The Annals of Applied Probability
2004, Vol. 14, No. 1, 489–516

AVERAGE AND DIFFUSION APPROXIMATION OF STOCHASTIC EVOLUTIONARY SYSTEMS IN AN ASYMPTOTIC SPLIT STATE SPACE

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Stochastic evolutionary systems of additive functional type, described by processes with locally independent increments, are considered with Markov switching in an asymptotic split state space having a stoppage state. The average and diffusion approximation limit theorems are established in both single and double merging. The proofs of these results are obtained using a singular perturbation approach of linear reducible–invertible operators and the tightness of processes. Particular cases of these systems including integral functionals, dynamic systems, storage processes and compound Poisson processes are also considered. The application of limit theorems in reliability and reward problems is discussed.

1. Introduction. Stochastic approximations, as average, diffusion and Poisson approximations, are interesting not only theoretically but also increasingly in practical systems modeling.

In the study of real systems, two problems usually arise. The first one is connected to the generally high complexity of the state space. The second one is connected to the fact that the local characteristics of the systems are not fixed but depend upon random factors.

Concerning the first problem, in order to be able to give analytical or numerical tractable models, the state space must be simplified via a reduction of the number of states. This is possible when some subsets are connected between them by small transition probabilities and the states within such subsets are asymptotically connected. This is typically the case of reliability and in most applications concerned with hitting time models, for which the state space is naturally cut in two subsets (the up states set and the down states set) [19, 23]. In this case, transitions between the subsets are slow compared with those within the subsets.

Concerning the second problem, we describe the random changes of local characteristics by a stochastic process, called a switching process [1–3]. In applications, switching processes could represent the environment [13, 25], or, in the particular case of dynamic reliability, the structure of the system [6].

Received March 2000; revised January 2002.

AMS 2000 subject classifications. Primary 60J55, 60B10, 60F17, 60K10; secondary 60G46, 60G60.

Key words and phrases. Stochastic evolutionary system, Markov process with locally independent increments, diffusion approximation, split state space, dynamic reliability, reward.

Usually, the switching process is assumed to be an ergodic process. Nevertheless, in many practical problems nonergodic stochastic switching processes have to be considered, for example, when the system is observed up to the hitting time to some subset of the state space. We are here interested in this case in order to solve reliability problems.

An interesting stochastic evolutionary system with Markov switching is the following:

$$\xi^\varepsilon(t) := \xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon)\big), \tag{1}$$

where for each $\varepsilon > 0$, the process $x^\varepsilon(t/\varepsilon)$, $t \geq 0$, is a switching process, with state space E^0, and $\eta^\varepsilon(t; x)$, $x \in E^0$, is a switched $\mathbb{R}^d$-valued stochastic process. There, $x^\varepsilon(t/\varepsilon)$, $t \geq 0$, is a nonergodic jump Markov process with one absorbing state, say 0. The additive functional $\xi^\varepsilon(t)$ gives the reward of the system up to time t or the cumulative sojourn time spent in a subset of states up to time t, and so on [19, 13, 18, 25, 26, 30]. If ζ^ε denotes the hitting time of $x^\varepsilon(t/\varepsilon)$ to state 0, then $\xi^\varepsilon(\zeta^\varepsilon)$ is the reward up to the system failure.

The operator-valued stochastic processes called random evolutions are powerful stochastic models for modeling real systems. Several stochastic models can be described as particular cases of a random evolution [4, 19, 13, 21, 22, 25, 27]. The stochastic evolutionary systems considered here are described by *processes with locally independent increments* with Markov nonergodic switching processes in an asymptotic split state space. Note that in the literature these Markov processes with locally independent increments have also been called "weakly differentiable Markov processes" [9], or "locally infinitely divisible processes" [8], or "piecewise–deterministic Markov processes" [5]. These processes are of increasing interest in the literature because of their importance in applications, for which they constitute an alternative to diffusion processes. It is worth noticing that such processes include strictly the independent increment processes. For their detailed presentation and applications see [5].

The underlying mathematical tools for the results obtained here are based on the theory of singular perturbed reducible–invertible operators and on the martingale characterization of Markov processes [7, 28, 29]. We obtain thus average and diffusion approximation limit theorems and give examples of application of these results to reliability problems. The theoretical results proven here can be applied in the reliability modeling of large state space systems with high reliability, as well as in maintenance modeling, in performance evaluation, and so on. In the abstract reliability setting the following partition of the state space $E^0 = E \cup \{0\}$ holds, where E contains the working or up states and 0 is the down state. In this case, reliability concerns the distribution of the hitting time to the state 0 [13, 20].

Moreover, functionals such as the stochastic integral functional (1) can be used for modeling the maintenance cost up to the system failure. Storage jump processes with Markov switching can be used for modeling the so-called *dynamic reliability* of systems [6]. Actually, dynamic reliability is a new and more general model than the classical reliability models.

Up to now, the only method used for obtaining numerical results in the study of dynamic reliability and other applied problems based on hitting time models is the Monte Carlo method (see [6] and the references therein). The stochastic approximation results presented here constitute an alternative to this well-known method [10–12, 14, 24].

Results presented in this paper extend the results previously obtained by the authors concerning diffusion approximation by the following points: the class of locally independent increment processes $\eta^\varepsilon(t;x)$, $t \geq 0$, $x \in E$, constitute a wider class than the independent increment processes class studied in [13]; the switching processes $x^\varepsilon(t/\varepsilon)$, $t \geq 0$, $\varepsilon > 0$, are nonergodic with stoppage time; we consider simultaneously an asymptotic split of the state space; the three level stochastic systems considered, that is, the switching processes, switched processes and additive functional all depend upon the parameter ε; finally, in [18] we have also studied Poisson approximation of systems (1) when the process $\eta^\varepsilon(t,x)$ is just a pure jump Markov process and $x^\varepsilon(t/\varepsilon)$ a semi-Markov process but via semimartingale techniques and compensative operator for semi-Markov process.

The paper is organized as follows. In Section 2, we give the general setting of the processes considered here. In Section 3, we give some particular cases of the considered random evolution as well as examples of their potential application to dynamic reliability. In Section 4, we give a general phase merging scheme, that is, a simplification of the state space with single and double merging, allowing us to obtain average limit results for switching processes. In Section 5, we present general averaging results for the stochastic systems. In Section 6, we present diffusion approximation results in single and double merging schemes under balance conditions. In Section 7, we obtain a differential equation satisfied by functionals of reward up to stoppage time of the limit process. Finally, in Section 8, we give the proofs of the theorems of the previous sections.

2. Preliminaries. Let us be given the Euclidean space $\mathbb{R}^d$ with the Borel σ-algebra $\mathscr{B}_d$ and the compact measurable space $(E, \mathscr{E})$. It is worth noting that slightly changed conditions allow including a locally compact space of values for the switched process. We consider the family of right continuous with left limits (cadlag) time-homogeneous Markov processes $\eta^\varepsilon(t;x)$, $t \geq 0$, $x \in E$, with locally independent increments in the series scheme and a small series parameter $\varepsilon > 0$. These processes depend on the phase state $x \in E$, take values in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, and their generators are given by

$$\begin{aligned} \mathbb{\Gamma}_\varepsilon(x)\varphi(u) &= a_\varepsilon(u;x)\varphi'(u) \\ &\quad + \varepsilon^{-1}\int_{\mathbb{R}^d}[\varphi(u+\varepsilon v)-\varphi(u)-\varepsilon v\varphi'(u)]\Gamma_\varepsilon(u,dv;x). \end{aligned} \tag{2}$$

A complete characterization of the above generator is given in [5]. It is worth noticing that the drift velocity of $a_\varepsilon(u;x)$ in (2) contains an initial drift and the drift due to the jumps. Note also that $\eta^\varepsilon(\cdot,\cdot)$ contains no diffusion part (see, e.g., [5, 8, 9]).

REMARK 1. It is understood that in the case where $d>1$, we have

$$v\varphi'(u)=\sum_{k=1}^{d} v_k \frac{\partial\varphi}{\partial u_k}(u).$$

The drift velocity $a_\varepsilon(u;x)$ and the measure of the random jumps $\Gamma_\varepsilon(u,dv;x)$ depend on the state $x\in E$ and on the series parameter $\varepsilon>0$. The family of time-homogeneous cadlag Markov jump processes $x^\varepsilon(t)$, $t\geq 0$, $\varepsilon>0$, in the same series scheme taking values in the state space $(E,\mathcal{E})$, is given by its generators

$$Q^\varepsilon\varphi(x)=q(x)\int_E P^\varepsilon(x,dy)[\varphi(y)-\varphi(x)], \tag{3}$$

where q is the intensity of jumps, which is a nonnegative element of the Banach space $\mathbf{B}(E)$ of real bounded functions defined on the state space E, with the sup-norm, that is, $\|\varphi\|:=\sup_{x\in E}|\varphi(x)|$.

The stochastic evolutionary system with Markov switching is represented as follows:

$$\xi^\varepsilon(t)=\xi^\varepsilon(0)+\int_0^t \eta^\varepsilon\bigl(ds;x^\varepsilon(s/\varepsilon)\bigr). \tag{4}$$

The regular Markov jump process can be defined by the Markov renewal process $(x_n^\varepsilon,\tau_n^\varepsilon,n\geq 0)$, and $\theta_n^\varepsilon:=\tau_n^\varepsilon-\tau_{n-1}^\varepsilon$, $n\geq 1$, given by the semi-Markov kernel [23]

$$\begin{aligned} Q^\varepsilon(x,B,t)&=\mathbb{P}(x_{n+1}^\varepsilon\in B,\theta_{n+1}^\varepsilon\leq t\,|\,x_n^\varepsilon=x)\\ &=P^\varepsilon(x,B)\bigl(1-e^{-q(x)t}\bigr). \end{aligned} \tag{5}$$

Let us introduce the counting process

$$\nu^\varepsilon(t):=\max\{n:\tau_n^\varepsilon\leq t/\varepsilon\}, \tag{6}$$

with the renewal moments

$$\tau_n^\varepsilon=\sum_{k=1}^{n}\theta_k^\varepsilon, \qquad n\geq 1,\ \tau_0^\varepsilon=0,$$

and the auxiliary processes

$$\tau^\varepsilon(t)=\tau^\varepsilon_{\nu^\varepsilon(t)}, \qquad \theta^\varepsilon(t)=t/\varepsilon-\tau^\varepsilon(t).$$

The evolutionary system (4) can be represented also in the following form:

$$\xi^\varepsilon(t):=\xi^\varepsilon(0)+\sum_{k=0}^{\nu^\varepsilon(t)-1}\eta^\varepsilon(\varepsilon\theta_{k+1}^\varepsilon;x_k^\varepsilon)+\eta^\varepsilon\bigl(\varepsilon\theta^\varepsilon(t);x^\varepsilon(t/\varepsilon)\bigr). \tag{7}$$

3. Particular cases and examples of stochastic evolutionary systems. Let us give here four typical evolutionary systems as particular cases of the above system (1).

1. *A stochastic integral functional* is determined by

$$\alpha^\varepsilon(t) := \int_0^t a_\varepsilon\big(x^\varepsilon(s/\varepsilon)\big)\,ds, \tag{8}$$

where $a_\varepsilon(x)$, $x \in E$, $\varepsilon > 0$, is a family of real-valued measurable functions such that

$$\int_0^t \big|a_\varepsilon\big(x^\varepsilon(s)\big)\big|\,ds < +\infty \qquad \text{a.s. } t \geq 0, \varepsilon > 0.$$

We will consider $a_\varepsilon(x) = a(x) + \varepsilon a_1(x)$, for all $x \in E$, $\varepsilon > 0$.

The corresponding generators (2) have the following form:

$$\mathbb{\Gamma}_\varepsilon(x)\varphi(u) = a_\varepsilon(x)\varphi'(u). \tag{9}$$

2. *A dynamical system* with Markov switching is determined by a solution of the evolutionary equation

$$\frac{d}{dt}U^\varepsilon(t) := a_\varepsilon\big(U^\varepsilon(t); x^\varepsilon(t/\varepsilon)\big). \tag{10}$$

The respective generators (2) have the following form:

$$\mathbb{\Gamma}_\varepsilon(x)\varphi(u) = a_\varepsilon(u; x)\varphi'(u). \tag{11}$$

3. *The storage jump process* with Markov switching is determined by the generators

$$\begin{aligned} \mathbb{\Gamma}_\varepsilon(x)\varphi(u) &= a_\varepsilon(u; x)\varphi'(u) \\ &\quad + \varepsilon^{-1}\int_{\mathbb{R}^d}[\varphi(u + \varepsilon v) - \varphi(u) - \varepsilon v\varphi'(u)]\Gamma_\varepsilon(u, dv; x), \end{aligned} \tag{12}$$

where $a_\varepsilon(u; x)$ is exactly the drift of the jump part, that is,

$$a_\varepsilon(u; x) = \int_{\mathbb{R}^d} v\Gamma_\varepsilon(u, dv; x).$$

4. *A compound Poisson process* with Markov switching is determined by the generators

$$\begin{aligned} \mathbb{\Gamma}_\varepsilon(x)\varphi(u) &= a_\varepsilon(x)\varphi'(u) \\ &\quad + \varepsilon^{-1}\int_{\mathbb{R}^d}[\varphi(u + \varepsilon v) - \varphi(u) - \varepsilon v\varphi'(u)]\Gamma_\varepsilon(dv; x), \end{aligned} \tag{13}$$

where

$$a_\varepsilon(x) = \int_{\mathbb{R}^d} v\Gamma_\varepsilon(dv; x).$$

The most used stochastic model for a dynamic reliability system is a coupled Markov process $\xi(t)$, $x(t)$, $t \geq 0$, on the product phase space $\mathbb{R}^d \times E$, (see [6]). The second component $x(t)$, $t \geq 0$, describes the evolution of the structure of the system with a stoppage time (time to failure). The first component $\xi(t)$, $t \geq 0$, takes values in $\mathbb{R}^d$, and especially in the dynamic system case it describes the operational parameters of the system like temperature, pressure, velocity, and so on, or the reward rate of the system. The lifetime of such a system is defined by $T = \inf\{t \geq 0 : (\xi(t), x(t)) \in B \times \{0\}\}$, where B is a critical region of $\mathbb{R}^d$, and the reliability is defined by

$$r(t) := \mathbb{P}(T > t), \qquad t \geq 0.$$

Of course, when $B = \mathbb{R}^d$, we get that T is equal to the stoppage time of $x(t)$, $t \geq 0$.

The pure-jump Markov process $x(t)$, $t \geq 0$, can be defined by the Markov renewal process $(x_n, \tau_n, n \geq 0)$, where $x_n = x(\tau_n)$, $n \geq 0$, is the imbedded Markov chain, and τ_n is the renewal jump moment of the Markov process $x(t)$, $t \geq 0$. As usual, let us define the sojourn times $\theta_k = \tau_k - \tau_{k-1}$, $k \geq 1$, the counting process $\nu(t) = \max\{n : \tau_n \leq t\}$ and the processes $\theta(t) := t - \tau(t)$, $\tau(t) := \tau_{\nu(t)}$.

EXAMPLE 1. The integral reward of the system is represented by the following integral functional:

$$\alpha_t = \int_0^t a(x(s))\, ds, \tag{14}$$

which can be written as follows:

$$\alpha_t = \sum_{k=1}^{\nu(t)} \theta_k a(x_{k-1}) + \theta(t) a(x(t)). \tag{15}$$

This representation is only due to the jump evolution of the process $x(t)$, $t \geq 0$, and can be interpreted as the reward on the interval $[0, t]$, the function $a(x)$, $x \in E$, being the reward rate per unit time in state x.

The reliability system $x(t)$, $t \geq 0$, defined on the phase space $E^0 = E \cup \{0\}$, with the subspace of working states E and the stoppage state $\{0\}$ has the stoppage time $\zeta := \inf\{t : x(t) = 0\}$. The total reward of the system up to stoppage time is

$$\alpha_\zeta = \int_0^\zeta a(x(s))\, ds.$$

EXAMPLE 2. The dynamical reward rate of the system is described by the solution of the evolutionary equation

$$\frac{d}{dt} U(t) = C(U(t), x(t)), \qquad U(0) = 0. \tag{16}$$

It can be represented as follows:

$$U(t) = \sum_{k=1}^{\nu(t)} \int_{\tau_{k-1}}^{\tau_k} C\bigl(U_{k-1}(s), x_{k-1}\bigr)\, ds + \int_{\tau(t)}^{t} C\bigl(U_{\tau(t)}(s), x_{\tau(t)}\bigr)\, ds, \tag{17}$$

where $U_k(t)$, $k \geq 0$, are determined by the recurrent equations

$$\frac{d}{dt} U_k(t) = C\bigl(U_k(t), x_k\bigr), \qquad \tau_k \leq t < \tau_{k+1}, k \geq 0, \tag{18}$$

with the initial values

$$U_k(\tau_k) = U_{k-1}(\tau_k), \qquad k \geq 1, U_0(0) = 0. \tag{19}$$

The representation (17) can be interpreted as the dynamical reward on the interval $[0, t]$ with the velocity of reward $C(U(s), x)$, in the state $x \in E$, which depends not only on state x of the system but also on the instant value of rate $U(s)$ at time $s \in [0, t]$.

EXAMPLE 3. The stochastic reward rate of the system is defined by the storage processes $\eta(t; x)$, $t \geq 0$, $x \in E$, with the generators

$$\mathbb{\Gamma}(x)\varphi(u) = \Lambda(u; x) \int_{\mathbb{R}^d} [\varphi(u + v) - \varphi(u)]\Gamma(u, dv; x). \tag{20}$$

The intensity of jumps $\Lambda(u; x)$ and the distribution function of jump values $\Gamma(u, dv; x)$ depend on the state of the system $x \in E$.

Particularly, a birth-and-death process $\eta(t; x)$, $t \geq 0$, $x \in E$, defined by the intensity $\lambda(u; x)$ and $\mu(u; x)$ of jumps $+1$ and -1, respectively, can be considered as a stochastic reward rate of a system $x(t)$, $t \geq 0$, with the number of working devices $\eta(t; x)$ in the state x. Of course, the true reward rate can be given by a functional of $\eta(t; x)$.

4. Phase merging scheme. The Markov switching processes $x^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$, are considered in a split state space

$$E^0 = E \cup \{0\}, \qquad E = \bigcup_{k=1}^{N} E_k, \qquad E_k \cap E_{k'} = \varnothing, \qquad k \neq k', \tag{21}$$

with absorbing state $\{0\}$.

The phase merging algorithm is considered under the following assumptions.

A1. The stochastic kernel in (3) is represented in the following form:

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B), \tag{22}$$

where the stochastic kernel $P(x, B)$ is coordinated with the splitting (21) as follows:

$$P(x, E_k) = \mathbf{1}_k(x) := \begin{cases} 1, & x \in E_k, \\ 0, & x \notin E_k. \end{cases} \tag{23}$$

The Markov supporting process $x(t)$, $t \geq 0$, on the state space $(E, \mathcal{E})$, determined by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \tag{24}$$

is supposed to be uniformly ergodic in every class E_k, $1 \leq k \leq N$, with the stationary distribution $\pi_k(dx)$, $1 \leq k \leq N$, satisfying the following relations:

$$\pi_k(dx)q(x) = q_k\rho_k(dx), \qquad q_k = \int_{E_k} \pi_k(dx)q(x),$$

$$\rho_k(B) = \int_{E_k} \rho_k(dx)P(x, B), \qquad \rho_k(E_k) = 1.$$

Define the projector Π, by

$$\Pi\varphi(x) := \sum_{k=1}^{N} \int_{E_k} \pi_k(dy)\varphi(y)\mathbf{1}_k(x), \tag{25}$$

where $\mathbf{1}_k(x) = 1$ for all $x \in E_k$, $1 \leq k \leq N$, and 0 otherwise. Let us denote by R_0 the potential operator of the jump Markov process defined by [13],

$$R_0 Q = Q R_0 = \Pi - I.$$

Define also the operator P, as follows:

$$P\varphi(x) = \sum_{k=1}^{N} \int_{E_k} P(x, dy)\varphi(y)\mathbf{1}_k(x).$$

The perturbing kernel $P_1(x, B)$ is a signed kernel and determines the stoppage probabilities

$$P^\varepsilon(x, \{0\}) = -\varepsilon P_1(x, E) =: \varepsilon p(x). \tag{26}$$

In fact, we consider here that the initial Markov process $x^\varepsilon(\cdot)$ is a perturbation of the ergodic Markov process $x(\cdot)$.

A2. The stationary exit probabilities verify

$$p_k = \int_{E_k} \rho_k(dx)P_1(x, E \setminus E_k) > 0, \qquad 1 \leq k \leq N, N \geq 2. \tag{27}$$

A3. The stationary stoppage probabilities

$$p_{k0} = \int_{E_k} \rho_k(dx)p(x), \qquad 1 \leq k \leq N, \tag{28}$$

verify $\max_{1 \leq k \leq N} p_{k0} > 0$.

Let the merging function be

$$m(x) = k, \qquad x \in E_k, 1 \leq k \leq N \quad \text{and} \quad m(0) = 0.$$

LEMMA 1 ([13]). *Under the above assumptions* A1–A3, *the weak convergence*

$$m\big(x^\varepsilon(t/\varepsilon)\big) \Rightarrow \hat{x}(t) \qquad as\ \varepsilon \to 0,$$

takes place. The limit merged Markov process $\hat{x}(t)$, $t \geq 0$, *on the merged state space* $\hat{E}^0 = \{0; 1, 2, \dots, N\}$ *is determined by the generator matrix*

$$\hat{Q}^0 = (\hat{q}^0_{kr}; 0 \leq k, r \leq N), \tag{29}$$

with entries

$$\hat{q}^0_{kr} = q_k p_{kr}, \qquad p_{kr} = \int_{E_k} \rho_k(dx) P_1(x, E_r), \qquad 1 \leq k, r \leq N, \tag{30}$$

$$p_{k0} = \int_{E_k} \rho_k(dx) p(x), \qquad 1 \leq k \leq N. \tag{31}$$

REMARK 2. The representations (22) and (26) and the relations (27) and (28) yield

$$p_{kr} \geq 0, \qquad r \neq k; \qquad p_{kk} < 0, \qquad 1 \leq k \leq N,$$

and the following identity takes place:

$$\sum_{r=0}^{k} p_{kr} = 0, \qquad 1 \leq k \leq N. \tag{32}$$

It is easy to verify that $\{0\}$ is an absorbing state of the merged Markov process $\hat{x}(t)$, $t \geq 0$, with the generator matrix $\hat{Q}^0$, given by relations (29)–(31). The intensity of stoppage is

$$\hat{q}^0_{k0} = q_k p_{k0}, \qquad 1 \leq k \leq N.$$

The transition probabilities of the embedded Markov chain are defined by

$$\hat{p}_{kr} = -p_{kr}/p_{kk}, \qquad r \neq k.$$

Due to (32) the following identity takes place:

$$\sum_{r \neq k} \hat{p}_{kr} = 1, \qquad 1 \leq k \leq N.$$

EXAMPLE 4. In the particular case where $N = 1$, the merged Markov process $\hat{x}(t)$, $t \geq 0$, has the merged state space $\hat{E}^0 = \{0, 1\}$ with absorbing state $\{0\}$. The intensity of the sojourn time in state $\{1\}$ is

$$q = \int_E \pi(dx) q(x).$$

The stoppage probability is

$$p = \int_E \rho(dx) p(x).$$

The time to stoppage (or to failure)

$$\hat{\zeta} := \max\{t : \hat{x}(t) = 0\}$$

has the exponential distribution with the parameter $\hat{\Lambda} = qp$, that is,

$$\hat{r}(t) := \mathbb{P}(\hat{\zeta} > t) = e^{-\hat{\Lambda} t},$$

which is the approximating reliability of the system.

EXAMPLE 5. Consider a three-state Markov process, $E^0 = \{0, 1, 2\}$, with generator matrix

$$Q^\varepsilon = \begin{pmatrix} 0 & 0 & 0 \\ \varepsilon\lambda & -(1+\varepsilon)\lambda & \lambda \\ \varepsilon\mu & \mu & -(1+\varepsilon)\mu \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & \mu & -\mu \end{pmatrix}}_{Q} + \varepsilon \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \lambda & -\lambda & 0 \\ \mu & 0 & -\mu \end{pmatrix}}_{Q_1}.$$

The transition matrix of the embedded Markov chain is

$$P^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 0 & 1-\varepsilon \\ \varepsilon & 1-\varepsilon & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P} + \varepsilon \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}}_{P_1}.$$

Now for the ergodic process $x(t)$, $t \geq 0$, taking values in $E = \{1, 2\}$, and generator Q, we have

$$\pi = \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu}\right).$$

And for the ergodic embedded Markov chain $x_n, n \geq 0$, we have

$$\rho = (1/2, 1/2).$$

Thus, as we have $p(1) = -P_1(1, E) = 1$ and $p(2) = -P_1(2, E) = 1$, the stoppage probability is $p = 1$.

On the other hand we have

$$q(1) = \lambda, \qquad q(2) = \mu.$$

Hence,

$$q = \pi_1 q(1) + \pi_2 q(2) = \frac{2\lambda\mu}{\lambda+\mu}$$

and

$$\hat{\Lambda} = qp = \frac{2\lambda\mu}{\lambda + \mu}.$$

The limit of the distribution of the normalized stoppage time is $\mathbb{P}(\hat{\zeta} > t) = \exp(-\hat{\Lambda}t)$.

The double merging algorithm can be used in the split phase space with $N > 1$ as given in the following lemma.

LEMMA 2. *Let the Markov jump process $x^\varepsilon(t)$, $t \ge 0$, on the split state space* (21) *be given by the generator* (3) *with the stochastic kernel represented as follows*:

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B) + \varepsilon^2 P_2(x, B), \tag{33}$$

where the stochastic kernel $P(x, B)$ satisfies condition (23) *and the perturbing kernel $P_1(x, B)$ satisfies the condition*

$$P_1(x, E) = 0.$$

The second perturbing kernel determines the stoppage probabilities

$$P^\varepsilon(x, \{0\}) = -\varepsilon^2 P_2(x, E) =: \varepsilon^2 p(x).$$

Then, under assumptions A2 *and* A3, *the weak convergence*

$$m\big(x^\varepsilon(t/\varepsilon)\big) \Rightarrow \hat{x}(t) \qquad \text{as } \varepsilon \to 0$$

takes place. The limit Markov merged process $\hat{x}(t)$, $t \ge 0$, on the merged state space $\hat{E} = \{1, 2, \dots, N\}$ is determined by the generating matrix $\hat{Q} = (\hat{q}_{kr}; 1 \le k, r \le N)$ with entries

$$\hat{q}_{kr} = q_k p_{kr} = q_k \int_{E_k} \rho_k(dx) P_1(x, E_r).$$

Under the additional condition of ergodicity of the merged Markov process $\hat{x}(t)$, $t \ge 0$, with stationary distribution $\hat{\pi} = (\hat{\pi}_k, 1 \le k \le N)$, the weak convergence

$$\hat{m}\big(x^\varepsilon(t/\varepsilon^2)\big) \Rightarrow \hat{\hat{x}}(t) \qquad \text{as } \varepsilon \to 0$$

takes place. The merging function $\hat{m}$ is defined by

$$\hat{m}(x) = \begin{cases} 1, & x \in E, \\ 0, & x = 0. \end{cases}$$

The limit double merged Markov renewal process $\hat{\hat{x}}(t)$, $t \ge 0$, is defined on the state space $\hat{\hat{E}} = \{1, 0\}$ by the intensity of sojourn time in state $\{1\}$, $\hat{\hat{q}} = \sum_{k=1}^N \hat{\pi}_k \hat{q}_k$, and the stoppage probability

$$\hat{\hat{p}} = \sum_{k=1}^N \hat{q}_k \hat{p}_{k0}, \qquad \hat{p}_{k0} := \int_{E_k} \rho_k(dx) p(x), \qquad 1 \le k \le N.$$

REMARK 3.

1. The Markov process $x^\varepsilon(t/\varepsilon)$, $t \geq 0$, in the phase merging scheme of Lemma 1 is determined by the generator
$$Q^\varepsilon = \varepsilon^{-1} Q + Q_1,$$
where Q is the generator given by (24), and
$$Q_1\varphi(x) = q(x) \int_E P_1(x, dy)\varphi(y).$$
2. The Markov process $x^\varepsilon(t/\varepsilon^2)$, $t \geq 0$, on the double merging scheme of Lemma 2 has the following generator:
$$Q^\varepsilon = \varepsilon^{-2} Q + \varepsilon^{-1} Q_1 + Q_2,$$
with the operator
$$Q_2\varphi(x) = q(x) \int_E P_2(x, dy)\varphi(y).$$
3. The phase merging algorithms are based on the solution of the singular perturbation problems [13].

For averaging and diffusion approximation results that follow in Sections 5 and 6, we need the following assumption.

ASSUMPTION A. Convergence in probability of the initial values of $\xi^\varepsilon(t)$, $m(x^\varepsilon(t/\varepsilon))$, $t \geq 0$, hold, that is,
$$\big(\xi^\varepsilon(0), m(x^\varepsilon(0))\big) \xrightarrow{P} \big(\xi(0), \hat{x}(0)\big),$$
and there exist a $c \in \mathbb{R}_+$, such that
$$\sup_{\varepsilon > 0} \mathbb{E}|\xi^\varepsilon(0)| \leq c < +\infty.$$

5. Average approximation scheme. In this section we will give two theorems for the averaging evolutionary system $\xi^\varepsilon(t)$ in single and double averaging of the switching Markov processes $x^\varepsilon(t)$, respectively.

In what follows the following Banach spaces will be used endowed by the corresponding sup-norms.

1. $\mathbf{B}$ is the Banach space of real-valued measurable bounded functions $\varphi(u, x)$, $u \in \mathbb{R}^d$, $x \in E$.
2. $\mathbf{B}^1 := C^1(\mathbb{R}^d \times E) \cap \mathbf{B}$ is the Banach space of continuously differentiable functions on $u \in \mathbb{R}^d$ uniformly on $x \in E$ with bounded first derivative.
3. $\mathbf{B}^2 := C^2(\mathbb{R}^d \times E) \cap \mathbf{B}$ is the Banach space of twice continuously differentiable functions on $u \in \mathbb{R}^d$ uniformly on $x \in E$ with bounded first two derivatives.

THEOREM 1 (Average approximation). *Let the stochastic evolutionary system $\xi^\varepsilon(t)$ be represented by*

$$\xi^\varepsilon(t) = \xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon)\big), \qquad t \ge 0, \varepsilon > 0. \tag{34}$$

Let the process $\eta^\varepsilon(t; x)$, $t \ge 0$, $x \in E$, $\varepsilon > 0$, be given by the generators (2). *Let that the switching Markov process $x^\varepsilon(t)$, $t \ge 0$, satisfies the phase merging condition of Lemma* 1.

Let the following conditions be valid.

C1. *The drift velocity $a(u; x)$ belongs to the Banach space $\mathbf{B}^1$, with*

$$a_\varepsilon(u; x) = a(u; x) + \theta^\varepsilon(u; x),$$

where $\theta^\varepsilon(u; x)$ goes to 0 *as $\varepsilon \to 0$ uniformly on $(u; x)$ and $\Gamma_\varepsilon(u, dv; x) \equiv \Gamma(u, dv; x)$ independent of ε.*

C2. *The operator*

$$\gamma_\varepsilon(x)\varphi(u) = \varepsilon^{-1} \int_{\mathbb{R}^d} [\varphi(u + \varepsilon v) - \varphi(u) - \varepsilon v \varphi'(u)]\Gamma(u, dv; x)$$

is negligible on $\mathbf{B}^1$,

$$\sup_{\varphi \in C^1(\mathbb{R}^d)} \|\gamma_\varepsilon(x)\varphi\| \to 0 \qquad \text{as } \varepsilon \to 0.$$

C3. *Assumption* A *holds.*

Then the stochastic evolutionary system $\xi^\varepsilon(t)$, $t \ge 0$, defined by relation (34), *converges weakly to the averaged stochastic system $\hat{U}(t \wedge \hat{\zeta})$*

$$\xi^\varepsilon(t) \Rightarrow \hat{U}(t \wedge \hat{\zeta}) \qquad \text{as } \varepsilon \to 0.$$

The limit process $\hat{U}(t)$, $t \ge 0$, is defined by a solution of the evolutionary equation

$$\frac{d}{dt}\hat{U}(t) = \hat{a}\big(\hat{U}(t), \hat{x}(t)\big), \qquad \hat{U}(0) = 0,$$

on the time interval $0 \le t \le \hat{\zeta}$ [$\hat{\zeta}$ is the stoppage time of the merged Markov process $\hat{x}(t)$, $t \ge 0$].

The averaged velocity is determined by

$$\hat{a}(u; k) = \int_{E_k} \pi_k(dx) a(u; x), \qquad 1 \le k \le N, \qquad \hat{a}(u; 0) = 0.$$

REMARK 4. There is no change in the result if we consider a dependent on ε measure of random jumps in the form $\Gamma_\varepsilon(u, dv; x) = \Gamma(u, dv; x) + \varepsilon\Gamma_1(u, dv; x)$.

REMARK 5. The operator $\gamma_\varepsilon(x)$ is the jump part after extraction of the drift part due to the jumps of the process $\eta^\varepsilon(\cdot,\cdot)$.

The following corollary gives particular results of Theorem 1, in the four cases described in Section 3.

COROLLARY 1.

1. *The stochastic integral functional* (8) *converges weakly as follows*:
$$\int_0^t a(x^\varepsilon(s/\varepsilon))\,ds \Rightarrow \int_0^{t\wedge\hat{\zeta}} \hat{a}(\hat{x}(s))\,ds \qquad \text{as } \varepsilon\to 0,$$
where
$$\hat{a}(k) = \int_{E_k} \pi_k(dx)a(x).$$
In the particular case where $N=1$, *the stochastic integral functional converges weakly*,
$$\int_0^t a\big(x^\varepsilon(s/\varepsilon)\big)\,ds \Rightarrow \hat{a}\cdot(t\wedge\hat{\zeta}) \qquad \text{as } \varepsilon\to 0, \qquad \hat{a} = \int_E \pi(dx)a(x).$$
2. *The dynamical system defined by* (10) *converges weakly to a dynamical system with a simplest switching process* $\hat{x}(t)$, $t\ge 0$, *instead of the initial one* $x^\varepsilon(t)$, $t\ge 0$.
3. *The storage jump process with Markov switching defined by the generators of* (12) *converges weakly as follows*:
$$\xi^\varepsilon(t) \Rightarrow \hat{V}(t\wedge\hat{\zeta}) \qquad \text{as } \varepsilon\to 0,$$
where the averaged process $\hat{V}(t)$, $t\ge 0$, *is determined by a solution of the evolutionary equation*
$$\frac{d}{dt}\hat{V}(t) = \hat{a}\big(\hat{V}(t),\hat{x}(t)\big), \qquad \hat{V}(0)=0.$$
4. *The compound Poisson process with Markov switching defined by the generators of* (13) *converges weakly as follows*:
$$\xi^\varepsilon(t) \Rightarrow \int_0^{t\wedge\hat{\zeta}} \hat{a}(\hat{x}(s))\,ds \qquad \text{as } \varepsilon\to 0.$$
Of course functions a *and* $\hat{a}$ *given in cases* 1 *and* 4 *are different functions from those given in Section* 3.

The following theorem concerns averaging result for the evolutionary system $\xi^\varepsilon(t)$ in the double merging scheme (33). For an ergodic double average of integral functionals see [16].

Define $\hat{\hat{\zeta}}$ the stoppage time of the process $\hat{\hat{x}}(t)$, by

$$\hat{\hat{\zeta}} = \min\{t : \hat{\hat{x}}(t) = 0\}.$$

THEOREM 2 (Double average). *Let the switching Markov process $x^\varepsilon(t)$, $t \geq 0$, satisfy the conditions of the double merging scheme* (*Lemma* 2). *Let the stochastic system be represented as follows*:

$$\xi^\varepsilon(t) = \xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon^2)\big),$$

where the processes $\eta^\varepsilon(t; x)$, $t \geq 0$, $x \in E$, are given by the generator of (2). *Let conditions* C1–C3 *of Theorem* 1 *be true*.

Then the weak convergence

$$\xi^\varepsilon(t) \Rightarrow \hat{\hat{U}}\big(t \wedge \hat{\hat{\zeta}}\big) \qquad \text{as } \varepsilon \to 0$$

takes place. The limit double averaged system $\hat{\hat{U}}(t)$, $t \geq 0$, is defined by a solution of the equation

$$\frac{d}{dt}\hat{\hat{U}}(t) = \hat{\hat{a}}\Big(\hat{\hat{U}}(t), \hat{\hat{x}}(t)\Big), \qquad \hat{\hat{U}}(0) = 0, \qquad \hat{\hat{a}}(u; 1) = \sum_{k=1}^{N} \hat{\pi}_k \hat{a}(u; k).$$

The stoppage time $\hat{\hat{\zeta}}$ has an exponential distribution with the parameter

$$\hat{\hat{\Lambda}} = qp \qquad (\text{see Lemma } 2).$$

REMARK 6. A result analogous to Corollary 1 can be obtained for the double merged process $\hat{\hat{x}}(t)$, $t \geq 0$, in the cases of stochastic integral (8), of the storage jump process with Markov switching (12) and of the compound Poisson process with Markov switching (13) [17].

6. Diffusion approximation scheme. The split state space $E^0 = E \cup \{0\}$ is considered for simplicity with $N = 1$. So, the supporting Markov process $x(t)$, $t \geq 0$, defined by the generator of (24) is uniformly ergodic on E with the stationary distribution $\pi(dx)$.

The main assumption in this section is that the balance condition says that the stationary average of the fast motion is equal to zero,

$$\hat{a}(u) = \int_E \pi(dx) a(u; x) \equiv 0. \tag{35}$$

We consider here the following additive functional

$$\xi^\varepsilon(t) = \xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon^2)\big),$$

THEOREM 3 (Diffusion approximation). *Let the processes* $\eta^{\varepsilon}(t;x)$, $t \geq 0$, $x \in E$, *be defined by the generators* (2) *with the drift velocity*

$$a_{\varepsilon}(u;x) = \varepsilon^{-1} a(u;x) + a_1(u;x) \tag{36}$$

and let the measures of random jumps be

$$\Gamma_{\varepsilon}(u, dv; x) = \varepsilon^{-1}\Gamma(u, dv; x) + \Gamma_1(u, dv; x). \tag{37}$$

Let the following conditions hold.

D1. *The drift velocity functions* $a(u;x)$ *and* $a_1(u;x)$ *belong to the Banach space* $\mathbf{B}^2$.

D2. *The operators*

$$\gamma_{\varepsilon}(x)\varphi(u) = \varepsilon^{-1}\int_{\mathbb{R}^d}\left[\varphi(u+\varepsilon v) - \varphi(u) - \varepsilon v\varphi'(u) - \frac{\varepsilon^2}{2}v^2\varphi''(u)\right]\Gamma_{\varepsilon}(u, dv; x)$$

are negligible on $\mathbf{B}^2$, *that is*,

$$\sup_{\varphi \in C^2(\mathbb{R}^d)} \|\gamma_{\varepsilon}(x)\varphi\| \to 0 \qquad \text{as } \varepsilon \to 0.$$

D3. *The switching Markov process* $x^{\varepsilon}(t)$, $t \geq 0$, *is defined by the generator* (3) *with the stochastic kernel*

$$P^{\varepsilon}(x, B) = P(x, B) + \varepsilon^2 P_1(x, B),$$

where the kernels $P(x, B)$ *and* $P_1(x, B)$ *satisfy assumptions* A1–A3 *of the phase merging scheme* (*Lemma* 1).

D4. *Assumption* A *holds*.

Then, *under the balance condition* (35), *the weak convergence*

$$\xi^{\varepsilon}(t) \Rightarrow \hat{\xi}(t \wedge \hat{\zeta}) \qquad \text{as } \varepsilon \to 0$$

takes place. *The limit diffusion process* $\hat{\xi}(t)$, $t \geq 0$, *is defined by the generator*

$$\hat{L}\varphi(u) = \hat{b}(u)\varphi'(u) + \tfrac{1}{2}\hat{B}(u)\varphi''(u). \tag{38}$$

The drift coefficient is defined by

$$\hat{b}(u) = \hat{a}_1(u) + \hat{b}_1(u),$$

where

$$\hat{a}_1(u) = \int_E \pi(dx)a_1(u;x) \quad \text{and} \quad \hat{b}_1(u) = \int_E \pi(dx)a(u;x)R_0 a'_u(u;x).$$

The covariance function is defined by

$$\hat{B}(u) = 2\int_E \pi(dx)[a(u;x)R_0 a(u;x) + C_0(u;x)],$$

where

$$C_0(u;x) = \frac{1}{2}\int_{\mathbb{R}^d} vv^*\Gamma(u,dv;x),$$

where v^* *is the transpose of the vector* v, *and* R_0 *is the potential operator of* Q (*see* [13]),

$$QR_0 = R_0 Q = \Pi - I.$$

The following corollary gives particular results of Theorem 3 in the four cases given in Section 3.

COROLLARY 2.

1. *The stochastic integral functional* (8) *converges weakly*,

$$\int_0^t a^\varepsilon\bigl(x^\varepsilon(s/\varepsilon^2)\bigr)\,ds \Rightarrow \hat{\xi}(t\wedge\hat{\zeta}) \qquad as\ \varepsilon\to 0.$$

   The limit process $\hat{\xi}(t)$, $t \ge 0$, *is a diffusion process with generator* (38), *where*

$$b(u) \equiv \hat{a}_1 = \int_E \pi(dx)a_1(x) \quad and \quad \hat{B}(u) \equiv \int_E \pi(dx)a(x)R_0 a(x).$$

2. *The dynamical system* (10) *converges weakly to a diffusion as in the above theorem.*
3. *The storage jump process with Markov switching defined by the generators* (12) *with*

$$a(u;x) = \int_E v\Gamma(u,dv;x) \quad and \quad a_1(u;x) = \int_E v\Gamma_1(u,dv;x)$$

   converges weakly as follows:

$$\xi^\varepsilon(t/\varepsilon^2) \Rightarrow \hat{\xi}(t\wedge\hat{\zeta}) \qquad as\ \varepsilon\to 0,$$

   where the limit process $\hat{\xi}(t)$, $t \ge 0$, *is a diffusion process with generator* (38), *where*

$$\hat{b}(u) = \int_E \pi(dx)b(u;x), \qquad \hat{B}(u) = \int_E \pi(dx)C_0(u;x),$$
$$b(u;x) = \int_{\mathbb{R}^d} v\Gamma(u,dv;x), \qquad C_0(u;x) = \frac{1}{2}\int_{\mathbb{R}^d} vv^*\Gamma(u,dv;x).$$

   In cases where $d > 1$, $C_0(u,x)$ *is a matrix function.*
4. *The compound Poisson process with Markov switching defined by generators* (13) *weakly converges*,

$$\xi^\varepsilon(t/\varepsilon^2) \Rightarrow \hat{\xi}(t\wedge\hat{\zeta}) \qquad as\ \varepsilon\to 0,$$

where the limit process $\hat{\xi}(t)$, $t \geq 0$, *is a diffusion process with generator* (38), *with drift*

$$\hat{a}_1 = \int_E \pi(dx) a_1(x), \qquad a_1(x) = \int_{\mathbb{R}^d} v \Gamma_1(dv; x),$$

and covariance function

$$\hat{B} = \int_E \pi(dx)[a_0(x) + C_0(x)], \qquad a_0(x) = a(x) R_0 a(x),$$

$$C_0(x) = \tfrac{1}{2} \int_{\mathbb{R}^d} v v^* \Gamma(dv; x).$$

A diffusion approximation in a phase double merging scheme (Lemma 2) is realized by the following theorem.

THEOREM 4 (Diffusion approximation in double merging scheme). *Let the switching Markov process* $x^\varepsilon(t)$, $t \geq 0$, *be defined by the generator* (3) *with the stochastic kernel*

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B) + \varepsilon^2 P_2(x, B),$$

where the kernels $P(x, B)$ *and* $P_k(x, B)$, $k = 1, 2$, *satisfy the conditions of Lemma* 2. *The processes* $\eta^\varepsilon(t; x)$, $t \geq 0$, $x \in E$, *are given by generators* (2) *with characteristics* (36) *and* (37).

Let conditions C1–C3 *of Theorem* 1 *hold. Then, under balance condition* (35) *the weak convergence*

$$\xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon^3)\big) \Rightarrow \hat{\hat{\xi}}\big(t \wedge \hat{\hat{\zeta}}\big) \qquad \text{as } \varepsilon \to 0$$

takes place. The limit diffusion process $\hat{\hat{\xi}}(t)$, $t \geq 0$, *is defined by the generator*

$$\hat{\hat{L}}\varphi(u) = \hat{\hat{b}}(u)\varphi'(u) + \hat{\hat{B}}(u)\varphi''(u). \tag{39}$$

The drift coefficient is defined by

$$\hat{\hat{b}}(u) = \hat{\hat{a}}_1(u) + \hat{\hat{b}}_1(u),$$

$$\hat{\hat{a}}_1(u) = \sum_{k=1}^N \hat{\pi}_k \hat{a}_1(u; k), \qquad \hat{a}_1(u; k) = \int_{E_k} \pi_k(dx) a_1(u; x),$$

$$\hat{\hat{b}}_1(u) = \sum_{k=1}^N \hat{\pi}_k b_1(u; k), \qquad b_1(u; k) = \hat{a}(u; k) \hat{R}_0 \hat{a}'_u(u; k).$$

The covariance function is defined by

$$\hat{\hat{B}}(u) = \sum_{k=1}^{N} \hat{\pi}_k \hat{B}_k(u),$$

$$\hat{B}_k(u) = \hat{a}(u;k)\hat{R}_0\hat{a}(u;k) + \hat{C}_0(u;k), \qquad \hat{C}_0(u;k) = \int_{E_k} \hat{\pi}_k(dx) C_0(u;x).$$

Here, the operators R_0 *and* $\hat{R}_0$ *are the potential operators for* Q *and* $\hat{Q}^0$, *respectively.*

COROLLARY 3.

1. *The stochastic integral functional* (8) *converges weakly*,

$$\int_0^t a^\varepsilon(x^\varepsilon(s/\varepsilon^3))\,ds \Rightarrow \hat{\hat{\xi}}(t \wedge \hat{\hat{\zeta}}) \qquad \text{as } \varepsilon \to 0.$$

   The limit process $\hat{\hat{\xi}}(t)$, $t \ge 0$, *is a diffusion process having generator* (39), *with*

$$\hat{\hat{b}}(u) \equiv \hat{\hat{a}}_1 = \sum_{k=1}^{N} \hat{\pi}_k \hat{a}_1(x), \qquad \hat{a}_1(k) = \int_{E_k} \pi_k(dx) a_1(x),$$

$$\hat{\hat{B}}(u) \equiv \sum_{k=1}^{N} \hat{\pi}_k \hat{B}_k, \qquad \hat{B}_k = \hat{a}(k)\hat{R}_0\hat{a}(k).$$

2. *The storage jump process with Markov switching defined by generators* (12) *weakly converges*,

$$\xi^\varepsilon(t/\varepsilon^3) \Rightarrow \hat{\hat{\xi}}(t \wedge \hat{\hat{\zeta}}) \qquad \text{as } \varepsilon \to 0,$$

   where the limit process $\hat{\hat{\xi}}(t)$, $t \ge 0$, *is a diffusion process with generator* (39), *where*

$$\hat{\hat{b}} \equiv 0, \qquad \hat{\hat{B}}(u) = \sum_{k=1}^{N} \hat{\pi}_k \hat{C}(u;k), \qquad \hat{C}(u;k) = \int_{E_k} \hat{\pi}_k(dx) C(u;x).$$

3. *The compound Poisson process with Markov switching defined by by generators* (13) *converges weakly*,

$$\xi^\varepsilon(t/\varepsilon^3) \Rightarrow \hat{\hat{\xi}}(t \wedge \hat{\hat{\zeta}}) \qquad \text{as } \varepsilon \to 0,$$

   where the limit process $\hat{\hat{\xi}}(t)$, $t \ge 0$, *is a diffusion process having generator* (39), *with*

$$\hat{\hat{b}} \equiv 0, \qquad \hat{\hat{B}} = \sum_{k=1}^{N} \hat{\pi}_k \hat{C}(k), \qquad \hat{C}(k) = \int_{E_k} \hat{\pi}_k(dx) C(x).$$

7. Reward of stochastic system per operating time. A function of particular interest in applications is the reward of operating time, that is, $\xi^\varepsilon \circ \zeta^\varepsilon$, where ζ^ε defined in the previous section is the stoppage time of the switching Markov process $x^\varepsilon(t)$.

Let us define the mean reward of operating time by the following relation:

$$W^\varepsilon(u) := \mathbb{E}\big[\xi^\varepsilon(\zeta^\varepsilon)|\xi^\varepsilon(0) = u\big], \tag{40}$$

and the limit reward by

$$\hat{W}(u) := \lim_{\varepsilon\to 0} W^\varepsilon(u) = \mathbb{E}\big[U(\hat{\zeta})|U(0) = u\big]. \tag{41}$$

Then we have (see Example 4),

$$\begin{aligned} \hat{W}(u) &= \hat{\Lambda}\int_0^\infty e^{-\hat{\Lambda}t}U(t)\,dt \\ &= u + \int_0^\infty e^{-\hat{\Lambda}t}U'(t)\,dt \\ &= \int_0^\infty e^{-\hat{\Lambda}t}\hat{a}(U(t))\,dt. \end{aligned} \tag{42}$$

The functional limit reward is

$$\begin{aligned} \tilde{\Phi}(u) &= \mathbb{E}\big[\Phi(U(\hat{\zeta}))|U(0) = u\big] \\ &= \hat{\Lambda}\int_0^\infty e^{-\hat{\Lambda}t}\Phi(U(t))\,dt \\ &= \hat{\Lambda}\int_0^\infty e^{-\hat{\Lambda}t}A_t\,dt\,\Phi(u), \end{aligned} \tag{43}$$

where the semigroup A_t is defined by

$$A_t\varphi(u) := \varphi(U(t)),$$

and $U(t)$ is a solution of the equation

$$\frac{d}{dt}U(t) = \hat{a}(U(t)), \qquad U(0) = u.$$

The generator of this semigroup, denoted by A, is defined by

$$A\varphi(u) := \hat{a}(u)\varphi'(u).$$

Thus

$$\tilde{\Phi}(u) = \hat{\Lambda}R_{\hat{\Lambda}}\Phi(u), \tag{44}$$

where $R_{\hat{\Lambda}}$ is the resolvent of the semigroup

$$\hat{\Lambda}R_{\hat{\Lambda}} = I + AR_{\hat{\Lambda}},$$

or

$$(\hat{\Lambda} - A)R_{\hat{\Lambda}} = I.$$

The following equation is verified by the functional reward:

$$\hat{a}(u)\hat{\Phi}'(u) - \hat{\Lambda}\hat{\Phi}(u) = \hat{\Lambda}\Phi(u). \tag{45}$$

An initial value can be

$$\hat{\Phi}(0) = \hat{\Lambda}\int_0^\infty e^{-\hat{\Lambda}t}\hat{a}(U_t^0)\,dt, \qquad U_0^0 = 0.$$

8. Proofs of theorems. We will prove only Theorems 1 and 3. The proofs of other theorems are similar.

8.1. *Proof of Theorem* 1. The proof of Theorem 1 will be realized by the following approach. First, we will establish the convergence of generators of some Markov processes by a singular perturbation of the linear operator technique. Second, we will prove the compactness property of the stochastic processes by proving the compact containment condition and the submartigale condition. Finally, we will establish the convergence results by applying Theorem 8.10 of [7] adapted to our conditions.

The generator of the coupled Markov processes $\xi^\varepsilon(t)$, $x^\varepsilon(t/\varepsilon)$, $t \ge 0$, in Theorem 1 is

$$L^\varepsilon = \varepsilon^{-1}Q + Q_1 + \mathbb{\Gamma}(x) + \gamma_\varepsilon(x) + \Xi^\varepsilon(x),$$

where operator Q is defined in (24), operator Q_1 in Remark 3, operator $\mathbb{\Gamma}(x)$ is defined by $\mathbb{\Gamma}(x)\varphi(u) = a(u;x)\varphi'(u)$, operator $\Xi^\varepsilon(x)$ is defined by $\Xi^\varepsilon(x)\varphi(u) = \theta^\varepsilon(u;x)\varphi'(u)$, and operator $\gamma_\varepsilon(x)$ is defined in condition C2 of Theorem 1. All the above operators are bounded ones. Of course, the operator $\gamma_\varepsilon(x) + \Xi^\varepsilon(x)$ is a negligible one on $\mathbf{B}^1$.

Let R_0 be the potential of the operator Q, that is, $R_0 = [Q + \Pi]^{-1} - \Pi$. Let $\overline{C}_0^2(\mathbb{R}^d \times \hat{E}^0)$ be the space of measurable bounded functions $\varphi(u, v)$ with compact support and twice continuously differentiable on the first argument.

LEMMA 3 ([13]). *The asymptotic representation of*

$$[\varepsilon^{-1}Q + Q_1 + \mathbb{\Gamma}(x)]\varphi^\varepsilon(u, x) = L\hat{\varphi} + \varepsilon\Theta^\varepsilon(x), \tag{46}$$

with $\varphi^\varepsilon(u, x) = \varphi(u, m(x)) + \varepsilon\varphi_1(u, x)$, $\hat{\varphi} = \hat{\varphi}(u, v) \in \overline{C}_0^2(\mathbb{R}^d \times \hat{E}^0)$, *is realized by*

$$L(x) = \hat{Q}_1 + \hat{\mathbb{\Gamma}}(x),$$

where contracted operators $\hat{Q}_1$, $\hat{\mathbb{\Gamma}}(x)$ *are defined by*

$$\Pi Q_1 \Pi = \hat{Q}_1 \Pi \quad \textit{and} \quad \Pi \mathbb{\Gamma}(x)\Pi = \hat{\mathbb{\Gamma}}(x)\Pi$$

and

$$\varphi_1 = R_0[L - L(x)]\varphi, \tag{47}$$

$$\Theta^\varepsilon(x) = \big(Q_1 + \mathbb{\Gamma}(x)\big)R_0\big[L - \big(Q_1 + \mathbb{\Gamma}(x)\big)\big]; \tag{48}$$

hence, $\Theta^\varepsilon(x)$ *is a bounded operator independent of* ε.

Let us consider the function $\varphi_0 : \mathbb{R} \to [1, +\infty)$ defined by $\varphi_0(u) = \sqrt{1+u^2}$; thus $\varphi_0'(u) = 2u/\sqrt{1+u^2}$. Hence,

$$|\varphi_0'(u)| \le 2 \le 2\varphi_0(u), \qquad |\varphi_0''(u)| \le 2 \le 2\varphi_0(u), \qquad u \in \mathbb{R}. \tag{49}$$

We obtain first the following inequality:

$$\begin{aligned} L^\varepsilon\varphi_0(u) &= \varepsilon^{-1}Q\varphi_0(u) + Q_1\varphi_0(u) + \mathbb{\Gamma}(x)\varphi_0(u) \\ &= a_\varepsilon(u;x)\varphi_0'(u) \le 2|a_\varepsilon(u;x)|\varphi_0(u) \\ &\le C_a\varphi_0(u). \end{aligned} \tag{50}$$

Now, by Lemma 4 of [17], we get the following compact containment condition.

LEMMA 4 [17]. *If*

$$Pb \in \mathbf{B}, \qquad b \in \mathbf{B}(E) \tag{51}$$

and

$$\mathbb{E}[|\xi^\varepsilon(0)|] \le c < +\infty, \tag{52}$$

then the family of processes (34)

$$\xi^\varepsilon(t) = \xi^\varepsilon(0) + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon)\big), \qquad t \ge 0, \varepsilon > 0, \tag{53}$$

satisfies the compact containment condition (*see* [7], *page* 129)

$$\lim_{l\to\infty} \sup_{\varepsilon>0} \mathbb{P}^\varepsilon\Big(\sup_{0\le t\le T} |\xi^\varepsilon(t)| \ge l\Big) = 0. \tag{54}$$

For any nonnegative function $\varphi \in C_0^\infty(\mathbb{R})$, we get the following inequality:

$$\begin{aligned} |L^\varepsilon\varphi(u)| &= |Q_1\varphi(u) + \mathbb{\Gamma}(x)\varphi(u)| \\ &\le q\varphi(u) + |a_\varepsilon(u;x)||\varphi'(u)| \\ &\le C_\varphi. \end{aligned} \tag{55}$$

Hence, similarly to Lemma 5 in [17], we get that the family of processes $\xi^\varepsilon(t)$, $t \ge 0$, $\varepsilon > 0$, is tight in $D_\mathbb{R}[0, T]$ for every $T > 0$.

The following convergence theorem is an adaptation of our conditions of Theorem 8.10, page 234 in [7]. See also [17]. Let us define the state space $V = \hat{E}^0 = \{0, 1, \dots, N\}$ of the merged process $\hat{x}(\cdot)$.

An algebra $\mathcal{A} \subset \overline{\mathcal{D}(E)}$ is called separating, if, whenever $P, Q \in \mathcal{P}(E)$ (the set of all probability measures on E), and $\int f\,dP = \int f\,dQ$ for $f \in \mathcal{A}$, we have $P = Q$ (see [7]).

THEOREM A. *Suppose the generator L of the coupled Markov process $\xi(t)$, $\hat{x}(t)$, $t \geq 0$, on the state space $\mathbb{R}^d \times V$, has at most one solution of a martingale problem in $D[0,\infty)$, and that the closure of the domain $\mathcal{D}(L)$ contains a separating algebra $\mathcal{A}$.*

Suppose the family of Markov processes $\xi^\varepsilon(t)$, $x^\varepsilon(t)$, $t \geq 0, \varepsilon > 0$ on $\mathbb{R}^d \times E$ defined by the generators $L^\varepsilon, \varepsilon > 0$, with domains $\mathcal{D}(L^\varepsilon)$ dense in $\overline{C}(\mathbb{R}^d \times E)$, satisfies the following conditions.

C1. *There exists a collection of functions $\varphi^\varepsilon(u,x) \in \overline{C}(\mathbb{R}^d \times E)$, such that the following uniform convergence takes place:*

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(u,x) = \varphi(u, m(x)) \in \overline{C}(\mathbb{R}^d \times V) \tag{56}$$

and such that for every $T > 0$,

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E}\bigl|\varphi^\varepsilon\bigl(u, x^\varepsilon(t)\bigr)\bigr| < +\infty. \tag{57}$$

C2. *The uniform convergence of generators*

$$\lim_{\varepsilon \to 0} L^\varepsilon \varphi^\varepsilon(u,x) = L\varphi(u, m(x)), \tag{58}$$

takes place, the functions $L^\varepsilon\varphi^\varepsilon$, $\varepsilon > 0$, are uniformly bounded, and $L\varphi \in \overline{C}(\mathbb{R}^d \times V)$.

C3. *The family of probability measures $(P^\varepsilon, \varepsilon > 0)$ corresponding to the switching merged processes $(\xi^\varepsilon(t), m(x^\varepsilon(t)), t \geq 0, \varepsilon > 0)$ is relatively compact.*

C4. *The convergence in probability of the initial values holds, that is,*

$$\bigl(\xi^\varepsilon(0), m\bigl(x^\varepsilon(0)\bigr)\bigr) \xrightarrow{P} (\xi(0), \hat{x}(0)) \qquad \text{as } \varepsilon \to 0,$$

with uniformly bounded expectation

$$\sup_{\varepsilon > 0} \mathbb{E}|\xi^\varepsilon(0)| \leq c < +\infty.$$

Then the weak convergence in $D_{\mathbb{R}^d \times V}[0,\infty)$

$$\bigl(\xi^\varepsilon(t), m\bigl(x^\varepsilon(t)\bigr)\bigr) \Rightarrow (\xi(t), \hat{x}(t)) \qquad \text{as } \varepsilon \to 0,$$

takes place.

Now we complete the proof of Theorem 1. It is easy to see that all the conditions of Theorem A are fulfilled and thus that the weak convergence stated in Theorem 1 holds. Namely, asymptotic representations (46)–(48) provide conditions C1 and C2 of Theorem A. Now, by (55) we can prove easily that $\varphi(\xi^\varepsilon(t)) + C_\varphi t, t \geq 0$, is a nonnegative $\mathcal{F}_t^\varepsilon$-submartingale, with $\mathcal{F}_t^\varepsilon = \sigma(\eta^\varepsilon(u), x^\varepsilon(u/\varepsilon), u \leq t)$. From this and the compact containment condition (Lemma 4), we get condition C3 (see [29, 17]).

It is worth noticing that the limit process $U(t \wedge \hat{\zeta})$ is a stopped process to time $\hat{\zeta}$ since the process $\hat{x}(\cdot)$ has one absorbing state.

8.2. *Proof of Theorem* 3. The generator of the coupled Markov processes $\xi^\varepsilon(t), x^\varepsilon(t/\varepsilon^2), t \geq 0$, in the case of Theorem 3 is

$$L^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}\mathbb{\Gamma}(x) + Q_1 + \mathbb{\Gamma}_1(x) + \mathbb{\Gamma}_0(x) + \gamma_\varepsilon(x) + \varepsilon\mathbb{B}_1(x),$$

where $\mathbb{\Gamma}(x)\varphi(u) = a(u;x)\varphi'(u)$, $\mathbb{\Gamma}_1(x)\varphi(u) = a_1(u;x)\varphi'(u)$, $\mathbb{\Gamma}_0(x)\varphi(u) = C_0(u;x)\varphi''(u)$, and $\mathbb{B}_1\varphi(u) = B_1(u;x)\varphi''(u)$, with $B_1(u;x) = \frac{1}{2}\int_{\mathbb{R}^d} vv^*\Gamma_1(u, dv;x)$. Of course, the operator $\gamma_\varepsilon(x) + \varepsilon\mathbb{B}_1(x)$ is a negligible one on $\mathbf{B}^1$.

Let us define the operator $Q_0 := Q_1 + \mathbb{\Gamma}_1(x) + \mathbb{\Gamma}_0(x) - \mathbb{\Gamma}(x)R_0\mathbb{\Gamma}(x)$, and the contracted operator $\hat{Q}_0$ by $\Pi Q_0 \Pi = \hat{Q}_0\Pi$. Then we have the following singular perturbation result for generator L^ε.

LEMMA 5. *Under the balance condition* $\Pi\mathbb{\Gamma}(x)\Pi\varphi = 0$ *and if* $\hat{Q}_0$ *is an reducible–invertible operator, the following asymptotic representation*:

$$[\varepsilon^{-2}Q + \varepsilon^{-1}\mathbb{\Gamma}(x) + Q_1 + \mathbb{\Gamma}_2(x)](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) = L\varphi + \varepsilon\theta^\varepsilon,$$

where $\mathbb{\Gamma}_2(x) := \mathbb{\Gamma}_1(x) + \mathbb{\Gamma}_0(x)$, *is realized by the vectors which are determined by the equation*

$$[\hat{Q}_1 + \hat{\mathbb{\Gamma}}_0 - \mathbb{\Gamma}(x)\widehat{R_0}\mathbb{\Gamma}(x)]\hat{\varphi} = \hat{L}\hat{\varphi},$$

and

$$\begin{aligned}
\varphi_1 &= -R_0\mathbb{\Gamma}(x)\varphi,\\
\varphi_2 &= R_0(L - Q_0)\varphi,\\
\theta^\varepsilon &= \big[\mathbb{\Gamma}(x) + \varepsilon\big(Q_1 + \mathbb{\Gamma}_0(x)\big)\big]\varphi_2 + \big(\mathbb{\Gamma}_0(x) + Q_1\big)\varphi_1.
\end{aligned}$$

The proof of Theorem 3 is based on the following theorem.

THEOREM B. *Let us consider the family of coupled Markov processes*

$$\xi^\varepsilon(t), x^\varepsilon(t/\varepsilon^2), \qquad t \geq 0, \varepsilon > 0, \tag{59}$$

a Markov process $\xi(t)$, $\hat{x}(t)$, $t \geq 0$, *of generator* L *with domain* $\mathcal{D}(L)$, *and an algebra* $\mathcal{A} \subset \overline{\mathcal{D}(L)}$ *that separate points. Consider also the test functions*

$$\varphi^\varepsilon(u,x) = \varphi(u, m(x)) + \varepsilon\varphi_1(u,x), \qquad \varphi \in \mathcal{A}.$$

Suppose that the following conditions are fulfilled:

C1. *The compact containment condition for the family* (59) *holds*.

C2. *For every* $T \in \mathbb{R}_+$, *we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{0\le t\le T} \big|\varphi^\varepsilon\big(\xi^\varepsilon(t), m\big(x^\varepsilon(t/\varepsilon^2)\big)\big) - \varphi\big(\xi^\varepsilon(t), m\big(x^\varepsilon(t/\varepsilon^2)\big)\big)\big|\Big] = 0 \tag{60}$$

C3. *and*

$$\sup_{\varepsilon>0} \mathbb{E}\big[\|L^\varepsilon \varphi^\varepsilon\|_{\infty,T}\big] < +\infty, \tag{61}$$

where $\|\varphi\|_{\infty,T} = \sup_{0\le t\le T} |\varphi(\xi(t), m(x(t/\varepsilon^2)))|$.

C4. *The convergence in probability of the initial values holds*, *that is*,

$$\big(\xi^\varepsilon(0), m(x^\varepsilon(0))\big) \xrightarrow{P} \big(\xi(0), \hat{x}(0)\big) \qquad \text{as } \varepsilon \to 0,$$

with uniformly bounded expectation

$$\sup_{\varepsilon>0} \mathbb{E}|\xi^\varepsilon(0)| \le c < +\infty.$$

Then

$$\big(\xi^\varepsilon(t), m\big(x^\varepsilon(t/\varepsilon^2)\big)\big) \Rightarrow \big(\xi(t), \hat{x}(t)\big) \qquad \text{as } \varepsilon \to 0.$$

PROOF. This is a compilation of our conditions of Theorem 9.4, page 145, and Corollary 8.6, page 231 of [7]. □

Let us first prove the compactness containment condition for processes (59).

LEMMA 6. *If* $\sup_{\varepsilon>0} \mathbb{E}|\xi^\varepsilon(0)| \le c < +\infty$, *then the family of stochastic processes* $\xi^\varepsilon(t)$, $t \ge 0$, $\varepsilon > 0$, *satisfies the compact containment condition*

$$\lim_{l\to\infty} \sup_{\varepsilon>0} \mathbb{P}^\varepsilon\Big(\sup_{0\le t\le T} |\xi^\varepsilon(t)| \ge l\Big) = 0. \tag{62}$$

PROOF. Let us consider the test functions $\varphi_0^\varepsilon(u,x) = \varphi_0(u) + \varepsilon\varphi_1(u,x)$, where $\varphi_0(u) = \sqrt{1+u^2}$.

From the asymptotic representation

$$L^\varepsilon \varphi_0^\varepsilon(u,x) = L\varphi_0 + \theta^\varepsilon \varphi_0$$

and the definition of the operator $\mathbb{\Gamma}^0(x) := -R_0\mathbb{\Gamma}(x)$, we get

$$\varphi_1 = -R_0\mathbb{\Gamma}(x)\varphi_0 = \mathbb{\Gamma}^0(x)\varphi_0.$$

Hence,

$$\varphi_0^\varepsilon(u,x) = \varphi_0(u) + \varepsilon\varphi_1(u,x) = [1 + \varepsilon\mathbb{\Gamma}^0(x)]\varphi_0(u).$$

From the boundness of the operator $R_0(x)$ and inequalities (49), we get

$$|\Gamma^0(x)\varphi_0(u)| \le C\varphi_0(u),$$

where C is a positive constant. And from this inequality we get

$$(1-\varepsilon C)\varphi_0 \le \varphi_0^\varepsilon(u,x) \le (1+\varepsilon C)\varphi_0. \tag{63}$$

Using the above inequalities, we can write the following relations:

$$\begin{aligned}
&\mathbb{P}\big(\varphi_0(\xi^\varepsilon(t)) \ge \varphi_0(l)\big)\\
&\quad = \mathbb{P}\big((1-\varepsilon C)\varphi_0(\xi^\varepsilon(t)) \ge (1-\varepsilon C)\varphi_0(l)\big)\\
&\quad \le \mathbb{P}\big(\varphi_0^\varepsilon\big(\xi^\varepsilon(t), x^\varepsilon(t)\big) \ge (1-\varepsilon C)\varphi_0(l)\big) && \text{[from (63)]}\\
&\quad \le 2\mathbb{E}\varphi_0^\varepsilon\big(\xi^\varepsilon(t), x^\varepsilon(t)\big)\big/\varphi_0(l) && (\text{for } \varepsilon \le 1/2C)\\
&\quad \le 4\mathbb{E}\varphi_0(\xi^\varepsilon(t))/\varphi_0(l) && \text{[from (63)]}\\
&\quad \le 4\mathbb{E}\varphi_0(\xi^\varepsilon(0))/\varphi_0(l)\\
&\quad \le 4\big(1+\mathbb{E}|\xi^\varepsilon(0)|\big)/\varphi_0(l),
\end{aligned}$$

which goes to 0 as $l \to +\infty$. □

For the other conditions of Theorem B, we can work as follows. The separating points algebra $\mathcal{A}$ considered here is $\overline{C}_0^2(\mathbb{R}^d \times V)$.

We have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} \big|\varphi^\varepsilon\big(\xi^\varepsilon(t), x^\varepsilon(t/\varepsilon^2)\big) - \varphi\big(\xi^\varepsilon(t), m\big(x^\varepsilon(t/\varepsilon^2)\big)\big)\big|\Big]\\
&\qquad = \varepsilon\mathbb{E}\sup_{0\le t\le T}\big|\varphi_1\big(\xi^\varepsilon(t), x^\varepsilon(t/\varepsilon^2)\big)\big| \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned} \tag{64}$$

On the other hand,

$$\begin{aligned}
&\sup_{0<\varepsilon\le\varepsilon_0} \mathbb{E}\big[\|L^\varepsilon\varphi^\varepsilon\|_{\infty,T}\big]\\
&\qquad \le \sup_{0<\varepsilon\le\varepsilon_0}\mathbb{E}\big[\|L\varphi\|_{\infty,T}\big] + \sup_{0<\varepsilon\le\varepsilon_0}\mathbb{E}\big[\|\theta^\varepsilon\|_{\infty,T}\big] < +\infty.
\end{aligned} \tag{65}$$

Now, by Theorem B, Lemma 6, relation (64) and inequality (65), the proof of Theorem 3 is achieved.

8.3. *Comments on the proofs of Theorems* 2 *and* 4. The proofs of Theorems 2 and 4 are similar to the previous ones.

It is easy to get the corresponding generators of the coupled Markov processes. There are the following:

$$L^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2 + \Gamma(x) + \gamma_\varepsilon(x),$$

for Theorem 2, and

$$L^\varepsilon = \varepsilon^{-3} Q + \varepsilon^{-2} Q_1 + \varepsilon^{-1} \mathbb{\Gamma}(x) + Q_2 + \mathbb{\Gamma}_1(x) + \gamma_\varepsilon(x),$$

for Theorem 4.

9. Concluding remarks. Results obtained in this paper can be used in order to develop numerical algorithmic settings for concret problems in reliability, replacement and more general problems concerning hitting times and functionals of type (1), (see, e.g., [6, 30]).

More general results concerning effects of the mode change of the switching and switched processes can be obtained by a similar way. A useful generalization is to consider semi-Markov switching process.

Acknowledgment. The authors are grateful to an anonymous referee for his useful comments that improved the presentation of this paper.

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