^2]\}, \end{aligned} \tag{4.26}$$

for appropriate C_{21}.

Let $G(x)$ denote the event on which $|X_s(x) - x| \le t^{1/4}$. Then, on $G(x)$,

$$(Y(x))^2 \le \psi_1(x), \tag{4.27}$$

where $\psi_1(x)$ is given in (4.24). Also, one has that

$$E[(Y(x))^2; G^c(x)] \le \sum_{|y|>t^{1/4}} K_s(y)(N_t(x) - N_t(x-y))^2 \le \psi_2(x). \tag{4.28}$$

Together, (4.26)–(4.28) imply that

$$E\big[e^{\theta Y(x)}\big] \le \exp\{C_{21}\theta^2 \psi(x)\},$$

which is (4.25). □

Conditioned on $\mathscr{I}$, the random variables $Y(x, j), (x, j) \in \mathscr{I}$, are independent, and, for each x and j, are distributed like $Y(x)$. Letting $Z(x), x \in \mathbb{Z}^d$, denote independent mean-2λ Poisson random variables, Lemma 4.2 therefore implies the following result.

COROLLARY 4.4. *Let t be sufficiently large. Then, for all s and $|\theta| \le C_{20} t^{d/2}$,*

$$E\left[\exp\left\{\theta \sum_{(x,j)\in\mathscr{I}} Y(x, j)\right\}\right] \le E\left[\exp\left\{C_{21}\theta^2 \sum_{x\in\mathbb{Z}^d} \psi(x) Z(x)\right\}\right] \tag{4.29}$$

holds a.s.

We now demonstrate (4.23). We do this by bounding the right side of (4.29) and applying Chebyshev's inequality.

PROOF OF (4.23). We first note that, by (4.14) and (4.24),

$$\sum_{x\in\mathbb{Z}^d} \psi_1(x) \le C_{16} t^{-(d+1)/2}. \tag{4.30}$$

Also, using $s \le t^{1/4}$, it follows from a standard large deviation estimate on K_s that

$$\sum_x \psi_2(x) = 2N_t(0) \sum_{|y|>t^{1/4}} K_s(y) \le e^{-C_{22} t^{1/4}} \tag{4.31}$$

for large t and appropriate C_{22}. So, by (4.30) and (4.31),

$$\sum_x \psi(x) \le 2C_{16} t^{-(d+1)/2} \tag{4.32}$$

for large t.

Since $Z(x)$ are independent mean-2λ Poisson random variables,

$$E\left[\exp\left\{C_{21}\theta^2 \sum_{x\in\mathbb{Z}^d} \psi(x)Z(x)\right\}\right] = \exp\left\{2\lambda\sum_x (e^{C_{21}\theta^2\psi(x)} - 1)\right\}. \tag{4.33}$$

For $\theta^2 \le C_{23}/\sup_x \psi(x)$, this is at most $\exp\{C_{24}\lambda\theta^2\sum_x \psi(x)\}$ for appropriate C_{24} (depending on C_{23}), which, by (4.32), is less than or equal to

$$\exp\{C_{25}\lambda\theta^2 t^{-(d+1)/2}\} \tag{4.34}$$

for large t and appropriate C_{25}. So, by Corollary 4.4 and (4.33)–(4.34),

$$E\left[\exp\left\{\theta \sum_{(x,\,j)\in\mathscr{I}} Y(x,\,j)\right\}\right] \le \exp\{C_{25}\lambda\theta^2 t^{-(d+1)/2}\}.$$

Setting $C_{23} = 50C_{16}$, $\theta = 5\varepsilon t^{(d+1)/4}$ and applying Chebyshev's inequality implies that

$$P\left(\sum_{(x,\,j)\in\mathscr{I}} Y(x,\,j) \ge \varepsilon t^{-d/4}\right) \le \exp\{-5\varepsilon^2(t^{1/4} - 5C_{25}\lambda)\}.$$

This gives (4.23) for large t. □

5. Approximation of ξ_t by $\xi_0 * N_t$. In this section, we demonstrate Proposition 2.1, which gives a uniform bound on $|\xi_t(D) - (\xi_0 * N_t)(D)|$ over rectangles $D \in \mathscr{D}_{T^{1/2}}$, for $t \in [T/M, T]$ and $M > 1$, where T is large. Our main tools for this are Propositions 2.3 and 2.4, which bound $|\xi_t(D) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(D)|$ and $|(\xi_0 * N_t)(0) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(0)|$, respectively. It is easy to extend the latter estimate from 0 to D. After combining these bounds, we will sum the exceptional probabilities over $D \in \mathscr{D}_{T^{1/2}}$ and $t \in \mathscr{S}_T$, where $\mathscr{S}_T$ is an appropriate lattice in $[T/M - 1, T]$. It is then not difficult to extend the bounds to all $t \in [T/M, T]$.

We first note that by Proposition 2.4, for large t,

$$\begin{aligned} &P\big(\big|(\xi_0 * N_t)(E) - (\xi_{t^{1/4}} * K_{t-t^{1/4}})(E)\big| \ge \varepsilon|E|t^{-d/4}\big) \\ &\qquad \le 4|E|\exp\big\{-(\varepsilon^2 \wedge 1)t^{1/4}\big\} \end{aligned} \tag{5.1}$$

for $|E| < \infty$ and $\varepsilon \ge 0$. Together with Proposition 2.3, this implies the following result. Recall that $v_t(\varepsilon) = \varepsilon \wedge t^{3d/16}$.

LEMMA 5.1. *For $d < 4$ and sufficiently large t,*

$$\begin{aligned} &P\big(|\xi_t(E) - (\xi_0 * N_t)(E)| \ge 2\varepsilon|E|t^{-d/4}\big) \\ &\qquad \le 4|E|\exp\big\{-(\varepsilon^2 \wedge 1)t^{1/4}\big\} + 6\exp\big\{-((\varepsilon v_t(\varepsilon)|E|t^{-9d/20}) \wedge t^{1/8})\big\} \end{aligned} \tag{5.2}$$

for all ε and $E \subset \mathbb{Z}^d$, with $|E| \le t^d$.

In order to derive Proposition 2.1, we rephrase (5.2) so that the bound on the right side does not depend on E. For E and t in the range of interest to us, the inequality simplifies to that given in (5.3) with a little work.

PROPOSITION 5.1. *For $d < 4$, $M > 1$ and sufficiently large T,*

$$P\big(\big|\xi_t(E) - (\xi_0 * N_t)(E)\big| \geq \varepsilon_1 T^{d/4}\big) \leq 8\exp\{-C_{26}((\varepsilon_1)^2 \wedge 1)T^{1/20}\}, \tag{5.3}$$

for appropriate $C_{26} > 0$ (depending on M), all $t \in [T/M, T]$, ε_1 and $E \subset \mathbb{Z}^d$, with $|E| \leq MT^{d/2}$.

PROOF. Setting $\varepsilon_1 = 2\varepsilon|E|(tT)^{-d/4}$, the left side of (5.2) can be written as

$$P(|\xi_t(E) - (\xi_0 * N_t)(E)| \geq \varepsilon_1 T^{d/4}).$$

This is the left side of (5.3). The first term on the right side of (5.2) is at most $4|E|\exp\{-C_{26}((\varepsilon_1)^2 \wedge 1)T^{1/4}\}$ for $t \geq T/M$, $|E| \leq MT^{d/2}$ and appropriate $C_{26} > 0$. For large T and $\varepsilon_1 \geq T^{-1/12}$, this is at most $\exp\{-C_{26}((\varepsilon_1)^2 \wedge 1)T^{1/12}\}$, which is dominated by the right side of (5.3), with the factor 2 instead of 8; the factor 2 there ensures that the inequality is trivial for $\varepsilon_1 < T^{-1/12}$. One can also check that

$$\varepsilon v_t(\varepsilon)|E|t^{-9d/20} \geq \frac{1}{4M}\varepsilon_1 v_t(\varepsilon_1)T^{d/20}.$$

Using this, it is easy to see that the second term on the right side of (5.2) is dominated by the right side of (5.3), with the factor 6, for large T. $\square$

Let $\mathscr{S}_T$ denote the set of all $t \in [T/M - 1, T]$ that are integer multiples of $b_T \overset{\text{def.}}{=} \exp\{-T^{1/41}\}$. By setting $\varepsilon_1 = \frac{1}{3}T^{-1/80}$ and summing over the exceptional probabilities obtained from (5.3), one obtains the following uniform bound over times $t \in \mathscr{S}_T$ and rectangles $D \in \mathscr{D}_{T^{1/2}}$.

PROPOSITION 5.2. *For $d < 4$,*

$$P\bigg(\sup_{t \in S_T}\sup_{D \in \mathscr{D}_{T^{1/2}}} |\xi_t(D) - (\xi_0 * N_t)(D)| \geq \tfrac{1}{3}T^{d/4-1/80}\bigg) \leq b_T \tag{5.4}$$

for sufficiently large T.

In order to deduce Proposition 2.1 from Proposition 5.2, we need to extend (5.4) to all $t \in [T/M, T]$. For this, it is enough to show that $|\xi_t(D) - \xi_{t'}(D)|$ and $|(\xi_0 * N_t)(D) - (\xi_0 * N_{t'})(D)|$ will both, with high probability, remain small simultaneously over all $|t - t'| < b_T$, for each given $t \in \mathscr{S}_T$ and $D \in \mathscr{D}_{T^{1/2}}$. Such bounds are provided by Lemmas 5.2 and 5.3.

LEMMA 5.2. *For all d, t and $D \in \mathscr{D}_{T^{1/2}}$,*

$$P\bigg(\sup_{t' \in [t, t+b_T]} |\xi_t(D) - \xi_{t'}(D)| \geq 2\bigg) \leq (b_T)^{3/2} \tag{5.5}$$

for sufficiently large T.

Since Lemma 5.3 will also be used in Section 7, it is stated somewhat more generally than needed here. We set $b_T^\delta = \exp\{-T^\delta\}$, and define $\mathscr{S}_T^\delta$ correspondingly.

LEMMA 5.3. *For all $d, M > 1, \delta > 0, t \in [T/M, T]$ and $D \in \mathscr{D}_{T^{1/2}}$,*

$$P\biggl(\sup_{t'\in[t,t+b_T^\delta]} |(\xi_0 * N_t)(D) - (\xi_0 * N_{t'})(D)| \ge \tfrac{1}{3} T^{d/4-1/80}\biggr) \le \exp\bigl\{-e^{T^\delta/4}\bigr\} \tag{5.6}$$

for sufficiently large T.

Summing up the exceptional probabilities in (5.5) and (5.6), for $\delta = 1/41$, $t \in \mathscr{S}_T$ and $D \in \mathscr{D}_{T^{1/2}}$, and combining the resulting bound with (5.4) implies that, for each $d < 4$ and M,

$$P\biggl(\sup_{t\in[T/M,\,T]} \sup_{D\in\mathscr{D}_{T^{1/2}}} |\xi_t(D) - (\xi_0 * N_t)(D)| \ge T^{d/4-1/80}\biggr) \le \exp\bigl\{-T^{1/42}\bigr\}$$

for sufficiently large T. This implies Proposition 2.1, as desired.

The conclusion in Lemma 5.2, that the probability of $\xi_{t'}(D)$ increasing or decreasing by more than 1 over a small time interval is very small, is not surprising. There are several steps that require a bit of estimation.

PROOF OF LEMMA 5.2. The A and B particles in D, for the state ξ_t, form a subset of the particles in D, for the state η_t. So, in order for at least two of these particles in D to leave D during $[t, t + b_T]$, under the process $\xi_{t'}$ (excluding annihilations), the same must be true under $\eta_{t'}$. The particles in $\eta_{t'}$ execute rate-d random walks. Since $\eta_t(D)$ is Poisson with mean $2\lambda|D|$,

$$E[(\eta_t(D))^2] = 4\lambda^2|D|^2 + 2\lambda|D| \le 4(\lambda^2 + \lambda)T^d \overset{\text{def.}}{=} \beta. \tag{5.7}$$

It is not difficult to see, using (5.7), that the probability of at least two jumps occuring over $[t, t + b_T]$, for those $\eta_t(D)$ particles starting in D, is at most $\beta(db_T)^2$. So, this is an upper bound on the probability of two particles of ξ_t leaving D by time $t + b_T$.

We also need an upper bound on the probability of at least two particles of ξ_t, in D^c, entering D by time $t + b_T$. If one restricts D^c to those sites within distance 2 (in the sum norm) of D, one obtains the same bound as above. On the other hand, the probability of a random walk moving k steps over this time period decays like $(db_T)^k/k!$. So, the expected number of particles starting from distance at least 3, which enter D by time $t + b_T$, is at most

$$2\lambda T^{d/2} \sum_{k=3}^{\infty} k^{d-1}(db_T)^k/k! \ll \beta(b_T)^2 \tag{5.8}$$

for large T. Applying Markov's inequality to this expectation and adding the resulting probability to the other two exceptional probabilities, we see that, for large T, the probability that

$$P\Big(\sup_{t'\in[t,\,t+b_T]}|\xi_t(D)-\xi_{t'}(D)|\geq 2\Big)\leq 3\beta(b_T)^2.$$

For large T, this is less than $(b_T)^{3/2}$. This implies (5.5). □

To demonstrate Lemma 5.3, we use moment generating functions and the independence of $\xi_0(x)$ at different x. Here, we abbreviate, and set $I^{\delta}_{t,T}=[t,t+b^{\delta}_T]$.

PROOF OF LEMMA 5.3. In order to demonstrate (5.6), it suffices to show that for given $\gamma>0$ and large enough T,

$$P\Big(\sup_{t'\in I^{\delta}_{t,T}}|(\xi_0*N_t)(0)-(\xi_0*N_{t'})(0)|\geq\gamma T^{-(d/4+1/80)}\Big)\leq\exp\{-e^{T^{\delta}/2}\},\tag{5.9}$$

since ξ_0 is translation invariant, and we can sum over $x\in D$. We will use the inequality

$$\sup_{t'\in I^{\delta}_{t,T}}|(\xi_0*N_t)(0)-(\xi_0*N_{t'})(0)|\leq\sum_{x\in\mathbb{Z}^d}|\xi_0(-x)|\sup_{t'\in I^{\delta}_{t,T}}|N_t(x)-N_{t'}(x)|\tag{5.10}$$

and the fact that $|\xi_0(-x)|$, $x\in\mathbb{Z}^d$, are dominated by independent Poisson random variables with mean 2λ. It follows that, for $\theta>0$,

$$\begin{aligned}&E\Big[\exp\Big\{\theta\sup_{t'\in I^{\delta}_{t,T}}|(\xi_0*N_t)(0)-(\xi_0*N_{t'})(0)|\Big\}\Big]\\&\qquad\leq\exp\Big\{2\lambda\sum_x\Big(\exp\Big\{\theta\sup_{t'\in I^{\delta}_{t,T}}|N_t(x)-N_{t'}(x)|\Big\}-1\Big)\Big\}.\end{aligned}\tag{5.11}$$

By (4.10), for large T, $t\in[T/M,T]$ and $x\in\mathbb{Z}^d$,

$$\sup_{t'\in I^{\delta}_{t,T}}|N_t(x)-N_{t'}(x)|\leq C_{27}b^{\delta}_T T^{-(d/2+1)}\Big(1+\frac{|x|^2}{T}\Big)e^{-|x|^2/4T}\tag{5.12}$$

for appropriate C_{27}. Summation of both sides of (5.12) implies that

$$\sum_x\sup_{t'\in I^{\delta}_{t,T}}|N_t(x)-N_{t'}(x)|\leq b^{\delta}_T.\tag{5.13}$$

The right side of (5.12) is at most b^{δ}_T. So, for $\theta=1/b^{\delta}_T$, the right side of (5.11) is at most

$$\exp\Big\{4\lambda(b^{\delta}_T)^{-1}\sum_x\sup_{t'\in I^{\delta}_{t,T}}|N_t(x)-N_{t'}(x)|\Big\},$$

which, by (5.13), is at most $e^{4\lambda}$. By Chebyshev's inequality, one obtains that for given $\gamma > 0$ and large T,

$$P\biggl(\sup_{t'\in I^{\delta}_{t,T}} |(\xi_0 * N_t)(0) - (\xi_0 * N_{t'})(0)| \geq \gamma T^{-(d/4+1/80)}\biggr)$$
$$\leq \exp\bigl\{4\lambda - \gamma(b^{\delta}_T)^{-1}T^{-(d/4+1/80)}\bigr\},$$

which, for large enough T, is at most $\exp\{-e^{T^{\delta}/2}\}$. This implies (5.9), and hence (5.6). □

We recall that the analog of Proposition 2.3, for $d \geq 4$, was given at the end of Section 3, after some minor changes in the argument, as Proposition 3.3. The other main result that has been employed in Section 5, Proposition 2.4, does not depend on d. By replacing Proposition 2.3 by Proposition 3.3, but otherwise reasoning the same as through Proposition 5.2, one obtains the analog of (5.4), with the bound $\frac{1}{3}T^{d/4-1/80}$ replaced by $\frac{1}{3}T^{d/2-81/80}$. Since neither Lemma 5.2 nor Lemma 5.3 depends on d, one thus obtains the following analog of Proposition 2.1, for $d \geq 4$.

PROPOSITION 5.3. *For $d \geq 4$ and $M > 0$,*

$$P\biggl(\sup_{t\in[T/M,\,T]}\sup_{D\in\mathscr{D}_{T^{1/2}}} |\xi_t(D) - (\xi_0 * N_t)(D)| \geq T^{d/2-81/80}\biggr) \leq \exp\{-T^{1/42}\} \tag{5.14}$$

for sufficiently large T.

Proposition 5.3 will be employed in [9]. Note that (2.4) and (5.14) are the same, if one formally sets $d = 4$ in both cases. (The arguments leading to Proposition 2.1, in fact, all hold for $d = 4$ as well.)

6. Upper bounds on $E[\xi^m_t(\cdot)]$. In this section, we demonstrate Proposition 2.2, which gives an upper bound on the expected number of particles of the minority type in cubes D_{R_T}, where the length R_T is chosen appropriately. We will use this bound in Section 7, together with Proposition 2.1, to obtain the desired estimates for ξ^A_t and ξ^B_t, which are given in Theorem 3. We also use Proposition 2.2 to compute the limiting density $\rho(t)$ as $t \to \infty$, for $d < 4$, in (1.8), which is an improvement of the upper and lower bounds given in (1.2).

In order to show Proposition 2.2, we rely heavily on a slight modification of Lemma 4.6 from [7]. For this, we set

$$R_T = \delta_1(T)T^{1/2} \quad \text{and} \quad r_T = T^{7/24}, \tag{6.1}$$

for appropriate $\delta_1(t)$ to be specified shortly. The above exponent 7/24 is itself not crucial, but needs to be slightly larger than 1/4. The required analog of Lemma 4.6 is given by Lemma 6.1 below. In contrast to our usual convention, we drop the assumption in the lemma that ξ_0 be the difference of two Poisson random fields.

LEMMA 6.1. *Assume $d < 4$, and that ξ_0 is translation invariant with $E[\xi_0^m(D_{R_T})] \geq L_1$, where $L_1 \geq C_{28}(R_T/r_T)^d$ for appropriate C_{28}. Assume that $\delta_1(T) \geq T^{-d/48}$. Then, for appropriate $C_{29} > 0$ [not depending on $\delta_1(\cdot)$] and large enough T,*

$$E[\xi_0^{\#}(D_{R_T})] - E[\xi_{R_T^2}^{\#}(D_{R_T})] \geq C_{29}L_1. \tag{6.2}$$

(We abbreviate $(R_T)^d$ by R_T^d and $(r_T)^d$ by r_T^d, here and later on.)

Lemma 6.1 says that if $E[\xi_0^m(D_{R_T})]$, the mean number of particles of the minority type in D_{R_T}, is not too small, then, on the average, the total number of particles lost in D_{R_T}, over the time interval $[0, R_T^2]$, must also be of this order of magnitude. The lemma, for $d < 4$, is identical to Lemma 4.6 of [7], if R_T and r_T are replaced by

$$R'_T = \delta_1 T^{1/2} \quad \text{and} \quad r'_T = \delta_2 T^{1/4}, \tag{6.3}$$

where δ_1 and δ_2 are constants. The purpose of choosing r_T as in (6.1), with $r_T \gg r'_T$, is to permit smaller values of L_1 when Lemma 6.1 (rather than Lemma 4.6) is applied. The condition $\delta_1(T) \geq T^{-d/48}$ is used at the end of the following sketch, as well as in the proof of Proposition 2.2.

SKETCH OF THE PROOF OF LEMMA 6.1. The proofs of the lemma and of Lemma 6.4 in [7] are almost the same. We begin here at the point where they diverge, referring the reader to [7] for the earlier part of the argument.

Denote the left side of (6.2) by u_T. The proofs of the two lemmas are identical up through (4.30) and (4.31) of [7], which state that

$$C_{30}u_T/L_1 \geq \begin{cases} R_T^2/r_T^2, & d = 1, \\ R_T^2/(r_T^2 \log r_T), & d = 2, \\ R_T^2/r_T^3, & d = 3, \end{cases} \tag{6.4}$$

for appropriate C_{30} and large T. Substitution for R_T and r_T, as in (6.1), gives the bounds

$$C_{30}u_T/L_1 \geq \begin{cases} (\delta_1(T))^2 T^{5/12}, & d = 1, \\ (\delta_1(T))^2 T^{5/12} \log T, & d = 2, \\ (\delta_1(T))^2 T^{1/8}, & d = 3. \end{cases} \tag{6.5}$$

Since $\delta_1(T) \geq T^{-d/48}$ is assumed, this implies that $u_T \geq (C_{30})^{-1}L_1$, which, setting $C_{29} = (C_{30})^{-1}$, gives (6.2). □

We now show Proposition 2.2. In addition to the lower bound $\delta_1(T) \geq T^{-d/48}$, where $R(T) = \delta_1(T)T^{1/2}$, we also assume here that $\delta_1(T) \to 0$ as $T \to \infty$. The reason for this is to be able to show, as in (2.6) of Proposition 2.2, that $(T^{-d/4}R_T^d)^{-1}\xi_{t_k}^m(D_{R_T})$ is negligible in the limit, for appropriate t_k. As we will see in Section 8, this will not be true if $\delta_1(T)$ is instead bounded away from 0. The major ingredients for the proof of Proposition 2.2 are Lemma 6.1 and (1.2). The main point of the argument is that the decrease in the expected number of particles given by (6.2) restricts the frequency with which (6.6) can occur.

PROOF OF PROPOSITION 2.2. Assume that

$$E[\xi_t^m(D_{R_T})] > C_1\delta_1(T)T^{-d/4}R_T^d \tag{6.6}$$

for large T and some t, where $C_1 \geq 1$ is fixed and will be chosen later. Setting $L_1 = C_1\delta_1(T)T^{-d/4}R_T^d$, it is not difficult to see that the assumptions of Lemma 6.1 are satisfied, if time t is replaced by 0: Clearly, ξ_t is translation invariant. Also, since $\delta_1(T) \geq T^{-d/48}$,

$$L_1 = C_1\delta_1(T)T^{-d/4}R_T^d \geq (R_T/r_T)^d.$$

Therefore, by (6.2),

$$E[\xi_t^\#(D_{R_T})] - E[\xi_{t+R_T^2}^\#(D_{R_T})] \geq C_1C_{29}\delta_1(T)T^{-d/4}R_T^d \tag{6.7}$$

for appropriate C_{29}.

Let $t_l' = T/2M + lR_T^2$, for $l = 0, 1, 2, \ldots$. One can select the values t_k appearing in (2.6), so that they form a subset of these t_l'. We argue inductively, setting $t_0 = T/2M$, and assuming that t_{k-1} has already been chosen. Let n_k denote the number of t_l' satisfying

$$t_l' - t_{k-1} \in [R_T^2, \delta_1(T)T - R_T^2], \tag{6.8}$$

where (6.6) holds, with $t = t_l'$. Also, set $t^* = t_{k-1} + \delta_1(T)T$. Since $E[\xi_t^\#(D_{R_T})]$ is decreasing in t, it follows from (6.7) that

$$\begin{aligned} E[\xi_{T/2M}^\#(D_{R_T})] &\geq E[\xi_{t_{k-1}}^\#(D_{R_T})] - E[\xi_{t^*}^\#(D_{R_T})] \\ &\geq C_1C_{29}n_k\delta_1(T)T^{-d/4}R_T^d. \end{aligned} \tag{6.9}$$

On the other hand, it follows from (1.2) that

$$E[\xi_{T/2M}^\#(D_{R_T})] = 2|D_{R_T}|\rho(T/2M) \leq C_{31}T^{-d/4}R_T^d, \tag{6.10}$$

where C_{31} depends on λ and M. So, comparing (6.9) and (6.10), one obtains that

$$n_k \leq C_{31}(C_1C_{29}\delta_1(T))^{-1}. \tag{6.11}$$

The number of t_l' satisfying (6.8) is at least

$$\delta_1(T)T/R_T^2 - 3 = (\delta_1(T))^{-1} - 3.$$

For $C_1 \overset{\text{def.}}{=} (2C_{31}/C_{29}) \vee 1$ and $\delta_1(T) < 1/6$, this is strictly larger than n_k. So, for this choice of C_1 and large T,

$$E[\xi_{t_\ell'}^m(D_{R_T})] \leq C_1\delta_1(T)T^{-d/4}R_T^d$$

holds for at least one t_l' satisfying (6.8). Setting t_k equal to this t_l' produces (2.6), as desired.

Since $\delta_1(T) \to 0$ as $T \to \infty$, it is easy to check that $t_1 \le T/M$ for large T. Also, since $t'_l - t'_{l-1} = R_T^2$, one has $t_{K-1} \le T$ and $t_K > T$ for some $K \le T^{d/24}$. So, for this choice of K, $[t_1, t_K] \supset [T/M, T]$ and $[t_1, t_{K-1}] \subset [T/2M, T]$, which completes the proof of the proposition. □

In the remainder of the section, we analyze the asymptotic density $\rho(t)$, for $d < 4$. Recall that the initial states of A and B are given by Poisson random fields with intensity λ. We already know, as in (1.2), that $\lambda^{-1/2}t^{d/4}\rho(t)$ is bounded above and below; here, we show convergence and identify the limit. The basic procedure is as follows: The density $\rho(t)$ is given by $E[\xi_t^A(D)]/|D| = E[\xi_t^B(D)]/|D|$. Using Proposition 2.2, the latter quantities can be approximated by $E[|\xi_t(D)|]/2|D|$, at appropriate t, when D is comparatively small. By applying Lemma 5.1 and some elementary tail estimates to $\xi_t(D)$ and $(\xi_0 * N_t)(D)$, one can show that $E[|\xi_t(D)|]$ is approximated by $E[|(\xi_0 * N_t)(D)|]$. Consequently, in order to compute the asymptotics of $\rho(t)$, it suffices to do the same for $E[|(\xi_0 * N_t)(D)|]/|D|$. This can be done using the central limit theorem.

We begin with several elementary estimates of ξ_t and $\xi_0 * N_t$.

LEMMA 6.2. *For all $E \subset \mathbb{Z}^d$, with $|E| < \infty$, and all $t \ge 1$,*

$$E[(\xi_t(E))^2] \le (\lambda^2 + \lambda)|E|^2 \tag{6.12}$$

and

$$E[((\xi_0 * N_t)(E))^2] \le C_{32}\lambda t^{-d/2}|E|^2 \tag{6.13}$$

for appropriate C_{32}. Suppose that $E_t \subset D_{\varepsilon(t)}$, where $\varepsilon(t) = o(t^{1/2})$. Then,

$$\lim_{t\to\infty} \sigma^2((\xi_0 * N_t)(E_t))/(t^{-d/2}|E_t|^2) = 2\lambda(4\pi)^{-d/2}. \tag{6.14}$$

PROOF. The random variable $|\xi_t(E)|$ is dominated by $\eta_t^{\#}(E)$, which is Poisson with mean $2\lambda|E|$. This implies (6.12). Since

$$(\xi_0 * N_t)(E) = \sum_{x\in\mathbb{Z}^d} \xi_0(x)N_t(E - x)$$

is the sum of independent mean-0 random variables with variances $2\lambda(N_t(E-x))^2$,

$$\sigma^2((\xi_0 * N_t)(E)) = 2\lambda\sum_x (N_t(E-x))^2. \tag{6.15}$$

The right side is at most $2\lambda|E|^2\sum_x(N_t(x))^2$. One can check that, for $t \ge 1$, this is at most $C_{32}\lambda t^{-d/2}|E|^2$ for appropriate C_{32}, which gives (6.13), since $(\xi_0 * N_t)(E)$ has mean 0. With a bit of estimation, one can also show that the right side of (6.15), with $E = E_t$, is asymptotically equal to

$$2\lambda|E_t|^2\int_{\mathbb{R}^d}(N_t(x))^2\,dx \sim 2\lambda(4\pi t)^{-d/2}|E_t|^2$$

for large t, which implies (6.14). □

The following result allows us to compare the expectations of $\xi_t(E)$ and $(\xi_0 * N_t)(E)$, for appropriate $|E|$. Its proof employs Lemma 5.1, (6.12) and (6.13).

LEMMA 6.3. *For $d < 4$, $E \subset \mathbb{Z}^d$ with $|E| \in [t^{19d/40}, t^d]$, and large t,*

$$E[|\xi_t(E) - (\xi_0 * N_t)(E)|] \le t^{-(d/4+1/200)}|E|. \tag{6.16}$$

PROOF. Let H_1 denote the set where $|\xi_t(E)| \ge t|E|$ and H_2 the set where $|(\xi_0 * N_t)(E)| \ge t|E|$. Also, let $Z = |\xi_t(E) - (\xi_0 * N_t)(E)|$. Then, one can check that

$$E[Z; H_1 \cup H_2] \le 2E[|\xi_t(E)|; H_1] + 2E[|(\xi_0 * N_t)(E)|; H_2]. \tag{6.17}$$

By (6.12) and the definition of H_1,

$$E[|\xi_t(E)|; H_1] \le (\lambda^2 + \lambda)t^{-1}|E|. \tag{6.18}$$

Similarly, by (6.13) and the definition of H_2,

$$E[|(\xi_0 * N_t)(E)|; H_2] \le C_{32}\lambda t^{-(d/2+1)}|E|. \tag{6.19}$$

So, by (6.17)–(6.19),

$$E[Z; H_1 \cup H_2] \le 2(C_{32} + 1)(\lambda^2 + \lambda)t^{-1}|E|. \tag{6.20}$$

Assume that $|E| \in [t^{19d/40}, t^d]$. Substitution of $\varepsilon = t^{-1/200}/6$ in Lemma 5.1 implies that

$$P(Z \ge \tfrac{1}{3}t^{-(d/4+1/200)}|E|) \le 7\exp\{-t^{d/80}\} \tag{6.21}$$

for large t. Let H_3 denote the random set in (6.21). Then, since $|Z| \le 2t|E|$ on $H_1^c \cap H_2^c$,

$$E[Z; H_1^c \cap H_2^c \cap H_3] \le 14t\exp\{-t^{d/80}\}|E|. \tag{6.22}$$

Also,

$$E[Z; H_3^c] \le \tfrac{1}{3}t^{-(d/4+1/200)}|E|. \tag{6.23}$$

Together, (6.20), (6.22) and (6.23) imply that

$$E[Z] \le t^{-(d/4+1/200)}|E|,$$

which is the desired inequality. □

By using (6.14) and the central limit theorem, we obtain the following limiting behavior for $E[|(\xi_0 * N_t)(E_t)|]$ and small $|E_t|$.

LEMMA 6.4. *Suppose that $E_t \subset \mathbb{Z}^d$, with $\phi \ne E_t \subset D_{\varepsilon(t)}$ and $\varepsilon(t) = o(t^{1/2})$. Then, for all d,*

$$\lim_{t\to\infty}(t^{-d/4}|E_t|)^{-1}E[|(\xi_0 * N_t)(E_t)|] = (4\lambda/\pi)^{1/2}(4\pi)^{-d/4}. \tag{6.24}$$

PROOF. Recall that

$$(\xi_0 * N_t)(E_t) = \sum_{x \in \mathbb{Z}^d} \xi_0(x) N_t(E_t - x),$$

where the summands are independent mean-0 random variables. By (6.14), for $E_t \subset D_{\varepsilon(t)}$ with $\varepsilon(t) = o(t^{1/2})$,

$$\lim_{t\to\infty} \sigma^2((\xi_0 * N_t)(E_t)/(t^{-d/4}|E_t|)) = 2\lambda(4\pi)^{-d/2}. \tag{6.25}$$

Since the third moment of a rate-λ Poisson random variable is $\lambda(\lambda^2 + 3\lambda + 1)$, and $\xi_0(x)$ is the difference of two such random variables,

$$E[|\xi_0(x) N_t(E_t - x)|^3] \le 40(\lambda^3 + \lambda)(N_t(E_t - x))^3.$$

Using this, one can check that for large t,

$$\sum_x E[(|\xi_0(x) N_t(E_t - x)|/(t^{-d/4}|E_t|))^3] \le 40(\lambda^3 + \lambda) t^{-d/4}, \tag{6.26}$$

which $\to 0$ as $t \to \infty$.

By the bounds, in (6.25) and (6.26), on the variances and third moments of $(t^{-d/4}|E_t|)^{-1}\xi_0(x)N_t(E_t - x)$, and by the Liapunov central limit theorem (see, e.g., [10], page 200), it follows that

$$(t^{-d/4}|E_t|)^{-1}(\xi_0 * N_t)(E_t) \Rightarrow (2\lambda)^{1/2}(4\pi)^{-d/4} Z_{0,1}, \tag{6.27}$$

where $Z_{0,1}$ is a normal random variable with mean 0 and variance 1, and $\Rightarrow$ denotes weak convergence. We also know from (6.13) [or (6.14)], that the second moments of $(t^{-d/4}|E_t|)^{-1}(\xi_0 * N_t)(E_t)$ are bounded in t. It follows from this and (6.27), that

$$\begin{aligned}\lim_{t\to\infty}(t^{-d/4}|E_t|)^{-1} E[|(\xi_0 * N_t)(E_t)|] &= (\lambda/\pi)^{1/2}(4\pi)^{-d/4}\int_{\mathbb{R}} |x| e^{-x^2/2}\, dx \\ &= (4\lambda/\pi)^{1/2}(4\pi)^{-d/4},\end{aligned}$$

which implies (6.24). $\square$

We note in passing that the assumption $\varepsilon = o(t^{1/2})$ in Lemma 6.4 was only used for the limit on the variance in (6.14). For $E_t \subset D_{t^{1/2}}$, it is not difficult to see, using (6.15), that a lower bound on the second moment of $(\xi_0 * N_t)(E)$ corresponding to (6.13), but with the inequality reversed, still holds. Using this, and reasoning as in Lemma 6.4, one can check that $(t^{-d/4}|E_t|)^{-1}(\xi_0 * N_t)(E_t)$ is bounded away from 0 in distribution (and not just in mean). In particular, the error bounds given in Theorem 3 are of smaller order of magnitude than the terms $\xi_t^A(D)$ and $\xi_t^B(D)$ there, for D of the same order as $D_{T^{1/2}}$ and $t \in [T/M, MT]$. This is what one would expect, because of (1.2).

Lemmas 6.3 and 6.4 imply the following limiting behavior for $E[|\xi_t(E_t)|]$, when $|E_t|$ is small.

COROLLARY 6.1. *Suppose $d < 4$, and that $E_t \subset \mathbb{Z}^d \cap D_{\varepsilon(t)}$, with $|E_t| \geq t^{19d/40}$ and $\varepsilon(t) = o(t^{1/2})$. Then,*

$$\lim_{t\to\infty} (t^{-d/4}|E_t|)^{-1} E[|\xi_t(E_t)|] = (4\lambda/\pi)^{1/2}(4\pi)^{-d/4}. \tag{6.28}$$

Using Corollary 6.1 and Proposition 2.2, we now compute the limiting density $\rho(t)$ for $d < 4$, which was given in (1.8).

PROPOSITION 6.1. *Assume that the initial distributions of A and B particles for the process ξ_t are given by independent Poisson random fields with intensity λ. Then, for $d < 4$,*

$$\lim_{t\to\infty} t^{d/4}\rho(t) = (\lambda/\pi)^{1/2}(4\pi)^{-d/4}. \tag{6.29}$$

PROOF. Since $\rho(t)$ is decreasing in t, it suffices to show (6.29) along a subsequence of times $u_1 < u_2 < \cdots$ with $\lim_{j\to\infty} u_j = \infty$ and $u_j - u_{j-1} = o(u_j)$. For all t and nonempty finite $E \subset \mathbb{Z}^d$, one has

$$\rho(t) = E[\xi_t^A(E) + \xi_t^B(E)]/2|E| = E[|\xi_t(E)|]/2|E| + E[\xi_t^m(E)]/|E|. \tag{6.30}$$

Our approach will be to use Proposition 2.2 to select $t = u_j$, so that the second term in (6.30) can be dropped in the limit, after scaling by $t^{-d/4}$. The limit in (6.29) will then follow from (6.28).

For given t, we set $T = t$ and $R_T = T^{23/48}$ in Proposition 2.2. By Proposition 2.2, one can then choose $s \in (t - t^{47/48}, t]$ such that, for large t,

$$s^{d/4} E[\xi_s^m(D_{s^{23/24}})]/|D_{s^{23/24}}| \leq 2C_1 s^{-1/48} \tag{6.31}$$

[since $\xi_s^m(D_{s^{23/24}}) \leq \xi_s^m(D_{t^{23/24}})$]. Employing such s and (6.31), it is easy to construct $u_1 < u_2 < \cdots$, with $\lim_{j\to\infty} u_j = \infty$ and $u_j - u_{j-1} = o(u_j)$, so that

$$\lim_{j\to\infty} u_j^{d/4} E[\xi_{u_j}^m(E_{u_j})]/|E_{u_j}| = 0, \tag{6.32}$$

where $E_t \overset{\text{def.}}{=} D_{t^{23/24}}$. Together with (6.30), (6.32) implies that

$$\lim_{j\to\infty} u_j^{d/4}\rho(u_j) = \lim_{j\to\infty} u_j^{d/4} E[|\xi_{u_j}(E_{u_j})|]/2|E_{u_j}|. \tag{6.33}$$

Along with (6.28), (6.33) implies that

$$\lim_{j\to\infty} u_j^{d/4}\rho(u_j) = (\lambda/\pi)^{1/2}(4\pi)^{-d/4}.$$

The limit (6.29) follows from this and the comment at the beginning of the proof. □

7. Approximation of ξ_t^A and ξ_t^B. In this section, we demonstrate Theorem 3, which, in $d < 4$, enables us to approximate $\xi_t^A(D)$ and $\xi_t^B(D)$ by $\sum_{x\in D}(\xi_0 * N_t)(x)^-$ and $\sum_{x\in D}(\xi_0 * N_t)(x)^+$, respectively. The bounds given in (2.2) and (2.3) of the theorem hold simultaneously over all times $t \in [T/M, MT]$ and rectangles $D \in \mathscr{D}_{MT^{1/2}}$, where M is fixed, off of an event which is of small probability when T is large. The main tools used in deriving Theorem 3 are Propositions 2.1 and 2.2. The statement in Proposition 2.1 is analogous to those in Theorem 3, except that here one approximates $\xi_t(D)$ by $(\xi_0 * N_t)(D)$. In order to derive the estimates in Theorem 3 from Proposition 2.1, one needs to show that, locally, the two particle types segregate, with the number of the minority type typically being negligible. Proposition 2.2 is employed for this.

One can break the reasoning required for this argument into two main parts. In Proposition 7.1, we will approximate $(\xi_0 * N_t)(D)^{\pm}$ by $\sum_{x\in D}(\xi_0 * N_t)(x)^{\pm}$ for small rectangles D. The reasoning here is straightforward, and is based on an estimate that shows $(\xi_0 * N_t)(x)$ does not fluctuate much locally.

We also need to approximate $\xi_t^A(D)$ and $\xi_t^B(D)$ by $(\xi_0 * N_t)(D)^-$ and $(\xi_0 * N_t)(D)^+$, again for small D. This involves bounding $\xi_t^m(D)$. Here, one needs to be more careful, since Proposition 2.2 only holds for certain times t_k, and bounds are given only on the expectation of $\xi_{t_k}^m(D)$. The probability estimates obtained by applying Markov's inequality to this expectation at each t_k are much weaker than the exceptional probabilities in Proposition 2.1, and one needs to work to keep these estimates small when summing over different events. One also needs to control the migration of particles over each interval $[t_k, t_{k+1})$. These difficulties are taken care of in the work leading up to Proposition 7.2.

Together, the reasoning from the last two paragraphs shows that $\xi_t^A(D)$ and $\xi_t^B(D)$ can be approximated by $\sum_{x\in D}(\xi_0 * N_t)(x)^-$ and $\sum_{x\in D}(\xi_0 * N_t)(x)^+$, for small D. Taking unions of such rectangles D, one obtains the corresponding estimates for all $D \in \mathscr{D}_{MT^{1/2}}$, as desired. As mentioned above, one needs to keep the exceptional probabilities which crop up under control.

Lemma 7.1 is the main technical estimate needed for Proposition 7.1. It is employed there and elsewhere in the section, with $t' = t$; it is employed in Section 9 with $t' \neq t$, but with $x' = x$. The argument is a straightforward application of moment generating functions.

LEMMA 7.1. *Let $|t - t'| \le t^{\alpha}$ and $|x - x'| \le t^{\alpha/2}$, where $\alpha \in [1/2, 1)$. Then, for appropriate $C_{33} > 0$,*

$$P(|(\xi_0 * N_t)(x) - (\xi_0 * N_{t'})(x')| \ge \varepsilon t^{-d/4}) \le 2\exp\{-C_{33}\varepsilon^2 t^{1-\alpha}\} \tag{7.1}$$

for large enough t and all $\varepsilon \in [0, 1]$.

PROOF. Fix t, t', x and x', and set

$$R(y) = N_t(x - y) - N_{t'}(x' - y).$$

One has for given θ,

$$E[\exp\theta\{(\xi_0 * N_t)(x) - (\xi_0 * N_{t'})(x')\}] \tag{7.2}$$
$$= \exp\left\{\lambda \sum_{y\in\mathbb{Z}^d} \left(e^{\theta R(y)} + e^{-\theta R(y)} - 2\right)\right\}.$$

Since $|t - t'| \le t^\alpha$ and $|x - x'| \le t^{\alpha/2}$, with $\alpha \le 1$, it follows from (4.10) and (4.13) that

$$|R(y)| \le C_{34} t^{(\alpha-d-1)/2}, \tag{7.3}$$

and from (4.11) and (4.14) that

$$\sum_{y\in\mathbb{Z}^d} (R(y))^2 \le C_{35} t^{\alpha-d/2-1}, \tag{7.4}$$

for appropriate C_{34} and C_{35}. By (7.3) and (7.4), (7.2) is, for $|\theta| \le C_{36} t^{(d+1-\alpha)/2}$ and given C_{36}, at most

$$\exp\left\{C_{37}\lambda\theta^2 \sum_y (R(y))^2\right\} \le \exp\left\{C_{35}C_{37}\lambda\theta^2 t^{\alpha-d/2-1}\right\} \tag{7.5}$$

for appropriate C_{37}. Since $\alpha \ge 1/2$ and $\varepsilon \le 1$, $\theta = \pm(2\lambda C_{35}C_{37})^{-1}\varepsilon t^{d/4+1-\alpha}$ satisfy the above bounds on $|\theta|$. Chebyshev's inequality, applied to (7.2) and (7.5) for both values of θ, implies that

$$P(|(\xi_0 * N_t)(x) - (\xi_0 * N_{t'})(x')| \ge \varepsilon t^{-d/4}) \le 2\exp\{-(4C_{35}C_{37}\lambda)^{-1}\varepsilon^2 t^{1-\alpha}\}.$$

This implies (7.1), with $C_{33} = (4C_{35}C_{37}\lambda)^{-1}$. □

We would like to replace the bound in (7.1), with $t' = t$, by one which simultaneously holds over $t \in [T/M, T]$ and $x \in D_{T^{1/2}}$, if ε is chosen not too small. Such an estimate follows directly from Lemma 7.1 and Lemma 5.3. (In some applications, t will remain fixed, and only x will be allowed to vary.)

LEMMA 7.2. *Let $\alpha \in [1/2, 1)$ and $\beta = (1-\alpha)/8$. Then, for all $M > 1$,*

$$P\left(\sup_{t\in[T/M,\,T]} \sup_{x\in D_{T^{1/2}}} \sup_{|x'-x|\le T^{\alpha/2}} |(\xi_0 * N_t)(x) - (\xi_0 * N_t)(x')| \ge T^{-d/4-\beta}\right) \le \exp\{-T^\beta\} \tag{7.6}$$

for sufficiently large T.

PROOF. Set $\varepsilon = MT^{-\beta}$, where $\alpha \in [1/2, 1)$ and $\beta \in (0, (1-\alpha)/8)$. One can show that the bound in (7.6), with $\frac{1}{2}\exp\{-T^\beta\}$ instead of $\exp\{-T^\beta\}$, holds for $t \in \mathscr{S}_T^\beta$, $x \in D_{T^{1/2}}$ and $|x' - x| \le T^{\alpha/2}$, by summing over the probabilities in (7.1). To extend the bound to all $[T/M, T]$ as in (7.6), one applies Lemma 5.3 with $D = \{x\}$. □

In Section 8, we will apply Lemma 7.2 in a somewhat different setting, where $(\xi_0 * N_t)(x) = \sum_{y\in\mathbb{Z}^d} \xi_0(y)N_t(x-y)$ has been extended to $x \in \mathbb{R}^d$. This slight generalization causes no changes in the statement of Lemma 7.2 or its proof.

Let $\mathscr{D}_R^r$ denote those rectangles contained in the cube D_R, for which the lengths of all sides are at most r. Using Lemma 7.2, we are able to compare $(\xi_0 * N_t)(D)^+$ with $\sum_{x\in D}(\xi_0 * N_t)(x)^+$ over such rectangles. Proposition 7.1 is the first main ingredient for demonstrating Theorem 7.3.

PROPOSITION 7.1. *Let $\alpha \in [1/2, 1)$ and $\beta = (1-\alpha)/8$. Then, for all $M > 1$,*

$$P\left(\sup_{t\in[T/M,\,T]} \sup_{D\in\mathscr{D}_{T^{1/2}}^{T^{\alpha/2}}} \left|(\xi_0 * N_t)(D)^- - \sum_{x\in D}(\xi_0 * N_t)(x)^-\right| \geq |D|T^{-d/4-\beta}\right) \leq \exp\{-T^\beta\} \tag{7.7}$$

and

$$P\left(\sup_{t\in[T/M,\,T]} \sup_{D\in\mathscr{D}_{T^{1/2}}^{T^{\alpha/2}}} \left|(\xi_0 * N_t)(D)^+ - \sum_{x\in D}(\xi_0 * N_t)(x)^+\right| \geq |D|T^{-d/4-\beta}\right) \leq \exp\{-T^\beta\} \tag{7.8}$$

for sufficiently large T.

PROOF. We consider just (7.8), since the argument for (7.7) is the same. Clearly, $(\xi_0 * N_t)(D)^+ \leq \sum_{x\in D}(\xi_0 * N_t)(x)^+$ always holds. For the other direction, we may assume that $(\xi_0 * N_t)(y) \geq 0$ for some $y \in D$. Then, on the nonexceptional set G in Lemma 7.2,

$$(\xi_0 * N_t)(x) > -T^{-d/4-\beta} \quad \text{for } x \in D,$$

for $\alpha \in [1/2, 1)$ and $\beta = (1-\alpha)/8$. It follows from this that, on G,

$$\begin{aligned}(\xi_0 * N_t)(D)^+ &\geq \sum_{x\in D}(\xi_0 * N_t)(x) = \sum_{x\in D}[(\xi_0 * N_t)(x)^+ - (\xi_0 * N_t)(x)^-] \\ &\geq \sum_{x\in D}(\xi_0 * N_t)(x)^+ - |D|T^{-d/4-\beta}.\end{aligned}$$

This implies (7.8). □

The second main ingredient for demonstrating Theorem 3 is to show that, for suitable small D, $\xi_t^A(D)$ and $\xi_t^B(D)$ can be approximated by $(\xi_0 * N_t)(D)^-$ and $(\xi_0 * N_t)(D)^+$. On account of Proposition 2.1, it will be enough to show that $\xi_t^A(D)$ and $\xi_t^B(D)$ can be approximated by $\xi_t(D)^-$ and $\xi_t(D)^+$. It will suffice to show that $\xi_t^m(D)$ is typically negligible. For this, we cover $D_{T^{1/2}}$ with disjoint cubes D^i, $i = 1, \ldots, I$, each of length $T^{\alpha/2}$, so that $\cup_i D^i \subset D_{6T^{1/2}}$ and $I \leq (6T^{(1-\alpha)/2})^d$. We will employ the following result.

PROPOSITION 7.2. *Let $d < 4$, $\alpha \in [1-2/1{,}000, 1)$, $\beta = (1-\alpha)/16$, and choose D^i as above. Then, for all $M > 1$,*

$$P\left(\sup_{t\in[T/M,\,T]} \sum_{i=1}^{I} \xi_t^m(D^i) \geq T^{d/4-\beta}\right) \leq T^{-\beta} \tag{7.9}$$

for large enough T.

The number of particles of minimum type, $\xi_t^m(E)$, is increasing with the set $E \subset \mathbb{Z}^d$. Also,

$$\xi_t^m(E) = \xi_t^A(E) - \xi_t(E)^- = \xi_t^B(E) - \xi_t(E)^+ \tag{7.10}$$

always holds. The following corollary is therefore an immediate consequence of (7.9).

COROLLARY 7.1. *Let $\alpha \in [1-2/1{,}000, 1)$, $\beta = (1-\alpha)/16$, and choose D^i as above. Then, for all $M > 1$,*

$$P\left(\sup_{t\in[T/M,\,T]} \sum_{i=1}^{I} \sup_{E\subset D^i} \left(\xi_t^A(E) - \xi_t(E)^-\right) \geq T^{d/4-\beta}\right) \leq T^{-\beta} \tag{7.11}$$

and

$$P\left(\sup_{t\in[T/M,\,T]} \sum_{i=1}^{I} \sup_{E\subset D^i} \left(\xi_t^B(E) - \xi_t(E)^+\right) \geq T^{d/4-\beta}\right) \leq T^{-\beta} \tag{7.12}$$

for large enough T.

Corollary 7.1, together with Proposition 2.1 and Proposition 7.1, provides a quick proof of Theorem 3. We therefore first show Theorem 3 and afterwards return to the argument for Proposition 7.2.

PROOF OF THEOREM 3. We will demonstrate the inequality for ξ_t^B; the argument for ξ_t^A is the same. By rescaling T, it suffices to show this over $t \in [T/M, T]$ and $D \in \mathscr{D}_{T^{1/2}}$ (after replacing the term 1/9,000 in the exponents with any larger power).

Let G_1 denote the nonexceptional set in (7.12) and G_2 the nonexceptional set in (2.4) of Proposition 2.1. Also, let $\alpha = 1 - 2/1{,}000$ and $\beta = (1-\alpha)/16 = 1/8{,}000$. For $D \in \mathscr{D}_{T^{1/2}}$, we set $E^i = D \cap D^i$. The sets E^i are always rectangles. So, by (2.4),

$$\sum_{i=1}^{I} |\xi_t(E^i)^+ - (\xi_0 * N_t)(E^i)^+| < IT^{d/4-1/80} \leq T^{d/4-\beta} \tag{7.13}$$

holds on G_2, for large T and $t \in [T/M, T]$. One can, of course, write $\xi_t^B(D)$ as $\sum_{i=1}^{I} \xi_t^B(E^i)$. It therefore follows from (7.13) that, on $G_1 \cap G_2$,

$$\left|\xi_t^B(D) - \sum_{i=1}^{I} (\xi_0 * N_t)(E^i)^+\right| < 2T^{d/4-\beta}, \tag{7.14}$$

for large T.

Let G_3 denote the nonexceptional set in (7.8) of Proposition 7.1. Setting $D = E^i$ there, for each i, and summing over i implies that

$$\left|\sum_{i=1}^{I}(\xi_0 * N_t)(E^i)^+ - \sum_{x\in D}(\xi_0 * N_t)(x)^+\right| < 6^d T^{d/4-2\beta}$$

on G_3. It follows from this and (7.14) that, on $G_1 \cap G_2 \cap G_3$,

$$\left|\xi_t^B(D) - \sum_{x\in D}(\xi_0 * N_t)(x)^+\right| < 3T^{d/4-\beta} \tag{7.15}$$

for large T. By (7.12), (2.4) and (7.8),

$$P((G_1 \cap G_2 \cap G_3)^c) \le 3T^{-\beta} \tag{7.16}$$

for large T. The bounds (7.15) and (7.16) imply (2.3), which is the desired result. □

In Section 9, we will employ the following result. It is an immediate consequence of Theorem 3 and Lemma 7.2. For $D \in \mathscr{D}_{T^\delta}$, $\delta < 1/2$, it gives the behavior of $\xi_T^A(D)$ and $\xi_T^B(D)$ in terms of $(\xi_0 * N_T)(0)^\pm$. [$|D|$ and δ will be chosen large enough so that the term $|D|T^{-(d+\varepsilon)/4}$ in $h(T, D)$ is dominant.]

COROLLARY 7.2. *Let $d < 4$, $\delta < 1/2$ and $\varepsilon = 1/2 - \delta$. Set $h(T, D) = T^{d/4-1/10{,}000} + |D|T^{-(d+\varepsilon)/4}$. Then, for sufficiently large T,*

$$\begin{aligned} &P(|\xi_T^A(D) - |D|(\xi_0 * N_T)(0)^-| \ge h(T, D) \text{ for some } D \in \mathscr{D}_{T^\delta}) \\ &\qquad \le T^{-1/10{,}000} \end{aligned} \tag{7.17}$$

and

$$\begin{aligned} &P(|\xi_T^B(D) - |D|(\xi_0 * N_T)(0)^+| \ge h(T, D) \text{ for some } D \in \mathscr{D}_{T^\delta}) \\ &\qquad \le T^{-1/10{,}000}. \end{aligned} \tag{7.18}$$

In Section 8, we will reformulate the approximations for ξ_t^A and ξ_t^B in (2.2) and (2.3) of Theorem 3 in terms of the convolutions $(\Phi * N_t)^-$ and $(\Phi * N_t)^+$, where Φ is white noise. This will lead to Theorem 4, which is a generalization of Theorem 1. To employ Theorem 3, we need to restate it in its scaled format. Recall (1.6), where ${}^T\hat{\xi}_t(x)$ is defined over $x \in \mathbb{Z}^d_{T^{1/2}}$. The convolution ${}^T\hat{\xi}_0 * N_t$, employed below, is also taken over $\mathbb{Z}^d_{T^{1/2}}$ and is defined in the obvious manner. Since $N_{Tt}(T^{1/2}x) = T^{-d/2}N_t(x)$ always holds, it is easy to check that

$$({}^T\hat{\xi}_0 * N_t)(x) = T^{d/4}(\xi_0 * N_{Tt})(T^{1/2}x). \tag{7.19}$$

One can therefore rewrite Theorem 3 as follows. (We assume here that $D \subset \mathbb{R}^d$, and explicitly write $D \cap \mathbb{Z}^d_{T^{1/2}}$ in the summation to avoid any ambiguity.)

THEOREM 3′. *For $d < 4$, and given $M > 1$,*

$$P\left(\sup_{t\in[1/M,M]}\sup_{D\in\mathscr{D}_M}\left|{}^T\hat{\xi}_t^A(D) - T^{-d/2}\sum_{x\in D\cap\mathbb{Z}^d_{T^{1/2}}}({}^T\hat{\xi}_0 * N_t)(x)^-\right| \geq T^{-1/9,000}\right) \leq T^{-1/9,000} \tag{7.20}$$

and

$$P\left(\sup_{t\in[1/M,M]}\sup_{D\in\mathscr{D}_M}\left|{}^T\hat{\xi}_t^B(D) - T^{-d/2}\sum_{x\in D\cap\mathbb{Z}^d_{T^{1/2}}}({}^T\hat{\xi}_0 * N_t)(x)^+\right| \geq T^{-1/9,000}\right) \leq T^{-1/9,000} \tag{7.21}$$

hold for sufficiently large T.

We point out that, here and in Theorem 3, it is possible to replace the summation in the formulas by the corresponding integrals. We avoid doing this, since ξ_0 (and ${}^T\hat{\xi}_0$) are discrete, which makes the definition of $(\xi_0 * N_t)(x)$ for nonlattice x less natural.

The remainder of the section is devoted to demonstrating Proposition 7.2. The main step for this is Proposition 7.3 below. We will first show how Proposition 7.2 follows from Proposition 7.3, and will then prove Proposition 7.3. We introduce the following notation. The set $\overline{D}^i$, $i = 1, \dots, I$, will denote all points (in $\mathbb{Z}^d$) within distance $\frac{1}{2}T^{\alpha/2}$ of D^i in the max norm. That is, $\overline{D}^i$ is the rectangle centered at the middle of D^i, and having length $2T^{\alpha/2}$. The times t_k, in Proposition 7.3 and later on, will be those given in Proposition 2.2 for $R_T = \lfloor T^{3\alpha/2-1}\rfloor$. (Recall that $\lfloor x\rfloor$ denotes the integral part of x.)

PROPOSITION 7.3. *Let $d < 4$, $\alpha \in [1 - 2/1,000, 1)$, $\beta = (1-\alpha)/16$, and choose $\overline{D}^i$ and t_k as above. Then, for all $M > 1$ and all $k = 1, \dots, K-1$,*

$$P\left(\sum_{i=1}^{I}\xi_{t_k}^m(\overline{D}^i) \geq T^{d/4-\beta}\right) \leq T^{-20\beta} \tag{7.22}$$

for large enough T.

In order to show Proposition 7.2 from Proposition 7.3, we will show that the probability is small that any particle moves from outside $\overline{D}^i$, at time t_k, to inside D^i, over times $[t_k, t_k + T^\gamma]$, for any i. (The constant γ will be chosen so that $\gamma < \alpha$ and $\cup_{k=1}^{K-1}[t_k, t_k + T^\gamma] \supset [T/M, T]$.) For this estimate, we let $W^{i,k}$ denote the number of particles (of either type) which violate this condition, for given i and k.

LEMMA 7.3. *For $\gamma \in (0, \alpha)$ and each k,*

$$P\left(\sum_{i=1}^{I} W^{i,k} \neq 0\right) \leq \exp\{-C_{38}T^{(\alpha-\gamma)\wedge(\alpha/2)}\} \tag{7.23}$$

for sufficiently large T and appropriate $C_{38} > 0$.

PROOF. Let X_t be a continuous time rate-d simple random walk in d dimensions, with $X_0 = 0$. Using moment generating functions and the reflection principle, it is not difficult to show that

$$P\Big(\sup_{t\le u}|X_t| \ge |x|\Big) \le 4d\exp\Big\{-C_{39}|x|\Big(\frac{|x|}{u}\wedge 1\Big)\Big\} \tag{7.24}$$

for each u and appropriate C_{39}. Moreover, at time t_k, each type of particle is dominated by a Poisson random field with intensity λ. Therefore, for given i and k,

$$P(W^{i,k}\ne 0) \le E[W^{i,k}] \le 2\lambda \sum_{|x|\ge T^{\alpha/2}} 4d\exp\Big\{-C_{39}\frac{|x|}{2}\Big(\frac{|x|}{2T^{\gamma}}\wedge 1\Big)\Big\}.$$

For given α and γ, with $\gamma < \alpha$, this is less than or equal to

$$\exp\{-C_{40}T^{\alpha/2}(T^{\alpha/2-\gamma}\wedge 1)\} = \exp\{-C_{40}T^{(\alpha-\gamma)\wedge(\alpha/2)}\}$$

for large T and appropriate C_{40}. Since $I \le (6T^{(1-\alpha)/2})^d$, this gives (7.23), for $C_{38} < C_{40}$, after summing over i. $\square$

We now demonstrate Proposition 7.2, assuming Proposition 7.3.

PROOF OF PROPOSITION 7.2. Let $\gamma \stackrel{\text{def.}}{=} (9/8)\alpha - 1/8 = 1 - 18\beta$. Since $\alpha < 1$, one has $\gamma < \alpha$. Lemma 7.3 implies that, for a given k, $k = 1,\dots,K-1$, no particles move from outside $\overline{D}^i$ to inside D^i, over $\mathscr{T}_k \stackrel{\text{def.}}{=} [t_k, t_k + T^{\gamma}]$, with overwhelming probability. Under this event,

$$\sum_{i=1}^{I}\xi_t^m(D^i) \le \sum_{i=1}^{I}\xi_{t_k}^m(\overline{D}^i) \quad \text{for } t\in[t_k, t_k+T^{\gamma}].$$

One can check that $(\alpha-\gamma)\wedge(\alpha/2) = 2\beta$. So, by Proposition 7.3 and Lemma 7.3,

$$P\Big(\sup_{t\in\mathscr{T}_k}\sum_{i=1}^{I}\xi_t^m(D^i) \ge T^{d/4-\beta}\Big) \le T^{-20\beta} + \exp\{-C_{38}T^{2\beta}\} \tag{7.25}$$

for large enough T and appropriate $C_{38} > 0$.

We wish to extend the range of t in (7.25) to $[T/M, T]$, in order to obtain (7.9). We first note $\mathscr{T}_k$ has length T^{γ}. By assumption, $R_T = \lfloor T^{3\alpha/2-1}\rfloor$, and so, for t_k chosen as in Proposition 2.2,

$$t_k - t_{k-1} \le T^{(3\alpha-1)/2} \quad \text{for all } k.$$

For γ chosen as above, $\gamma > (3\alpha-1)/2$, and so $t_k - t_{k-1} \ll T^{\gamma}$ for large T. Also, $[T/M, T] \subset [t_1, t_K]$, where K is as in Proposition 2.2.

It follows from these observations, that an appropriate collection $\mathscr{T}_{k_1}, \mathscr{T}_{k_2}, \dots$ of at most $2T^{1-\gamma} = 2T^{18\beta}$ of these intervals covers $[T/M, T]$. Applying these

$\mathscr{T}_{k_i}$ in (7.25) and summing the probabilities implies that

$$P\left(\sup_{t\in[T/M,\,T]}\sum_{i=1}^{I}\xi_t^m(D^i)\geq T^{d/4-\beta}\right)\leq 4T^{-2\beta}$$

for large enough T. This gives (7.9). $\square$

We now demonstrate Proposition 7.3. The proposition would be a simple application of Proposition 2.2 and Chebyshev's inequality if one replaced the upper bound $T^{-20\beta}$ in (7.22) by a multiple of $T^{-7\beta}$. The bound $T^{-7\beta}$ is too coarse, however, to apply in the proof of Proposition 7.2, since it needs to be applied over $2T^{18\beta}$ events.

One can get around this problem by covering each of the cubes $\overline{D}^i$ with disjoint cubes $D^{i,j}$, $j=1,\dots,J$, each of length $\lfloor T^{\alpha'/2}\rfloor$, where $\alpha'=3\alpha-2$. We do this with $\cup_j D^{i,j}$ contained in the cube having the same center as $\overline{D}^i$ but with length $6T^{\alpha/2}$, so that $J\leq(7T^{(\alpha-\alpha')/2})^d=(7T^{1-\alpha})^d$, and we center each $D^{i,j}$ so that it is a translate (in $\mathbb{Z}^d$) of $D_{\lfloor T^{\alpha'/2}\rfloor}$. We will apply Proposition 2.2 to each of these cubes $D^{i,j}$, which gives smaller bounds on the exceptional probabilities than if we applied it to $\overline{D}^i$ directly. We will then show that the fluctuation of $\xi_{t_k}^m(D^{i,j})$ over $j=1,\dots,J$, for fixed i, is small enough so that one retains the improved bounds in (7.22) as well, when one replaces $\sum_{i,j}\xi_{t_k}^m(D^{i,j})$ by $\sum_i\xi_{t_k}^m(\overline{D}^i)$.

Before presenting the proof of Proposition 7.3, we first give two preliminary lemmas. The first lemma says that the fluctuation in $\xi_t(D^{i,j})$, between different $D^{i,j}$ with the same i, will typically be small.

LEMMA 7.4. *Let $d<4$, $\alpha\in[1-2/1{,}000,1)$, $\beta=(1-\alpha)/16$ and choose $D^{i,j}$ as above. Then, for all $M>1$,*

$$P\left(\max_j\xi_t(D^{i,j})-\min_j\xi_t(D^{i,j})\geq 3T^{d(2\alpha'-1)/4-2\beta}\right)\leq 2\exp\{-T^{2\beta}\}\tag{7.26}$$

for sufficiently large T, $t\in[T/M,T]$ and all i.

PROOF. Let G_1 denote the nonexceptional set in (7.6), and fix i. Since $|D^{i,j}|=\lfloor T^{\alpha'/2}\rfloor^d$ for all j,

$$\max_j(\xi_0*N_t)(D^{i,j})-\min_j(\xi_0*N_t)(D^{i,j})<T^{d(2\alpha'-1)/4-2\beta}\tag{7.27}$$

on G_1 for each $t\in[T/M,T]$. Let G_2 denote the nonexceptional set in (2.4). Then, for each j,

$$|\xi_t(D^{i,j})-(\xi_0*N_t)(D^{i,j})|<T^{d/4-1/80}\tag{7.28}$$

holds on G_2. Since $\alpha\geq 1-2/1{,}000$, one can check that the bound in (7.27) is larger than that in (7.28). So, (7.27) and (7.28) imply that

$$\max_j\xi_t(D^{i,j})-\min_j\xi_t(D^{i,j})<3T^{d(2\alpha'-1)/4-2\beta}\tag{7.29}$$

on $G_1 \cap G_2$. By (7.6) and (2.4),

$$P((G_1 \cap G_2)^c) \le \exp\{-T^{2\beta}\} + \exp\{-T^{1/42}\} \le 2\exp\{-T^{2\beta}\} \tag{7.30}$$

for large T. The bound in (7.26) follows from (7.29) and (7.30). □

Let H_i denote the set of realizations where, for a given t, $\xi_t(D^{i,j}) \ge 0$ holds for all j or $\xi_t(D^{i,j}) \le 0$ holds for all j. The next lemma says that, on H_i^c, the quantities $\xi_t^m(\overline{D}^i)$ and $\sum_j \xi_t^m(D^{i,j})$ will typically be close.

LEMMA 7.5. *Let $d < 4$, $\alpha \in [1 - 2/1{,}000, 1)$, $\beta = (1-\alpha)/16$, and choose $\overline{D}^i$ and $D^{i,j}$ as above. Then, for all $M > 1$,*

$$P\biggl(\xi_t^m(\overline{D}^i) - \sum_{j=1}^{J} \xi_t^m(D^{i,j}) \ge 3 \cdot 7^d T^{d(2\alpha-1)/4-2\beta}; H_i^c\biggr) \le 2\exp\{-T^{2\beta}\} \tag{7.31}$$

for sufficiently large T, $t \in [T/M, T]$, and all i.

PROOF. On H_i^c, $\max_j \xi_t(D^{i,j}) > 0$ and $\min_j \xi_t(D^{i,j}) < 0$. So, on H_i^c,

$$\max_j |\xi_t(D^{i,j})| < \max_j \xi_t(D^{i,j}) - \min_j \xi_t(D^{i,j}).$$

It therefore follows from Lemma 7.4, that

$$P\Bigl(\max_j |\xi_t(D^{i,j})| \ge 3T^{d(2\alpha'-1)/4-2\beta}; H_i^c\Bigr) \le 2\exp\{-T^{2\beta}\}. \tag{7.32}$$

That is, the numbers of A and B particles are almost the same over each $D^{i,j}$.

One always has that

$$\begin{aligned} \xi_t^m(\overline{D}^i) &\le \xi_t^m\biggl(\bigcup_j D^{i,j}\biggr) = \biggl(\sum_j \xi_t^A(D^{i,j})\biggr) \wedge \biggl(\sum_j \xi_t^B(D^{i,j})\biggr) \\ &\le \sum_j (\xi_t^A(D^{i,j}) \vee \xi_t^B(D^{i,j})) = \sum_j (\xi_t^m(D^{i,j}) + |\xi_t(D^{i,j})|), \end{aligned} \tag{7.33}$$

since $\overline{D}^i \subset \cup_j D^{i,j}$ and the sets $D^{i,j}$, $j = 1, \dots, J$ are disjoint. Off of H_i and the exceptional set in (7.32), this is

$$< \sum_j \xi_t^m(D^{i,j}) + 3JT^{d(2\alpha'-1)/4-2\beta} < \sum_j \xi_t^m(D^{i,j}) + 3 \cdot 7^d T^{d(2\alpha-1)/4-2\beta}. \tag{7.34}$$

Together with (7.32), (7.33) and (7.34) imply (7.31). □

We now prove Proposition 7.3, and hence complete the proof of Theorem 3. The argument combines Proposition 2.2 with Lemma 7.5.

PROOF OF PROPOSITION 7.3. Fix M, and let t_k, $k = 1, \dots, K-1$, be chosen as in Proposition 2.2. We apply (2.6) to each $D^{i,j}$, $i = 1, \dots, I$ and $j = 1, \dots, J$, choosing $R_T = \lfloor T^{\alpha'/2} \rfloor$, and using the translation invariance of ξ_t. Since $\delta_1(T) \le T^{(\alpha'-1)/2} = T^{-24\beta}$, one has

$$E\left[\sum_{i,j} \xi_{t_k}^m(D^{i,j})\right] \le C_1 IJT^{d(2\alpha'-1)/4-24\beta} \le C_1 42^d T^{d/4-24\beta}, \tag{7.35}$$

for large T and appropriate C_1. By Markov's inequality, this implies

$$P\left(\sum_{i,j} \xi_{t_k}^m(D^{i,j}) \ge C_1 42^d T^{d/4-2\beta}\right) \le T^{-22\beta}. \tag{7.36}$$

We wish to replace $\sum_{i,j} \xi_{t_k}^m(D^{i,j})$ with $\sum_i \xi_{t_k}^m(\overline{D}^i)$, in (7.34). On the set H_i defined above Lemma 7.5, for a given t_k,

$$\xi_{t_k}^m(\overline{D}^i) \le \sum_j \xi_{t_k}^m(D^{i,j}) \tag{7.37}$$

clearly holds, since the minimum type over each $D^{i,j}$, $j = 1, \dots, J$, is the same. On H_i^c, we can employ Lemma 7.5 (with the value of M being twice the value chosen here). Summing the exceptional probabilities in (7.31) over $i = 1, \dots, I$, and combining this with (7.37), gives

$$\begin{aligned} &P\left(\sum_i \xi_{t_k}^m(\overline{D}^i) - \sum_{i,j} \xi_{t_k}^m(D^{i,j}) \ge 3 \cdot 42^d T^{d/4-2\beta}\right) \\ &\qquad \le 2 \cdot 6^d T^{d(1-\alpha)/2} \exp\{-T^{2\beta}\} \end{aligned} \tag{7.38}$$

for large T and $k = 1, \dots, K-1$. Together with (7.36), (7.38) implies that

$$P\left(\sum_i \xi_{t_k}^m(\overline{D}^i) \ge T^{d/4-\beta}\right) \le T^{-20\beta},$$

which is (7.22). □

8. Convergence of ${}^T\hat{\xi}_t$ to $(2\lambda)^{1/2}(\Phi * N_t)$. Theorem 1 states that ${}^T\hat{\xi}_t$ converges to $(2\lambda)^{1/2}(\Phi * N_t)$ as $T \to \infty$, where Φ is the mean-0 generalized Gaussian random field with covariance given by (1.3). In the mathematical physics and other literature, Φ is referred to as white noise. In this section, we first discuss white noise and its connection with Brownian sheet. We then demonstrate Theorem 4, which is a generalization of Theorem 1.

Brownian sheet is the higher dimensional analog of Brownian motion. Brownian sheet $W(x)$, with $x = (x_1, \dots, x_d) \in [0, \infty)^d$, is the real-valued Gaussian process with mean 0 and covariances

$$E[W(x)W(y)] = \prod_{j=1}^d (x_j \wedge y_j). \tag{8.1}$$

A version of this process exists where almost all realizations are continuous in x; we will, from now on, automatically choose this version. Various more refined sample path properties of Brownian sheet have been investigated in [16] and the references given there.

One can extend the domain of W from $[0,\infty)^d$ to $\mathbb{R}^d$. One can do this by employing 2^d independent copies $W^1,\dots,W^{2d}$, of W, each defined on $[0,\infty)^d$, where each W^n is identified with a different one of the 2^d orthants. Writing $x^{\mathrm{pos}}=(|x_1|,\dots,|x_d|)$, where $x=(x_1,\dots,x_d)$, one can extend $W(x)$ to $x\in\mathbb{R}^d$ by setting $W(x)=W^n(x^{\mathrm{pos}})$, where W^n is the copy identified with the orthant containing x. As before, almost all realizations of $W(x)$ will be continuous in x.

Let $D\subset\mathbb{R}^d$ be a finite rectangle. Denote by x and y the vertices where all of the coordinates are maximized, respectively, minimized, and, for z any vertex of D, let $\nu(z)$ denote the number of coordinates z shares with y. We set

$$\Phi(D)=\sum_z(-1)^{\nu(z)}W(z). \tag{8.2}$$

When $y=0$, one has $\Phi(D)=W(x)$. The operator Φ defines a mean-0 generalized Gaussian random field, with covariance satisfying

$$E[\Phi(D_1)\Phi(D_2)]=|D_1\cap D_2| \tag{8.3}$$

for pairs of rectangles D_1 and D_2. This is the same expression as (1.3). Thus, (8.2) gives a representation for white noise in terms of Brownian sheet. One may check (8.3) by decomposing D_1 and D_2 into unions of rectangles in the different orthants, and then writing these as differences of rectangles, with each rectangle having the origin as a vertex. One then applies the formula (8.1) to each such pair. The white noise Φ given by (8.2) is almost surely continuous in D as its vertices are varied; we shall henceforth assume this continuity for Φ.

We note that for W defined here, $W(x)=0$ for any $x\in\mathbb{R}^d$ with at least one coordinate equal to 0. Thus, $W(x)$ is "centered" at 0. One can recenter $W(x)$ at any given point y by setting

$$W^y(x)=W(x)-\sum_{j=1}^d g_j(x),$$

where each $g_j(x)$, $j=1,\dots,d$, is an appropriate random function which is constant in its jth coordinate. [First set $g_1(x)=W(x)$ for each x sharing its first coordinate with y, then set $g_2(x)=W(x)-g_1(x)$ for each x sharing its second coordinate with y, etc.] Replacing W by W^y does not change the corresponding operator Φ. [For instance, subtracting $g_j(x)$ from $W(x)$ does not change $\Phi(D)$ in (8.2), since the effect, on the right side, on pairs of vertices differing only in the jth coordinate, cancels out due to the factor $\nu(z)$.]

There exists a unique process V, with domain $\mathbb{Z}^d$ and centered at 0, which corresponds to ξ_0 as W does to Φ. That is,

$$\xi_0(D)=\sum_z(-1)^{\nu(z)}V(z) \tag{8.4}$$

for all rectangles D having vertices $z\in\mathbb{Z}^d$. When x and y are chosen as above (8.2), with $y=0$, one has $\xi_0(D)=V(x)$. As in the definition for ${}^T\hat{\xi}$ in (1.6), we set

$${}^T\widehat{V}(z)=V(T^{1/2}z)/T^{d/4}, \tag{8.5}$$

for $z \in \mathbb{Z}^d_{T^{1/2}}$. It then follows that

$$ {}^T\hat{\xi}_0(D) = \sum_z (-1)^{\nu(z)}\, {}^T\widehat{V}(z) \tag{8.6}$$

for all D having vertices $z \in \mathbb{Z}^d_{T^{1/2}}$.

We want to be able to compare ${}^T\widehat{V}$ with W, when $T \to \infty$. For this, we need to define V at nonlattice points, which we do by interpolating. The most natural way is to use the following scheme. For $d = 1$, the interpolation between y and $y+1$ will be linear. For $d = 2$ and $x \in y + (0, 1]^2$, with $y = (y_1, y_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2)$, set

$$\begin{aligned} V(x) = V(y) &+ \big[V(y + (1,0)) - V(y)\big](x_1 - y_1) \\ &+ [V(y + (0,1)) - V(y)](x_2 - y_2) \\ &+ \big[V(y + (1,1)) - V(y + (1,0)) - V(y + (0,1)) + V(y)\big] \\ &\quad \times (x_1 - y_1)(x_2 - y_2). \end{aligned}$$

This interpolation is linear along the sides of $y + [0,1]^2$, and has a correction term that is proportional to the area of the rectangle given by the opposing vertices y and x. The interpolation for $d > 2$ is defined analogously, with the new volume term, for each added dimension, corresponding to the right side of (8.4).

We will also find it useful to extend $\xi_0(y)$ to all of $\mathbb{R}^d$. We do this by setting ${}_e\xi_0(x) = \xi_0(y)$ for $x \in y - [0,1)^d$ and $y \in \mathbb{Z}^d$. One then has, by (8.4),

$$ {}_e\xi_0(x) = \frac{\partial^d V(x)}{\partial x_1 \cdots \partial x_d}$$

for x with noninteger coordinates. Since ${}_e\xi_0(y)$ will serve the role of a density, the "extension" ${}^T_e\hat{\xi}_0$ to $\mathbb{R}^d$ needs to be scaled differently than ${}^T\hat{\xi}_0$ to be useful. For $x \in \mathbb{R}^d$, we set

$$ {}^T_e\hat{\xi}_0(x) = T^{d/4}\, {}_e\xi_0(T^{1/2}x); \tag{8.7}$$

at $x \in \mathbb{Z}^d_{T^{1/2}}$, one has ${}^T_e\hat{\xi}_0(x) = T^{d/2}\, {}^T\hat{\xi}_0(x)$. This scaling gives

$$ {}^T_e\hat{\xi}_0(x) = \frac{\partial^d\, {}^T\widehat{V}(x)}{\partial x_1 \cdots \partial x_d} \tag{8.8}$$

for x with coordinates not in $\mathbb{Z}_{T^{1/2}}$.

Since W is defined on $\mathbb{R}^d$, convolution with respect to N_t will be interpreted as an integral over $\mathbb{R}^d$, that is,

$$ (W * N_t)(x) = \int_{\mathbb{R}^d} W(x-y) N_t(y)\, dy \qquad \text{for } x \in \mathbb{R}^d. \tag{8.9}$$

Since the growth of $|W(x)|$ can be controlled as $|x| \to \infty$, the integral is almost surely well defined and finite. (One can employ estimates similar to those in the proofs of Lemmas 8.2 and 8.3.) We define ${}^T\widehat{V} * N_t$ in the same way. We will also employ the convolutions $W * N'_t$ and ${}^T\widehat{V} * N'_t$, where

$$N'_t(x) \overset{\text{def.}}{=} \frac{\partial^d N_t(x)}{\partial x_1 \cdots \partial x_d} = (-1)^d \frac{x_1 \cdots x_d}{t^d} N_t(x). \tag{8.10}$$

Using (8.8)–(8.10) and integrating by parts in each direction, one can check that

$$({}^T\widehat{V} * N'_t)(x) = ({}^T_e\hat{\xi}_0 * N_t)(x) \quad \text{for } x \in \mathbb{R}^d. \tag{8.11}$$

It follows from (8.2) that

$$(\Phi * N_t)(D) = \sum_z (-1)^{\nu(z)} (W * N_t)(z), \tag{8.12}$$

where z are the vertices of the rectangle D. Using (8.1), one can check that $\Phi * N_t$ scales according to

$$(\Phi * N_t)(D) = M^{-d/4} (\Phi * N_{Mt})(M^{1/2} D), \tag{8.13}$$

for $M > 0$. We write $(\Phi * N_t)(x)$ for the density of $(\Phi * N_t)(D)$. Then,

$$(\Phi * N_t)(x) = (W * N'_t)(x); \tag{8.14}$$

(8.14) can be thought of as a formal d-fold integration by parts. It follows from (8.14) that $(\Phi * N_t)(x)$ is continuous in (t, x) for almost all realizations. One can check that since W is a family of Gaussian random variables, so is $\Phi * N$. A simple computation shows that, for any (t_1, x_1) and (t_2, x_2),

$$E[(\Phi * N_{t_1})(x_1)(\Phi * N_{t_2})(x_2)] = \int_{\mathbb{R}^d} N_{t_1}(x_1 - y) N_{t_2}(x_2 - y)\, dy. \tag{8.15}$$

For $t_1 = t_2 = t$, this equals $(4\pi t)^{-d/2} \exp\{-|x_1 - x_2|^2/4t\}$. In particular, $\sigma^2(\Phi * N_t)(x) = (4\pi t)^{-d/2}$.

The main result in this section is Theorem 4, which is a stronger version of Theorem 1. The main tools for demonstrating the theorem are Theorem 3′ of Section 7 and Proposition 8.1. Let $\overline{D}_M = [-M/2, M/2]^d$, for $M > 0$. (We will use this notation throughout the remainder of the section.) Proposition 8.1 states that $({}^T_e\hat{\xi}_0 * N_t)(x)$, with $(t, x) \in (0, 1] \times \overline{D}_1$, converges weakly to $(2\lambda)^{1/2}(\Phi * N_t)(x)$ as $T \to \infty$. Convergence is with respect to the uniform topology on compact sets of $C((0, 1] \times \overline{D}_1)$, the space of continuous functions on $(0, 1] \times \overline{D}_1$.

PROPOSITION 8.1. *For all d,*

$$({}^T_e\hat{\xi}_0 * N_\cdot)(\cdot) \Rightarrow (2\lambda)^{1/2} (\Phi * N_\cdot)(\cdot) \quad \text{as } T \to \infty, \tag{8.16}$$

on $(0, 1] \times \overline{D}_1$.

Since the demonstration of Proposition 8.1 requires several steps, it will be postponed until the latter part of the section. The proposition has two useful corollaries. The first will be employed in Section 9.

COROLLARY 8.1. *For all d,*

$$T^{d/4}(\xi_0 * N_T)(0) \Rightarrow (2\lambda)^{1/2}(4\pi)^{-d/4} Z_{0,1} \quad \text{as } T \to \infty, \tag{8.17}$$

where $Z_{0,1}$ has the standard normal distribution.

PROOF. By the observations immediately above and below (8.15), $(\Phi * N_1)(0)$ is normally distributed with mean 0 and variance $(4\pi)^{-d/2}$. Also, using (8.7), one can check that

$$({}_e^T\hat{\xi}_0 * N_1)(0) = T^{d/4}({}_e\xi_0 * N_T)(0). \tag{8.18}$$

So, (8.17) will follow from (8.16) and (8.18) once we show that

$$T^{d/4}[(\xi_0 * N_T)(0) - ({}_e\xi_0 * N_T)(0)] \to 0 \quad \text{as } T \to \infty. \tag{8.19}$$

(These two convolutions are not identical, since $\xi_0 * N_T$ is a sum over $\mathbb{Z}^d$, whereas ${}_e\xi_0 * N_T$ is an integral over $\mathbb{R}^d$.) Since ${}_e\xi_0(x) = \xi_0(y)$ for $x \in y - [0,1)^d$ and $y \in \mathbb{Z}^d$, $({}_e\xi_0 * N_T)(0)$ is the average of $(\xi_0 * N_T)(z)$ over $z \in [0,1)^d$. By Lemma 7.2, with $\alpha = 1/2$, all of these values are, off of a set of probability $\exp\{-T^{1/16}\}$, within $T^{-d/4-1/16}$ of $(\xi_0 * N_T)(0)$, and hence so is $({}_e\xi_0 * N_T)(0)$. This implies (8.19). □

The space $C((0,1]\times D_1)$ admits a complete, separable metric. Consequently, weak convergence in (8.16) implies the corresponding convergence in probability, if, for each T, ξ_0 and Φ are coupled appropriately (see, e.g., [5], Theorem 3.3). That is,

$$({}_e^T\hat{\xi}_0 * N_t)(x) - (2\lambda)^{1/2}(\Phi * N_t)(x) \to 0 \quad \text{in probability as } T \to \infty,$$

uniformly in (t, x) on compact sets. Integration of $({}_e^T\hat{\xi}_0 * N)(x)$ and $(\Phi * N)(x)$ over rectangles $D \subset D_1$, in the d space variables, immediately produces the following result. By $({}_e^T\hat{\xi} * N_t)^{\pm}(D)$ and $(\Phi * N_t)^{\pm}(D)$, we mean the functions $({}_e^T\hat{\xi} * N_t)(x)^{\pm}$, respectively, $(\Phi * N_t)(x)^{\pm}$, integrated over $x \in D$.

COROLLARY 8.2. *Fix d, $M > 1$ and $\varepsilon > 0$. For each T, there exist copies of ξ_0 and Φ, so that*

$$\lim_{T\to\infty} P\left(\sup_{t\in[1/M,1]}\sup_{D\in\mathscr{D}_1}\left|({}_e^T\hat{\xi}_0 * N_t)^{-}(D) - (2\lambda)^{1/2}(\Phi * N_t)^{-}(D)\right| > \varepsilon\right) = 0 \tag{8.20}$$

and

$$\lim_{T\to\infty} P\left(\sup_{t\in[1/M,1]}\sup_{D\in\mathscr{D}_1}\left|({}_e^T\hat{\xi}_0 * N_t)^{+}(D) - (2\lambda)^{1/2}(\Phi * N_t)^{+}(D)\right| > \varepsilon\right) = 0. \tag{8.21}$$

Theorem 3′, in Section 7, states that, for large T in $d < 4$, ${}^T\hat{\xi}_t^A(D)$ and ${}^T\hat{\xi}_t^B(D)$ are, over $t \in [1/M, M]$ and $D \in \mathscr{D}_M$, uniformly approximated by $\sum_{x\in D\cap\mathbb{Z}^d_{T^{1/2}}}({}^T\hat{\xi}_0 * N_t)(x)^-$, respectively, $\sum_{x\in D\cap\mathbb{Z}^d_{T^{1/2}}}({}^T\hat{\xi}_0 * N_t)(x)^+$. Putting this together with Corollary 8.2 produces the following uniform approximations on $\xi_t^A(D)$ and $\xi_t^B(D)$.

THEOREM 4. *Let $d < 4$, and fix $M > 1$ and $\varepsilon > 0$. For each T, there exist copies of ξ_0 and Φ, so that*

$$\lim_{T\to\infty} P\Big(\sup_{t\in[1/M,\,M]}\sup_{D\in\mathscr{D}_M}\big|{}^T\hat{\xi}^A_t(D)-(2\lambda)^{1/2}(\Phi*N_t)^-(D)\big|>\varepsilon\Big)=0 \tag{8.22}$$

and

$$\lim_{T\to\infty} P\Big(\sup_{t\in[1/M,\,M]}\sup_{D\in\mathscr{D}_M}\big|{}^T\hat{\xi}^B_t(D)-(2\lambda)^{1/2}(\Phi*N_t)^+(D)\big|>\varepsilon\Big)=0. \tag{8.23}$$

By fixing t and D, one obtains Theorem 1 as a special case of Theorem 4. One can also phrase Theorem 4 in terms of weak convergence, if one wishes. In (8.22) and (8.23), it suffices to consider those D with a vertex at the origin; the four quantities in (8.22), (8.23) can each be written as a function of the opposite vertex. The limits can then be formulated in terms of weak convergence, with respect to the uniform topology on compact sets, of continuous functions from $(0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}^2$.

We now demonstrate Theorem 4. The main estimates that are needed are supplied by Theorem 3′ and Corollary 8.2. One also needs to do some tedious but straightforward comparisons to coordinate these estimates.

PROOF OF THEOREM 4. Since the arguments are the same for both parts, we will just show (8.23). Note that

$$ {}^T\hat{\xi}_t(D)=M^{-d/4}\,{}^{T/M}\hat{\xi}_{Mt}(M^{1/2}D)\quad\text{for } M>1. \tag{8.24}$$

Using (8.13) and (8.24), it is enough to show (8.23) for $t\in[1/M,1]$ and $D\in\mathscr{D}_1$. We claim, it suffices to show this for $D'\in\mathscr{D}_1$, with D' having vertices in $\mathbb{Z}^d_{T^{1/2}}$; we denote the set by $\mathscr{D}_{1,\,T^{1/2}}$. To see this, note that for any $D\in\mathscr{D}_1$, there is a $D'\in\mathscr{D}_{1,\,T^{1/2}}$, with $D'\cap\mathbb{Z}^d_{T^{1/2}}=D\cap\mathbb{Z}^d_{T^{1/2}}$, so that the volume of the symmetric difference $D\triangle D'$ is at most $2d/T^{1/2}$. Since $(\Phi*N_t)(x)$ is continuous in (t,x) (for almost all realizations), it is bounded on $[1/M,1]\times D_1$, and so the same is true for $(\Phi*N_t)^+(D\triangle D')/|D\triangle D'|$, for $|D\triangle D|>0$. Replacing D by D', in (8.23), thus changes the second term by a random multiple of $T^{-1/2}$, and leaves the first term unchanged.

The display (8.23), with $D\in\mathscr{D}_{1,\,T^{1/2}}$ replacing $D\in\mathscr{D}_1$, will follow immediately from Theorem 3′ and Corollary 8.2, once we have shown that

$$\begin{aligned}\sup_{t\in[1/M,1]}\sup_{D\in\mathscr{D}_{1,\,T^{1/2}}}\Big|T^{-d/2}&\sum_{y\in D\cap\mathbb{Z}^d_{T^{1/2}}}({}^T\hat{\xi}_0*N_t)(y)^+\\ &-\int_D({}^T_e\hat{\xi}_0*N_t)(x)^+\,dx\Big|\to 0\end{aligned} \tag{8.25}$$

in probability as $T\to\infty$. The reasoning here is similar to that for (8.19), in the proof of Corollary 8.1. The convolutions on the left and on the right are somewhat different, since ${}^T_e\hat{\xi}_0$ on the left is defined on $\mathbb{Z}^d_{T^{1/2}}$ (and so $*$ is defined as a sum), whereas ${}^T\hat{\xi}_0$ on the right has been extended to $\mathbb{R}^d$ (and so $*$ is defined an an integral). Since ${}_e\xi_0(x) = \xi_0(y)$ for $x \in y - [0,1)^d$ and $y \in \mathbb{Z}^d$, one can check that $({}^T_e\hat{\xi}_0 * N_t)(x)$ is the average of $({}^T\hat{\xi}_0 * N_t)(z)$ over $z \in x + [0, T^{-1/2})^d$. By (7.19) and Lemma 7.2, with $\alpha = 1/2$, all of these values are, off of a set of probability $\exp\{-T^{1/16}\}$ (not depending on t or x), within $T^{-1/16}$ of $({}^T\hat{\xi}_0 * N_t)(y)$, and hence so is $({}^T_e\hat{\xi}_0 * N_t)(x)$. Integration over D produces an error that is at most $T^{-1/16}$. This implies (8.25). □

We now set out to show Proposition 8.1. The basic idea is as follows. By the invariance principle in Proposition 8.2, ${}^T\widehat{V}(\cdot) \Rightarrow (2\lambda)^{1/2}W(\cdot)$ as $T \to \infty$. Convolution by N' will be a continuous map if one truncates the tail of N', and the error involved in this truncation can be made as small as desired. By employing a standard weak convergence result, one can therefore show that

$$({}^T\widehat{V} * N'_\cdot)(\cdot) \Rightarrow (2\lambda)^{1/2}(W * N'_\cdot)(\cdot) \tag{8.26}$$

on $(0,1] \times \overline{D}_1$. On account of (8.11) and (8.14), this is equivalent to (8.16).

In Proposition 8.2, the domains of ${}^T\widehat{V}$ and W are each $\mathbb{R}^d$. Convergence is with respect to the uniform topology on compact sets of $C(\mathbb{R}^d)$.

PROPOSITION 8.2. *For all d,*

$$ {}^T\widehat{V}(\cdot) \Rightarrow (2\lambda)^{1/2}W(\cdot) \quad \text{as } T \to \infty. \tag{8.27}$$

The proof of Proposition 8.2, for general d, is similar to the proof for $d = 1$, which is a special case of the standard invariance principle. Rather than go into the proof in detail, we will briefly discuss related results in [13]. We will also summarize the key steps for $d = 1$, in [4], and will indicate how they extend to general d.

For $d = 1$, Proposition 8.2 is just a special case of the invariance principle, since $\xi_0(x)$, $x \in \mathbb{Z}$, are i.i.d. with mean 0 and variance 2λ. For $d = 2$, the analog of (8.27), on $[0,1]^2$, is shown in Theorem 3 of [13] for i.i.d. random variables with finite variance. By intersecting $\overline{D}_M$, $M > 0$, with each of the four quadrants and treating each of them separately, the problem in Proposition 8.2 for $d = 2$ reduces to this setting. The extension of the invariance principle to $d \geq 3$ is briefly discussed in [13]. Theorem 3 in [13] is an application of an earlier result in the paper on the convergence of Banach-valued random variables to the corresponding Banach-valued Brownian motion. In the proof of the theorem, the dimension of the parameter space is effectively lowered from 2 to 1 by treating the evolution of the process along one of the directions as the state of a corresponding process, whose state space consists of continuous functions on $[0,1]$ with the uniform topology. The extension, from d to $d+1$, involves a similar induction argument. It is sketched in [13].

If one wishes, one can instead show Proposition 8.2 directly, using reasoning corresponding to that given in [4] for the one-dimensional case. The two main steps are as follows. One first shows convergence of the joint distributions. This part proceeds as in $d = 1$, with the extension of the dimension not affecting the argument. One then needs to show tightness of the sequence of probability measures. For this, one can apply the analog of Theorem 8.3 of [4]. The main condition in Theorem 8.3 is that, for fixed $\varepsilon > 0$, the probability of an oscillation, of size at least ε, occurring over any interval of length δ, converges to 0 as $\delta \to 0$. The analog of this condition in our setting, where such intervals are replaced by cubes of length δ, is not difficult to show. Since $\xi_0(x)$ is the difference of two independent mean-λ Poisson distributions at each x, it is symmetric. So, using the reflection principle (as in Lemma 8.3 below), one can show that the probability of a large fluctuation occurring in a cube is only a fixed multiple of the probability of ${}^T\widehat{V}$ attaining a large value at one of the vertices. One can show this is small by employing second moments together with Chebyshev's inequality (as in Lemma 8.2).

We will break the work in showing (8.26), and hence Proposition 8.1, into the following two steps. Lemma 8.1 is the analog of (8.26), with the integral associated with $*$ restricted to the domain $\overline{D}_M$. Convergence is with respect to the uniform topology on compact sets for $C((0,1] \times D_1)$.

LEMMA 8.1. *For all d and M,*

$$\int_{\overline{D}_M} {}^T\widehat{V}(y) N'_\cdot(\cdot - y)\, dy \Rightarrow (2\lambda)^{1/2} \int_{\overline{D}_M} W(y) N'_\cdot(\cdot - y)\, dy \quad \textit{as} \quad T \to \infty. \tag{8.28}$$

The other step says that the contribution to the integral associated with $*$ is insignificant off of the set $\overline{D}_M$.

LEMMA 8.2. *For all d and $\varepsilon > 0$,*

$$P\left(\sup_{t\in[0,1]} \sup_{x\in\overline{D}_1} \int_{\overline{D}_M^c} |{}^T\widehat{V}(y) N'_t(x-y)|\, dy \geq \varepsilon \right) \to 0 \tag{8.29}$$

uniformly in T as $M \to \infty$. Similarly,

$$P\left(\sup_{t\in[0,1]} \sup_{x\in\overline{D}_1} \int_{\overline{D}_M^c} |W(y) N'_t(x-y)|\, dy \geq \varepsilon \right) \to 0 \tag{8.30}$$

as $M \to \infty$.

The proof of Proposition 8.1 is immediate from Lemmas 8.1 and 8.2. Together, (8.28)–(8.30) imply that

$$\int_{\mathbb{R}^d} {}^T\widehat{V}(y) N'_\cdot(\cdot - y)\, dy \Rightarrow (2\lambda)^{1/2} \int_{\mathbb{R}^d} W(y) N'_\cdot(\cdot - y)\, dy, \tag{8.31}$$

where $\Rightarrow$ denotes weak convergence with respect to the uniform topology on compact subsets of $C((0,1] \times \overline{D}_1)$. This is (8.26), which is equivalent to (8.16).

PROOF OF LEMMA 8.1. For fixed $M > 0$, let Ξ denote the linear map from the space of continuous functions on $\overline{D}_M$ to the space of continuous functions on $(0,1] \times \overline{D}_1$, which is given by

$$(\Xi(g))(t,x) = \int_{\overline{D}_M} g(y) N'_t(x-y)\, dy.$$

Denote by Ξ^L the map Ξ where the domain of $\Xi(g)$ is restricted to $[1/L, 1] \times \overline{D}_1$. We write $\|\cdot\|_{\overline{D}_M}$ and $\|\cdot\|_{[1/L,1]\times\overline{D}_1}$ for the uniform metrics on the spaces of continuous functions on $\overline{D}_M$ and $[1/L, 1] \times \overline{D}_1$, respectively. Since

$$\|\Xi^L(g)\|_{[1/L,1]\times\overline{D}_1} \le C(L)\|g\|_{\overline{D}_M}$$

for all g and appropriate $C(L)$, the map Ξ^L is continuous. Hence, so is Ξ.

We know from Proposition 8.2 that ${}^T\widehat{V}(\cdot) \Rightarrow (2\lambda)^{1/2}W(\cdot)$ as $T \to \infty$. Since Ξ is continuous, it follows from a standard result on weak convergence that

$$(\Xi({}^T\widehat{V}))(\cdot,\cdot) \Rightarrow (2\lambda)^{1/2}(\Xi(W))(\cdot,\cdot) \quad \text{as } T \to \infty. \tag{8.32}$$

(See, e.g., Theorem 5.1 of [4].) The limit (8.32) is equivalent to (8.28). $\square$

In order to demonstrate Lemma 8.2, we first need the following bounds on $\sup_{|y|\le j} |{}^T\widehat{V}(y)|$, $j \in \mathbb{Z}^+$, and $\sup_{|y|\le r} |W(y)|$, $r \in \mathbb{R}^+$. Both bounds are repeated applications of the reflection principle, and employ Chebyshev's inequality with the second moments of ${}^T\widehat{V}$ and W. Here, $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^d$ and $|\cdot|_\infty$ denotes the max norm on $\mathbb{R}^d$.

LEMMA 8.3. *For all $d, \varepsilon > 0$ and $j \in \mathbb{Z}^+$,*

$$P\Big(\sup_{|y|_\infty \le j} |{}^T\widehat{V}(y)| \ge \varepsilon\Big) \le 2\cdot 4^d P({}^T\widehat{V}(\mathbf{1}j) \ge \varepsilon). \tag{8.33}$$

Similarly, for all $r > 0$,

$$P\Big(\sup_{|y|_\infty \le r} |W(y)| \ge \varepsilon\Big) \le 2\cdot 4^d P(W(\mathbf{1}r) \ge \varepsilon). \tag{8.34}$$

PROOF. Since the proofs are similar, we demonstrate just (8.34). We first note that it is enough to show

$$P\Big(\sup_{y\in H_r} W(y) \ge \varepsilon\Big) \le 2^d P(W(\mathbf{1}r) \ge \varepsilon), \tag{8.35}$$

where H_r is the set of y with $|y|_\infty \le r$ and having nonnegative coordinates. (The different orthants contribute an additional factor 2^d, and the absolute value contributes the factor 2.)

In order to show (8.35), we repeatedly apply the reflection principle. Let H_r^1 denote the subset of points $y \in H_r$, $y = (y_1,\dots,y_d)$, with $y_1 = r$. We will show that

$$P\Big(\sup_{y\in H_r} W(y) \ge \varepsilon\Big) \le 2P\Big(\sup_{y\in H_r^1} W(y) \ge \varepsilon\Big). \tag{8.36}$$

To obtain (8.36), set

$$Y_1 = \inf\{y_1\colon W(y) = \varepsilon \text{ for some } y \in H_r\} \wedge r. \tag{8.37}$$

Also, let $Y = (Y_1,\dots,Y_d)$ denote the smallest point in H_r at which this occurs (ordering y_1 first, then $y_2,\dots,$ down through y_d). One has $W(Y) = \varepsilon$ unless

$Y_1 = r$. Set $Y' = (r, Y_2, \dots, Y_d)$. When the first coordinate is varied, with the other coordinates remaining fixed, the increments of $W(y)$ are independent, and so it follows by symmetry that

$$P(W(Y') \geq W(Y)) \geq 1/2. \tag{8.38}$$

This implies (8.36).

Proceeding inductively, one can continue in the same manner as above. Starting on H_r^{i-1}, the $d-i+1$-dimensional face, where the first $i-1$ coordinates are all fixed and equal r, one can reflect, in the ith direction, to obtain

$$P\Bigl(\sup_{y\in H_r^{i-1}} W(y) \geq \varepsilon\Bigr) \leq 2P\Bigl(\sup_{y\in H_r^{i}} W(y) \geq \varepsilon\Bigr). \tag{8.39}$$

Continuing until $i = d$, where $H_r^d = \{\mathbf{1}r\}$, one obtains (8.35) after putting the inequalities together.

We point out that we are implicitly employing the strong Markov property here. To justify its application for $i = 1$, for example, let $W_1(y_1)$, $y_1 \in [0, r]$, denote the process whose value at y_1 is the map W^1 from $[0, r]^{d-1}$ to $\mathbb{R}$, with

$$W^1(y_2, \dots, y_d) = W(y_1, \dots, y_d),$$

that is, $W_1(y_1)$ is given by the "slice" of W taken where the first coordinate is y_1. Assign the uniform topology on $C([0, r]^{d-1})$ to these states. Then, one can check that $W_1(y_1)$ is Feller continuous, and is hence strong Markov. The justification for the other steps is analogous. □

Employing Lemma 8.3, we now show Lemma 8.2. This will complete the proof of Proposition 8.1.

PROOF OF LEMMA 8.2. Since the proofs of (8.29) and (8.30) are similar, we will do just (8.30). We first note that $W(\mathbf{1}j)$ is normally distributed with second moment j^d. So, by Chebyshev's inequality,

$$P(W(\mathbf{1}j) \geq e^j) \leq j^d/e^{2j}. \tag{8.40}$$

Set $E_j = \{y: |y|_\infty \in (j-1, j]\}$. It follows from (8.34) and (8.40) that

$$P\Bigl(\sup_{y\in E_j} |W(y)| \geq e^j\Bigr) \leq 2(4j)^d/e^{2j}. \tag{8.41}$$

For $x \in \overline{D}_1$, $y \in E_j$ and $j \geq 4$, one has $|x-y|_\infty \geq j/2$. It is therefore easy to check that, for large enough j,

$$\sup_{t\in[0,1]} \sup_{\substack{x\in\overline{D}_1\\ y\in E_j}} |N_t'(x-y)| < e^{-j^2/9}. \tag{8.42}$$

Together with (8.41), (8.42) implies that

$$P\Bigl(\sup_{t\in[0,1]} \sup_{x\in\overline{D}_1} \int_{E_j} |W(y)N_t'(x-y)|\,dy \geq (2j)^d e^{j-j^2/9}\Bigr) \leq 2(4j)^d/e^{2j} \tag{8.43}$$

for large enough j. Since $\cup_{j=\lfloor M\rfloor/2}^{\infty} E_j \supset \overline{D}_M^c$, it follows from (8.43) that

$$P\left(\sup_{t\in[0,1]}\sup_{x\in\overline{D}_1}\int_{\overline{D}_M^c}|W(y)N_t'(x-y)|dy \geq e^{-M^2/40}\right) \leq 2^{-M},$$

for large enough M. This clearly implies (8.30). □

9. Local behavior of ξ_t. In this section, we are interested in the local (or microscopic) behavior of ξ_t for large t, after space has been appropriately scaled. On account of (1.8) [or (1.2)], the scaling given in (1.10), by $\check{\xi}_t(E) = \xi_t(t^{1/4}E)$, is the right scaling. The goal here is to demonstrate Theorem 2, which states that $\check{\xi}_t$ converges to a mixture of Poisson random fields.

We will divide the work needed for Theorem 2 into two main steps. In Proposition 9.1, we break the evolution of ξ_r, $r \in [0, t]$, into the time intervals $[0, s]$ and $[s, t]$, where $t = s + s^\alpha$ and $\alpha > 0$. (Later on, we will choose α to be slightly less than 1.) We examine there the behavior of ${}_s\tilde{\eta}_r$, which will be a slight variant of the process ${}_s\eta_r$ defined in Section 2, where the annihilation between particles is suppressed starting at time s. The interval $[s, t]$ has been chosen so that it is (1) long enough so that particles will mix locally to form Poisson random fields, but (2) short enough so that the density changes insignificantly. In Proposition 9.2, we restore the annihilation between particles over $[s, t]$. On account of (2), ξ_t and ${}_s\tilde{\eta}_t$, will typically be the same locally. It will follow from Proposition 9.2, that the Laplace functionals of ξ_t converge to the desired limits, which implies Theorem 2.

In order to show Propositions 9.1 and 9.2, we employ the following two lemmas. For Lemma 9.1, we partition $\mathbb{R}^d$ by cubes $D^1, D^2, \ldots$, each having length $\lfloor s^\beta \rfloor$, where $\beta \in (0, \alpha/2)$. We then have $|D^i| = \lfloor s^\beta\rfloor^d$ for each i; the exact choice of the translates does not matter. (Later on, β will be slightly less than 1/2, with $\beta = \alpha - 1/2$.) The following bound limits the local fluctuations of $K_{s^\alpha}(x)$ as x varies. [Recall that $K_t(x)$ is the random walk kernel introduced in Section 2.]

LEMMA 9.1. *Fix d. For $\alpha > 0$, $\beta \in (0, \alpha/2)$ and large s,*

$$\sum_{i=1}^{\infty}\max_{y,\,y'\in D^i}|K_{s^\alpha}(x-y) - K_{s^\alpha}(x-y')| \leq C_{41}s^{-\alpha/2-(d-1)\beta} \tag{9.1}$$

holds for all x and appropriate C_{41}.

PROOF. It follows without difficulty from (4.13), that for large s,

$$\sum_{i=1}^{\infty}\max_{y,\,y'\in D^i}|N_{s^\alpha}(x-y) - N_{s^\alpha}(x-y')| \leq C_{42}s^{-\alpha/2-(d-1)\beta} \tag{9.2}$$

holds for all x and appropriate C_{42}. It also follows immediately from (4.8) that

$$\sum_{i=1}^{\infty} \max_{y \in D^i} |N_{s^\alpha}(x-y) - K_{s^\alpha}(x-y)| \le C_{11} s^{-\alpha/2-d\beta} \tag{9.3}$$

for all x. Together, (9.2) and (9.3) imply (9.1). $\square$

We will employ Lemma 9.1 in Proposition 9.1 in the form of the following corollary. We need the following terminology. Set

$$m^i(x) = \min_{y \in D^i} K_{s^\alpha}(x-y), \qquad M^i(x) = \max_{y \in D^i} K_{s^\alpha}(x-y), \tag{9.4}$$

where D^i, $i = 1, 2, \ldots,$ are given above. Also, let I denote the smallest set of indices of these cubes so that $D_{2s^\gamma} \subset \cup_{i \in I} D^i$, where $\gamma \in (\alpha/2, 1]$ is fixed. (Later on, γ will be slightly less than 1/2.)

COROLLARY 9.1. *Fix d. For $\alpha > 0$, $\beta \in (\alpha/4, \alpha/2)$, $\gamma \in (\alpha/2, 1]$ and large s,*

$$(1 - C_{43} s^{-\alpha/2+\beta}) s^{-d\beta} \le \sum_{i \in I} m^i(x) \le \sum_{i=1}^{\infty} M^i(x) \le (1 + C_{43} s^{-\alpha/2+\beta}) s^{-d\beta} \tag{9.5}$$

holds for all $x \in D_{s^\gamma}$ and appropriate C_{43}.

PROOF. We consider the lower bound. Let $A^i(x)$ denote the average of $K_{s^\alpha}(x-y)$ over $y \in D^i$. By (9.1),

$$\sum_{i=1}^{\infty} (A^i(x) - m^i(x)) \le C_{41} s^{-\alpha/2-(d-1)\beta} \tag{9.6}$$

holds for large s. On the other hand, since $|D^i| \le s^{d\beta}$ for all i,

$$\sum_{i=1}^{\infty} A^i(x) \ge s^{-d\beta} \sum_{y \in \mathbb{Z}^d} K_{s^\alpha}(y) = s^{-d\beta}. \tag{9.7}$$

Also, using a simple large deviations estimate, one can check that

$$\sum_{i \notin I} A^i(x) \le 2 s^{-d\beta} \sum_{y \notin D_{s^\gamma}} K_{s^\alpha}(y) \le \exp\{-C_{44} s^{2\gamma-\alpha}\}$$

for large s, $x \in D_{s^\gamma}$ and appropriate C_{44}. Since $\gamma > \alpha/2$, this last term goes to 0 quickly as $s \to \infty$. Together, the above three estimates imply the lower bound in (9.5). Only (9.6) and (9.7) are needed for the upper bound, which follows in the same manner with the inequalities reversed. (The upper bound holds for all x.) $\square$

The process ${}_s\tilde{\eta}_r$ alluded to earlier is the same as ${}_s\eta_r$, except that at time s, one kills all of the particles outside $\cup_{i \in I} D^i$, where D^i and I are specified before Corollary 9.1. Over $(s, t]$, the process evolves without interaction between particles. The following lemma says that this modification will typically not affect the configuration of particles in D_{s^γ} at time t, which contains the regions we are interested in.

LEMMA 9.2. *Fix d. For $\alpha > 0$ and $\gamma \in (\alpha/2, 1]$,*

$$P({}_s\tilde{\eta}_t^A(x) \neq {}_s\eta_t^A(x) \text{ for some } x \in D_{s^\gamma}) \leq \exp\{-C_{45}s^{2\gamma-\alpha}\} \tag{9.8}$$

and

$$P({}_s\tilde{\eta}_t^B(x) \neq {}_s\eta_t^B(x) \text{ for some } x \in D_{s^\gamma}) \leq \exp\{-C_{45}s^{2\gamma-\alpha}\} \tag{9.9}$$

for large s and appropriate C_{45}.

PROOF. Since $D_{2s^\gamma} \subset \cup_{i\in I} D^i$, it suffices, for each case, to calculate an upper bound on the expected number of particles in $D_{2s^\gamma}^c$ at time s, which are in D_{s^γ} at time t, for the process η_r. One can then apply Markov's inequality. The configurations of A and B particles at time s are dominated by Poisson random fields with intensity λ. So, the argument reduces to elementary large deviation estimates on the probability of a particle moving distance greater than s^γ over the time interval $[s, s+s^\alpha]$. The estimates required here are similar to those in Lemma 7.3. □

Proposition 9.1 provides information on the behavior near 0 of ${}_s\tilde{\eta}_t$. (The replacement of ${}_s\eta_t$ by ${}_s\tilde{\eta}_t$ simplifies the reasoning somewhat.) The main tools in the proof of Proposition 9.1 are Corollaries 7.2 and 9.1. Corollary 7.2 allows us to approximate $\xi_s^A(D^i)$ and $\xi_s^B(D^i)$ by $\lfloor s^\beta \rfloor^d (\xi_0 * N_s)(0)^-$ and $\lfloor s^\beta \rfloor^d \times (\xi_0 * N_s)(0)^+$ for all $i \in I$, since both β and γ will be slightly less than 1/2. On account of Corollary 9.1, if one ignores annihilations over $(s, t]$, the probabilities of such particles being at a given site $x \in D_{s^\gamma}$ at time t do not depend much on their exact locations within D^i at time s. Together with some approximation, this behavior will imply (9.10) and (9.11) as $t \to \infty$.

PROPOSITION 9.1. *Assume that $d < 4$, and set $\alpha = 1 - 10^{-5}$ and $\gamma = 1/2 - 10^{-6}$. Then, for $f \in C_c^+(\mathbb{R}^d)$,*

$$\begin{aligned} &E\left[\exp\left\{-\sum_{x\in\mathbb{Z}^d} f(x/t^{1/4})\,{}_s\tilde{\eta}_t^A(x)\right\}\,\middle|\,\mathscr{F}_0\right] \\ &\quad - \exp\left\{t^{d/4}(\xi_0 * N_t)(0)^- \int_{\mathbb{R}^d}(e^{-f(x)}-1)\,dx\right\} \\ &\quad \to 0 \quad \text{in probability as } t\to\infty \end{aligned} \tag{9.10}$$

and

$$\begin{aligned} &E\left[\exp\left\{-\sum_{x\in\mathbb{Z}^d} f(x/t^{1/4})\,{}_s\tilde{\eta}_t^B(x)\right\}\,\middle|\,\mathscr{F}_0\right] \\ &\quad - \exp\left\{t^{d/4}(\xi_0 * N_t)(0)^+ \int_{\mathbb{R}^d}(e^{-f(x)}-1)\,dx\right\} \\ &\quad \to 0 \quad \text{in probability as } t\to\infty. \end{aligned} \tag{9.11}$$

PROOF. We will demonstrate just (9.11), since the argument for (9.10) is the same. Set $\beta = 1/2-10^{-5}$, and let $D^1, D^2, \ldots$ denote the cubes of length $\lfloor s^\beta \rfloor$ and I the set of indices that were introduced above. Then, $D^I \overset{\text{def.}}{=} \cup_{i\in I} D^i \subset D_{3s^\gamma}$. Also, set $\varepsilon = 10^{-6}$, and let H_s denote the set of realizations where

$$|\xi_s^B(D^i) - \lfloor s^\beta \rfloor^d (\xi_0 * N_s)(0)^+| < s^{d\beta-(d+\varepsilon/2)/4} \tag{9.12}$$

for all $i \in I$. Since $\gamma < 1/2$, it follows from Corollary 7.2 that

$$P(H_s) \to 1 \quad \text{as } s \to \infty. \tag{9.13}$$

Using (9.13), we first obtain upper bounds for the left side of (9.11). Since the particles of ${}_s\tilde{\eta}_r$ execute independent random walks over $(s, t]$, and ${}_s\tilde{\eta}_s = \xi_s$ on D^I,

$$\begin{aligned} &E\left[\exp\left\{-\sum_{x\in\mathbb{Z}^d} f(x/t^{1/4})\,{}_s\tilde{\eta}_t^B(x)\right\}\,\middle|\,\mathscr{F}_0\right] \\ &\quad = \prod_{y\in D^I}\left[\sum_{x\in\mathbb{Z}^d} \exp\{-f(x/t^{1/4})\}K_{s^\alpha}(x-y)\right]^{\xi_s^B(y)} \\ &\quad = \prod_{y\in D^I}\left[1+\sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)K_{s^\alpha}(x-y)\right]^{\xi_s^B(y)}. \end{aligned} \tag{9.14}$$

Define $m^i(x)$ as in (9.4) and set $Z_s = [\lfloor s^\beta \rfloor^d (\xi_0 * N_s)(0) - s^{d\beta-(d+\varepsilon/2)/4}]^+$. On H_s, Z_s is a lower bound of $\xi_s^B(D^i)$, for all $i \in I$. Grouping all B particles for each D^i together, one can therefore check that, on H_s, (9.14) is less than or equal to

$$\begin{aligned} &\prod_{i\in I}\left[1+\sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)m^i(x)\right]^{Z_s} \\ &\quad \le \prod_{i\in I}\exp\left\{Z_s\sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)m^i(x)\right\} \\ &\quad = \exp\left\{Z_s\sum_{x\in\mathbb{Z}^d}\left[(\exp\{-f(x/t^{1/4})\}-1)\sum_{i\in I}m^i(x)\right]\right\}. \end{aligned} \tag{9.15}$$

By applying the lower bound for $\sum_{i\in I} m^i(x)$ in Corollary 9.1, with $x \in D_{s^\gamma}$, one obtains the upper bound

$$\exp\left\{(1 - C_{43}s^{-\alpha/2+\beta})s^{-d\beta}Z_s\sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)\right\}$$

for large t. [Since $f(\cdot)$ has compact support, the values of $m^i(x)$ for $x \notin D_{s^\gamma}$ do not matter.] Substituting in for Z_s, one can check that this is at most

$$\begin{aligned} \exp\Bigg\{&(1 - C_{46}s^{-\alpha/2+\beta})[(\xi_0 * N_s)(0) - s^{-(d+\varepsilon/2)/4}]^+ \\ &\times \sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)\Bigg\}, \end{aligned} \tag{9.16}$$

for appropriate C_{46}.

Since f is continuous and has compact support,

$$t^{-d/4}\sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)\to\int_{\mathbb{R}^d}(e^{-f(x)}-1)\,dx\quad\text{as } t\to\infty. \tag{9.17}$$

One has $\beta<\alpha/2$ and $\varepsilon>0$, and so one can use this to write (9.16) as

$$\exp\left\{[c_{1,s}t^{d/4}(\xi_0*N_s)(0)+c_{2,s}]^+\int_{\mathbb{R}^d}(e^{-f(x)}-1)\,dx\right\}, \tag{9.18}$$

where $c_{1,s}\to 1$ and $c_{2,s}\to 0$ as $s\to\infty$. Since $f(x)\ge 0$ for all x, (9.18) is asymptotically equivalent to the expression obtained by dropping the terms $c_{1,s}$ and $c_{2,s}$. So, combining (9.13)–(9.18), one sees that

$$\begin{aligned}&\Bigg[E\Bigg[\exp\Bigg\{-\sum_x f(x/t^{1/4})_s\tilde{\eta}_t^B(x)\Bigg\}\Bigg|\mathscr{F}_s\Bigg]\\ &\quad-\exp\Bigg\{t^{d/4}(\xi_0*N_s)(0)^+\int_{\mathbb{R}^d}(e^{-f(x)}-1)\,dx\Bigg\}\Bigg]^+\\ &\quad\to 0\quad\text{in probability as } t\to\infty.\end{aligned} \tag{9.19}$$

Taking the conditional expectation of the left side of (9.19) with respect to $\mathscr{F}_0$ produces the same expression as in (9.19), but with $\mathscr{F}_0$ replacing $\mathscr{F}_s$. Moreover, it follows from Lemma 7.1 that

$$P(|(\xi_0*N_t)(0)^+-(\xi_0*N_s)(0)^+|\ge t^{-(d+1-\alpha)/4})\le 2\exp\{-C_{33}t^{(1-\alpha)/2}\} \tag{9.20}$$

for large t; note that $\alpha<1$. Together, (9.19)–(9.20) imply that

$$\begin{aligned}&\Bigg[E\Bigg[\exp\Bigg\{-\sum_x f(x/t^{1/4})_s\tilde{\eta}_t^B(x)\Bigg\}\Bigg|\mathscr{F}_0\Bigg]\\ &\quad-\exp\{t^{d/4}(\xi_0*N_t)(0)^+\int_{\mathbb{R}^d}(e^{-f(x)}-1)\,dx\}\Bigg]^+\\ &\quad\to 0\quad\text{in probability as } t\to\infty.\end{aligned} \tag{9.21}$$

This is the desired upper bound for the left side of (9.11).

We still need to show the analog of (9.21), but with $[\cdot]^-$ replacing $[\cdot]^+$ on the left side of (9.21). The argument for this direction is essentially the same as before. We define $M^i(x)$ as in (9.4), and set $Z_s'=[\lfloor s^\beta\rfloor^d(\xi_0*N_s)(0)+s^{d\beta-(d+\varepsilon/2)/4}]^+$. Reasoning as in (9.14) through the first line of (9.15), one obtains that, on H_s,

$$\begin{aligned}&E\Bigg[\exp\Bigg\{-\sum_{x\in\mathbb{Z}^d}f(x/t^{1/4})_s\tilde{\eta}_t^B(x)\Bigg\}\Bigg|\mathscr{F}_s\Bigg]\\ &\quad\ge\prod_{i\in I}\Bigg[1+\sum_{x\in\mathbb{Z}^d}(\exp\{-f(x/t^{1/4})\}-1)M^i(x)\Bigg]^{Z_s'}.\end{aligned} \tag{9.22}$$

Note that the process ${}_s\tilde{\eta}_t$ rather than ${}_s\eta_t$ is needed for (9.22), because the above product is restricted to $i \in I$. Since f has compact support, one can check, using a standard version of the local central limit theorem, that

$$\sum_{x\in\mathbb{Z}^d} (\exp\{-f(x/t^{1/4})\} - 1) M^i(x) \le C_{47} s^{d(1-2\alpha)/4}$$

for large t and appropriate C_{47}. The right side of (9.22) is therefore at least

$$\exp\biggl\{(1 + C_{47}s^{d(1-2\alpha)/4}) Z'_s \sum_{x\in\mathbb{Z}^d} \Bigl[(\exp\{-f(x/t^{1/4})\} - 1) \sum_{i\in I} M^i(x)\Bigr]\biggr\},$$

which is the analog of the last line in (9.15). Since $\alpha > 1/2$, the term $C_{47}s^{d(1-2\alpha)/4}$ is negligible.

From here on, the arguments leading to (9.21) can be copied, with the upper bound for $\sum_{i=1}^{\infty} M^i(x)$ in Corollary 9.1, and Lemma 7.1 being applied. In place of (9.21), one obtains

$$\begin{aligned} &\biggl[E\biggl[\exp\biggl\{-\sum_{x\in\mathbb{Z}^d} f(x/t^{1/4})\, {}_s\tilde{\eta}_t^B(x)\biggr\} \biggm| \mathscr{F}_0\biggr] \\ &\qquad - \exp\{t^{d/4}(\xi_0 * N_t)(0)^+ \int_{\mathbb{R}^d} (e^{-f(x)} - 1)\, dx\}\biggr]^- \\ &\qquad \to 0 \quad \text{in probability as } t \to \infty. \end{aligned} \tag{9.23}$$

Together, (9.21) and (9.23) imply (9.11).

In Proposition 9.2, we replace ${}_s\tilde{\eta}_t$, in (9.10) and (9.11), with ξ_t; we also examine the joint behavior of ξ_t^A and ξ_t^B. In addition to Proposition 9.1, we employ Lemma 9.2, which allows us to compare ${}_s\tilde{\eta}_t$ with ${}_s\eta_t$. On account of (1.8), the decrease in the density $\rho(t)$ is smooth, and so comparison of ${}_s\eta_t$ with ξ_t is also not difficult; the reasoning for this follows [2]. Together, these results will imply (9.25).

PROPOSITION 9.2. *Assume that $d < 4$. Then, for $f = (f_1, f_2)$, with $f_i \in C_c^+(\mathbb{R}^d)$,*

$$\begin{aligned} &E\biggl[\exp\biggl\{-\sum_{x\in\mathbb{Z}^d} (f_1(x/t^{1/4})\xi_t^A(x) + f_2(x/t^{1/4})\xi_t^B(x))\biggr\} \biggm| \mathscr{F}_0\biggr] \\ &\quad - \exp\biggl\{t^{d/4}\biggl[(\xi_0 * N_t)(0)^- \int_{\mathbb{R}^d} (e^{-f_1(x)} - 1)\, dx \\ &\qquad + (\xi_0 * N_t)(0)^+ \int_{\mathbb{R}^d} (e^{-f_2(x)} - 1)\, dx\biggr]\biggr\} \to 0 \\ &\qquad\qquad \textit{in probability as } t \to \infty. \end{aligned} \tag{9.25}$$

PROOF. We first compare ${}_s\tilde{\eta}_t$ and ξ_t. Recall that $t = s + s^{\alpha}$; as in Proposition 9.1, we set $\alpha = 1 - 10^{-5}$. It therefore follows from (1.8) that

$$t^{d/4}(\rho(s) - \rho(t)) \le C_{48} t^{-10^{-5}} \tag{9.26}$$

for large t and appropriate C_{48}. Consequently, for given $M > 0$,

$$E[{}_s\eta_t^B(D_{Mt^{1/4}})] - E[\xi_t^B(D_{Mt^{1/4}})] \le 2C_{48}M^d t^{-10^{-5}}. \tag{9.27}$$

The particles of ξ_t form a subset of those of ${}_s\eta_t$. Therefore, by (9.27) and Markov's inequality,

$$P({}_s\eta_t^B(x) \neq \xi_t^B(x) \text{ for some } x \in D_{Mt^{1/4}}) \to 0 \quad \text{as } t \to \infty. \tag{9.28}$$

The analogous limit holds for A particles as well. It follows from this and Lemma 9.2 that

$$\begin{aligned} &P({}_s\tilde{\eta}_t^A(x) \neq \xi_t^A(x) \text{ for some } x \in D_{Mt^{1/4}}) \to 0 \quad \text{as } t \to \infty, \\ &P({}_s\tilde{\eta}_t^B(x) \neq \xi_t^B(x) \text{ for some } x \in D_{Mt^{1/4}}) \to 0 \quad \text{as } t \to \infty. \end{aligned} \tag{9.29}$$

We now derive (9.25) from (9.10) and (9.11). Let G_t denote the event where $(\xi_0 * N_t)(0) \ge 0$. It follows from (9.10) and (9.11) that, for $f_i \in C_c^+(\mathbb{R}^d)$,

$$\begin{aligned} &1_{G_t} \sum_{x\in\mathbb{Z}^d} f_1(x/t^{1/4})_s\tilde{\eta}_t^A(x) \to 0 \quad \text{in probability as } t \to \infty, \\ &1_{G_t^c} \sum_{x\in\mathbb{Z}^d} f_2(x/t^{1/4})_s\tilde{\eta}_t^B(x) \to 0 \quad \text{in probability as } t \to \infty, \end{aligned} \tag{9.30}$$

where 1_G denotes the indicator function of the event G. Consequently, by (9.29),

$$\begin{aligned} &1_{G_t} \sum_{x\in\mathbb{Z}^d} f_1(x/t^{1/4})\xi_t^A(x) \to 0 \quad \text{in probability as } t \to \infty, \\ &1_{G_t^c} \sum_{x\in\mathbb{Z}^d} f_2(x/t^{1/4})\xi_t^B(x) \to 0 \quad \text{in probability as } t \to \infty. \end{aligned} \tag{9.31}$$

So, in order to demonstrate (9.25), it suffices to show the analogous limit,

$$\begin{aligned} &E\left[\exp\left\{-1_{G_t^c} \sum_{x\in\mathbb{Z}^d} f_1(x/t^{1/4})\xi_t^A(x) - 1_{G_t} \sum_{x\in\mathbb{Z}^d} f_2(x/t^{1/4})\xi_t^B(x)\right\} \middle| \mathscr{F}_0\right] \\ &\quad - \exp\left\{t^{d/4}\left[(\xi_0 * N_t)(0)^- \int_{\mathbb{R}^d}(e^{-f_1(x)} - 1)\,dx \right.\right. \\ &\qquad\qquad \left.\left. + (\xi_0 * N_t)(0)^+ \int_{\mathbb{R}^d}(e^{-f_2(x)} - 1)\,dx\right]\right\} \to 0 \\ &\qquad\qquad\qquad\qquad \text{in probability as } t \to \infty. \end{aligned} \tag{9.32}$$

On G_t, the left side of (9.32) reduces to the left side of (9.11), if f_2 is replaced by f and $\xi_t^B(x)$ by ${}_s\tilde{\eta}_t^B(x)$; similarly, on G_t^c, the left side of (9.32) reduces to the left side of (9.10). So, (9.32) follows from (9.10), (9.11) and (9.29). This demonstrates the proposition. □

We now demonstrate Theorem 2. We know from Corollary 8.1 that

$$t^{d/4}(\xi_0 * N_t)(0) \Rightarrow b_d Z_{0,1} \quad \text{as } t \to \infty, \tag{9.33}$$

where $Z_{0,1}$ has a standard normal distribution, and $b_d = (2\lambda)^{1/2}(4\pi)^{-d/4}$. Taking expectations in (9.25), and substituting in (9.33) implies that

$$E\left[\exp\left\{-\sum_{x\in\mathbb{Z}^d}\left(f_1(x/t^{1/4})\xi_t^A(x)+f_2(x/t^{1/4})\xi_t^B(x)\right)\right\}\right] \tag{9.34}$$
$$\to E\left[\exp\left\{b_d\left(Z_{0,1}^-\int_{\mathbb{R}^d}(e^{-f_1(x)}-1)dx+Z_{0,1}^+\int_{\mathbb{R}^d}(e^{-f_2(x)}-1)\,dx\right)\right\}\right]$$

as $t\to\infty$, for $f_i\in C_c^+(\mathbb{R}^d)$. One can rescale ξ_t as in (1.10), setting $\check{\xi}_t(E) = \xi_t(t^{1/4}E)$. One can also rewrite the left side of (9.34), viewing $\check{\xi}_t^A$ and $\check{\xi}_t^B$ as random measures on $\mathbb{R}^d$. Doing this, one can rephrase (9.34) as

$$E\left[\exp\left\{-\int_{\mathbb{R}^d}f_1(x)\check{\xi}_t^A(dx)-\int_{\mathbb{R}^d}f_2(x)\check{\xi}_t^B(dx)\right\}\right] \tag{9.35}$$
$$\to E\left[\exp\left\{b_d\left(Z_{0,1}^-\int_{\mathbb{R}^d}(e^{-f_1(x)}-1)\,dx+Z_{0,1}^+\int_{\mathbb{R}^d}(e^{-f_2(x)}-1)\,dx\right)\right\}\right]$$

as $t\to\infty$. The right side of (9.35) is the Laplace functional of a convex combination of Poisson random fields with two types of particles, where the intensities are given by $b_dZ_{0,1}^-$ and $b_dZ_{0,1}^+$. Letting F denote the distribution function of $b_dZ_{0,1}$, we can write this random field as $\mathscr{P}_F$, as in (1.11). It follows from (9.35) that the pair $(\check{\xi}_t^A,\check{\xi}_t^B)$ converges weakly to $\mathscr{P}_F$, on the Borel measures on $\mathbb{R}^d$ with finite mass on compact subsets. That is,

$$(\check{\xi}_t^A,\check{\xi}_t^B)\Rightarrow\mathscr{P}_F\quad\text{as } t\to\infty. \tag{9.36}$$

The limit in (9.36) is the same as that in (1.12). This completes the proof of Theorem 2. □

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