*The Annals of Applied Probability*
1994, Vol. 4, No. 1, 184–193

# GENERAL BRANCHING PROCESSES IN VARYING ENVIRONMENT[1]

BY HARRY COHN AND PETER JAGERS

*University of Melbourne, and Chalmers University of Technology and Gothenburg University*

A conditioning and martingale method is used to relate the asymptotics of uniformly integrable general branching processes in varying environment to the behaviour of their expectations.

**1. Introduction.** Branching processes are naturally analyzed in two steps. First you take the expectations, then the process itself. The first step may be trivial for Galton–Watson or birth-and-death type processes, but certainly not so for general processes, where you would have to resort to Markov renewal theory or similar approaches. The second step relates the process itself to its expectation, nowadays usually by martingale methods.

One such approach is due to Cohn (1985). It can be viewed as based upon the following lemma [cf. Cohn (1985), Theorem 3.1]. (Note that the concept of weak $L^1$-convergence referred to is that of functional analysis: $X_n \to X$ weakly in $L^1$ means that $E[X_nY] \to E[XY]$ for all bounded measurable $Y$.)

LEMMA 1. *Let $\{X_n\}$ be a sequence of uniformly integrable random variables on a space $(\Omega, \mathscr{F}, \mathbb{P})$, which also carries a filtration $\{\mathscr{F}_k\}$ generating $\mathscr{F}$. Assume that for all $k$ $\mathbb{E}[X_n|\mathscr{F}_k] \overset{P}{\to}$ some $Y_k$ where $\overset{P}{\to}$ stands for convergence in probability. Then $\{Y_k, \mathscr{F}_k\}$ is a martingale. It converges a.s. and in $L^1$ to a limit $X$. Further, $X_n \to X$ weakly in $L^1$. If there is a set $A \in \mathscr{F}$ such that $X_n \geq X$ on $A$ and the contrary on its complement, then the convergence holds even in $L^1$.*

COROLLARY 2. *If the $X_n$ and $X$ are indicator functions and $\{\mathscr{F}_k\}$ generates $\mathscr{F}$, then $\mathbb{E}[X_n|\mathscr{F}_k] \overset{P}{\to}$ some $Y_k$, for all $k$, implies that $X_n \overset{L_1}{\to} X$ where $\overset{L_1}{\to}$ stands for convergence in $L_1$.*

PROOF. The corollary is obvious since indicators are bounded and hence uniformly integrable. As the set $A$ we can choose $\{X = 0\}$.

The lemma itself follows from weak $L^1$-compactness being equivalent to uniform integrability [Dunford–Pettis; cf. Neveu (1965), page 118]. The strong

Received April 1992; revised June 1993.
[1]Supported by the Australian Research Council and the Swedish Natural Sciences Research Council.
*AMS* 1991 *subject classifications*. Primary 60J80; secondary 60F25.
*Key words and phrases.* Branching process, martingale, varying environment.

$L^1$-convergence is there since

$$\mathbb{E}[|X_n - X|] = \mathbb{E}[X_n; A] - \mathbb{E}[X; A] + \mathbb{E}[X; A'] - \mathbb{E}[X_n; A'] \to 0,$$

where the prime denotes the complement. □

We shall use this lemma to relate the asymptotics of branching processes in varying environment to that of their expectations. The case treated will be so general that many different types of mean behaviour will be included.

To make the object of study precise observe that a general branching population, as in Jagers (1989), can be at least partly defined without any reference to probability measures. Following this procedure, let

$$I = \bigcup_{n=0}^{\infty} N^n, \qquad N^0 = \{0\}, \qquad N = \{1,2,\dots\}$$

denote the Ulam–Harris space of all possible individuals. Denote the *life space* of all perceivable individual life paths by $(\Omega, \mathscr{A})$ and assume that a *reproduction process* is defined on it. This reproduction process should tell the mother's age at bearing and the *type* of the child, the latter being an element of a *type space* $(S, \mathscr{S})$. Thus, on the life space there is a sequence of maps $(\tau(k), \sigma(k))$, $k = 1, 2, \dots, 0 \le \tau(1) \le \tau(2) \le \cdots \le \infty$, $\sigma(k)$: $\Omega \to [0, \infty]$, with the interpretation that $\tau(k)$ is the mother's age at giving birth to her $k$th child and $\sigma(k)$ is that child's type. Of course, if $\tau(k)(\omega) < \tau(k+1)(\omega) = \infty$, then a mother leading the life path $\omega$ will never obtain more than $k$ children.

The whole population space is $(S \times \Omega^I, \mathscr{S} \times \mathscr{A}^I)$ and on this space the birth time and type of an individual $x \in I$ are denoted by $\tau_x$ and $\sigma_x$, respectively. [Note that there is a slight change in notation from Jagers (1989).] These are inductively given from a starting type $\sigma_0 \in S$ and $\tau_0 = 0$.

The easiest way to introduce an environment might be through an *environment space*, $E$. It is then supposed that the environment varies in a deterministic fashion according to some function of time; call it $\phi$: $\mathbb{R}_+ \to E$. The ancestor 0 is born at time 0 when the environment is $\phi(0)$. If her type is $\sigma_0$, her life path will be chosen by a probability kernel determined by the couple $\sigma_0$, $\phi(0)$. More generally, if an individual of type $s$ is born at time $t$, her life path will be chosen according to the kernel $P(s, \phi(t), \cdot)$. Note that by the abstract form of the environment space this formulation includes the possibility of environmental changes during an individual's life influencing her career and, in particular, her reproduction.

We shall not speak more about this dependence on the environment, or its variation, but only require that it is such that a sequence of conditions can be checked. Certainly this will be the case for a fixed [Jagers (1989)] or a periodically varying environment [Jagers and Nerman (1985)]. To formulate things consider a *random characteristic*, that is, a way of measuring individual contributions to the population, giving weight zero to yet unborn individuals and uninfluenced by the individual's progenitors' lives [cf. Jagers

(1989)] and define the $\chi$-counted population $z_t^\chi$ by

$$z_t^\chi = \sum_{x \in I} \chi(t - \tau_x, \sigma_x, S_x),$$

$S_x$ denoting the daughter process of $x$, that is, allowing the lives of $x$ and all her progeny to influence the characteristic. For short we allow ourselves to write $\chi_x(t - \tau_x)$ for $\chi(t - \tau_x, \sigma_x, S_x)$ and to talk about it as the characteristic pertaining to $x$. As function of this (age) argument, the characteristic must be cadlag. Not to complicate arguments, we also take it to be nonnegative and bounded.

Now introduce the notation $m_t^\chi(u, s)$ for the expected value at time $t + u$ of a $\chi$-counted population started from a newborn $s$-individual at time $u$. Strictly speaking, if the subscript $s$ denotes start from an ancestor 0 of type $s \in S$ and the superscript indicates environment dependence, this is

$$m_t^\chi(u, s) = \mathbb{E}_s^{\phi(u+\cdot)}[z_t^\chi].$$

For the particular characteristic $\chi(a) = 1_{R_+}(a)$, counting all individuals with nonnegative ages $a$, that is, born, we delete the superscript, writing simply $m_t$. Similarly, we omit reference to the starting time $u = 0$ and the starting type $\sigma_0$.

There seems to exist no literature about general, time inhomogeneous branching processes. Generation-dependent reproduction in single-type populations was analyzed by Edler (1978). In the (one-type) Galton–Watson case, where time and generation inhomogeneity coincide, there is a more extensive literature; compare with D'Souza and Biggins (1992). Cohn and Hering (1983) discussed Markov branching in the continuous time case.

**2. The limit of a process normed by its mean.** The following conditions are to be used.

CONDITION 1. The mean of the process, as counted by the characteristic, should tend to infinity: $m_t^\chi \to \infty$, as $t \to \infty$.

CONDITION 2. The limiting ratio $h(u, s) := \lim_{t \to \infty} m_{t-u}^\chi(u, s)/m_t^\chi$ exists for all $u, s$. It is strictly positive and satisfies $\sup_{u \geq t, s \in S} h(u, s) \to 0$, as $t \to \infty$.

CONDITION 3. The random variables $w_t^\chi := z_t^\chi/m_t^\chi$ are uniformly integrable over $t$, but also over starting time $u$ and starting type.

To show how the conditions combine with Lemma 1 to yield the process asymptotics, without too much of technicalities, we consider first weak $L^1$-convergence. Write $mx$ to denote $x$'s mother, $x \in I$, and let $I_u$ be the *coming generation* at time $u$:

$$I_u = \{x \in I; \tau_{mx} \leq u < \tau_x < \infty\}$$

[Nerman (1981)]. Write $\mathscr{F}_u$ for the *pre-$I_u$ $\sigma$-algebra*, that is, the $\sigma$-algebra generated by the complete lives of all individuals neither in $I_u$ nor stemming from it. With $\prec$ symbolizing the partial order of descent, so that $x \prec y$ means that $y$ stems from $x$, we obtain for $u < t$ that

$$
\begin{aligned}
z_t^\chi &= \sum_{\tau_x \le u} \chi_x(t - \tau_x) + \sum_{x \in I_u} \sum_{x \prec y} \chi_y(t - \tau_y) \\
&= \sum_{\tau_x \le u} \chi_x(t - \tau_x) + \sum_{x \in I_u} z_{t-\tau_x}^\chi \circ S_x .
\end{aligned} \tag{1}
$$

By the branching property, the conditional expectation of the normed form of this is

$$\mathbb{E}[w_t^\chi | \mathscr{F}_u] = \sum_{\tau_x \le u} \chi_x(t - \tau_x)/m_t^\chi + \sum_{x \in I_u} m_{t-\tau_x}^\chi / m_t^\chi . \tag{2}$$

Since the characteristic is taken as bounded, Conditions 1 and 2 imply that

$$\mathbb{E}[w_t^\chi | \mathscr{F}_u] \to \sum_{x \in I_u} h(\tau_x, \sigma_x),$$

as $t \to \infty$. By Lemma 1 and the uniform integrability of Condition 3,

$$\sum_{x \in I_u} h(\tau_x, \sigma_x)$$

is a martingale with a limit $w^\chi$ almost surely and in $L^1$, and the weak $L^1$-convergence $w_t^\chi \to w^\chi$, as $t \to \infty$, follows.

THEOREM 3. *Under the stated three conditions, $w_t^\chi \xrightarrow{L_1} w^\chi$, as $t \to \infty$.*

For the proof we shall need the following result on weighted sums of independent random variables.

THEOREM 4. *Let $\{\xi_i^{(n)};\ 1 \le i \le n,\ n \ge 1\}$ be a triangular array of uniformly integrable random variables with mean 1 such that for each $n$ $\{\xi_i^{(n)};\ 1 \le i \le n\}$ are independent. Write $\{a_i^{(n)};\ 1 \le i \le n,\ n \ge 1\}$ for a set of positive numbers such that $\lim_{n \to \infty} \max_{1 \le i \le n} a_i^{(n)}/A_n = 0$, where $A_n = \sum_{i=1}^n a_i^{(n)}$. Assume that $\lim_{n \to \infty} A_n = \infty$. Then $U_n = (\sum_{i=1}^n a_i^{(n)} \xi_i^{(n)})/A_n$ converges in probability to 1.*

The proof of this theorem may be carried out as in Jamison, Orey and Pruitt (1965), Theorem 1. In fact we shall use the following corollary.

COROLLARY 5. *Suppose that $\{\alpha_i^{(n)};\ 1 \le i \le n,\ n \ge 1\}$ are some positive numbers with $\lim_{n \to \infty} \max_{1 \le i \le n} \alpha_i^{(n)} = 0$ and that $\alpha := \lim_{n \to \infty} \sum_{i=1}^n \alpha_i^{(n)}$ exists. Then $T_n = \sum_{i=1}^n \alpha_i^{(n)} \xi_i^{(n)}$ converges in probability to $\alpha$ as $n \to \infty$.*

PROOF. Take $a_i^{(n)} = \alpha_i^{(n)}/\max_{1 \le j \le n} \alpha_j^{(n)}$ and apply Theorem 4 if $\alpha > 0$. The case $\alpha = 0$ follows from $\mathbb{E}[|T_n|] \to 0$. □

PROOF OF THEOREM 3. Write $w_u^\chi(\tau_x, \sigma_x) := z_u^\chi \circ S_x / m_u^\chi$ to denote a normed process started $u$ time units ago at time $\tau_x$ from an ancestor of type $\sigma_x$. Then in view of (1), the finiteness of the set $I_u$ and Conditions 1 and 2,

$$
\begin{aligned}
w_t^\chi - &\sum_{x \in I_u} h(\tau_x, \sigma_x) w_{t-\tau_x}^\chi(\tau_x, \sigma_x) \\
&= w_t^\chi - \sum_{x \in I_u} \frac{m_{t-\tau_x}^\chi}{m_t^\chi} w_{t-\tau_x}^\chi(\tau_x, \sigma_x) \\
&\quad + \sum_{x \in I_u} \left( \frac{m_{t-\tau_x}^\chi}{m_t^\chi} - h(\tau_x, \sigma_x) \right) w_{t-\tau_x}^\chi(\tau_x, \sigma_x) \\
&\to 0
\end{aligned}
\tag{3}
$$

in probability, as $t \to \infty$, for any given $u$ and thus fixed $I_u$. On the other hand, for any $\varepsilon > 0$,

$$
\lim_{u \to \infty} \mathbb{P}\left( \left| \sum_{x \in I_u} h(\tau_x, \sigma_x) \xi_x - \sum_{x \in I_u} h(\tau_x, \sigma_x) \right| > \varepsilon \,\middle|\, \mathscr{I}_u \right) = 0 \quad \text{a.s.} \tag{4}
$$

for any random variables $\{\xi_x,\ x \in I_u\}$ independent of $\mathscr{I}_u$ and belonging to a set of uniformly integrable variables with mean 1. Indeed, $\{h(\tau_x, \sigma_x),\ x \in I_u\}$ act as constants when conditioning on $\mathscr{I}_u$, so that Corollary 5 applies provided that $\# I_u \to \infty$ as $u \to \infty$, where $\# I_u$ stands for the cardinality of $I_u$. However, in view of the latter part of Condition 2,

$$
\# I_u \geq \sum_{x \in I_u} h(\tau_x, \sigma_x) \Big/ \sup_{\substack{u \geq t \\ s \in S}} h(t, s) \to \infty,
$$

unless the a.s. convergent martingale $\sum_{x \in I_u} h(\tau_x, \sigma_x) \to 0$, as $u \to \infty$.

Since, trivially,

$$
\mathbb{E}\left[ \sum_{x \in I_u} h(\tau_x, \sigma_x) \xi_x \,\middle|\, \mathscr{I}_u \right] = \sum_{x \in I_u} h(\tau_x, \sigma_x),
$$

the Markov inequality makes (4) hold in this case as well. By the dominated convergence theorem the conditional statements expressed by (3) and (4) hold unconditionally as well, and imply that $\{w_t^\chi\}$ converges in probability to $w^\chi$ as $t \to \infty$. The $L^1$-convergence follows now from Condition 3. □

For the general branching process we consider, the classical extinction-or-explosion dichotomy is far from evident. Part of it is a consequence of the preceding theorem.

COROLLARY 6. *Under the conditions of Theorem* 3, $\mathbb{P}(w^\chi > 0) > 0$ *and* $z_t^\chi \xrightarrow{P} \infty$, *given that* $w^\chi > 0$.

In the one-type Galton–Watson case the dichotomy was established by D'Souza and Biggins (1992).

**3. The limit distribution is continuous.** We shall need a result for independent random variables due to Rogozin (1961) [cf. also Petrov (1975), Chapter 3].

LEMMA 7. *Let* $X_1, \dots, X_n$ *be independent random variables,* $S_n = X_1 + \cdots + X_n$ *and* $p_k = \sup_y P(X_k = y)$. *Then*

$$\sup_y P(S_n = y) \le A\left(\sum_{k=1}^{n} (1 - p_k)\right)^{-1/2},$$

*where* $A$ *is an absolute constant.*

Rogozin's result is in fact stated for discrete random variables, but it also applies to the case where distribution functions may have a continuous component.

We shall consider two conditions that guarantee the continuity of the limit distribution of $w_t^{\chi}$.

CONDITION 4. The quantity $q_t = \mathbb{P}(z_t^{\chi} = 0)$ is bounded away from 0 uniformly over $t$, but also over starting time $u$ and starting type.

CONDITION 5. The random variables $(w_t^{\chi})^2$ are uniformly integrable and their expectations are bounded away from 1, uniformly over $t$, but also over starting time $u$ and starting type.

THEOREM 8. *Suppose that in addition to the assumptions of Theorem* 3 *one of Conditions* 4 *and* 5 *holds. Then the distribution function of* $w^{\chi}$ *is continuous, except possibly at* 0.

PROOF. As is easy to see from Theorem 3, $w^{\chi}$, the limit random variable of $\{w_t^{\chi}\}$, admits the representation

$$w^{\chi} = \sum_{x \in I_u} h(\tau_x, \sigma_x) w^{\chi}(\tau_x, \sigma_x), \tag{5}$$

where $\{w^{\chi}(\tau_x, \sigma_x)\}$ are independent random variables given $\mathscr{I}_u$. Write

$$p_x := \sup_{v \ge 0} \mathbb{P}(w^{\chi}(\tau_x, \sigma_x) = v | \mathscr{I}_u).$$

It is clear that under Condition 4 that

$$\sup_{v > 0} \mathbb{P}(w^{\chi}(\tau_x, \sigma_x) = v | \mathscr{I}_u)$$

is bounded away from 1. Also,

$$\mathbb{P}(w^{\chi}(\tau_x, \sigma_x) = 0 | \mathscr{I}_u)$$

is bounded away from 1 in view of the uniform integrability (Condition 3), which precludes a null limit for variables with expectation 1. As far as Condition 5 is concerned, it is easy to see that random variables with variances bounded away from zero and uniformly integrable in $L^2$ cannot converge to a constant in probability. Thus $p_x$ is bounded away from 1 in either case.

Notice that for any nonzero constant $a$ and random variable $V$, $\sup_v P(V = v) = \sup_v P(aV = v)$. Since the $h(\tau_x, \sigma_x)$ act as constants when conditioning on $\mathcal{I}_u$, the lemma implies that

$$1_{\{w^{\chi} > 0\}}\mathbb{P}(w^{\chi} = c|\mathcal{I}_u) \le 1_{\{w^{\chi} > 0\}}A\biggl(\sum_{x\in I_u}(1 - p_x)\biggr)^{-1/2}, \tag{6}$$

where $1_E$ is the indicator of the set $E$.

Since

$$\sum_{x\in I_u} h(\tau_x, \sigma_x) \to w^{\chi}$$

almost surely, it follows from Condition 2—as in the proof of Theorem 3—that $\#I_u \to \infty$ a.s. on $\{w^{\chi} > 0\}$. The $p_x$ are bounded away from 1, and thus the right-hand side of (6) must tend to zero as $u \to \infty$. The theorem follows from taking expectations. □

**4. The asymptotic composition.** The preceding section has dealt with $w_t^{\chi}$ and its relation to its mean. Another interesting topic is the relation, as time passes, between processes counted by different characteristics, and in particular between an arbitrary $z_t^{\chi}$ and the total population $y_t$, that is, for the particular choice of characteristic $\chi = 1_{R_+\times S}$. In order to approach it we formulate two further conditions, using the shorthand $m_t^A$ for $m_t^{1_{R_+\times A}}$, $A \in \mathcal{S}$ and $m_t = m_t^S$.

CONDITION 6. *As $t \to \infty$ within some unbounded subset $T \subseteq \mathbb{R}_+$, the probability measures $m_{t-v-du}^{ds}(v, r)/m_{t-v}(v, r)$ converge in total variation to some $\lambda_{v,r}\,(du \times ds)$.*

CONDITION 7. *The limit $f_v(u, s) := \lim_{t\in T, t\to\infty} \mathbb{E}_{t-v-u,s}[\chi(u)]$ exists.*

Typically, the subset $T$ would be $\mathbb{R}_+$ or some lattice, reflecting a diurnal or seasonal rhythm. Note that, since characteristics are still bounded, the conditions imply

$$\begin{aligned}
&m_{t-v}^{\chi}(v, r)/m_{t-v}(v, r)\\
&\quad = \int_{[0,t-v]\times S} \mathbb{E}_{u,s}[\chi(t - v - u)]\, m_{du}^{ds}(v, r)/m_{t-v}(v, r)\\
&\quad = \int_{[0,t-v]\times S} \mathbb{E}_{t-v-u,s}[\chi_0(u)]\, m_{t-v-du}^{ds}(v, r)/m_{t-v}(v, r)\\
&\quad \to \int_{R_+\times S} f_v(u, s)\,\lambda_{v,r}(du \times ds) \qquad \text{as } t \in T \to \infty,
\end{aligned}$$

by the strength of convergence in total variation.

We write just $\lambda$ for $\lambda_{0,\sigma_0}$ and $f$ for $f_0$.

THEOREM 9. *Under all the given conditions,*

$$z_t^{\chi}/y_t \to \int_{R_+\times S} f(u,s)\lambda(du\times ds)$$

*in* $L^1$ *on* $\{w := w^{1_{R_+}} > 0\}$, *as* $t\to\infty$ *in* $T$.

PROOF. Recall the argument preceding Theorem 1:

$$\begin{aligned}
\mathbb{E}[z_t^{\chi}/m_t|\mathscr{F}_u] &= \sum_{x\in I_u} m_{t-\tau_x}^{\chi}(\tau_x,\sigma_x)/m_t\\
&= \sum_{x\in I_u}\int_{[0,t-\tau_x]\times S}\mathbb{E}_{v,s}[\chi(t-\tau_x-v)]m_{dv}^{ds}(\tau_x,\sigma_x)/m_{t-\tau_x}(\tau_x,\sigma_x)\\
&\qquad\times\big(m_{t-\tau_x}(\tau_x,\sigma_x)/m_t\big)\\
&\to \sum_{x\in I_u}\int_{R_+\times S} f(v,s)\lambda_{\tau_x,\sigma_x}(dv\times ds)h(\tau_x,\sigma_x)
\end{aligned}$$

as $t\to\infty$. When then $u\to\infty$, this must converge to

$$w^{\chi}\lim_{t\to\infty} m_t^{\chi}/m_t = w^{\chi}\int_{R_+\times S} f(v,s)\lambda(dv\times ds).$$

The random component of the converging expression does not contain any reference to the characteristic $\chi$, and the latter is assumed to satisfy Condition 7, which obviously holds for the characteristic $1_{R_+}$ that counts the total population. Hence, it follows that $w^{\chi}=w$. The rest is direct:

$$\frac{z_t^{\chi}}{y_t} = \frac{z_t^{\chi}}{m_t^{\chi}}\frac{m_t}{y_t}\frac{m_t^{\chi}}{m_t} \to \int_{R_+\times S} f(v,s)\lambda(dv\times ds)$$

in probability and also in $L^1$, because $\chi$ is bounded. □

## 5. Examples.

5.1. *Single and Multitype Galton–Watson processes in varying environment.* In the case of a single-type Galton–Watson process [see Jagers (1975)], the data consist of the offspring variables $\{X_n\}$ whose distribution functions may vary with generations. It is easy to see that the $h$ functions in Condition 2 reduce to $(E(X_1)\cdots E(X_n))^{-1}$, which tend to 0 by Condition 1. Thus, in this case Condition 2 follows from Condition 1. For a close investigation of this case, refer to D'Souza and Biggins (1992).

For $p$-type Galton–Watson processes in varying environment, the crucial Condition 2 reduces to classical weak ergodicity (and some properties of the limits) for positive matrices. This is expressed via ratios of entries of products of matrices, and as in Condition 2 such quantities arise from expectations of variables associated with the branching process. Indeed, if $M_n(i, j)$ is the expected number of offspring of type $j$ produced by one individual of type $i$ of the $n$th generation and $M_n = (M_n(i, j))$ is the corresponding matrix, then the expected population sizes of various types at time $n$ generated by one individual of the $k$th generation will be given by products of the form ${}^kM^n := M_k \cdots M_n$. For classical and modern conditions ensuring weak ergodicity, we refer to Cohn and Nerman (1990), and for a more detailed analysis of the convergence of processes in this case, to a forthcoming paper by the same authors.

5.2. *Single-type general processes in constant environment.* Consider the classical case of supercritical general branching processes with all individuals of the same type [cf. Jagers and Nerman (1984)]. If the reproduction function, $\mu$, now just a measure on $\mathbb{R}_+$ without any suffix for time-dependence, is nonlattice, then established theory works to yield the process limits precisely under the famed $x \log x$ condition [Jagers and Nerman (1984), Theorem 7.2]. This goes for characteristics whose expectations, normed by $e^{\alpha t}$, are directly Riemann integrable. If we want to get rid of this, we have to impose an absolutely continuous component in the reproduction function. Then the conditions considered here hold.

Condition 1 holds trivially by the exponential growth of the mean of well-behaved supercritical branching processes,

$$m_t^\chi \sim e^{\alpha t}\mathbb{E}[\hat{\chi}(\alpha)]/\alpha\beta,$$

where $\alpha$ is the Malthusian parameter, the caret denotes the Laplace transform and

$$\beta = \int_0^\infty te^{-\alpha t}\mu(dt) < \infty.$$

In Condition 2 the function $h(u, s)$ reduces to $e^{-\alpha u}$. The uniform integrability follows from the $x \log x$ condition, and convergence in total variation of mean ratios is precisely where the renewal theorem is needed in the form requiring a diffuse reproduction function.

5.3. *Abstract type space and constant environment.* If individuals can be of various types, then conditions trivially satisfied in the one-type case typically impose a uniformity requirement. Of course, the process is still assumed to be supercritical, to ensure Condition 1. In the terminology of Jagers (1989), Condition 2 holds if the eigenfunction $h$ is bounded, since our ratio limit $h(u, s)$ now takes the form

$$h(u, s) = e^{-\alpha u}h(s)/h(\sigma_0)$$

in Jagers' terminology, where $\sigma_0$ is the starting type. Condition 3 follows from $x \log x$ and a condition of uniformity in the starting type [cf. Jagers (1989) and Condition 7 from Markov renewal arguments (under suitable assumptions], which can also be used to investigate Condtion 4 or 5.

5.4. *Periodic environment.* Jagers and Nerman (1985) studied one-type general branching processes, such that an individual born $s$ o'clock would have a life law, and in particular a reproduction function, determined by $s$. (This has a background in the diurnal rhythms of certain cell kinetics.) In our terminology, this means that $\mu_u$ would have period 1. The assumptions made in 1986 include boundedness of the eigenfunction involved and a condition (3.2) leading to the uniformities asked for in the mean asymptotics. The probabilistic theory in that work was developed under $L^2$-assumptions, stronger than $x \log x$.

**Acknowledgment.** We thank one of the referees for suggestions that led to a simpler proof of Theorem 3.

## REFERENCES

COHN, H. (1985). A martingale approach to supercritical branching processes. *Ann. Probab.* **13** 1179–1191.

COHN, H. and HERING, H. (1983). Inhomogeneous Markov branching processes: supercritical case. *Stochastic Process. Appl.* **14** 79–91.

COHN, H. and NERMAN, O. (1990). On products of nonnegative matrices. *Ann. Probab.* **18** 1806–1815.

D'SOUZA, J. C. and BIGGINS, J. D. (1992). The supercritical Galton–Watson process in varying environments. *Stochastic Process. Appl.* **42** 39–47.

EDLER, L. (1978). Strict supercritical generation-dependent Crump–Mode–Jagers branching processes. *Adv. in Appl. Probab.* **10** 744–763.

JAGERS, P. (1975). *Branching Processes with Biological Applications*. Wiley, New York.

JAGERS, P. (1989). General branching processes as Markov fields. *Stochastic Process. Appl.* **32** 183–212.

JAGERS, P. and NERMAN, O. (1984). The growth and composition of branching populations. *Adv. in Appl. Probab.* **16** 221–259.

JAGERS, P. and NERMAN, O. (1985). Branching processes in periodically varying environment. *Ann. Probab.* **13** 254–268.

JAMISON, B., OREY, S. and PRUITT, W. (1965). Convergence of weighted averages of independent random variables. *Z. Wahrsch. Verw. Gebiete* **4** 40–44.

LOÈVE, M. (1977). *Probability Theory*, 4th ed. Springer, New York.

NERMAN, O. (1981). On the convergence of supercritical general (C–M–J) branching processes. *Z. Wahrsch. Verw. Gebiete* **57** 365–395.

NEVEU, J. (1965). *Mathematical Foundations of the Calculus of Probability*. Holden-Day, San Francisco.

PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer, Berlin.

ROGOZIN, B. A. (1961). An estimate for concentration functions. *Theory Probab. Appl.* **6** 96–99.

DEPARTMENT OF STATISTICS
UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA, 3052
AUSTRALIA

DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY
AND GOTHENBURG UNIVERSITY
S-412 96 GÖTEBORG
SWEDEN