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ON CRAMÉR-LIKE ASYMPTOTICS FOR RISK PROCESSES WITH STOCHASTIC RETURN ON INVESTMENTS

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We consider a classical risk process compounded by another independent process. Both of these component processes are assumed to be Lévy processes. We show asymptotically that as initial capital y increases the ruin probability will essentially behave as $y^{-\kappa}$, where κ depends on one of the component processes.

1. Introduction. This paper supplements Paulsen (1998a) by giving Cramér-like asymptotics for the ruin probability for a risk process in a stochastic economic environment. Let P and R be independent Lévy processes, where P can be regarded as a risk process in a world without economic factors and R is the process that describes return on investments. Then compounded assets at time t equal

$$Y_t = y + P_t + \int_0^t Y_{s-}\, dR_s, \tag{1.1}$$

where $P_0 = R_0 = 0$.

A major problem in classical risk theory is to find the probability of eventual ruin, that is, the probability that assets ever become negative. Let $T_y = \inf\{t : Y_t < 0\}$ with $T_y = \infty$ if Y never becomes negative. Then T_y is the time of ruin, and $\psi(y) = P(T_y < \infty)$ is the probability that ruin will ever occur. We will prove that $\psi(y)$ behaves essentially like $y^{-\kappa}$, where $\kappa > 0$ typically depends on R only, the exception being when P has very heavy-tailed negative jumps, in which case κ depends on P only. When R is dominating, our result was basically conjectured by Paulsen and Gjessing (1997).

Similar, but somewhat weaker and different results have been obtained independently by Kalashnikov and Norberg (2002), Nyrhinen (2001) and Frolova, Kabanov and Pergamenshchikov (2002). All these papers deal with the case when the effect of R dominates that of P, so that κ depends on R only. Our proof for this case is merely an extension of the example in Section 3 of Nyrhinen (2001); this extension is based on methods used in Paulsen (1998a). For the case when the effect of P dominates that of R, we shall use results from Grey (1994), Gjessing and Paulsen (1997) as well as Klüppelberg and Stadtmüller (1997).

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2. The model. We will assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ satisfying the usual conditions; that is, $\mathcal{F}_t$ is right-continuous and P-complete.

As in Paulsen (1998a), let P and R be independent Lévy processes w.r.t. the filtration $\mathbf{F}$. We interpret P_t as the surplus at time t in a world with no return on investments, and for notational convenience we let $P_0 = 0$. The interpretation of R is that one unit of money invested at time 0 will be worth $\mathcal{E}(R)_t$ at time t, where $\mathcal{E}(R)$ is the Doléans–Dade exponential of R. Furthermore, $R_0 = 0$.

Total compounded assets is then given by (1.1), and the solution is [see, e.g., Paulsen (1998a), which also contains a discussion of inflation within this framework]

$$Y_t = e^{\tilde{R}_t}(y + Z_t). \tag{2.1}$$

Here

$$\tilde{R}_t = \log \mathcal{E}(R)_t = R_t - \tfrac{1}{2}\langle R^c, R^c\rangle_t + \sum_{s\le t}\bigl(\log(1+\Delta R_s) - \Delta R_s\bigr) \tag{2.2}$$

and

$$Z_t = \int_0^t e^{-\tilde{R}_s}\, dP_s. \tag{2.3}$$

Note that $Y_t < 0$ is equivalent to $Z_t < -y$.

For $\varepsilon_P > 0$, the Lévy–Itô representation of P takes the form

$$\begin{aligned} P_t = pt + \sigma_P W_{P,t} &+ \int_0^t \int_{\{|x|\le \varepsilon_P\}} x\bigl(\mu_P(ds,dx) - K_P(dx)\,ds\bigr) \\ &+ \int_0^t \int_{\{|x|>\varepsilon_P\}} x\mu_P(ds,dx), \end{aligned} \tag{2.4}$$

where W_P is a Brownian motion, μ_P is the random measure associated with the jumps of P and K_P is the compensator of μ_P, that is, a (nonstochastic) Radon measure with $K_P(\{0\}) = 0$ and $\int_{-\infty}^{\infty}(x^2 \wedge 1)\, K_P(dx) < \infty$. If the jumps are summable, ε_P can be set equal to 0, and if they are integrable, it can be set equal to ∞. Similarly, for $0 < \varepsilon_R < 1$,

$$\begin{aligned} R_t = rt + \sigma_R W_{R,t} &+ \int_0^t \int_{\{|x|\le \varepsilon_R\}} x\bigl(\mu_R(ds,dx) - K_R(dx)\,ds\bigr) \\ &+ \int_0^t \int_{\{|x|>\varepsilon_R\}} x\mu_R(ds,dx). \end{aligned} \tag{2.5}$$

If ΔR_t can be smaller than or equal to -1, ruin will occur with probability 1. To exclude this trivial case, we assume that

$$K_R\bigl((-\infty,-1]\bigr) = 0. \tag{2.6}$$

As for ε_P, ε_R can be equal to 0 if the jumps are summable, and equal to ∞ if they are integrable. Otherwise, it should be between 0 and 1 since integrability can fail at $x = -1$.

REMARK 2.1. In Paulsen (1998a) ε_P and ε_R were set equal to 1 and $1/2$, respectively, but when applicable the values 0 and ∞ will usually make the formulas easier for applications, hence the change here. For an example, see Example 3.1. However, unless explicitly stated, we shall always have that $0 < \varepsilon_P < \infty$ and $0 < \varepsilon_R < 1$.

It will be assumed throughout that

$$\int_{\{|x|>\varepsilon_R\}} |\log(1+x)| K_R(dx) < \infty. \tag{2.7}$$

Then a few calculations give

$$\tilde{R}_t = \tilde{r}t + \sigma_R W_{R,t} + \int_0^t \int_{-1}^{\infty} \log(1+x)\big(\mu_R(ds,dx) - K_R(dx)\,ds\big), \tag{2.8}$$

where

$$\begin{aligned} \tilde{r} = r - \tfrac{1}{2}\sigma_R^2 &+ \int_{\{|x|\le\varepsilon_R\}} \big(\log(1+x) - x\big) K_R(dx) \\ &+ \int_{\{|x|>\varepsilon_R\}} \log(1+x) K_R(dx). \end{aligned} \tag{2.9}$$

Note that $\tilde{r}$ does not depend on ε_R, whereas this is not the case for r.

It is quite common to start out with the $\tilde{R}$ process instead of R; this is done, for example, in Kalashnikov and Norberg (2002) and in Nyrhinen (2001). In that case it is easy to see that

$$R_t = \tilde{R}_t + \tfrac{1}{2}\langle \tilde{R}^c, \tilde{R}^c\rangle_t + \sum_{s\le t}(e^{\Delta \tilde{R}_s} - \Delta\tilde{R}_s - 1). \tag{2.10}$$

Since we have assumed (2.7), the Lévy–Itô representation of $\tilde{R}$ using its own jump measures $\mu_{\tilde{R}}$ and $K_{\tilde{R}}$ becomes

$$\tilde{R}_t = \tilde{r}t + \sigma_R W_{R,t} + \int_0^t \int_{-\infty}^{\infty} x\big(\mu_{\tilde{R}}(ds,dx) - K_{\tilde{R}}(dx)\,ds\big), \tag{2.11}$$

and, in particular, (2.7) is equivalent to $\tilde{R}$ being a special semimartingale.

3. Asymptotic ruin probabilities. We shall use the same model and assumptions as in Section 2, and, in particular, (2.6) and (2.7) are assumed to hold. Then it is proved in Paulsen (1998a) that, under some weak additional assumptions, $\psi(y) = 1$ if $\tilde{r} < 0$, and if $R \neq 0$ then $\psi(y) = 1$ also when $\tilde{r} = 0$. Therefore, we shall assume in the rest of the paper that $\tilde{r} > 0$. Furthermore, in order for ruin to happen it is necessary that

$$P(P_t < 0) > 0. \tag{3.1}$$

The following result is proved in Paulsen (1998a).

THEOREM 3.1. *Let* Y, P *and* R *be given by* (1.1), (2.4) *and* (2.5), *respectively. Also let* $\tilde{r}$ *be given by* (2.9) *and assume that* $\tilde{r} > 0$. *Furthermore, assume that* (2.6), (2.7) *and* (3.1) *hold and, in addition, that either*

(i) $\int_{-\infty}^{\infty} \log(1 + |x|) K_P(dx) < \infty$ *and* $\int_{-1}^{-\varepsilon_R} \big(\log(1 + x)\big)^4 K_R(dx) < \infty$

or

(ii) $\int_{\{|x| > \varepsilon_P\}} |x| K_P(dx) < \infty$ *and* $\int_{-1}^{-\varepsilon_R} (1 + x)^{-2} K_R(dx) < \infty$

hold. Then

$$\psi(y) = \frac{H(-y)}{E[H(-Y_{T_y}) | T_y < \infty]}. \tag{3.2}$$

Here H *is the* (*continuous*) *distribution function of the a.s. finite random variable*

$$Z_\infty = \int_0^\infty e^{-\tilde{R}_s}\, dP_s, \tag{3.3}$$

where $\tilde{R}$ *is given by* (2.8). *Finally*, $\psi(y) < 1$ *unless* $P_t = pt$, $R_t = rt$ *and* $p < -ry$.

Using the measure $K_{\tilde{R}}$, $\int_{-1}^{-\varepsilon_R} (\log(1 + x))^4 K_R(dx) < \infty$ is equivalent to $\int_{-\infty}^{-\varepsilon_R} x^4 K_{\tilde{R}}(dx) < \infty$ and $\int_{-1}^{-\varepsilon_R} (1 + x)^{-2} K_R(dx) < \infty$ is equivalent to $\int_{-\infty}^{-\varepsilon_R} e^{-2x} K_{\tilde{R}}(dx) < \infty$.

EXAMPLE 3.1. Frequently for the jump-diffusion models appearing in mathematical finance, R is of the form

$$R_t = rt + \sigma_R W_{R,t} + \sum_{i=1}^{N_{R,t}} S_{R,i},$$

where the sum is a compound Poisson process; that is, the $\{S_{R,i}\}$ are i.i.d. and independent of the Poisson process N_R, the latter having intensity λ_R. This is the same as (2.5) with $\varepsilon_R = 0$ and $K_R(dx) = \lambda_R F_R(dx)$, where $F_R(x) = P(S_R \le x)$ and S_R is generic for the $S_{R,i}$. Now $\tilde{R}$ becomes

$$\tilde{R}_t = \tilde{r}t + \sigma_R W_{R,t} + \sum_{i=1}^{N_{R,t}} \log(1 + S_{R,i}) - \lambda_R E\big[\log(1 + S_R)\big]t,$$

where $\tilde{r} = r - 1/2\sigma_R^2 + \lambda_R E[\log(1 + S_R)]$. Furthermore, $K_{\tilde{R}}((-\infty, x]) = K_R((-1, e^x - 1])$, and if F_R has a density f_R, then $K_{\tilde{R}}(dx) = \lambda_R e^x f_R(e^x - 1)\, dx$. In the famous Merton jump-diffusion model (1976), it is assumed that $\log(1 + S_R)$ is normally distributed, and it is easy to check that in this case either condition on K_R in Theorem 3.1 is satisfied.

Under the assumptions of Theorem 3.1, it is shown in Gjessing and Paulsen (1997) that Z_∞ satisfies the random equation

$$Z_\infty \stackrel{d}{=} AZ_\infty + B, \qquad Z_\infty \text{ is independent of } (A, B), \tag{3.4}$$

where

$$A = e^{-\tilde{R}_T} \quad \text{and} \quad B = \int_0^T e^{-\tilde{R}_t}\, dP_t$$

for any stopping time T. Here $X \stackrel{d}{=} Y$ means that X and Y have the same distribution. Formally, the integrand in B should be $e^{-\tilde{R}_{t-}}$, but since P and R are independent, it makes no difference.

Define ν_κ by

$$\nu_\kappa = -\log E[e^{-\kappa \tilde{R}_1}], \tag{3.5}$$

that is, $E[e^{-\kappa \tilde{R}_t}] = e^{-\nu_\kappa t}$. For later use, let us collect a few facts about ν_κ. They are well known and not very difficult to prove.

LEMMA 3.1. *Assume that $\tilde{R}$ given by* (2.11) *is nondeterministic and satisfies $\sigma_R \ne 0$ or $K_{\tilde{R}}((-\infty, 0)) > 0$ [equivalently $K_R((-1, 0)) > 0$]. Then $\lim_{\kappa\to\infty} \nu_\kappa = -\infty$ and $\nu_\kappa > -\infty$ implies that $\nu_\alpha > -\infty$ for $0 < \alpha < \kappa$. Let $\kappa_2 = \sup\{\kappa \ge 0 : \nu_\kappa > -\infty\}$. Then ν_κ is continuous and concave on $(0, \kappa_2)$ and $\lim_{\kappa\to\kappa_2} \nu_\kappa = \nu_{\kappa_2}$ (possibly $-\infty$). Furthermore, if $\tilde{r} > 0$ and $\kappa_2 > 0$, there exists a β with $0 < \beta \le \kappa_2$ so that $\nu_\kappa > 0$ on $(0, \beta)$.*

If $\nu_\kappa > -\infty$, using Itô's formula on $Y_t = e^{-\kappa \tilde{R}_t}$, it is not hard to prove that

$$\nu_\kappa = \kappa\tilde{r} - \tfrac{1}{2}\kappa^2\sigma_R^2 - \int_{-\infty}^{\infty} (e^{-\kappa x} - 1 + \kappa x) K_{\tilde{R}}(dx).$$

Also, using that $\Delta\tilde{R}_t = \log(1+\Delta R_t)$, we get, in terms of r and K_R,

$$\nu_\kappa = \kappa r - \tfrac{1}{2}\kappa(\kappa+1)\sigma_R^2 - \int_{\{|x|\le\varepsilon_R\}} \big((1+x)^{-\kappa} - 1 + \kappa x\big) K_R(dx) \\ - \int_{\{|x|>\varepsilon_R\}} \big((1+x)^{-\kappa} - 1\big) K_R(dx).$$

It follows that a necessary and sufficient condition for $\nu_\kappa > -\infty$ is

$$\int_{-1}^{-\varepsilon_R} (1+x)^{-\kappa} K_R(dx) < \infty \quad \text{or, equivalently,} \quad \int_{-\infty}^{-\varepsilon_R} e^{-\kappa x} K_{\tilde{R}}(dx) < \infty; \tag{3.6}$$

that is, negative jumps in investment returns should not be too large.

We are now ready for the main result of this paper.

THEOREM 3.2. *Let the conditions of Theorem* 3.1 *hold.*

(a) *Assume there exists a* $\kappa_0 > 0$ *so that* $\nu_{\kappa_0} = 0$. *Assume, in addition, that for some* $\varepsilon > 0$,

$$\nu_{\kappa_0+\varepsilon} > -\infty \quad \text{and} \quad \int_{\{|x|>\varepsilon_P\}} |x|^{\kappa_0+\varepsilon} K_P(dx) < \infty,$$

and finally that the distribution of R_T *has an absolutely continuous component when* T *is uniformly distributed on* $[0, 1]$ *and independent of* R. *Then*

$$y^{\kappa_0}\psi(y) = C + o(y^{-\varepsilon}) \qquad \text{as } y \to \infty.$$

Here

$$C = \frac{1}{\kappa_0 m h} E\big[\big((AZ_\infty + B)^-\big)^{\kappa_0} - \big((AZ_\infty)^-\big)^{\kappa_0}\big],$$

where (A, B, Z_∞) *are as in* (3.4), $m = E[A^{\kappa_0}\log A]$ *and* $h = \lim_{y\to\infty} h(y)$ *with* $h(y) = E[H(-Y_{T_y}) | T_y < \infty]$. *Also* $0 < C < \infty$. *Finally*, $x^- = -\max\{0, x\}$.

(b) *Assume* $K_P((-\infty, -x]) \sim x^{-\kappa_1} L(x)$ *as* $x \to \infty$, *where* L *is slowly varying*. *Assume also that* $\nu_{\kappa_1+\varepsilon} > 0$ *for some* $\varepsilon > 0$. *Then*

$$\psi(y) \sim \frac{1}{h(y)} \frac{1}{\nu_{\kappa_1}} y^{-\kappa_1} L(y) \qquad \text{as } y \to \infty,$$

where $h(y)$ *is as in part* (a).

For the proof we will need two lemmas.

LEMMA 3.2. *Assume that* $\nu_\alpha > 0$ *and let* T *be exponentially distributed with intensity* γ; *that is*, $f_T(t) = 1_{\{t>0\}}\gamma e^{-\gamma t}$ *with* T *and* R *independent. Then*

$$E\Big[\sup_{t\le T} e^{-\alpha\tilde{R}_t}\Big] < \infty.$$

PROOF. Let $p > 1$ and define the martingale N by

$$N_t = \exp\left(-\frac{\alpha}{p}\tilde{R}_t + \nu_{\alpha/p}t\right). \tag{3.7}$$

Then $1 \leq E[N_t^p] = \exp\{-(\nu_\alpha - p\nu_{\alpha/p})t\}$; hence $p\nu_{\alpha/p} - \nu_\alpha > 0$. Now choose $p > 1$ so small that $p\nu_{\alpha/p} - \nu_\alpha < \gamma$. Then by conditioning on T and using Doob's inequality,

$$E\left[\sup_{t\leq T} e^{-\alpha\tilde{R}_t}\right] \leq E\left[\sup_{t\leq T} N_t^p\right] \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty E[N_t^p]\gamma e^{-\gamma t}\,dt < \infty.$$

The proof of the next lemma is similar and is omitted.

LEMMA 3.3. *Assume that* $-\infty < \nu_\alpha < 0$ *and let* T *satisfy* $E[e^{(\varepsilon-\nu_\alpha)T}] < \infty$ *for some* $\varepsilon > 0$ *with* T *and* R *independent. Then*

$$E\left[\sup_{t\leq T} e^{-\alpha\tilde{R}_t}\right] < \infty.$$

PROOF OF THEOREM 3.2. For part (a) we will make extensive use of Nyrhinen (2001), actually following closely the example in Section 3 of that paper. Let $T_1, T_2, \ldots$ be i.i.d. uniformly distributed random variables, also independent of P and R. From Lemma 3.3 it follows that

$$E\left[\sup_{t\leq T} e^{-(\kappa_0+\varepsilon)\tilde{R}_t}\right] < \infty.$$

Let $V_n = T_1 + \cdots + T_n$ with $V_0 = 0$. Also define, for $n = 1, 2, \ldots,$

$$A_n = e^{-(\tilde{R}_{V_n} - \tilde{R}_{V_{n-1}})}, \qquad B_n = e^{\tilde{R}_{V_{n-1}}} \int_{V_{n-1}}^{V_n} e^{-\tilde{R}_t}\,dP_t$$

and

$$L_n = e^{\tilde{R}_{V_n}} \inf_{V_{n-1}<t\leq V_n} \left\{-\int_t^{V_n} e^{-\tilde{R}_s}\,dP_s\right\}.$$

Then the (A_n, B_n, Z_n) are i.i.d., and by letting

$$\bar{Z}_n = B_1 + A_1B_2 + \cdots + A_1\cdots A_{n-1}B_n + A_1\cdots A_nL_n,$$

it is not hard to see that

$$\bar{Z}_n = \inf_{V_{n-1}<t\leq V_n} Z_s;$$

hence, $\psi(y) = P(\bar{Z}_n < -y \text{ for some } n)$. Therefore, by Nyrhinen [(2001), Theorem 2] and the discussion following it, we must prove that

$$E[|B|^{\kappa_0+\varepsilon}] < \infty \quad \text{and} \quad E\left[\left|\inf_{t\leq T}(Z_t - Z_T)\right|^{\kappa_0+\varepsilon}\right] < \infty.$$

But $B \stackrel{d}{=} Z_T$, and hence it is sufficient to prove that

$$E\Big[\sup_{t\le T}|Z_t|^{\kappa_0+\varepsilon}\Big]<\infty.$$

This is accomplished provided

$$E\bigg[\bigg(\int_0^T e^{-\tilde{R}_s}\,ds\bigg)^{\kappa_0+\varepsilon}\bigg]<\infty, \tag{3.8}$$

$$E\bigg[\sup_{t\le T}\bigg|\int_0^t e^{-\tilde{R}_s}\,dW_{P,s}\bigg|^{\kappa_0+\varepsilon}\bigg]<\infty, \tag{3.9}$$

$$E\bigg[\sup_{t\le T}\bigg|\int_0^t\int_{\{|x|\le\varepsilon_P\}} e^{-\tilde{R}_{s-}}x\big(\mu_P(ds,dx)-K_P(dx)\,ds\big)\bigg|^{\kappa_0+\varepsilon}\bigg]<\infty, \tag{3.10}$$

$$E\bigg[\sup_{t\le T}\bigg|\int_0^t\int_{\{|x|>\varepsilon_P\}} e^{-\tilde{R}_{s-}}x\mu_P(ds,dx)\bigg|^{\kappa_0+\varepsilon}\bigg]<\infty. \tag{3.11}$$

Using Lemma 3.3, the proof of (3.8)–(3.10) is exactly as the proof of (4.12)–(4.14) in Paulsen (1998a). Also writing

$$\int_0^t\int_{\{|x|>\varepsilon_P\}} e^{-\tilde{R}_{s-}}x\mu_P(ds,dx)=\sum_{i=1}^{N_t}e^{-\tilde{R}_{U_i}}S_i,$$

where N is a Poisson process, $U_1,U_2,\dots$ are the times of jump of N and $E[|S|^{\kappa_0+\varepsilon}]<\infty$, we get, for (3.11),

$$\begin{aligned}
&E\bigg[\sup_{t\le T}\bigg|\int_0^t\int_{\{|x|>\varepsilon_P\}} e^{-\tilde{R}_{s-}}x\mu_P(ds,dx)\bigg|^{\kappa_0+\varepsilon}\bigg]\\
&\qquad\le E\Big[\sup_{t\le T}e^{-(\kappa_0+\varepsilon)\tilde{R}_t}\Big]E\bigg[\bigg(\sum_{i=1}^{N_T}|S_i|\bigg)^{\kappa_0+\varepsilon}\bigg].
\end{aligned}$$

But

$$E\bigg[\bigg(\sum_{i=1}^{N_T}|S_i|\bigg)^{\kappa_0+\varepsilon}\bigg]\le E[N_T^{\kappa_0+\varepsilon}]E[|S|^{\kappa_0+\varepsilon}]<\infty.$$

By Nyrhinen (2001) it remains to show that $\bar{z}=-\infty$, where $\bar{z}=\inf\{z: P(\inf_n Z_n<z)>0\}$. But this is obvious from (3.1) and the fact that P is a Lévy process. We know from Nyrhinen (2001) that $C>0$, so it follows from Theorem 3.1 and Theorem 3.2 in Gjessing and Paulsen (1997) that C is of the stated form. This implies, in particular, that $\lim_{y\to\infty}E[H(-Y_{T_y})|T_y<\infty]$ must exist.

To prove part (b), using Theorem 3.1, we have to show that

$$H(-y) \sim \frac{1}{\nu_{\kappa_1}} y^{-\kappa_1} L(y) \qquad \text{as } y \to \infty.$$

To do so, we shall use (3.4) with $T = \inf\{t > 0 : |\Delta P_t| > \varepsilon_P\}$ ($\varepsilon_P > 0$). Then T is exponentially distributed with intensity γ say, so, by Lemma 3.2,

$$E\left[\sup_{t \le T} e^{-(\kappa_1+\varepsilon)\tilde{R}_t}\right] < \infty. \tag{3.12}$$

For B as in (3.4), we have $B = Q + e^{-\tilde{R}_T} S$, where

$$\begin{aligned} Q = p\int_0^T e^{-\tilde{R}_t}\,dt + \sigma_R \int_0^T e^{-\tilde{R}_t}\,dW_{R,t} \\ + \int_0^T \int_{\{|x| \le \varepsilon_P\}} e^{-\tilde{R}_{t-}}\bigl(\mu_P(dt,dx) - K_P(dx)\,dt\bigr). \end{aligned}$$

As in the proof of part (a), using (3.12), it follows that $E[|Q|^{\kappa_1+\varepsilon}] < \infty$; hence $P(Q < -y) = o(y^{-\kappa_1})$ as $y \to \infty$. Furthermore, by assumption, $P(S < -y) = \gamma^{-1} K_P((-\infty, -y])$, and S is independent of $(Q, e^{-\tilde{R}_T})$. We can now use Lemma 2 in Grey (1994); the only difference is that $I_1(t)$ defined there is now $o(t^{-\kappa_1})$. Thus, using that $E[e^{-\kappa_1 \tilde{R}_T}] = \gamma/(\gamma + \nu_{\kappa_1})$, this lemma gives

$$P(B < -y) = P(Q + e^{-\tilde{R}_T} S < -y) \sim \frac{1}{\gamma + \nu_{\kappa_1}} K_P\bigl((-\infty, -y]\bigr).$$

Finally, Theorem 1 in Grey (1994) yields

$$\begin{aligned} P(Z_\infty < -y) &\sim \frac{1}{1 - \gamma/(\gamma + \nu_{\kappa_1})} \frac{1}{\gamma + \nu_{\kappa_1}} y^{-\kappa_1} L(y) \\ &= \frac{1}{\nu_{\kappa_1}} y^{-\kappa_1} L(y) \qquad \text{as } y \to \infty. \end{aligned}$$

REMARK 3.2. (a) In part (a) of the theorem it was assumed that $\tilde{R}_T$, or, equivalently, R_T, has an absolutely continuous component when T is uniformly distributed and independent of R. Using a random instead of a fixed time gives a little more generality. Assume, for instance, that $R_t = rt + \sum_{i=1}^{N_{R,t}} S_{R,i}$, where the sum is a compound Poisson process and S_R has an arithmetic distribution. If $r \neq 0$, then $\tilde{R}_T$ is absolutely continuous distributed, while, for fixed t, $\tilde{R}_t$ has no absolutely continuous component. If $r = 0$ we do not know if our result holds, but Theorem 2 of Nyrhinen (2001) gives at least the large-deviation-type result

$$\lim_{y \to \infty} y^{-1} \log \psi(y) = -\kappa_0.$$

(b) Since $Y_{T_y} \le 0$,

$$0 < H(0) \le \liminf_{y\to\infty} h(y) \le \limsup_{y\to\infty} h(y) \le 1.$$

In part (a) we saw that $h = \lim_{y\to\infty} h(y)$ exists, but we do not know this for part (b). Because of the heavy-tailed negative jumps of P, we conjecture that h exists and equals 1, in which case

$$\psi(y) \sim \frac{1}{\nu_{\kappa_1}} y^{-\kappa_1} L(y) \qquad \text{as } y \to \infty. \tag{3.13}$$

In Section 4 we will show that (3.13) holds in a generic special case, thus strongly suggesting it holds in the general case.

There are a few cases not treated in Theorem 3.2 (or even in Theorem 3.1), when the negative jumps of R, or, equivalently, $\tilde{R}$, can become too large in absolute value. They may not be terribly interesting, but for completeness we shall end this section by looking at the most prevalent of them. It will be assumed that R_T has an absolutely continuous component.

Assume first that the conditions of Theorem 3.1 hold and let κ_2 be as in Lemma 3.1. Assume that $\kappa_2 > 0$ and that $\nu_{\kappa_2} > 0$. Notice that this implies that $E[|B|^{\kappa_2+\delta}] = \infty$ for all $\delta > 0$, where $B \stackrel{d}{=} Z_T$ as in the proof of Theorem 3.2. Assume also that $\int_{\{|x|>\varepsilon_P\}} |x|^{\kappa_2+\varepsilon} K_P(dx) < \infty$ for some $\varepsilon > 0$. Let $\tilde{R}^{(\gamma)}$ be as $\tilde{R}$, except that $\tilde{r}$ is replaced by $\gamma\tilde{r}$, and let $\nu_\kappa^{(\gamma)} = -\log E[e^{-\kappa \tilde{R}_1^{(\gamma)}}]$. Then since $e^{-\kappa x} - 1 + \kappa x \ge 0$, it is clear that, for any δ with $0 \le \delta < \kappa_2$, there exists a γ between 0 and 1 so that $\nu_{\kappa_2-\delta}^{(\gamma)} = 0$. However, for $\gamma < 1$, $\tilde{R}^{(\gamma)}$ is stochastically dominated by $\tilde{R}$; hence $\psi(y) \le \psi^{(\gamma)}(y) \stackrel{\text{def}}{=} P(\tilde{R}_t^{(\gamma)} < 0 \text{ for some } t)$. But $\tilde{R}^{(\gamma)}$ satisfies the conditions of Theorem 3.2, so $\psi^{(\gamma)}(y) \sim c^{(\gamma)} y^{-(\kappa_2-\delta)}$ for some $c^{(\gamma)} > 0$. On the other hand, we can let $\tilde{R}^{[m]}$ be as $\tilde{R}$, except that $K_{\tilde{R}}^{[m]}(A) = K_{\tilde{R}}(A \cap (-m, \infty))$ for any Borel set A. Then $\tilde{R}$ is stochastically dominated by $\tilde{R}^{[m]}$ and hence $\psi(y) \ge \psi^{[m]}(y) \stackrel{\text{def}}{=} P(\tilde{R}_t^{[m]} < 0 \text{ for some } t)$. Now for any $\delta > 0$ we can choose m so that $\nu_{\kappa_2+\delta}^{[m]} = 0$, and then, by Theorem 3.2, $\psi^{[m]}(y) \sim c^{[m]} y^{-(\kappa_2+\delta)}$ for some $c^{[m]} > 0$. Hence we get the Kalashnikov–Norberg-type result: for all $\delta > 0$ there exist positive constants C_1 and C_2 so that

$$C_1 y^{-(\kappa_2+\delta)} \le \psi(y) \le C_2 y^{-(\kappa_2-\delta)}.$$

If $\kappa_2 = 0$, the second argument above can still be used to conclude that, for any $\delta > 0$, there is a $C > 0$ so that

$$\psi(y) \ge C y^{-\delta}. \tag{3.14}$$

In the unlikely case that K_R does not satisfy the conditions of Theorem 3.1, for instance, if $\int_{-1}^{-\varepsilon_R} (\log(1+x))^4 K_R(dx) = \infty$ while $\int_{-\infty}^{-\varepsilon_R} \log(1+|x|) K_P(dx) < \infty$, we can let $\tilde{R}$ stochastically dominate $\tilde{R}^{[m]}$ as above and conclude that (3.14) holds.

Our final odd case is when $\nu_{\kappa_0} = 0$, $\nu_{\kappa_0+\varepsilon} > -\infty$ for some $\varepsilon > 0$ and $K_P((-\infty, -x]) \sim x^{-\kappa_0} L(x)$ as $x \to \infty$. We can then dominate P by $P^{[m]}$, where $P^{[m]}$ is as P, except that $K_P^{[m]}(A) = K_P(A \cap (-m, \infty))$ for any Borel set A, and also dominate $\tilde{R}$ by $\tilde{R}^{(\gamma)}$ as above to conclude that, for all $\delta > 0$, there exist a positive C_1 and a positive C_2 independent of δ so that

$$C_1 y^{-\kappa_0} \le \psi(y) \le C_2 y^{-(\kappa_0-\delta)}.$$

4. A special case. In this section we consider the special case

$$P_t = pt - \sum_{i=1}^{N_t} S_i \quad \text{and} \quad R_t = rt + \sigma W_t, \tag{4.1}$$

where $\sum_{i=1}^{N_t} S_i$ is a compound Poisson process with intensity λ and jump distribution F. We assume that S is positive so that $F(0) = 0$. Furthermore, W is a standard Brownian motion independent of the income process P. It is assumed that $\tilde{r} = r - \frac{1}{2}\sigma^2 > 0$ and that $E[\log^+ S] < \infty$; hence, by Theorem 3.1, $\psi(y) < 1$.

It is easy to see that $\kappa_0 = 2\tilde{r}/\sigma^2$, and if, for some $\varepsilon > 0$, $E[S^{\kappa_0+\varepsilon}] < \infty$, it follows from Theorem 3.2(a) that $\psi(y) \sim C y^{-\kappa_0}$ for some positive C. This result is also proved in the example in Section 3 in Nyrhinen (2001). Clearly, $\kappa_0 \to \infty$ as $\sigma \to 0$, so our result is not useful when $\sigma = 0$, a fact that is well known. Indeed, in this case if $0 < \alpha_0 \stackrel{\text{def}}{=} \sup\{\alpha > 0 : E[e^{\alpha S_P}] < \infty\} < \infty$, then roughly $\psi(y) \sim C e^{-\alpha_0 y}$. If both σ and α_0 equal 0, the asymptotic behavior of ψ can take various forms; a general survey is found in Paulsen (1998b) and additional results can be found in Asmussen (1998).

Assume now as in Theorem 3.2(b) that $\bar{F}(x) = 1 - F(x) \sim x^{-\kappa_1} L(x)$, where $\kappa_1 < \kappa_0$. Then, by Theorem 3.2(b),

$$\psi(y) \sim \frac{1}{h(y)} \frac{\lambda}{\nu_{\kappa_1}} y^{-\kappa_1} L(y) \qquad \text{as } y \to \infty.$$

Here $K_P((-\infty, -x]) = \lambda \bar{F}(x)$, hence the slight difference from Theorem 3.2(b). Again $h(y) = E[H(-Y_{T_y}) | T_y < \infty]$. The conjecture (3.13) now becomes

$$\psi(y) \sim \frac{\lambda}{\nu_{\kappa_1}} y^{-\kappa_1} L(y) \qquad \text{as } y \to \infty. \tag{4.2}$$

When $\sigma = 0$ so that $\nu_{\kappa_1} = r\kappa_1$, both Klüppelberg and Stadtmüller (1998) and Asmussen (1998) proved that (4.2) holds, so in this case the conjecture [see Remark 3.2(b)] $\lim_{y\to\infty} h(y) = 1$ is correct. We will show that it is generally correct for the model (4.1) as well. To this end, we will rewrite an integrodifferential equation for the survival probability in a way similar to that in Sundt and Teugels (1995). So consider the integrodifferential equation

$$\tfrac{1}{2}\sigma^2 y^2 \phi''(y) + (ry + p)\phi'(y) = \lambda\phi(y) - \lambda \int_0^y \phi(y - x)\, dF(x). \tag{4.3}$$

Paulsen and Gjessing (1997) proved that if (4.3) has a smooth solution satisfying the boundary condition $\lim_{y\to\infty}\phi(y)=1$, then $\phi(y)=1-\psi(y)$, that is, the probability of survival. Later Wang and Wei (2001) proved that under some conditions the survival probability really is a smooth solution of (4.3) with the boundary condition $\lim_{y\to\infty}\phi(y)=1$.

Integrating (4.3) from 0 to y and doing the necessary changes of variables and integration by parts brings it into the form

$$\begin{aligned}\tfrac{1}{2}\sigma^2y^2\phi'(y)+\big((r-\sigma^2)y+p\big)\phi(y)\\ =p\phi(0)+\int_0^y\phi(y-x)\big(\lambda\bar{F}(x)+r-\sigma^2\big)\,dx.\end{aligned}\tag{4.4}$$

Introducing, as in Sundt and Teugels (1995), the probability distribution function

$$G(y)=\frac{\phi(y)-\phi(0)}{1-\phi(0)}=1-\frac{\psi(y)}{\psi(0)}$$

brings (4.4) into the form

$$\begin{aligned}&\frac{1}{2}\sigma^2y^2G'(y)+\big((r-\sigma^2)y+p\big)G(y)\\ &\qquad=\int_0^y G(y-x)\big(\lambda\bar{F}(x)+r-\sigma^2\big)\,dx+\lambda\frac{\phi(0)}{1-\phi(0)}\int_0^y\bar{F}(x)\,dx.\end{aligned}$$

Assume finally that $\eta=E[S]$ is finite and set $F_I(y)=\eta^{-1}\int_0^y\bar{F}(x)\,dx$. Then the above equation becomes

$$\begin{aligned}\tfrac{1}{2}\sigma^2y^2G'(y)+\big((r-\sigma^2)y+p\big)G(y)\\ =\rho G*F_I(y)+(r-\sigma^2)\int_0^y G(x)\,dx+KF_I(y),\end{aligned}\tag{4.5}$$

where $\rho=\lambda\eta$ and $K=\rho\phi(0)/(1-\phi(0))$. Note that (4.4) and (4.5) are consequences of (4.3) and are valid for any distribution function F with $F(0)=0$. Thus these equations are of independent interest.

PROPOSITION 4.1. *Assume the model* (4.1) *with* $\bar{F}(x)\sim x^{-\kappa_1}L(x)$ *as* $x\to\infty$, *where* $\kappa_1<\kappa_0$. *Assume furthermore that* $\phi(y)=1-\psi(y)$ *solves* (4.3). *Then* (4.2) *holds*, *where*

$$\nu_{\kappa_1}=\kappa_1\tilde{r}-\tfrac{1}{2}\kappa_1^2\sigma^2=\tfrac{1}{2}\kappa_1\sigma^2(\kappa_0-\kappa_1).$$

PROOF. The expression for ν_{κ_1} is easy to verify. In the proof we will make use of arguments and methods from Klüppelberg and Stadtmüller (1998), hereafter abbreviated as K&S. As there, assume that $\eta=E[S]<\infty$; the case with $\eta=\infty$

can be dealt with as explained by K&S. For any distribution function F_0 with $F_0(0) = 0$ and for any $q \geq 0$, define the L_q-transform by

$$L_q F_0(s) = \int_0^\infty e^{-sy}(sy)^q \, dF_0(y).$$

An easy integration by parts shows that (4.5) can be written as

$$\tfrac{1}{2}\sigma^2 y^2 G'(y) + (r - \sigma^2)\int_0^\infty x \, dG(x) + pG(y) = \rho G * F_I(y) + K F_I(y).$$

Taking the L_q-transform on both sides, some integration by parts yields

$$\begin{aligned} &\frac{1}{s}\left(\frac{1}{2}\sigma^2 L_{q+2}G(s) + \left(r - \frac{q+2}{2}\sigma^2\right)L_{q+1}G(s)\right) + pL_q G(s) \\ &\qquad = \rho L_q G * F_I(y) + K L_q F_I(y). \end{aligned} \tag{4.6}$$

By Theorem 3.2(b), $\bar{G}(y) = 1 - G(y) \sim \lambda(\psi(0)\nu_{\kappa_1} h(y))^{-1} y^{-\kappa_1} L(y)$. Since $1/h(y)$ lies in the compact set $[1, 1/H(0)]$, by considering convergent subsequences, it follows from Proposition 3.2 in K&S that $L_{q+2}G(s) \sim (q + 1 - \kappa_1)L_{q+1}G(s)$. Therefore the left-hand side of (4.6) can be replaced with

$$\left(\tilde{r} - \frac{1}{2}\sigma^2\kappa_1\right)\frac{1}{s}L_{q+1}G(s) + pL_q G(s).$$

But then (4.6) is of the same form as (2.11) in K&S; hence, by Corollary 2.4 there,

$$\psi(y) \sim \frac{\lambda}{\kappa_1(\tilde{r} - \frac{1}{2}\sigma^2\kappa_1)} y^{-\kappa_1} L(y).$$

This ends the proof. $\square$

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