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A NOTE ON KRYLOV'S L_p-THEORY FOR SYSTEMS OF SPDES

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Abstract We extend Krylov's L_p-solvability theory to the Cauchy problem for systems of parabolic stochastic partial differential equations (SPDEs). Some additional integrability and regularity properties are also presented.

Keywords Stochastic partial differential equations, Cauchy problem

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1 Introduction

A comprehensive theory of second order quasi-linear parabolic stochastic differential equations in Bessel classes $H_p^s(\mathbf{R}^d)$ was developed by N. V. Krylov in [1], [2]. This theory applies to a large class of important equations, including equations of nonlinear filtering, stochastic heat equation with nonlinear noise term, etc.. The main results of the theory are sharp in that they could not be improved under the same assumptions.

In this paper we extend Krylov's L_p-theory to parabolic systems of quasilinear stochastic PDEs. Specifically, we are considering the system of equations

$$\begin{aligned} \partial_t u^l = \partial_i(a^{ij}(t,x)\partial_j u^l) + & D^l(\mathbf{u},t,x) \\ & + [\sigma^k(t,x)\partial_k u^l + Q^l(\mathbf{u},t,x)] \cdot \dot{W}, \qquad (1.1) \\ u^l(0,x) = u_0^l(x), l = 1,\ldots,d; x \in \mathbf{R}^d & \end{aligned}$$

where W is a cylindrical Wiener process in a Hilbert space. In (1.1) and everywhere below the summation with respect to the repeated indices is assumed.

Among other reasons this research was motivated by our interest in stochastic Fluid Mechanics (see e.g. [6], [7]). While the results below do not apply directly to stochastic Navier-Stokes equations, they provide us with important estimates for solutions of suitable approximation to the latter.

The structure of the paper is as follows.

In Section 2 we present a simple and straightforward construction of stochastic integrals for H_p^s-valued integrands (for related results see [3], [4]). In this Section we also derive an Ito formula for L_p-norms of H_p^s-valued semimartingales.

In Section 3 we present some auxiliary results about pointwise multipliers in H_p^s needed for the derivation of apriori estimates for (1.1) (see Lemma 8). We give a more precise version of Krylov's Lemma 5.2 in [2] with an estimate that gives a positive answer to Krylov's question raised in Remark 6.5 (see [2]).

In Section 4, following Krylov's ideas, we derive the main results about the existence and uniqueness of solutions to equation (1.1). The results of the last subsection, in particular those concerning the regularity of solutions (Proposition 1, Corollary 3, Corollary 4) are new not only for systems but also for the scalar equations considered in [1], [2]. In addition, in Section 4, we obtain some new integrability properties of the solution (Proposition 2-3, Corollary 3-4).

To conclude the Introduction, we outline some notation which will be used throughout the paper. $\mathbf{R}^d$ denotes d-dimensional Euclidean space with elements $x = (x_1, \ldots, x_d)$; if $x, y \in \mathbf{R}^d$, we write

$$(x,y) = \sum_{i=1}^d x_i y_i,\ |x| = \sqrt{(x,x)}.$$

Let us fix a separable Hilbert space Y. The scalar product of $x, y \in Y$ will be denoted by $x \cdot y$.

If u is a function on $\mathbf{R}^d$, the following notational conventions will be used for its partial derivatives: $\partial_i u = \partial u/\partial x_i, \partial^2_{ij} = \partial^2 u/\partial x_i \partial x_j$, $\partial_t u = \partial u/\partial t, \nabla u = \partial u = (\partial_1 u, \ldots, \partial_d u)$, and $\partial^2 u = (\partial^2_{ij} u)$ denotes the Hessian matrix of second derivatives. Let $\alpha = (\alpha_1, ..., \alpha_d)$ be a multi-index, then $\partial^\alpha_x = \Pi^d_{i=1} \partial^{\alpha_i}_{x_i}$.

Let $C^\infty_0 = C^\infty_0(\mathbf{R}^d)$ be the set of all infinitely differentiable functions on $\mathbf{R}^d$ with compact support.

For $s \in (-\infty, \infty)$, write $\Lambda^s = \Lambda^s_x = \left(1 - \sum^d_{i=1} \partial^2/\partial x^2_i\right)^{s/2}$.

For $p \in [1, \infty)$ and $s \in (-\infty, \infty)$, we define the space $H^s_p = H^s_p(\mathbf{R}^d)$ as the space of generalized functions u with the finite norm

$$|u|_{s,p} = |\Lambda^s u|_p,$$

where $|\cdot|_p$ is the L_p norm. Obviously, $H^0_p = L_p$. Note that if $s \geq 0$ is an integer, the space H^s_p coincides with the Sobolev space $W^s_p = W^s_p(\mathbf{R}^d)$.

If $p \in [1, \infty)$, and $s \in (-\infty, \infty)$, $H^s_p(Y) = H^s_p(\mathbf{R}^d, Y)$ denotes the space of $Y-$valued functions on $\mathbf{R}^d$ so that the norm $||g||_{s,p} = |\,|\Lambda^s g|_Y\,|_p < \infty$. We also write $L_p(Y) = L_p(\mathbf{R}^d, Y) = H^0_p(Y) = H^0_p(\mathbf{R}^d, Y)$.

Obviously, the spaces $C^\infty_0, H^s_p\left(\mathbf{R}^d\right)$, and $H^s_p(\mathbf{R}^d, Y)$ can be extended to vector functions (denoted with bold-faced letters). For example, the space of all vector functions $\mathbf{u} = (u^1, \ldots, u^d)$ such that $\Lambda^s u^l \in L_p$, $l = 1, \ldots, d$, with the finite norm

$$|\mathbf{u}|_{s,p} = (\sum_l |u^l|^p_{s,p})^{1/p}$$

we denote by $\mathbb{H}^s_p = \mathbb{H}^s_p(\mathbf{R}^d)$. Similarly, we denote by $\mathbb{H}^s_p(Y) = \mathbb{H}^s_p(\mathbf{R}^d, Y)$ the space of all vector functions $g = (g^l)_{1 \leq l \leq d}$, with Y-valued components g^l, $1 \leq l \leq d$, so that $||g||_{s,p} = (\sum_l |g^l|^p_{s,p})^{1/p} < \infty$. The set of all infinitely differentiable vector-functions $u = (u^1, \ldots, u^d)$ on $\mathbf{R}^d$ with compact support will be denoted by $\mathbb{C}^\infty_0$.

When $s = 0$, $\mathbb{H}^s_p(Y) = \mathbb{L}_p(Y) = \mathbb{L}_p(\mathbf{R}^d, Y)$. Also, in this case, the norm $||g||_{0,p}$ is denoted more briefly by $||g||_p$. To forcefully distinguish L_p-norms in spaces of $Y-$valued functions, we write $||\cdot||_p$, while in all other cases a norm is denoted by $|\cdot|$.

The duality $\langle \cdot, \cdot \rangle_s$ between $\mathbb{H}^s_q\left(\mathbf{R}^d\right)$, and $\mathbb{H}^{-s}_p\left(\mathbf{R}^d\right)$ where $p \geq 2$ and $q = p/\left(p-1\right)$ is defined by

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_s = \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{s,p} = \sum^d_{i=1} \int_{\mathbf{R}^d} \left[\Lambda^s \phi^i\right](x) \, \Lambda^{-s} \psi^i(x) \, dx, \boldsymbol{\phi} \in \mathbb{H}^s_q, \boldsymbol{\psi} \in \mathbb{H}^{-s}_p.$$

If $\mathbf{f} \in \mathbb{H}^s_q\left(\mathbf{R}^d, Y\right)$ and $\boldsymbol{\phi} \in \mathbb{H}^{-s}_p\left(\mathbf{R}^d\right)$ where $p \geq 2$ and $q = p/\left(p-1\right)$, we write

$$\langle \mathbf{f}, \boldsymbol{\phi} \rangle_{s,Y} = \langle \mathbf{f}, \boldsymbol{\phi} \rangle_{s,p,Y} = \sum^d_{l=1} \int_{\mathbf{R}^d} \left[\Lambda^s f^l(x)\right] \Lambda^{-s} \phi^l(x) \, dx \; .$$

Obviously, the function $\boldsymbol{\phi} \longrightarrow \langle \mathbf{f}, \boldsymbol{\phi} \rangle_{s,Y}$ is a linear mapping from H^{-s}_p into Y and $|\langle \mathbf{f}, \boldsymbol{\phi} \rangle_s|_Y \leq ||\mathbf{f}||_{s,q} \, |\boldsymbol{\phi}|_{-s,p}$.

Similar notation, $\langle \phi, \psi \rangle_s$ and $\langle f, \phi \rangle_{s,Y}$, will be used for scalar functions.

2 Stochastic integrals

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a filtration $\mathbb{F}$ of right continuous σ-algebras $(\mathcal{F}_t)_{t\geq 0}$. All the $\sigma-$algebras are assumed to be $\mathbf{P}-$completed. Let $W(t)$ be an $\mathbb{F}$-adapted cylindrical Brownian motion in Y. In this section we will construct a natural stochastic integral with respect to $W(t)$ for $\mathbb{F}$-adapted $H_p^s(\mathbf{R}^d, Y)$-valued integrands.

Let $p \geq 2, s \in (-\infty, \infty)$. Then $\mathcal{I}_{s,p}$ denotes the set of all measurable $\mathbb{F}$-adapted $\mathbb{H}_p^s(Y)$-valued functions such that for every t,

$$\int_0^t ||\mathbf{g}(r)||_{s,p}^p \, dr < \infty \quad \mathbf{P} - \text{a.s.}$$

If $\mathbf{g} \in \mathcal{I}_{m,p}$ then for every and $\boldsymbol{\phi} \in \mathbb{H}_q^{-m}$ where $q = p/(p-1)$, we can define a stochastic integral

$$\mathbf{M}_t(\boldsymbol{\phi}) = \int_0^t \langle \mathbf{g}(r), \boldsymbol{\phi} \rangle_{s,Y} \cdot dW(r).$$

Indeed, by Hölder inequality,

$$\begin{aligned} &\int_0^t \left| \langle \mathbf{g}(r), \boldsymbol{\phi} \rangle_{s,Y} \right|_Y^2 dr \leq \int_0^t ||\mathbf{g}(r)||_{s,p}^2 \, ||\boldsymbol{\phi}||_{-s,q}^2 \, dr \leq \\ &C \left(\int_0^t ||\mathbf{g}(r)||_{s,p}^p \, dr \right)^{(p-2)/p} ||\boldsymbol{\phi}||_{-s,q}^{2p/(p-2)} < \infty \quad \mathbf{P} - \text{a.s.} \end{aligned} \tag{2.1}$$

Owing to (2.1), the stochastic integral $\int_0^t \langle \mathbf{g}(r), \boldsymbol{\phi} \rangle_{s,Y} \cdot dW(r)$ is well defined (see e.g. [9] or [5]). Of course the integral above is defined as a linear functional on $\mathbb{H}_q^{-s}$. In fact, it can be characterized more precisely. Specifically, the following result holds.

Theorem 1 *If* $\mathbf{g} \in \mathcal{I}_{s,p}, p \geq 2$, *then there is a unique* $\mathbb{H}_p^s(Y)$*-valued continuous martingale* $\mathbf{M}(t) = \int_0^t \mathbf{g}(r) \cdot dW(r)$ *such that for all* $\boldsymbol{\phi} \in \mathbb{H}_q^{-s}$,

$$\left\langle \int_0^t \mathbf{g}(r) \cdot dW(r), \boldsymbol{\phi} \right\rangle_s = \int_0^t \langle \mathbf{g}(r), \boldsymbol{\phi} \rangle_{s,Y} \cdot dW(r) \quad \forall t > 0, \mathbf{P} - a.s. \tag{2.2}$$

Moreover, for each $T > 0$ *there exists a constant* C *so that for each stopping time* $\tau \leq T$,

$$\mathbf{E} \sup_{r \leq \tau} |\mathbf{M}(r)|_{s,p}^p \leq C\mathbf{E} \int_0^\tau ||\mathbf{g}(r)||_{s,p}^p \, dr.$$

To prove the Theorem we will need the following technical result.

Lemma 1 *Assume* $\mathbf{g} \in \mathcal{I}_{s,p}$. *Then there is a sequence of* $\mathbb{F}$*-adapted* $\mathbb{H}_p^s(\mathbf{R}^d)$*-valued processes* $\mathbf{g}_n(r) = \mathbf{g}_n(r,x)$ *such that* $\mathbf{P}$*-a.s.* $\mathbf{g}_n(r,x)$ *is smooth in* x *and for each* n,

$$\sup_x |\mathbf{g}_n(s,x)|_Y \leq C_n ||\mathbf{g}(r)||_{s,p}, \;\; ||\mathbf{g}_n(r)||_{s,p} \leq ||\mathbf{g}(r)||_{s,p},$$

and

$$\int_0^t ||\mathbf{g}_n(r) - \mathbf{g}(r)||_{s,p}^p \, dr \to 0, \;\; as \; n \to \infty.$$

$\mathbf{P}$*-a.s. for all* t .

Proof If we have two $\mathbb{H}_p^s(Y)$-valued continuous martingales $\mathbf{M}_1(t)$, $\mathbf{M}_2(t)$ satisfying (2.2), then $\mathbf{P}$-a.s. for all t and $\boldsymbol{\phi} \in \mathbb{H}_q^{-s}$,

$$\langle \mathbf{M}_1(t) - \mathbf{M}_2(t), \boldsymbol{\phi} \rangle_s = 0,$$

and the uniqueness follows. Let φ be a nonnegative function so that $\varphi \in C_0^\infty(\mathbf{R}^d)$ and $\int \varphi \, dx = 1$. For $\varepsilon > 0$, write $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. Set

$$\mathbf{g}_n(r,x) = \int \Lambda^{-s}\varphi_{1/n}(x-y)\Lambda^s \mathbf{g}(r,y)\, dy.$$

Note that $\mathbf{g}_n$ is a smooth bounded Y-valued function. Moreover, by Hölder inequality,

$$|\partial_x^\alpha \mathbf{g}_n(r,x)|_Y \le C'_{1/n,\alpha}(\int |\Lambda^s \mathbf{g}(r,y)|_Y^p \, dy)^{1/p} \left(\int |\partial_x^\alpha \Lambda^{-s}\varphi_{1/n}(x-y)|^q dy \right)^{1/q} \le$$

$$\le C_{1/n,\alpha} ||\mathbf{g}(r)||_{s,p},$$

It is readily checked that for all r, ω, and $p \ge 2$, we have the following:

(a) $||\mathbf{g}_n(r,\cdot)||_{s,p} \le ||\mathbf{g}(r,\cdot)||_{s,p}$,

(b) $||\mathbf{g}_n(r) - \mathbf{g}(r)||_{s,p} \to 0$ as $n \to \infty$.

Indeed,

$$||\mathbf{g}_n(r,\cdot)||_{s,p} = \left|\left|\Lambda^s \int \Lambda^{-s}\varphi_{1/n}(x-y)\Lambda^s \mathbf{g}(r,y)\, dy\right|\right|_p =$$

$$\left|\left|\int \varphi_{1/n}(x-y)\Lambda^s \mathbf{g}(r,y)\, dy\right|\right|_p \le ||\Lambda^s \mathbf{g}(r,\cdot)||_p = ||\mathbf{g}(r,\cdot)||_{s,p}.$$

Analogously, one can prove (b). Now, the statement follows by Lebesgue's dominated convergence theorem. □

Proof of Theorem 1 Let $\mathbf{g}_n$ be a sequence from Lemma 1. Since for every x and t, $\int_0^t |\mathbf{g}_n(r,x)|_Y^2 dr < \infty$ P-a.s., the stochastic integral

$$\mathbf{M}_n(t,x) = \int_0^t \mathbf{g}_n(r,x) \cdot dW(r),$$

is well defined for each x (see e.g. [9] or [5]). It is not difficult to show that for every x,

$$\Lambda^s \mathbf{M}_n(t,x) = \int_0^t \Lambda^s \mathbf{g}_n(r,x) \cdot dW_r \tag{2.3}$$

P-a.s. By the Burkhölder-Davis-Gundy and Minkowski's inequality, for each stopping time $\tau \le T$

$$\mathbf{E}\sup_{r\le\tau} |\mathbf{M}_n(r)|_{s,p}^p = \mathbf{E}\sup_{r\le\tau} |\Lambda^s \mathbf{M}_n(r)|_p^p \le \tag{2.4}$$

$$C\mathbf{E}\int_0^\tau ||\Lambda^s \mathbf{g}_n(r)||_p^p \, dr = C\mathbf{E}\int_0^\tau ||\mathbf{g}_n(r)||_{s,p}^p \, dr$$

and

$$\mathbf{E}\sup_{r\le\tau} |\mathbf{M}_n(r) - \mathbf{M}_{n'}(r)|_{s,p}^p \le C\mathbf{E}\int_0^\tau ||\mathbf{g}_n(r) - \mathbf{g}_{n'}(r)||_{s,p}^p \, dr.$$

Firstly, we prove the existence of a continuous in t $\mathbb{H}_p^s$-valued modification of $\mathbf{M}_n(t,x)$. Let

$$\mathbf{g}_{n,k}^s(r,x) = 1_{\{|x|\leq k\}} 1_{\{\|\mathbf{g}(r,\cdot)\|_{s,p}\leq k\}} \mathbf{g}_n(r,x)$$

Let $\tau \leq T$ be a stopping time such that

$$\mathbf{E}\int_0^\tau ||\mathbf{g}(r,\cdot)||_{s,p}^p \, dr < \infty.$$

Define

$$\mathbf{M}_{n,k}(t,x) = \int_0^{t\wedge\tau} \mathbf{g}_{n,k}^s(r,x)\cdot dW_r, \; \mathbf{M}_{n,k}^s(t,x) = \int_0^{t\wedge\tau} \Lambda^s \mathbf{g}_{n,k}^s(r,x)\cdot dW_r.$$

Then, for all $u \leq t \leq T$,

$$\mathbf{E}[(\int |\mathbf{M}_{n,k}^s(t,x) - \mathbf{M}_{n,k}^s(u,x)|^p \, dx)^2]$$

$$\leq C_k \mathbf{E} \int |\mathbf{M}_{n,k}^s(t,x) - \mathbf{M}_{n,k}^s(u,x)|^{2p} \, dx$$

$$\leq C_k \int \mathbf{E}(\int_{u\wedge\tau}^{t\wedge\tau} |\Lambda^s \mathbf{g}_{n,k}^s(r,x)|_Y^2 \, dr)^p \, dx \leq C_k |t-u|^p,$$

and by Kolmogorov's criterion, $\mathbf{M}_{n,k}^s$ has a continuous $\mathbb{L}_p$-valued modification. On the other hand,

$$\mathbf{E}\sup_{r\leq\tau} |\mathbf{M}_{n,k}(r,\cdot) - \mathbf{M}_n(r,\cdot)|_{s,p}^p = \mathbf{E}\sup_{r\leq\tau} |\mathbf{M}_{n,k}^s(r,\cdot) - \Lambda^s \mathbf{M}_n(r,\cdot)|_p^p \leq$$

$$C\mathbf{E}\int_0^\tau ||\Lambda^s(\mathbf{g}_{n,k}^s(r,\cdot) - \mathbf{g}_n(r,\cdot))||_p^p \, dr \to 0,$$

as $k \to \infty$. So, $\Lambda^s \mathbf{M}_n$ has an $\mathbb{L}_p$-valued continuous modification or, equivalently, $\mathbf{M}_n$ has an $\mathbb{H}_{s,p}$-valued continuous modification. By (2.3) we have that for all $t > 0, \boldsymbol{\phi} \in \mathbb{H}_q^{-s}$,

$$\langle \mathbf{M}_n(t), \boldsymbol{\phi}\rangle_s = \int_0^t \langle \mathbf{g}_n(r), \boldsymbol{\phi}\rangle_{s,Y} \cdot dW_r \quad \forall t > 0, \mathbf{P}-a.s.$$

for every $t > 0$, $\mathbf{P}$-a.s.

Now, by (2.4), $\mathbf{M}_n$ is a Cauchy sequence. Making $n \uparrow \infty$ on both sides of the equality we complete the proof. □

Remark 1 *For $p \in [1,2)$, the stochastic integral with the properties above does not exist (see [12]).*

Lemma 2 *Let $p \geq 2$. Let $\mathbf{g} \in \mathcal{I}_{0,p}$ and*

$$\int_0^t ||\mathbf{g}(r)||_1^2 \, dr < \infty \; \forall t > 0, \; \mathbf{P}-a.s.$$

Then for every $t > 0$, $\mathbf{P}$-a.s. one has

$$\int \int_0^t \mathbf{g}(r,x) \cdot dW(r)\, dx = \lim_{m\to\infty} \int \phi_m(x) \left(\int_0^t \mathbf{g}(r,x) \cdot dW(r) \right) dx$$
$$= \int_0^t (\int \mathbf{g}(r,x)\, dx) \cdot dW(r),$$

where $\phi_m \in C_0^\infty$ is any uniformly bounded sequence converging pointwise to 1.

Proof By Theorem 1,

$$\mathbf{Z}_m(t) = \int \phi_m(x) \int_0^t \mathbf{g}(r,x) \cdot dW(r)\, dx = \int_0^t (\int \mathbf{g}(r,x)\phi_m(x)\, dx) \cdot dW(r)$$

for every $t > 0$, $\mathbf{P}$-a.s.
Let τ be a stopping time such that

$$\mathbf{E} \int_0^\tau ||\mathbf{g}(r)||_1^2\, dr < \infty.$$

Then

$$\mathbf{E} \sup_{r\le\tau} |\mathbf{Z}_n(r) - \mathbf{Z}_m(r)|^2 \le C\mathbf{E} \int_0^\tau ||\mathbf{g}(r)(\phi_n - \phi_m)||_1^2\, dr \to 0,$$

as $n, m \to \infty$, and the statement follows. □

Now we can prove the Ito formula for the L_p-norm of a semimartingale.

Lemma 3 *Let $p \ge 2$. Set*

$$\mathbf{u}(t,x) = \mathbf{u}_0(x) + \int_0^t \mathbf{a}(r,x)\, dr + \int_0^t \mathbf{b}(r,x) \cdot dW(r) \tag{2.5}$$

where $\mathbf{b} \in \mathcal{I}_{0,p}$, $\mathbf{u}_0$ is an $\mathcal{F}_0$-measurable $\mathbb{L}_p$-valued random variable, and $\mathbf{a}$ is an $\mathbb{F}$-adapted $\mathbb{H}_p^{n-1}$-valued process where $n = 0$ or 1. If $\mathbf{u}(t)$ is continuous $\mathbb{L}_p$-valued process and

$$\int_0^t (|\mathbf{a}(r)|_{n-1,p}^p + |\mathbf{u}(r)|_{1-n,p}^p)\, dr < \infty$$

for all $t > 0$, $\mathbf{P}$-a.s., then

$$|\mathbf{u}(t)|_p^p = |\mathbf{u}_0|_p^p + p \int_0^t \langle |\mathbf{u}(r)|^{p-2}\mathbf{u}(r), \mathbf{a}(r)\rangle_{1-n}\, dr$$

$$+ p \int_0^t (\int |\mathbf{u}(r,x)|^{p-2}(\mathbf{u}(r,x), \mathbf{b}(r,x))\, dx\, dW(r) \tag{2.6}$$

$$+ \frac{p}{2} \int_0^t (\int [(p-2)|\mathbf{u}(r,x)|^{p-4} u^i(r,x) u^j(r,x)$$

$$+ |\mathbf{u}(r,x)|^{p-2}\delta_{ij}]\mathbf{b}^i(r,x) \cdot \mathbf{b}^j(r,x)\, dx)\, dr$$

for all $t > 0$, $\mathbf{P}$-a.s..

Proof We remark that all the integrals in (2.6) are well defined. For example, let us prove that the duality $\langle |\mathbf{u}(r)|^{p-2}\mathbf{u}(r), \mathbf{a}(r)\rangle_{1-n}$ makes sense if $n=0$. Since $\mathbf{a}(r) \in \mathbb{H}_p^{-1}$, there exist functions $\mathbf{a}_i(r) \in \mathbb{L}_p$ so that $\mathbf{a}(r) = \sum_{i=0}^d \partial_i \mathbf{a}_i\,(r)$ where $\partial_0 = 1$. Now it is not difficult to see that

$$\langle |\mathbf{u}(r)|^{p-2}\mathbf{u}(r), \mathbf{a}(r)\rangle_1 = \sum_{i=0}^d \langle |\mathbf{u}(r)|^{p-2}\mathbf{u}(r), \partial_i \mathbf{a}_i\,(r)\rangle_1 = \\ -\sum_{i=0}^d \langle \partial_i |\mathbf{u}(r)|^{p-2}\mathbf{u}(r), \mathbf{a}_i\,(r)\rangle_0. \tag{2.7}$$

The right hand side of the equality is finite owing to the obvious equality

$$\partial_i(|\mathbf{u}|^{p-2}u^l) = (p-2)|\mathbf{u}|^{p-4}u^m \partial_i u^m u^l + |\mathbf{u}|^{p-2}\partial_i u^l).$$

Let $\varphi \in C_0^\infty$ be a non-negative function such that $\int \varphi\, dx = 1$. For $\varepsilon > 0$, write

$$\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon),\ \mathbf{u}_\varepsilon(t,x) = \int \varphi_\varepsilon(x-y)\mathbf{u}(t,y)\, dy = \mathbf{u}(t) * \varphi_\varepsilon(x).$$

Similarly, we write $\mathbf{b}_\varepsilon(t) = \mathbf{b}(t) * \varphi_\varepsilon(x)$, $\mathbf{u}_{0,\varepsilon} = \mathbf{u}_0 * \varphi_\varepsilon(x)$. Let $\mathbf{a}_\varepsilon(t,x)$ $= \langle \mathbf{a}(t), \varphi_\varepsilon(x-\cdot)\rangle$. For all x and t, we have

$$\mathbf{u}_\varepsilon(t,x) = \mathbf{u}_{0,\varepsilon}(x) + \int_0^t \mathbf{a}_\varepsilon(r,x)\, ds + \int_0^t \mathbf{b}_\varepsilon(r,x) \cdot dW\,(r) \quad \mathbf{P}-a.s.$$

Let $\phi \in C_0^\infty, \phi = 1$ on $\{|x| \le 1\}$, $\phi = 0$ on $\{|x| \ge 2\}$. Then $\phi_m(x) = \phi(x/m)$ is a uniformly bounded sequence converging pointwise to 1. By Ito formula, we have

$$\begin{aligned} |\mathbf{u}_\varepsilon(t,x)|^p\phi_m(x) = |\mathbf{u}_{0,\varepsilon}(x)|^p\phi_m(x) + \\ \int_0^t p|\mathbf{u}_\varepsilon(r,x)|^{p-2}(\mathbf{u}_\varepsilon(r.x), \mathbf{a}_\varepsilon(r,x))\phi_m(x)\, dr \\ + \int_0^t p\phi_m(x)|\mathbf{u}_\varepsilon(r,x)|^{p-2}(\mathbf{u}_\varepsilon(r,x), \mathbf{b}_\varepsilon(r,x)) \cdot\, dW\,(r) \\ + \frac{p}{2}\int_0^t \phi_m(x)[(p-2)|\mathbf{u}_\varepsilon(r,x)|^{p-4}u_\varepsilon^i(r,x)u_\varepsilon^j(r,x) \\ + |\mathbf{u}_\varepsilon(r,x)|^{p-2}\delta_{ij}]b_\varepsilon^i(r,x) \cdot b_\varepsilon^j(r,x)\, ds.. \end{aligned} \tag{2.8}$$

Also,

$$\sup_{r \le t,x} |\mathbf{u}_\varepsilon(r,x)| + \sup_{rst} |\mathbf{u}(r)|_p < \infty \quad \forall t > 0,\ \mathbf{P}-a.s.$$

and

$$\int_0^t (|\mathbf{a}_\varepsilon(r) - \mathbf{a}(r)|_{n-1,p}^p + |\mathbf{u}_\varepsilon(r) - \mathbf{u}(r)|_{1-n,p}^p + |\mathbf{b}_\varepsilon(r) - \mathbf{b}(r)|_p^p)\, dr \to 0,$$

$$|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|_p^p \to 0,$$

as $\varepsilon \to 0$. We complete the proof by taking integrals of both sides of (2.8) and passing to the limit as $\varepsilon \to 0$, and then as $m \to \infty$. $\square$

3 Pointwise multipliers in $\mathbb{H}^s_p$

If $u \in H^s_p(Y)$ (resp. $\mathbf{u} \in \mathbb{H}^s_p(Y)$), then

$$|u|_{s,p} = |\Lambda^s u|_p = |\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}u]|_p,$$

(resp. $|\mathbf{u}|_{s,p} = |\Lambda^s \mathbf{u}|_p = |\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}\mathbf{u}]|_p$) where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is the inverse Fourier transform:

$$\mathcal{F}f(\xi) = (2\pi)^{-d/2}\int e^{-i\xi\cdot x} f(x)\,dx,\ \mathcal{F}^{-1}f(x) = (2\pi)^{-d/2}\int e^{i\xi\cdot x} f(\xi)\,d\xi.$$

Define the operators

$$\begin{aligned} \tilde{\Lambda}_s\mathbf{u} &= \begin{cases} \mathcal{F}^{-1}[(1+|\xi|^s)\mathcal{F}\mathbf{u}], & \text{if } s \geq 0, \\ \mathcal{F}^{-1}[(1+|\xi|^{|s|})^{-1}\mathcal{F}\mathbf{u}], & \text{if } s < 0, \end{cases} \\ \dot{\Lambda}^s\mathbf{u} &= \mathcal{F}^{-1}(|\xi|^s\mathcal{F}\mathbf{u}),\ s \geq 0. \end{aligned}$$

Consider the norms on $\mathbb{H}^s_p(Y)$

$$\begin{aligned} ||\mathbf{u}||_{\hat{}\, s,p} &= ||\mathbf{u}||_p + ||\dot{\Lambda}^s\mathbf{u}||_p, \quad \text{if } p \in [1,\infty], s \geq 0.\ , \\ |\mathbf{u}|_{\tilde{}\, s,p} &= |\tilde{\Lambda}^s\mathbf{u}|_p,\ p \in (1,\infty), s \in (-\infty,\infty). \end{aligned}$$

Now we prove the equivalence of the norms $|f|_{s,p}$, $|f|_{\tilde{}\, s,p}$.

Lemma 4 *The norms $||\mathbf{u}||_{s,p}$, and $||\mathbf{u}||_{\tilde{}\, s,p}$ are equivalent for $p \in (1,\infty)$, $s \in (-\infty,\infty)$, and $||\mathbf{u}||_{s,p}$, and $||\mathbf{u}||_{\hat{}\, s,p}$ are equivalent for $p \in [1,\infty], s \geq 0$.*

Proof For each multiindex μ and $s \geq 0$, we have

$$\begin{aligned} &|\partial^\mu_\xi \frac{1+|\xi|^s}{(1+|\xi|^2)^{s/2}}| \leq C_\mu|\xi|^{-|\gamma|}, \\ &|\partial^\mu_\xi \frac{(1+|\xi|^2)^{s/2}}{1+|\xi|^s}| \leq C_\mu|\xi|^{-|\gamma|}. \end{aligned} \tag{3.1}$$

Therefore, the equivalence of $|\mathbf{u}||_{s,p}$ and $||\mathbf{u}||_{\tilde{}\, s,p}$ for $p \in (1,\infty)$ follows from Theorem 6.1.6 in [10].

The part of the statement regarding the case $s > 0, p \in [1,\infty]$ follows by Theorem 6.3.2 in [10]. $\square$

Remark 2 *For $s \in (0,2]$, $\mathbf{f} \in \mathbb{C}_0^\infty(Y)$, denote*

$$\bar\partial^s \mathbf{f}(x) = -\mathcal{F}^{-1}[|\xi|^s \mathcal{F}\mathbf{f}(\xi)](x).$$

It is well known (and easily seen) that there is a constant $N = N(s) > 0$ such that

$$\bar\partial^s \mathbf{f}(x) = N(s) \int [\mathbf{f}(x+y) - \mathbf{f}(x) - (\nabla \mathbf{f}(x), y)(1_{\{|y|\le 1\}} 1_{\{s=1\}} + 1_{\{1<s<2\}})] \frac{dy}{|y|^{d+s}},$$

$$\bar\partial^2 \mathbf{f}(x) = \Delta \mathbf{f}(x),$$

i.e. $\bar\partial^s$ is the generator of s-stable stochastic process.

Indeed, for $w = \xi/|\xi|$, we have

$$\mathcal{F}[\int [\mathbf{f}(\cdot + y) - \mathbf{f}(\cdot) - (\nabla \mathbf{f}(\cdot), y)(1_{\{|y|\le 1\}} 1_{\{s=1\}} + 1_{\{1<s<2\}})] \frac{dy}{|y|^{d+s}}]$$

$$= \hat{\mathbf{f}}(\xi) \int [e^{i(\xi,y)} - 1 - i(\xi, y)(1_{\{|y|\le 1\}} 1_{\{s=1\}} + 1_{\{1<s<2\}})] \frac{dy}{|y|^{d+s}}$$

$$= -|\xi|^s \hat{\mathbf{f}}(\xi) \int [1 - \cos(w,y)] \frac{dy}{|y|^{d+s}} = -c(s)|\xi|^s \hat{\mathbf{f}}(\xi)$$

where $c(s)$ is a positive constant depending on s.

Lemma 5 *Let $\delta \in (0,1)$. Then for each $p \in [1,\infty]$ there is a constant C so that for all $\mathbf{u} \in \mathbb{H}_p^s(Y)$, $z \in \mathbf{R}^d$*

$$||\mathbf{u}(\cdot + z) - \mathbf{u}(\cdot)||_{s,p} \le C||\dot\Lambda^\delta \mathbf{u}||_{s,p}|z|^\delta \le C||\mathbf{u}||_{s+\delta,p}|z|^\delta.$$

Proof Indeed, there is a constant N so that for any $x, z \in \mathbf{R}^d, \mathbf{u} \in \mathbb{C}_b^\infty(Y)$

$$\mathbf{u}(x+z) - \mathbf{u}(x) = N \int k^{(\delta)}(z,y) \bar\partial^\delta \mathbf{u}(x-y)\, dy \tag{3.2}$$

where $k^{(\delta)}(z,y) = |y+z|^{-d+\delta} - |y|^{-d+\delta}$. One can easily see this by taking Fourier transform of (3.2) (see [11], Chapter II, section 2). Also, it can be easily seen, that for some constant C

$$\int |k^{(\delta)}(z,y)| dy = C|z|^\delta. \tag{3.3}$$

Using Minkowsky's inequality we obtain from (3.2), (3.3) the desired estimate. □

Also, we will need some spaces of Y-valued continuous functions. For $m = 1, 2, 3, \ldots$, we define

$$C^m(Y) = \{u : \partial^\alpha u \text{ is uniformly continuous on } \mathbf{R}^d \text{ for all } |\alpha| \le m\},$$

with the norm $||u||_{C^m} = \sum_{|\alpha|\le m} ||\partial^\alpha u||_\infty$. For a non-integer $s > 0$, we define

$$\mathcal{C}^s(Y) = \{u \in C^{[s]} : ||u||_{\mathcal{C}^s} = ||u||_{C^{[s]}} + \sum_{|\alpha|=[s]} \sup_{x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|_Y}{|x-y|^{\{s\}}} < \infty\},$$

where $s = [s] + \{s\}, s$ is an integer and $0 \le \{s\} < 1$. For an integer $s > 0$, we denote

$$\mathcal{C}^s(Y) = \{u \in C^{s-1} : ||u||_{\mathcal{C}^s} = ||u||_{C^{s-1}} + \sum_{|\alpha|=[s]^-} \sup_{x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|_Y}{|x-y|} < \infty,$$

where $s = [s]^- + 1$. If $Y = \mathbf{R}^d$, we write simply $C^m, \mathcal{C}^s$.

Lemma 6 *Let $s > 0$. Then*
a) $H^s_\infty(Y) \subseteq \mathcal{C}^s(Y)$, if s is not an integer;
b) $\mathcal{C}^{s+\varepsilon}(Y) \subseteq H^s_\infty(Y)$ for each $\varepsilon > 0$.

Proof For an non-integer s, $\mathcal{C}^s$ is Zygmund's space (see Theorem 2.5.7 and Corollary 2.5.12 in [8]). Therefore the statement a) follows by Theorem 6.2.4 in [10].
Let $s \in (0, 2], u \in \mathcal{C}^{s+\varepsilon}(Y)$. We can assume that $s + \varepsilon$ is not an integer and $s < 2$. By Remark 2,

$$|-\dot{\Lambda}^s u|_Y = |\bar{\partial}^s u|_Y = N(s)|\int_{|y|\le 1} ... + \int_{|y|>1} ...|_Y$$

$$\le C[||u||_{\mathcal{C}^{s+\varepsilon}} + ||u||_\infty + |\nabla u|_\infty 1_{\{|\alpha|>1\}}].$$

So, the statement b) follows by Lemma 4. □

Define

$$B^s(Y) = \begin{cases} H^s_\infty(Y), & \text{if } s > 0 \text{ is not an integer,} \\ \mathcal{C}^s(Y), & \text{if } s > 0 \text{ is an integer,} \\ L_\infty(Y), & \text{if } s = 0, \end{cases}$$

and denote the corresponding norms by $|\cdot|_{B^s}$. If $Y = \mathbf{R}^d$, we write simply B^s. The main statement we need is the following Lemma.

Lemma 7 *a) Let $a \in B^{|s|}(Y), s \in (-\infty, \infty), p \in (1, \infty)$. Then there is a constant N so that*

$$||a\mathbf{u}||_{s,p} \le N||a||_{B^{|s|}} |\mathbf{u}|_{s,p}$$

for all $\mathbf{u} \in \mathbb{H}^s_p$, where $a\mathbf{u} = (au^1, \ldots, au^d)$;
b) Assume, $p \in (1, \infty), \kappa > 0$ and

$$a \in \begin{cases} B^s(Y), & \text{if } s \ge 0, \\ B^{|s|+\kappa}(Y), & \text{if } s < 0. \end{cases}$$

Let $\bar{a}_s = |a|_{B^s}$ *if* $s \geq 0$ *and* $\bar{a}_s = |a|_{B^{|s|+\kappa}}$ *if* $s < 0$.
Then for every s *there exist constants* $s_0 < s$ *and* N *such that*

$$||a\mathbf{u}||_{s,p} \leq N(||a||_\infty |\mathbf{u}|_{s,p} + \bar{a}_s |\mathbf{u}|_{s_0,p})$$

for all $\mathbf{u} \in \mathbb{H}^s_p$.
Moreover,

$$\tilde{\Lambda}^s(a\mathbf{u}) = a\tilde{\Lambda}^s\mathbf{u} + \mathbf{H}_s(a,\mathbf{u}), \text{ if } s \neq 2m+1\,(m=0,1,\ldots),$$

$$\partial_i\dot{\Lambda}^{s-1}(a\mathbf{u}) = a(\partial_i\dot{\Lambda}^{s-1}\mathbf{u}) + \mathbf{H}_s(a,\mathbf{u}), \text{ if } s = 2m+1\,(m=0,1,\ldots)$$

where $||\mathbf{H}_s(a,\mathbf{u})||_p \leq C\bar{a}_s|\mathbf{u}|_{s_0,p}$.

Proof Let $s \in (0,2)$, $\mathbf{u} \in \mathbb{C}_0^\infty, a \in C_b^\infty(Y)$ (a and all its derivatives are bounded). Then, by Remark 2,

$$\dot{\Lambda}^s(a\mathbf{u}) = a\dot{\Lambda}^s\mathbf{u} + \mathbf{u}\dot{\Lambda}^s a - \int [\mathbf{u}(x+y) - \mathbf{u}(x)](a(x+y) - a(x))\,\frac{dy}{|y|^{d+s}}. \tag{3.4}$$

By Minkowski's inequality,

$$||\int [\mathbf{u}(\cdot+y) - \mathbf{u}(\cdot)](a(\cdot+y) - a(\cdot))\,\frac{dy}{|y|^{d+s}}||_p$$

$$\leq \int |\mathbf{u}(\cdot+y) - \mathbf{u}(\cdot)|_p ||a(\cdot+y) - a(\cdot)||_\infty \frac{dy}{|y|^{d+s}}.$$

If $s \in (0,2)$ and $s \neq 1$, we have by Lemma 5 for each $s_0 \in ((s-1)^+, s)$

$$\int |\mathbf{u}(\cdot+y) - \mathbf{u}(\cdot)|_p ||a(\cdot+y) - a(\cdot)||_\infty \frac{dy}{|y|^{d+s}}$$

$$\leq ||a||_{B^{|s|\wedge 1}} |\dot{\Lambda}^{s_0}\mathbf{u}|_p \int_{|y|\leq 1} \frac{dy}{|y|^{d+s-(s\wedge 1)-s_0}} + 4||a||_\infty |\mathbf{u}|_p \int_{|y|>1} \frac{dy}{|y|^{d+s}}.$$

In the case $s=1$, we have $\partial_i(a\mathbf{u}) = a\partial_i\mathbf{u} + \mathbf{u}\partial_i a$ and

$$||\nabla(a\mathbf{u})||_p \leq ||a||_\infty |\nabla\mathbf{u}|_p + ||\nabla a||_\infty |\mathbf{u}|_p.$$

In the case $s=2$, we have $\Delta(a\mathbf{u}) = a\Delta\mathbf{u} + \mathbf{u}\Delta a + 2(\nabla a)(\nabla\mathbf{u})$ and

$$||\Delta(a\mathbf{u})||_p \leq ||a||_\infty |\Delta\mathbf{u}|_p + ||\Delta a||_\infty |\Delta\mathbf{u}|_p + 2||\nabla a||_\infty |\nabla\mathbf{u}|_p.$$

Therefore both parts of our statement hold for $s \in [0,2]$. For an arbitrary $s > 2$, we can find a positive integer m so that $s = 2m + r, r \in (0,2]$. If $r \neq 1$,

$$\dot{\Lambda}^s(a\mathbf{u}) = \dot{\Lambda}^r\dot{\Lambda}^{2m}(a\mathbf{u}) = \dot{\Lambda}^r(a\dot{\Lambda}^{2m}\mathbf{u}) + \dot{\Lambda}^r\mathbf{h}$$

where $\mathbf{h}$ is a linear combinations of the products in the form $(\partial^\nu \mathbf{u})(\partial^\mu a)$, where $\nu \neq \mathbf{0}$, and $|\nu| + |\mu| = 2m$ According to the previous estimates, there is $s_0 \in ((r-1)^+, r)$ so that

$$||\dot{\Lambda}^r \mathbf{h}||_p \leq C||a||_{B^{|s|}} |\mathbf{u}|_{2m-1+s_0}.$$

On the other hand, by (3.4)

$$\dot{\Lambda}^r(a\dot{\Lambda}^{2m}\mathbf{u}) = a\dot{\Lambda}^s\mathbf{u}+\dot{\Lambda}^r a\dot{\Lambda}^{2m}\mathbf{u} + \tilde{\mathbf{h}}$$

where $||\tilde{\mathbf{h}}||_p \leq C||a||_{B^{s\wedge 1}} |\dot{\Lambda}^{s_0+2m}\mathbf{u}|_p$. If $s = 2m+1, m = 1, 2, \ldots$, then

$$\partial_i \dot{\Lambda}^{2m}(a\mathbf{u}) = a\partial_i \dot{\Lambda}^{2m}\mathbf{u} + \mathbf{H}_s$$

where $||\mathbf{H}_s||_p \leq C||a||_{B^s} |\mathbf{u}|_{s-1,p}$.
So, we found that for each $s > 0$, there is a constant C so that

$$||a\mathbf{u}||_{s,p} \leq C||a||_{B^s} |\mathbf{u}|_{s,p}.$$

Since the multiplication by a is selfadjoint operation, by duality, obviously, follows that for each $s \in (-\infty, \infty)$ we have for some C

$$||a\mathbf{u}||_{s,p} \leq C||a||_{B^{|s|}} |\mathbf{u}|_{s,p} \tag{3.5}$$

If $s = -2m < 0, m = 1, 2, \ldots, \mathbf{u} \in \mathbb{H}_p^s$, then $\mathbf{u} = (1-\Delta)^m \mathbf{h}, \mathbf{h} \in \mathbb{L}_p$. Then it is easy to check that

$$a\mathbf{u} = (1-\Delta)^m (a\mathbf{h}) - \mathbf{H}, \tag{3.6}$$

and the function $\mathbf{H}$ is a linear combinations of the products in the form $(\partial^\nu \mathbf{h})(\partial^\mu a)$, where $\mu \neq \mathbf{0}$, and $|\nu| + |\mu| = 2m$. Since $\partial^\mu a \in B^{|s|+\kappa-|\mu|}$, using (3.5) we obtain

$$||(\partial^\nu \mathbf{h})(\partial^\mu a)||_{s,p} \leq ||(\partial^\nu \mathbf{h})(\partial^\mu a)||_{s-\kappa+|\mu|,p} \leq C|(\partial^\nu \mathbf{h})|_{s-\kappa+|\mu|} ||(\partial^\mu a)||_{B^{-s+\kappa-|\mu|}}$$

$$\leq C||\partial^\mu a||_{B^{|s|+\kappa-|\mu|}} |\partial^\nu \mathbf{h}|_{s+|\mu|-\kappa,p} \leq$$

$$\leq C||a||_{B^{|s|+\kappa}} |\mathbf{h}|_{-\kappa,p} \leq C||a||_{B^{|s|+\kappa}} |\mathbf{u}|_{s-\kappa,p}.$$

So, by (3.6)

$$(1-\Delta)^s(a\mathbf{u}) = a(1-\Delta)^s\mathbf{u} - (1-\Delta)^s\mathbf{H},$$

$||(1-\Delta)^s\mathbf{H}||_p = ||\mathbf{H}||_{s,p} \leq C||a||_{B^{|s|+\kappa}} |\mathbf{u}|_{s-\kappa,p}$, and

$$||a\mathbf{u}||_s \leq C(||a||_\infty |\mathbf{u}|_s + ||a||_{B^{|s|+\kappa}} |\mathbf{u}|_{s-\kappa}).$$

If $s < 0$ is not an integer, then there is a positive integer m so that $s = -2m - r, r \in (0, 2)$. Let $\mathbf{u} \in \mathbb{H}_p^s$. Then $\mathbf{u} = \mathbf{h} + \dot{\Lambda}^r \dot{\Lambda}^{2m}\mathbf{h}$, $\mathbf{h} \in \mathbb{L}_p$. Let $\tilde{\mathbf{h}} = \dot{\Lambda}^{2m}\mathbf{h}$. We have $\tilde{\mathbf{h}} \in \mathbb{H}_p^{-2m}$ and

$$a\dot{\Lambda}^r\tilde{\mathbf{h}} = \dot{\Lambda}^r(a\tilde{\mathbf{h}}) - (\dot{\Lambda}^r a)\tilde{\mathbf{h}} -$$

$$\int (\tilde{\mathbf{h}}(x+y) - \tilde{\mathbf{h}}(x))(a(x+y) - a(x))\frac{dy}{|y|^{d+r}}, \tag{3.7}$$

$$a\tilde{\mathbf{h}} = a\dot{\Lambda}^{2m}\mathbf{h} = \dot{\Lambda}^{2m}(a\mathbf{h}) - \mathbf{g},$$

where $\mathbf{g}$ is a linear combination of $(\partial^\nu \mathbf{h})(\partial^\mu a), |\mu|+|\nu|=2m, \mu\neq \mathbf{0}$. Since

$$||\dot{\Lambda}^r(\partial^\nu \mathbf{h}\,\partial^\mu a)||_{s,p} \leq ||\partial^\nu \mathbf{h}\,\partial^\mu a||_{s+r,p} \leq ||\partial^\nu \mathbf{h}\,\partial^\mu a||_{s+r+|\mu|-\kappa,p}$$

$$\leq C||\partial^\mu a||_{B^{|s|-r-|\mu|+\kappa}}|\mathbf{h}|_{s+|\mu|+|\nu|+r-\kappa,p}$$

$$= C||\partial^\mu a||_{B^{|s|-r-|\mu|+\kappa}}|\mathbf{h}|_{-\kappa,p} \leq C||a||_{B^{|s|-r+\kappa}}|\mathbf{h}|_{-\kappa,p},$$

we have $||\dot{\Lambda}^r \mathbf{g}||_{s,p} \leq C||a||_{B^{|s|-r+\kappa}}|\mathbf{h}|_{-\kappa}$.
Also by (3.5),

$$||(\dot{\Lambda}^r a)\tilde{\mathbf{h}}||_s \leq C||(\dot{\Lambda}^r a)\tilde{\mathbf{h}}||_{-2m-\kappa} \leq C||\dot{\Lambda}^r a||_{B^{2m+\kappa}}|\tilde{\mathbf{h}}|_{-2m-\kappa} \leq C||\dot{\Lambda}^r a||_{B^{2m+\kappa}}|\mathbf{u}|_{s-\kappa}.$$

Fix $\kappa' \in (0,\kappa)$. Let $\delta_1 = \max\{r-1,0\}+\kappa, \varepsilon_1 = \delta_1 - \kappa', \varepsilon_2 = \min\{1,r\}$. By Lemma 5 and (3.5) and using Minkowsky's inequality, we have

$$||\int (\tilde{\mathbf{h}}(\cdot+y)-\tilde{\mathbf{h}}(\cdot))(a(\cdot+y)-a(\cdot))\frac{dy}{|y|^{d+r}}||_{s,p}$$

$$\leq C\int ||\tilde{\mathbf{h}}(\cdot+y)-\tilde{\mathbf{h}}(\cdot))(a(\cdot+y)-a(\cdot))||_{-2m-\delta_1,p}\frac{dy}{|y|^{d+r}}$$

$$\leq \int |\tilde{\mathbf{h}}(\cdot+y)-\tilde{\mathbf{h}}(\cdot)|_{-2m-\delta_1}||a(\cdot+y)-a(\cdot)||_{B^{2m+\delta_1}}\frac{dy}{|y|^{d+r}}$$

$$\leq C[\int_{|y|\leq 1}|\tilde{\mathbf{h}}|_{-2m-\delta_1+\varepsilon_1}||a||_{B^{2m+\delta_1+\varepsilon_2}}|y|^{\varepsilon_1+\varepsilon_2-d-\gamma}\,dy$$

$$+\int_{|y|>1}|\tilde{\mathbf{h}}|_{-2m-\delta_1}||a||_{B^{2m+\delta_1}}]\frac{dy}{|y|^{d+r}}].$$

So,

$$||\int (\tilde{\mathbf{h}}(x+y)-\tilde{\mathbf{h}}(x))(a(x+y)-a(x))\frac{dy}{|y|^{d+r}}||_{s,p}$$

$$\leq C|\tilde{\mathbf{h}}|_{-2m-\kappa'}||a||_{B^{s+\kappa}} \leq C|\mathbf{u}|_{s-\kappa'}||a||_{B^{s+\kappa}}.$$

Thus, according to (3.7),

$$a\mathbf{u} = a\tilde{\Lambda}^{-s}\mathbf{h} = \tilde{\Lambda}^{-s}(a\mathbf{h})+\mathbf{G},$$

where $||\mathbf{G}||_{s,p} \leq C||a||_{B^{|s|+\kappa}}|\mathbf{h}|_{-\kappa,p} \leq C||a||_{B^{|s|+\kappa}}|\mathbf{u}|_{s-\kappa,p}$. Therefore,

$$\tilde{\Lambda}^s(a\mathbf{u}) = a\mathbf{h}+\tilde{\Lambda}^s\mathbf{G} = a\tilde{\Lambda}^s\mathbf{u}+\tilde{\Lambda}^s\mathbf{G},$$

and $||\tilde{\Lambda}^s\mathbf{G}||_p \leq C||\mathbf{G}||_{s,p} \leq C||a||_{B^{|s|+\kappa}}|\mathbf{u}|_{s-\kappa,p}$. □

4 Systems of SPDEs in Sobolev spaces

As in the previous Section, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a filtration $\mathbb{F}$ of right continuous σ-algebras $(\mathcal{F}_t)_{t\geq 0}$. All the $\sigma-$algebras are assumed to be $\mathbf{P}-$completed. Let $W(t)$ be an $\mathbb{F}$-adapted cylindrical Brownian motion in Y. Let $s \in (-\infty, \infty)$. For $\mathbf{v} \in \mathbb{H}_p^{s+1}$, let $\mathbf{Q}(\mathbf{v}, t) = \mathbf{Q}(\mathbf{v}, t, x)$ be a predictable $\mathbb{H}_p^s(Y)$-valued function and $\mathbf{D}(\mathbf{v}, t) = \mathbf{D}(\mathbf{v}, t, x)$ a predictable $\mathbb{H}_p^{s-1}$-valued function. Let $a = a(t) = (a^{ij}(t,x))_{1\leq i,j\leq d}$ be a symmetric $\mathbb{F}$-adapted matrix. Let $\sigma = \sigma(t) = (\sigma^k(t,x))_{1\leq k\leq d}$ be $\mathbb{F}$-adapted vector function with Y-valued components σ^k, and let $\mathbf{u}_0 = \left(u_0^l\right)_{1\leq l\leq d}$ be an $\mathcal{F}_0-$measurable $\mathbb{H}_p^{s+1-2/p}-$valued function so that $\mathbf{E}\,||\mathbf{u}_0||_{s+1-2/p,p}^p < \infty$. Everywhere in this section it is assumed that $p \geq 2$.

Consider the following nonlinear system of equations on $[0,\infty)$:

$$\partial_t \mathbf{u}(t,x) = \partial_i(a^{ij}(t,x)\partial_j \mathbf{u}) + \mathbf{D}(\mathbf{u}, t, x) +$$

$$[\sigma^k(t,x)\partial_k \mathbf{u}(t,x) + \mathbf{Q}(\mathbf{u}, t, x)] \cdot \dot{W}, \tag{4.1}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x)$$

where $\mathbf{u}(t) = \mathbf{u}(t,x) = (u^k(t,x))_{1\leq k\leq d}$.

The following assumptions will be used in the future:

A. For all $t \geq 0$, $x, \lambda \in R^d$,

$$K|\lambda|^2 \geq [a^{ij}(t,x) - \frac{1}{2}\sigma^i(t,x)\cdot\sigma^j(t,x)]\lambda^i\lambda^j \geq \delta|\lambda|^2,$$

where K, δ are fixed strictly positive constants.

A1(s,p). For all $t, x, y, \mathbf{P}$-a.s.

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_Y \leq K|x-y|$$

and

$$\begin{cases} |a^{ij}(t)|_{B^s} \leq K, & \text{if } s > 1, \\ |a(t,x)| \leq K, & \text{if } -1 < s \leq 1, \\ |a^{ij}(t)|_{B^{-s+\varepsilon}} \leq K, & \text{if } s \leq -1, \end{cases}$$

where $\varepsilon \in (0,1)$.

The Y-valued function $\sigma(t,x)$ is $\mathbf{P}$-a.s. continuously differentiable in x and for all i, t

$$\begin{cases} ||\sigma^i(t)||_{B^s} \leq K, & \text{if } s \geq 1, \\ |\sigma^i(t,x)|_Y \leq K, & \text{if } s \in (-1,1), \\ ||\sigma^i(t)||_{B^{-s+\varepsilon}} \leq K, & \text{if } s \leq -1, \end{cases}$$

where $\varepsilon \in (0,1)$.

$\mathbf{A2}(s,p)$ For $\mathbf{v} \in \mathbb{H}_p^{s+1}$, $\mathbf{Q}(\mathbf{v},t)=\mathbf{Q}(\mathbf{v},t,x)$ is a predictable $\mathbb{H}_p^s(Y)$-valued function and $\mathbf{D}(\mathbf{v},t)=\mathbf{D}(\mathbf{v},t,x)$ is a predictable $\mathbb{H}_p^{s-1}$-valued function, and $\mathbf{P}$-a.s. for each t

$$\int_0^t (|\mathbf{D}(\mathbf{0},r)|_{s-1,p}^p + ||\mathbf{Q}(\mathbf{0},r)||_{s,p}^p)\, dr < \infty \ \forall t>0, \mathbf{P}-a.s.$$

where $\mathbf{0}=(0,\ldots,0)$.

$\mathbf{A3}(s,p)$. For every $\varepsilon>0$, there exists a constant K_ε such that for any $\mathbf{u},\mathbf{v} \in \mathbb{H}_p^{s+1}$,

$$|\mathbf{D}(\mathbf{u},t,x)-\mathbf{D}(\mathbf{v},t,x)|_{s-1,p} + ||\mathbf{Q}(\mathbf{u},t,x)-\mathbf{Q}(\mathbf{v},t,x)||_{s,p} \le$$

$$\varepsilon|\mathbf{u}-\mathbf{v}|_{s+1,p} + K_\varepsilon|\mathbf{u}-\mathbf{v}|_{s-1,p} \quad \mathbf{P}-a.s.$$

Given a stopping time τ, we consider a stochastic interval

$$[[0,\tau]] = \begin{cases} [0,\tau(\omega)], & \text{if } \tau(\omega)<\infty, \\ [0,\infty), & \text{otherwise.} \end{cases}$$

Definition 1 *Given a stopping time τ, an $\mathbb{H}_p^s(\mathbf{R}^d)$-valued $\mathbb{F}-$adapted function $\mathbf{u}(t)$ on $[0,\infty)$ is called an $\mathbb{H}_p^s$-solution of equation (4.1) in $[[0,\tau]]$ if it is strongly continuous in t with probability 1,*

$$\mathbf{u}(t\wedge\tau)=\mathbf{u}(t), \int_0^{t\wedge\tau} |\mathbf{u}(s)|_{s+1,p}^p\, ds < \infty \ \forall t>0, \mathbf{P}-a.s., \tag{4.2}$$

and the equality

$$\begin{aligned} \mathbf{u}(t\wedge\tau) = \mathbf{u_0} + \textstyle\int_0^{t\wedge\tau} \left[\partial_i(a^{ij}(r)\partial_j\mathbf{u}) + \mathbf{D}(\mathbf{u},r)\right] dr + \\ \textstyle\int_0^{t\wedge\tau} [\sigma^k(r)\partial_k\mathbf{u}\,(r) + \mathbf{Q}(\mathbf{u},r)]\cdot dW\,(r) \end{aligned} \tag{4.3}$$

holds in $\mathbb{H}_p^{s-1}(\mathbf{R}^d)$ for every $t>0$, $\mathbf{P}-a.s.$

If $\tau=\infty$, we simply say $\mathbf{u}$ is an $\mathbb{H}_p^s$-solution of equation (4.1).

Sometimes, when the context is clear, instead of "$\mathbb{H}_p^s$-solution" we will simply say "solution".

It is readily checked that all the integrals in 4.3 are well defined. For example, let us consider the stochastic integral. Since ∂_i is a bounded operator from $\mathbb{H}_p^s$ into $\mathbb{H}_p^{s-1}$(see [8]), by Lemma 7 and Assumption $\mathbf{A1}(s,p)$, we have $\left|\left|\sigma^k(r)\partial_k\mathbf{u}\,(r)\right|\right|_{s,p} \le C\,||\mathbf{u}\,(r)\,||_{s+1,p}$ for $r\le\tau$ $\mathbf{P}$-a.s. By assumptions $\mathbf{A2}(s,p)$,$\mathbf{A3}(s,p)$,

$$\int_0^{t\wedge\tau} ||\mathbf{Q}(\mathbf{u},r)||_{s,p}^p\, dr \le C\int_0^{t\wedge\tau} (||\mathbf{Q}(\mathbf{0},r)||_{s,p}^p + |\mathbf{u}\,(r)\,|_{s+1,p}^p)dr.$$

Thus, $[\sigma^k(r)\partial_k\mathbf{u}+\mathbf{Q}(\mathbf{u},r)]1_{\{r<\tau\}} \in \mathcal{I}_{s,p}$, and the integral is defined by Theorem 1.

Remark 3 *It is not difficult to show that (4.3) can be replaced by the equality*

$$\begin{aligned} \left\langle u^l(t\wedge\tau),\phi^l\right\rangle_s = \left\langle u_0^l,\phi^l\right\rangle_s + \textstyle\int_0^{t\wedge\tau} -\left\langle (a^{ij}(r)\partial_i u^l), \partial_j\phi^l\right\rangle_s + \\ \langle\Lambda^{-1}D^l(\mathbf{u},r),\Lambda\phi^l\rangle_s dr + \textstyle\int_0^{t\wedge\tau}\langle\sigma^k(r)\partial_k u^l + Q^l(\mathbf{u},r),\phi^l\rangle_{s,Y}\cdot dW\,(r) \\ \forall t>0, \mathbf{P}-a.s. \end{aligned} \tag{4.4}$$

which holds for all $\boldsymbol{\phi}=(\phi^l)_{1\le l\le d}$ such that $\phi^l\in C_0^\infty, l=1,\ldots,d,$.

Indeed, owing to (4.3), we have

$$\begin{aligned}\left\langle u^l(t\wedge\tau),\phi^l\right\rangle_{s-1} &= \left\langle u^l_0,\phi^l\right\rangle_{s-1} + \textstyle\int_0^{t\wedge\tau}\left\langle \partial_j(a^{ij}(r)\partial_i u^l) + D^l(\mathbf{u},r),\phi^l\right\rangle_{s-1} dr \\ &+ \textstyle\int_0^{t\wedge\tau}\left\langle \sigma^k(r)\partial_k u^l + Q^l(\mathbf{u},r),\phi^l\right\rangle_{s-1,Y}\cdot dW(r) \quad \forall t>0, \mathbf{P}-a.s.\end{aligned} \tag{4.5}$$

On the other hand, since $\mathbf{u}\in\mathbb{H}^s_p$, $\mathbf{u_0}\in\mathbb{H}^{s+1-2/p}_p$, and $\mathbf{P}$-a.s. for $r\le\tau$, $\sigma^k(r)\partial_k\mathbf{u}+Q^l(\mathbf{u},r)\in\mathbb{H}^s_p$,we have that $\left\langle u^l(t),\phi^l\right\rangle_{s-1} = \left\langle u^l(t),\phi^l\right\rangle_s$, $\left\langle u^l_0,\phi^l\right\rangle_{s-1} = \left\langle u^l_0,\phi^l\right\rangle_{s+1-2/p}$, and,

$$\left\langle \sigma^k(r)\partial_k u^l + Q^l(\mathbf{u},r),\phi^l\right\rangle_{s-1,Y} = \left\langle \sigma^k(r)\partial_k u^l + Q^l(\mathbf{u},r),\phi^l\right\rangle_{s,Y}.$$

It is readily checked that $dr\times d\mathbf{P}$-a.s.

$$\left\langle \partial_j(a^{ij}(r)\partial_i u^l),\phi^l\right\rangle_{s-1} = -\left\langle a^{ij}(r)\partial_i u^l,\partial_j\phi^l\right\rangle_{s-1} =$$

$$-\left\langle \Lambda^s(a^{ij}(r)\partial_i u^l),\Lambda^{-s}\partial_j\phi^l\right\rangle_0 = -\left\langle (a^{ij}(r)\partial_i u^l),\partial_j\phi^l\right\rangle_s$$

Note that to prove the first equality one should first establish it for smooth functions and then prove it in the general case by approximations. Thus, (4.5) implies (4.4). Now by reversing the order of our arguments one could easily show that (4.3) follows from (4.4).

The basic result of this Section is given in the following

Theorem 2 *Let $s\in(-\infty,\infty), p\ge 2$. Let **A**, **A1**$(s,p)$-**A3**$(s,p)$ be satisfied and $|\mathbf{u}_0|^p_{s+1,p}<\infty$ $\mathbf{P}$-a.s. Then for each stopping time τ the Cauchy problem (1.1) has a unique $\mathbb{H}^s_p$-solution in $[[0,\tau]]$. Moreover, for each $T>0$, there is a constant C such that for each stopping time $\bar\tau\le T\wedge\tau$,*

$$\begin{aligned}\mathbf{E}[\sup_{r\le\bar\tau}|\mathbf{u}(r)|^p_{s,p} + \int_0^{\bar\tau}|\partial^2\mathbf{u}(r)|^p_{s-1,p}\,ds] &\le C\mathbf{E}[|\mathbf{u}_0|^p_{s+1-2/p,p} \\ &+ \int_0^{\bar\tau}(|\mathbf{D}(\mathbf{0},r)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0},r)||^p_{s,p})\,dr].\end{aligned}$$

The Theorem will be proved in several steps. We begin with a simple particular case.

Theorem 3 *(cf Theorem 4.10 in [2]). Let $s\in(-\infty,\infty), p\ge 2$. Assume **A**, **A1**$(s,p)$-**A3**$(s,p)$. Suppose that $\mathbf{D}$ and $\mathbf{Q}$ are independent of $\mathbf{u}$, a^{ij} and σ^k are independent of x, and $\mathbf{u}_0=\mathbf{0}$.*

Then for each stopping time τ there is a unique $\mathbb{H}^s_p-$solution $\mathbf{u}$ of equation (4.1) in $[[0,\tau]]$. Moreover,

(i) for each stopping time $\bar\tau\le\tau$,

$$\mathbf{E}\int_0^{\bar\tau}|\partial^2\mathbf{u}(r)|^p_{s-1,p}\,dr \le N\mathbf{E}\int_0^{\bar\tau}(|\mathbf{D}(r)|^p_{s-1,p} + ||\mathbf{Q}(r)||^p_{s,p})\,dr, \tag{4.6}$$

where $N=N(d,p,\delta,K)$ does not depend on $\tau,\bar\tau$;

(ii) for each finite T and each stopping time $\bar{\tau} \le T \wedge \tau$

$$\mathbf{E}\sup_{r\le\bar{\tau}} |\mathbf{u}(r)|^p_{s,p} \le e^T C\mathbf{E}\int_0^{\bar{\tau}} (|\mathbf{D}(r)|^p_{s-1,p} + ||\mathbf{Q}(r)||^p_{s,p})\, dr], \tag{4.7}$$

where $C = C(d,p,\delta,K)$ does not depend on T and $\bar{\tau},\tau$.

Proof The statement is a straightforward corollary of the results of [2]. Indeed, owing to our assumptions one can treat each component u^l of $\mathbf{u}$ separately. The the statement regarding the existence follows directly by Theorem 4.10 in [2] considering $\mathbf{D}(r) = \mathbf{D}(r)1_{[[0,\tau]]}(r)$, and $\mathbf{Q}(r) = \mathbf{Q}(r)1_{[[0,\tau]]}(r)$. According to Lemma 4.7 in [2], the uniqueness is an obvious consequence of the deterministic heat equation result. In particular, we obtain (4.7) by taking $\lambda = 1/p$ in (4.26) in [2]. □

To prove Theorem 3 in the general case, we will rely on the two fundamental techniques: partition of unity and the method of continuity. The same technology was used in [2] for scalar equations. The next step is to derive a priori L_p-estimates for a solution of (4.1).

Lemma 8 *Assume **A**, **A1**(s,p)-**A3**(s,p). Suppose that $\mathbf{u}$ is an $\mathbb{H}^s_p-$ solution of equation (4.1) in $[[0,\tau]]$ with $\mathbf{u}_0$= $\mathbf{0}$.*
Then for each T there is a constant $C = C(d,p,\delta,K,T)$ such that for each stopping time $\bar{\tau} \le T \wedge \tau$,

$$\begin{aligned}&\mathbf{E}[\sup_{r\le\bar{\tau}} |\mathbf{u}(r)|^p_{s,p} + \textstyle\int_0^{\bar{\tau}} |\partial^2\mathbf{u}(r)|^p_{s-1,p}\, dr] \le \\ &C\mathbf{E}\textstyle\int_0^{\bar{\tau}} (|\mathbf{D}(\mathbf{0},r)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0},r)||^p_{s,p})\, dr.\end{aligned} \tag{4.8}$$

Proof In order to use Theorem 8 we start with a standard partition of unity. Let $\psi \in C_0^\infty(\mathbf{R})$, be $[0,1]$-valued and such that $\psi(s) = 1$, if $|s| \le 5/8$, and $\psi(s) = 0$, if $|s| > 6/8$. For an arbitrary but fixed $\kappa > 0$ there we choose m such that $\kappa < 2^{-m}$. Consider a grid in $\mathbf{R}^d$ consisting of $x_k = k2^{-m}$, $k = (k_1,\dots,k_d) \in \mathbf{Z}^d$, where $\mathbf{Z}$ is the set of all integers. Given $k \in \mathbf{Z}^d$, we define a function on $\mathbf{R}^d$:

$$\bar{\eta}_k(x) = \prod_{l=1}^d \psi((x^l - x^l_k)2^m).$$

Notice that $0 \le \bar{\eta}_k \le 1, \bar{\eta}_k = 1$ in the cube $v_k = \{x : |x^l - x^l_k| \le (5/8)2^{-m}, l = 1,\dots,d\}$, and $\bar{\eta}_k = 0$ outside the cube $V_k = \{x : |x^l - x^l_k| \le (6/8)2^{-m}, l = 1,\dots,d\}$. Obviously,

1. $\cup_k v_k = \mathbf{R}^d$ and
$$1 \le \sum_k 1_{V_k} \le 2^d;$$

2. For all multiindices γ
$$|\partial^\gamma \bar{\eta}_k| \le N(d,|\gamma|)2^{m|\gamma|} < N(d)\kappa^{-|\gamma|}.$$

Denote

$$\eta_k(x) = \bar{\eta}_k(x)(\sum_k \bar{\eta}_k(x))^{-1}, k = 1, \ldots$$

Obviously, $\sum_k \eta_k = 1$ in $\mathbf{R}^d$ and for all k and multiindices μ,

$$|\partial^\mu \eta_k| \leq N(d, |\mu|)\kappa^{-|\mu|},$$

and for each $p \geq 1, \mu$

$$\sum_k \eta_k(x)^p \leq N(p, d), \sum_k |\partial^\mu \eta_k|^p \leq N(p.d, |\mu|)\kappa^{-p|\mu|}. \tag{4.9}$$

So, by Lemma 6.7 in [2], for any n there exist constants $c = c(d, p, \kappa)$, $C = C(d, p, \kappa)$ such that for all $\mathbf{f} \in \mathbb{H}_p^n, \mathbf{g} \in \mathbb{H}_p^n(Y)$

$$c|\mathbf{f}|_{n,p}^p \leq \sum_k |\eta_k \mathbf{f}|_{n,p}^p \leq C|\mathbf{f}|_{n,p}^p, \tag{4.10}$$

$$c||\mathbf{g}||_{n,p}^p \leq \sum_k ||\eta_k \mathbf{g}||_{n,p}^p \leq C||\mathbf{g}||_{n,p}^p.$$

Multiplying (4.1) by η_k, we obtain

$$\partial_t(\mathbf{u}\eta_k) = \partial_i(a^{ij}(t, x_k)\partial_j(\eta_k \mathbf{u})) + \mathbf{D}_k(\mathbf{u}, t, x) + \tag{4.11}$$

$$+ [\sigma^i(t, x_k)\partial_i(\eta_k \mathbf{u}) + \mathbf{Q}_k(\mathbf{u}, t, x)] \cdot \dot{W},$$

where

$$\mathbf{D}_k(\mathbf{u}, t, x) = \eta_k[\mathbf{D}(\mathbf{u}, t, x) + \partial_i(a^{ij}(t, x) - a^{ij}(t, x_k)\partial_j \mathbf{u})]$$

$$- \partial_i(a^{ij}(t, x_k)\partial_j \eta_k \mathbf{u}) - a^{ij}(t, x_k)\partial_i \eta_k \partial_j \mathbf{u},$$

$$\mathbf{Q}_k(\mathbf{u}, t, x) = \eta_k[\mathbf{Q}(\mathbf{u}, t, x) + (\sigma^i(t, x) - \sigma^i(t, x_k))\partial_i \mathbf{u}]$$

$$- \sigma^i(t, x_k)\partial_i \eta_k \mathbf{u}.$$

We have

$$\sum_k |\eta_k \partial_i(a^{ij}(t) - a^{ij}(t, x_k)\partial_j \mathbf{u}(t))|_{s-1,p}^p$$

$$\leq 2^{p-1} \sum_k |\partial_i \eta_k(a^{ij}(t) - a^{ij}(t, x_k))\partial_j \mathbf{u}(t)|_{s-1,p}^p$$

$$+ 2^{p-1} \sum_k |\partial_i[\eta_k(a^{ij}(t) - a^{ij}(t, x_k))\tilde{\eta}_k \partial_j \mathbf{u}(t)]|_{s-1,p}^p,$$

where $\tilde{\eta}_k(x) = \bar{\eta}_k(5x/6)$ (notice $\tilde{\eta}_k(x) = 1$ in V_k and $\tilde{\eta}_k(x) = 0$ if there is l such that $|x^l - x^l_k| > 0.9 \cdot 2^{-m}$). According to Lemma 7, there is a constant C and $s_0 < s$ such that

$$\sum_k |\partial_i[\eta_k(a^{ij}(t) - a^{ij}(t, x_k))\tilde{\eta}_k \partial_j \mathbf{u}(t)]|^p_{s-1,p}$$

$$\leq \sum_k |\eta_k(a^{ij}(t) - a^{ij}(t, x_k))\tilde{\eta}_k \partial_j \mathbf{u}(t)|^p_{s,p}$$

$$\leq C \sum_k [\sup_{x,k} |\tilde{\eta}_k(a^{ij}(t) - a^{ij}(t, x_k))|^p |\partial \mathbf{u}(t)|^p_{s,p}$$

$$+ |\eta_k \partial_j \mathbf{u}(t)|^p_{s_0-1,p}].$$

Similarly, by Lemma 7 there is $s_0 < s$ so that

$$\sum_k ||\eta_k(\sigma^i(t) - \sigma^i(t, x_k))\partial_i \mathbf{u}(t)||^p_{s,p}$$

$$= \sum_k ||\tilde{\eta}_k(\sigma^i(t) - \sigma^i(t, x_k))\eta_k \partial_i \mathbf{u}(t)||^p_{s,p}$$

$$\leq C \sum_k [\sup_x ||\tilde{\eta}_k(\sigma^i(t) - \sigma^i(t, x_k))||^p |\eta_k \partial_i \mathbf{u}(t)|^p_{s,p}$$

$$+ |\eta_k \partial_i \mathbf{u}(t)|^p_{s_0,p}].$$

It follows by the assumptions, (4.10), Lemma 7 and interpolation theorem (see Lemma 6.7 in [2]) that for each ε there is $\kappa > 0$ and a constant $C = C(\varepsilon, \kappa, d, p, \delta, K)$ such that

$$\sum_k |\mathbf{D}_k(\mathbf{u}, t)|^p_{s-1,p} \leq \varepsilon |\partial^2 \mathbf{u}(t)|^p_{s-1,p} + C(|\mathbf{u}(t, \cdot)|^p_{s-1,p} + |\mathbf{D}(\mathbf{0}, t)|^p_{s-1,p}),$$

$$\sum_k ||\mathbf{Q}_k(\mathbf{u}, t, \cdot)||^p_{s,p} \leq \varepsilon |\partial^2 \mathbf{u}(t)|^p_{s-1,p} + C(|\mathbf{u}(t)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0}, t)||^p_{s,p}).$$

Choosing ε sufficiently small, applying (4.10) and Theorem 3 to $\eta_k \mathbf{u}$ (it is a solution to the equation (4.11), we obtain that

(i) for each stopping time $\bar{\tau} \leq \tau$

$$\mathbf{E} \int_0^{\bar{\tau}} |\partial^2 \mathbf{u}(t)|^p_{s-1,p} \, dt \leq N \mathbf{E} \int_0^{\bar{\tau}} (|\mathbf{u}(t)|^p_{s-1,p} + |\mathbf{D}(\mathbf{0}, t)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0}, t)||^p_{s,p}) \, dt$$

where $N = N(p, d, \delta, K)$ does not depend on τ.

(ii) for each $T > 0$ and each stopping time $\bar{\tau} \le T \wedge \tau$

$$\mathbf{E}\sup_{t\le\bar{\tau}} |\mathbf{u}(t)|^p_{s,p} \le Ne^T\mathbf{E}\int_0^{\bar{\tau}} (|\mathbf{u}(t)|^p_{s-1,p} + |\mathbf{D}(\mathbf{0},t)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0},t)||^p_{s,p})\, dt$$

Fix an arbitrary $\bar{\tau} \le T \wedge \tau$ such that

$$\mathbf{E}[\sup_{t\le\bar{\tau}} |\mathbf{u}(t)|^p_{s,p} + \int_0^{\bar{\tau}} (|\mathbf{u}(t)|^p_{s-1,p} + |\mathbf{D}(\mathbf{0},t)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0},t)||^p_{s,p})\, dt] < \infty.$$

Then for each $t \le T$

$$\mathbf{E}\sup_{r\le t\wedge\bar{\tau}} |\mathbf{u}(r)|^p_{s,p} \le Ne^T\mathbf{E}\int_0^t \sup_{\bar{r}\le r\wedge\bar{\tau}} |\mathbf{u}(\bar{r})|^p_{s,p}\, dr$$

$$+\mathbf{E}\int_0^{\bar{\tau}} |\mathbf{D}(\mathbf{0},t)|^p_{s-1,p} + ||\mathbf{Q}(\mathbf{0},t)||^p_{s,p}\, dt,$$

and the statement follows by Gronwall's inequality. □

Now we can prove the uniqueness of a solution to equation (4.1).

Corollary 1 *Assume* ***A, A1****(s,p)*-***A3****(s,p). Then for each stopping time τ there is at most one $\mathbb{H}^s_p$−solution to (4.1) in $[[0,\tau]]$.*

Proof If $\mathbf{u}_1,\mathbf{u}_2$ are solutions to (4.1), then $\mathbf{v} = \mathbf{u}_2 - \mathbf{u}_1$ satisfies on $[[0,\tau]]$ the equation

$$\partial_t\mathbf{v}(t,x) = \partial_i(a^{ij}(t,x)\partial_j\mathbf{v}) + \mathbf{D}(\mathbf{v}+\mathbf{u}_1,t,x) - \mathbf{D}(\mathbf{u}_1,t,x)$$

$$+ [\sigma^k(t,x)\partial_k\mathbf{v}(t,x) + \mathbf{Q}(\mathbf{v}+\mathbf{u}_1,t,x) - \mathbf{Q}(\mathbf{u}_1,t,x)]\cdot\dot{W},$$

$$\mathbf{v}(0,x) = \mathbf{0}.$$

Applying Lemma 8 to this equation we get $\mathbf{v} = \mathbf{0}$ $\mathbf{P}$-a.s. □

Remark 4 *In fact the uniqueness of the solution can be proved in a larger functional class, similar to the one of Theorem 5.1 in [2]. For the sake of simplicity we will not address this problem in the present paper.*

To complete the proof of Theorem 2 we apply the standard method of continuity (cf. Theorem 5.1 in [2]).

Proof of Theorem 2.

(Existence) Without any loss of generality we can assume $\mathbf{u}_0 = \mathbf{0}$ (see Proof of Theorem 5.1 in [2]) and $\tau = \infty$. Now, let us take $\lambda \in [0,1]$ and consider the equation

$$\begin{aligned} \partial_t \mathbf{u}(t,x) = \partial_i[\lambda\delta_{ij} + (1-\lambda)a^{ij}\partial_j \mathbf{u}] + (\mathbf{1}-\lambda)\,\mathbf{D}(\mathbf{u},\mathbf{t},\mathbf{x}) + \\ +(1-\lambda)[\sigma^k\partial_k \mathbf{u} + \mathbf{Q}(\mathbf{u},t,x)]\cdot \dot{W} \end{aligned} \tag{4.12}$$

with zero initial condition. By Lemma 8 the a priori estimate (4.8) holds with the same constant C for all λ. Assume that for $\lambda = \lambda_0$ and any $\mathbf{D}, \mathbf{Q}$ satisfying $\mathbf{A3}(n,p)$, equation (4.12) has a unique solution.

For other $\lambda \in [0,1]$ we rewrite (4.12) as follows:

$$\begin{aligned} \partial_t \mathbf{u}(t,x) = \partial_i[(\lambda_0\delta_{ij} + (1-\lambda_0)a^{ij})\partial_j \mathbf{u}] + (1-\lambda_0)\,\mathbf{D}(\mathbf{u},t,x) \\ +(\lambda-\lambda_0)\left(\partial_i[(\delta_{ij}-a^{ij})\partial_j \mathbf{u}] - \mathbf{D}(\mathbf{u},t,x)\right) \\ +(1-\lambda_0)[\sigma^k\partial_k \mathbf{u} + \mathbf{Q}(\mathbf{u},t,x)]\cdot \dot{W} \\ -(\lambda-\lambda_0)\,[\sigma^k\partial_k \mathbf{u} + \mathbf{Q}(\mathbf{u},t,x)]\cdot \dot{W} \end{aligned}$$

This equation can be solved by iterations. Specifically, take $\mathbf{u}_0 = \mathbf{0}$ and write

$$\begin{aligned} \partial_t \mathbf{u}_{k+1}(t,x) = \partial_i[(\lambda_0\delta_{ij} + (1-\lambda_0)a^{ij})\partial_j \mathbf{u}_{k+1}] + (1-\lambda_0)\,\mathbf{D}(\mathbf{u}_{k+1},t,x) \\ +(\lambda-\lambda_0)\left(\partial_i[(\delta_{ij}-a^{ij})\partial_j \mathbf{u}_k] - \mathbf{D}(\mathbf{u}_k,\mathbf{t},\mathbf{x})\right) \\ +(1-\lambda_0)[\sigma^i\partial_i \mathbf{u}_{k+1} + \mathbf{Q}(\mathbf{u}_{k+1},t,x)]\cdot \dot{W} \\ -(\lambda-\lambda_0)\,[\sigma^i\partial_i \mathbf{u}_k + \mathbf{Q}(\mathbf{u}_k,t,x)]\cdot \dot{W} \end{aligned} \tag{4.13}$$

So, for $k \geq 1$, $\bar{\mathbf{u}}_{k+1} = \mathbf{u}_{k+1} - \mathbf{u}_k$ is a solution of the equation

$$\begin{aligned} \partial_t \bar{\mathbf{u}}_{k+1}(t,x) = \partial_i[(\lambda_0\delta_{ij} + (1-\lambda_0)a^{ij})\partial_j \bar{\mathbf{u}}_{k+1}] + (1-\lambda_0)\,[\mathbf{D}(\mathbf{u}_k + \bar{\mathbf{u}}_{k+1},t,x) - \\ \mathbf{D}(\mathbf{u}_k,t,x)] + (\lambda-\lambda_0)(\partial_i[(\delta_{ij}-a^{ij})\partial_j \bar{\mathbf{u}}_k] - [\mathbf{D}(\mathbf{u}_k,t,x) - \mathbf{D}(\mathbf{u}_{k-1},t,x)]) \\ +(1-\lambda_0)[\sigma^i\partial_i \bar{\mathbf{u}}_{k+1} + \mathbf{Q}(\mathbf{u}_k + \bar{\mathbf{u}}_{k+1},t,x) - \mathbf{Q}(\mathbf{u}_k,t,x)]\cdot \dot{W} \\ -(\lambda-\lambda_0)\,[\sigma^i\partial_i \bar{\mathbf{u}}_k + \mathbf{Q}(\mathbf{u}_k,t,x) - \mathbf{Q}(\mathbf{u}_{k-1},t,x)]\cdot \dot{W} \end{aligned}$$

By Lemma 8, for each $T > 0$ there is a constant $C = C(d,p,\delta,K,T)$ such that for all stopping times $\tau \leq T$

$$\mathbf{E}[\sup_{r\leq\tau}|\bar{\mathbf{u}}_{k+1}(r)|^p_{s,p}+\int_0^\tau|\partial^2\bar{\mathbf{u}}_{k+1}(r)|^p_{s-1,p}\,dr\,]$$

$$\leq C'|\lambda-\lambda_0|^p\mathbf{E}\int_0^\tau(|\partial\bar{\mathbf{u}}_k(r)|^p_{s,p}+|\partial^2\bar{\mathbf{u}}_k(r)|^p_{s-1,p})\,dr$$

$$\leq C|\lambda-\lambda_0|^p\mathbf{E}[\sup_{r\leq\tau}|\bar{\mathbf{u}}_k(r)|^p_{s,p}+\int_0^\tau|\partial^2\bar{\mathbf{u}}_k(r)|^p_{s-1,p}\,dr].$$

Fix an arbitrary stopping time $\tau \leq T$ such that

$$I(\tau)=\mathbf{E}[\sup_{r\leq\tau}|\mathbf{u}_1(r)|^p_{s,p}+\int_0^\tau|\partial^2\mathbf{u}_1(r)|^p_{s-1,p}\,dr]<\infty,$$

Notice $\mathbf{u}_1$ and τ do not depend on λ (only on λ_0). Let $|\lambda-\lambda_0|<C^{-1/p}/2$. Then

$$\mathbf{E}[\sup_{r\leq\tau}|\bar{\mathbf{u}}_{k+1}(r)|^p_{s,p}+\int_0^\tau|\partial^2\bar{\mathbf{u}}_{k+1}(r)|^p_{s-1,p}\,dr]^{1/p}\leq(1/2)^kI(\tau)^{1/p}.$$

and $(\mathbf{u}_k)$ is a Cauchy sequence on $[0,\tau]$. Therefore, there is a continuous in t and $\mathbb{H}^s_p$-valued process $\mathbf{u}$ such hat

$$\mathbf{E}[\sup_{r\leq\tau}|\mathbf{u}_k(r)-\mathbf{u}(r)|^p_{s,p}+\int_0^\tau|\partial^2(\mathbf{u}_k(r)-\mathbf{u}(r))|^p_{s-1,p}\,dr]\to 0,$$

as $k\to\infty$. Obviously $\mathbf{u}$ is a solution to (4.12) on $[0,\tau]$. Since τ is any stopping time such that $I(\tau)$ is finite, it follows that we have a solution for any $|\lambda-\lambda_0|<C^{-1/p}/2$ (assuming we have one for λ_0). For $\lambda=1$ it does exist by Theorem 3. So, in finite number of steps starting with $\lambda=1$, we get to $\lambda=0$. This proves the statement.

Corollary 2 *(cf. Corollary 5.11 in [2]) Assume* ***A**, **A1**(s,p)-**A3**(s,p). Assume further **A1**(s,q)-**A3**(s,q) for $q\geq 2$, and suppose that $|\mathbf{u}_0|_{s+1-2/p,p}+|\mathbf{u}_0|_{s+1-2/q,q}<\infty$ $\mathbf{P}$-a.s. Then the $\mathbb{H}^s_p-$solution $\mathbf{u}$ of equation (4.1) is also an $\mathbb{H}^s_q-$solution of the equation.*
Moreover, for each $T>0$, there is a constant C such that for each stopping time $\tau\leq T$,

$$\mathbf{E}[\sup_{r\leq\tau}|\mathbf{u}(r)|^l_{s,l}+\int_0^\tau|\partial^2\mathbf{u}(r)|^l_{s-1,l}\,dr] \tag{4.14}$$

$$\leq C\mathbf{E}[|\mathbf{u}_0|^l_{s+1-2/l,l}+\int_0^\tau(|\mathbf{D}(\mathbf{0},r)|^l_{s-1,l}+||\mathbf{Q}(\mathbf{0},r)||^l_{s,l})\,dr],$$

$l=p,q$.

Proof We follow the lines of the proof of the Theorem 2 by introducing the parameter $\lambda \in [0,1]$ and considering the equation (4.12). We can assume that $\mathbf{u}_0 = \mathbf{0}$. The statement holds true for $\lambda = 1$ by Lemma 5.11 in [2] applied to each component of $\mathbf{u}$. If it is true for λ_0, then (4.13) defines a sequence $\mathbf{u}_k$ of $\mathbb{H}^s_p$-valued continuous processes that are $\mathbb{H}^s_q$-valued and continuous as well, and

$$\int_0^t (|\partial^2 \mathbf{u}_k(r)|^l_{s-1,l}\, dr < \infty, l = p, q$$

$\mathbf{P}$-a.s. for all t.

For each $T > 0$, there are constants $C_l = C(d, l, \delta, K, T), l = p, q$ such that for all stopping times $\tau \leq T$,

$$\mathbf{E}[\sup_{r\leq\tau} |\bar{\mathbf{u}}_{k+1}(r)|^l_{s,l} + \int_0^\tau |\partial^2 \bar{\mathbf{u}}_{k+1}(r)|^l_{s-1,l}\, dr]$$

$$\leq C'|\lambda - \lambda_0|^p \mathbf{E} \int_0^\tau ((|\bar{\mathbf{u}}_k(r)|^p_{s,p} + |\partial^2 \bar{\mathbf{u}}_k(r)|^p_{s-1,p})\, dr$$

$$\leq C_l|\lambda - \lambda_0|^p \mathbf{E}[\sup_{r\leq\tau} |\bar{\mathbf{u}}_k(r)|^l_{s,l} + \int_0^\tau |\partial^2 \bar{\mathbf{u}}_k(r)|^l_{s-1,l}\, dr],$$

$l = p.q$. Fix an arbitrary stopping time $\tau \leq T$ such that

$$I(\tau) = \mathbf{E}[\sup_{r\leq\tau}(|\mathbf{u}_1(r)|^p_{s,p} + |\mathbf{u}_1(r)|^q_{s,q} + \int_0^\tau (|\partial^2 \mathbf{u}_1(r)|^p_{s-1,p} + |\partial^2 \mathbf{u}_1(r)|^q_{s-1,q})\, dr] < \infty.$$

Let $C = \max\{C_p, C_q\}, |\lambda - \lambda_0| < C^{-1/p}/2$. Then

$$\mathbf{E}[\sup_{r\leq\tau} |\bar{\mathbf{u}}_{k+1}(r)|^l_{s,l} + \int_0^\tau |\partial^2 \bar{\mathbf{u}}_{k+1}(r)|^l_{s-1,l}\, dr]^{1/p} \leq (1/2)^k I(\tau)^{1/p},$$

$l = p, q$. Therefore, there is a continuous in t and $\mathbb{H}^s_p \cap \mathbb{H}^s_q$-valued process $\mathbf{u}$ such hat

$$\mathbf{E}[\sup_{r\leq\tau} |\mathbf{u}_k(r) - \mathbf{u}(r)|^l_{s,l} + \int_0^\tau |\partial^2(\mathbf{u}_k(r) - \mathbf{u}(r))|^l_{s-1,l}\, dr] \to 0,$$

$l = p, q$, and the statement follows. □

4.1 Some estimates

Unfortunately, if s is positive, Assumption $\mathbf{A3}(s, p)$ is rarely satisfied for equations of Mathematical Physics even in the scalar case (see Example 1 below). The following Proposition as well Corollary 4 below help to circumvent this problem in many important cases.

Proposition 1 *Assume that for each* $\mathbf{v} \in \mathbb{H}_p^{s+1}$, $\mathbf{Q}(\mathbf{v},t)$ *is a predictable* $\mathbb{H}_p^{s+1}$*-valued process and* $\mathbf{D}(\mathbf{v},t)$ *is a predictable* $\mathbb{H}_p^{s}$*-valued process. Let* ***A**,* ***A1***$(s+1,p)$*,* ***A2***$(s+1,p)$*,* ***A3***(s,p) *be satisfied,* $|\mathbf{u}_0|_{s+2,p} < \infty$ *with probability one, and for all* $t > 0, \mathbf{v} \in \mathbb{H}_p^{s+1}$,

$$| \, |\mathbf{Q}(\mathbf{v},t)| \, |_{s+1,p} \le | \, |\mathbf{Q}(\mathbf{0},t)| \, |_{s+1,p} + C|\mathbf{v}|_{s+1,p}.$$

$$|\mathbf{D}(\mathbf{v},t)|_{s,p} \le |\mathbf{D}(\mathbf{0},t)|_{s,p} + C|\mathbf{v}|_{s+1,p}.$$

Suppose also that

$$\int_0^t (| \, |\mathbf{Q}(\mathbf{0},r)| \, |_{s+1,p}^p + |\mathbf{D}(\mathbf{0},r)|_{s,p}^p) \, dr < \infty$$

$\mathbf{P}$*-a.s. for all* t*. Then (4.1) has a unique continuous* $\mathbb{H}_p^{r+1}$*- solution.*
Moreover, for each $T > 0$ *there is a constant* C *such that for each stopping time* $\tau \le T$*,*

$$\begin{aligned} \mathbf{E}1_A[\sup_{r\le\tau} |\mathbf{u}(r)|_{s+1,p}^p + \int_0^\tau |\partial^2\mathbf{u}(r)|_{s,p}^p \, dr] &\le C\mathbf{E}1_A[|\mathbf{u}_0|_{s+2,p}^p \\ &+ \int_0^\tau (|\mathbf{D}(\mathbf{0},r)|_{s,p}^p + | \, |\mathbf{Q}(\mathbf{0},r)| \, |_{s+1,p}^p) \, dr]. \end{aligned}$$

Proof Since the assumptions **A, A1**(s,p)-**A3**(s,p) are satisfied, the existence and uniqueness of $\mathbb{H}_p^s$- solution is guaranteed by Theorem 2 By the same Theorem, the linear equation

$$\partial_t\xi(t,x) = \partial_i(a^{ij}(t,x)\partial_j\xi(t,x)) + \mathbf{D}(\mathbf{u},t,x) +$$

$$[\sigma^k(t,x)\partial_k\xi(t,x) + \mathbf{Q}(\mathbf{u},t,x)] \cdot \dot{W},$$

$$\xi(0,x) = \mathbf{u}_0(x),$$

has a unique $\mathbb{H}_p^{s+1}$-solution. Thus, $\xi = \mathbf{u}$ $\mathbf{P}$-a.s.. Moreover, for each T there is a constant C such that for all stopping times $\tau \le T$,

$$\mathbf{E}[\sup_{r\le t\wedge\tau} |\mathbf{u}(r)|_{s+1,p}^p + \int_0^{t\wedge\tau} |\partial^2\mathbf{u}(r)|_{s,p}^p \, dr] \le C\mathbf{E}[|\mathbf{u}_0|_{s+2,p}^p + \int_0^{t\wedge\tau} (|\mathbf{u}(r)|_{n+1,p}^p$$

$$+ |\mathbf{D}(\mathbf{0},r)|_{s,p}^p + ||\mathbf{Q}\,(\mathbf{0},r)||_{s+1,p}^p) \, dr].$$

Now the estimate of the statement follows by Gronwall's inequality. □

Example 1 *Let us consider the following scalar equation:*

$$\partial_t u = \Delta u + D(u) + u \cdot \dot{W},$$

$$u(0,x) = 0$$

*where $W(t)$ is a one-dimensional Wiener process, $D(u) = \partial[f(u(x))](= \partial f(u(x))\partial u(x))$ and f is a scalar Lipschitz function on R^1. Then **A3(1,p)** would require the following estimate:*

$$|D(u) - D(v)|_p = |\nabla f(u(x))\partial u(x)) - \nabla f(v(x))\partial v(x)|_p$$

$$\leq \varepsilon |u - v|_{2,p} + K_\varepsilon |u - v|_p,$$

which is false in general even if ∇f is Lipschitz.

On the other hand, the assumptions of the Proposition are satisfied for $n = 0$. Indeed,

$$|D(u)|_p = |\nabla f(u)\partial u|_p \leq C|\partial u|_p$$

where C is the Lipcshitz constant of f.

Now,since ∂ is a bounded operator from $\mathbb{H}^s_p$ into $\mathbb{H}^{s+1}_p$, we have

$$|D(u) - D(v)|_{-1,p} = |\partial[f(u)] - \partial[f(v)]|_{-1,p} \leq$$

$$C|f(u) - f(v)|_p \leq C'|u - v|_p \leq$$

$$\varepsilon|u - v|_{1,p} + K_\varepsilon|u - v|_{-1,p}.$$

*(The latter inequality follows from Remark 5.5 in [2].) Thus assumption **A3(0,p)** is verified and we are done.*

Proposition 2 *Let $s \in (-\infty, \infty)$, $p \geq 2$. Assume **A, A1**$(s, 2)$-**A3**$(s, 2)$. Suppose $|\mathbf{u}_0|_{s+1,2} < \infty$ $\mathbf{P}$-a.s. Assume further that*

$$\int_0^t (|\mathbf{D}(\mathbf{0},r)|^p_{s-1,2} + |\,|\mathbf{Q}(\mathbf{0}, r)|\,|^p_{s,2})\, dr < \infty$$

$\mathbf{P}$-a.s. for all t.

Then for each $T > 0$, there is a constant C such that for each stopping time $\tau \leq T$,

$$\mathbf{E}[\sup_{r\leq\tau} |\mathbf{u}(r)|^p_{s,2} + \int_0^\tau |\mathbf{u}(r)|^{p-2}_{s,2}|\nabla\mathbf{u}(r)|^2_{s,2}\, dr \leq C\mathbf{E}[|\mathbf{u}_0|^p_{s+1,2} + \int_0^\tau (|\mathbf{D}(\mathbf{0}, r)|^p_{s-1,2}$$

$$+ |\,|\mathbf{Q}(\mathbf{0}, r)|\,|^p_{s,2})\, dr].$$

Proof Since the assumptions of Theorem 2 are satisfied, there is a unique $\mathbb{H}^s_2$- solution $\mathbf{u}\,(t, x)$ of equation (4.1). Let $s \neq 2m + 1$, $m = 0, 1, \ldots$. Then $\tilde{\mathbf{u}} = \tilde{\Lambda}^s\mathbf{u}$ is $\mathbb{L}_2$-valued continuous and satisfies the equation

$$\partial_t\tilde{\mathbf{u}}(t, x) = \tilde{\Lambda}^s[\partial_i(a^{ij}(t, x)\partial_j\mathbf{u}) + \mathbf{D}(\mathbf{u}, t, x)]+$$

$$\tilde{\Lambda}^s[\sigma^k(t, x)\partial_k\mathbf{u}(t, x) + \mathbf{Q}(\mathbf{u}, t, x)] \cdot \dot{W},$$

$$\tilde{\mathbf{u}}(0, x) = \tilde{\mathbf{u}}_0(x),$$

where $\tilde{\mathbf{u}}_0 = \tilde{\Lambda}^s \mathbf{u}_0$. On the other hand, by Lemma 7,

$$\tilde{\Lambda}^s \partial_i (a^{ij}(t,x)\partial_j \mathbf{u}) = \partial_i \tilde{\Lambda}^s (a^{ij}(t,x)\partial_j \mathbf{u})$$

$$= \partial_i (a^{ij}(t,x)\partial_j \tilde{\Lambda}^s \mathbf{u}) + \partial_i \mathbf{H}_s(a^{ij}, \partial_j \mathbf{u}), \tag{4.15}$$

$$\tilde{\Lambda}^s [\sigma^k(t,x)\partial_k \mathbf{u}(t,x)] = \sigma^k(t,x)\partial_k \tilde{\Lambda}^s \mathbf{u}(t,x) + \mathbf{H}(\sigma^k, \partial_k \mathbf{u}),$$

and

$$|\mathbf{H}_s(a^{ij}, \partial_j \mathbf{u})|_2 + |\mathbf{H}(\sigma^k, \partial_k \mathbf{u})|_2 \le C|\nabla \mathbf{u}(t)|_{s_0,2},$$

($s_0 < s$). By interpolation theorem, for each ε there is a constant C_ε so that

$$|\mathbf{H}_s(a^{ij}, \partial_j \mathbf{u})|_2 + |\mathbf{H}(\sigma^k, \partial_k \mathbf{u})|_2 \le \varepsilon |\nabla \mathbf{u}(t)|_{s,2} + C_\varepsilon |\nabla \mathbf{u}(t)|_{s-1,2}. \tag{4.16}$$

Applying Ito formula, we obtain

$$|\tilde{\mathbf{u}}(t)|_2^p = |\tilde{\mathbf{u}}(0)|_2^p + p\int_0^t |\tilde{\mathbf{u}}(r)|_2^{p-2} \langle \tilde{\mathbf{u}}(r), \tilde{\Lambda}^s(\mathbf{D}(\mathbf{u}(r), r)\rangle_{1,2}\, ds$$

$$- p\int_0^t |\tilde{\mathbf{u}}(r)|_2^{p-2} \int a^{ij}(r)\partial_i \tilde{u}^l(r))\partial_j \tilde{u}^l(r)\, dx\, dr$$

$$- p\int_0^t |\tilde{\mathbf{u}}(r)|_2^{p-2} \int \partial_i \tilde{u}^l(r) H_s^l(a^{ij}(r), \partial_j \mathbf{u}(r))\, dx\, dr$$

$$+ p\int_0^t |\tilde{\mathbf{u}}(r)|_2^{p-2} (\int \tilde{u}^l(r)\tilde{b}^l(r)\, dx) \cdot dW_r$$

$$+ \frac{p}{2}\int_0^t |\tilde{\mathbf{u}}(r)|_2^{p-2} \int \tilde{b}^i(r) \cdot \tilde{b}^i(r)\, dx\, ds$$

$$+ \frac{p}{2}(p-2)\int_0^t |\tilde{\mathbf{u}}(r)|_2^{p-4} |\int \tilde{u}^l(r)\tilde{b}^l(r)\, dx|_Y^2\, dr,$$

where $\tilde{b}^k(r) = \sigma^i(r)\partial_i \tilde{u}^k(r) + H_s^l(\sigma^k, \partial_k \mathbf{u}) + \tilde{\Lambda}^s Q^k(\mathbf{u}, r)$. Notice

$$\tilde{\Lambda}^s(\mathbf{D}(\mathbf{u}(r), r)) \in \mathbb{H}_q^{-1}, \tilde{\Lambda}^s Q^k(\mathbf{u}, r) \in \mathbb{L}_2,$$

and, by $\mathbf{A3(}s,2)$, for each ε there is a constant C_ε so that

$$
\begin{aligned}
|\tilde\Lambda^s(\mathbf{D}(\mathbf{u}(r),r))|_{-1,2} &\le C|\mathbf{D}(\mathbf{u}(r),r)|_{s-1,2}\\
&\le \varepsilon|\mathbf{u}(r)|_{s+1,2}+C_\varepsilon(|\mathbf{u}(r)|_{s-1,2}+|\mathbf{D}(\mathbf{0},r)|_{s-1,2}), \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
||\tilde\Lambda^s Q^k(\mathbf{u},r)||_2 &\le C||Q^k(\mathbf{u},r)||_{s,2}\\
&\le \varepsilon|\mathbf{u}(r)|_{s+1,2}+C_\varepsilon(|\mathbf{u}(r)|_{s-1,2}+||\mathbf{Q}(\mathbf{0},r)||_{s,2}),
\end{aligned}
$$

$$
|\int \tilde u^l(r)\tilde b^l(r)\,dx|_Y \le \varepsilon|\mathbf{u}(r)|_{s+1,2}|\mathbf{u}(r)|_{s.2}+C_\varepsilon(|\mathbf{u}(r)|_{s,2}^2+|\mathbf{u}(r)|_{s,2}||\mathbf{Q}(\mathbf{0},r)||_{s,2}.
$$

So, $y(t)=|\tilde{\mathbf{u}}(t)|_2^p$ is a semimartingale:

$$
y(t)=y(0)+\int_0^t h(r)\,dr+\int_0^t g(r)\cdot dW(r)\,,
$$

where $h(r)$, $g(r)$ are measurable $\mathbb{F}$-adapted (g is Y-valued). Let $c(r)=|\tilde{\mathbf{u}}(r)|_2^{p-2}\int|\nabla\tilde{\mathbf{u}}(r)|^2\,dx$. Since

$$
\begin{aligned}
N(\tilde{\mathbf{u}}(r)) &= -|\tilde{\mathbf{u}}(r)|_2^{p-2}\int a^{ij}(r)\partial_i\tilde u^l(r)\partial_j\tilde u^l(r)\,dx\\
&\quad+\frac{1}{2}|\tilde{\mathbf{u}}(r)|_2^{p-2}\int \sigma^k(r)\cdot\sigma^l(r)\partial_k\tilde u^i(r)\partial_l\tilde u^i(r)\\
&\le -\delta|\tilde{\mathbf{u}}(r)|_2^{p-2}\int|\nabla\tilde{\mathbf{u}}(r)|^2\,dx=-\delta c(r),
\end{aligned}
$$

using (4.15)-(4.17), we find easily that for each ε there is a constant C_ε so that

$$
h(r)\le(-\delta+\varepsilon)c(r)+C_\varepsilon(y(r)+f(r)), \qquad (4.18)
$$

$$
\begin{aligned}
|g(r)|_Y &\le \varepsilon|\mathbf{u}(r)|_{s,2}^{p-1}|\mathbf{u}|_{s+1,2}+C_\varepsilon(y(r)+y(r)^{1-1/p}l(r)^{1/p}),\\
|g(r)|_Y^2 &\le \varepsilon^2|\mathbf{u}(r)|_{s,2}^{2p-2}|\mathbf{u}|_{s+1,2}^2+C_\varepsilon(y(r)^2+y(r)^{2-2/p}l(r)^{2/p})\\
&\le \varepsilon^2 y(r)c(r)+C_\varepsilon(y(r)^2+y(r)^{2-2/p}l(r)^{2/p}),
\end{aligned}
$$

where

$$
f(r)=|\mathbf{D}(\mathbf{0},r)|_{s-1,2}^p,\, l(r)=|\,|\mathbf{Q}(\mathbf{0},r)|\,|_{s,2}^p.
$$

Let $v<t\le T, t-v\le 1/4$, and $\tilde\tau$ be a stopping time such that $\sup_{r\le\tilde\tau}y(r)$ is bounded and

$$
\mathbf{E}\int_0^{\tilde\tau}(f(r)+l(r))\,dr<\infty.
$$

Fix an arbitrary stopping time τ. Let $\bar{\tau} = \tau \wedge \tilde{\tau}$. Then by Burkhölder's inequality and (4.18)

$$\mathbf{E} \sup_{v \le r \le t} y(r \wedge \bar{\tau}) \le \mathbf{E} y(v \wedge \bar{\tau}) + C_\varepsilon \mathbf{E}[\sup_{v \le r \le t} y(r \wedge \bar{\tau})(t - v) +$$

$$- \int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} (\delta - \varepsilon) c(r)\, dr + \int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} f(r)\, dr$$

$$+ (\int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} [\varepsilon^2 y(r) c(r) + C_\varepsilon (y(r)^2 + y(r)^{2(1-1/p)} l(r)^{2/p})]\, dr)^{1/2}].$$

For each ε there is a constant C_ε independent of T such that

$$(\int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} y(r)^{2(1-1/p)} l(r)^{2/p})\, dr)^{1/2} \le \sup_{v \le r \le t} y(r \wedge \bar{\tau})^{1-1/p} (\int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} l(r)^{2/p}\, dr)^{1/2}$$

$$\le \varepsilon \sup_{v \le r \le t} y(r \wedge \bar{\tau}) + C_\varepsilon (\int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} l(r)^{2/p}\, dr)^{p/2}$$

$$\le \varepsilon \sup_{v \le r \le t} y(r \wedge \bar{\tau}) + C_\varepsilon \int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} l(r)\, dr.$$

Also,

$$(\int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} \varepsilon^2 y(r) c(r)\, dr)^{1/2} \le \varepsilon \sup_{v \le r \le t} y(r)^{1/2} (\int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} c(r)\, dr)^{1/2}$$

$$\le 2\varepsilon [\sup_{v \le r \le t} y(r) + \int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} c(r)\, dr].$$

So, there is a constant C such that

$$\mathbf{E} \sup_{v \le r \le t} y(r \wedge \bar{\tau}) \le C \mathbf{E}[y(v \wedge \bar{\tau}) + \sup_{v \le r \le t} y(r \wedge \bar{\tau})(t - v)^{1/2} + \int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} (f(r) + l(r))\, dr],$$

and we can find ε_0 such that for $|t - v| \le \varepsilon_0$

$$\mathbf{E} \sup_{v \le r \le t} y(r \wedge \bar{\tau}) \le C \mathbf{E}[y(v \wedge \bar{\tau}) + \int_{v \wedge \bar{\tau}}^{t \wedge \bar{\tau}} (f(r) + l(r))\, dr].$$

Now, the estimate easily follows. □

Now we derive similar estimates for $|\mathbf{u}(t)|_{s,q}^p, q \ge 2$.

Proposition 3 *Let $s \in \{0,1,\ldots\}, q \geq 2, |\mathbf{u}_0|_{s+1-2/q,q} < \infty$ $\mathbf{P}$-a.s., and **A**, **A1**(s,q)-**A3**(s,q) hold. Assume further that $p \geq q$, $a^{ij} \in B^{s\vee 2}$, if $s \geq 1$, and*

$$| \, |\mathbf{Q}(\mathbf{v},t)| \, |_{s,q} \leq | \, |\mathbf{Q}(\mathbf{0},t)| \, |_{s,q} + C|\mathbf{v}|_{s,q}.$$

$$|\mathbf{D}(\mathbf{v},t)|_{s-1,q} \leq |\mathbf{D}(\mathbf{0},t)|_{s-1,q} + C|\mathbf{v}|_{s,q},$$

$$\int_0^t (|\mathbf{D}(\mathbf{0},r)|^p_{s-1,q} + | \, |\mathbf{Q}(\mathbf{0},r)| \, |^p_{s,q}) \, dr < \infty$$

$\mathbf{P}$-a.s. for all t.

Then Theorem 2 holds. Moreover, for each $T > 0$, there is a constant C such that for each stopping time $\tau \leq T$,

$$\mathbf{E} \sup_{r \leq \tau} |\mathbf{u}(r)|^p_{s,q} \leq C\mathbf{E}[|\mathbf{u}_0|^p_{s,q} + \int_0^\tau (|\mathbf{D}(\mathbf{0},r)|^p_{s-1,q} + | \, |\mathbf{Q}(\mathbf{0},r)| \, |^p_{s,q}) \, dr]. \tag{4.19}$$

Proof Since the assumptions of Theorem 2 are satisfied, there is a unique $\mathbb{H}^s_q$- solution of equation (4.1). Let α be a multiindex such that $|\alpha| \leq s$. Then $\mathbf{u}_\alpha = \partial^\alpha \mathbf{u}$ is $\mathbb{L}_q$-valued continuous and satisfies the equation

$$\partial_t \mathbf{u}_\alpha(t,x) = \partial^\alpha[\partial_i(a^{ij}(t,x)\partial_j \mathbf{u}) + \mathbf{D}(\mathbf{u},t,x)] +$$

$$\partial^\alpha[\sigma^k(t,x)\partial_k \mathbf{u}(t,x) + \mathbf{Q}(\mathbf{u},t,x)] \cdot \dot{W},$$

$$\mathbf{u}_\alpha(0,x) = \mathbf{u}_{0,\alpha}(x),$$

where $\mathbf{u}_{0,\alpha} = \partial^\alpha \mathbf{u}_0$. Define

$$\mathbf{G}_\alpha = \sum_{\nu+\mu=\alpha, |\nu|\geq 1} \partial^\nu a^{ij}(t)\partial_j \partial^\mu \mathbf{u}(t),$$

$$\tilde{\mathbf{G}}_\alpha = \sum_{\nu+\mu=\alpha, |\nu|\geq 1} \partial^\nu \sigma^k(t)\partial_k \partial^\mu \mathbf{u}(t).$$

Differentiating the product, we obtain

$$\partial^\alpha \partial_i(a^{ij}(t)\partial_j \mathbf{u}(t)) = \partial_i(a^{ij}(t)\partial_j \mathbf{u}_\alpha(t)) + \partial_i \mathbf{G}_\alpha(t) \tag{4.20}$$

$$\partial^\alpha[\sigma^k(t)\partial_k \mathbf{u}(t)] = \sigma^k(t)\partial_k \mathbf{u}_\alpha(t) + \tilde{\mathbf{G}}_\alpha(t).$$

and

$$|\mathbf{H}_\alpha(t)|_q + ||\tilde{\mathbf{H}}_\alpha(t)||_q \leq C|\mathbf{u}(t)|_{s-1,q}, \tag{4.21}$$

Let $\tilde{b}^k_\alpha(r) = \sigma^i(r)\partial_i u^k_\alpha(r) + \tilde{G}^l_\alpha(r) + \partial^\alpha(Q^k(\mathbf{u},r)), 1 \le k \le d$. Applying Ito formula, we obtain that $y_\alpha(t) = |\mathbf{u}_\alpha(t)|_q^p$ is a semimartingale:

$$y_\alpha(t) = y_\alpha(0) + \int_0^t h_\alpha(r)\,dr + \int_0^t g_\alpha(r)\cdot dW(r),$$

where

$$\begin{aligned}
h_\alpha(r) = p|\mathbf{u}_\alpha(r)|_q^{p-q}\{&\langle|\mathbf{u}_\alpha(r)|^{q-2}\mathbf{u}_\alpha(r), \partial^\alpha(\mathbf{D}(\mathbf{u}(r),r)\rangle_{0,q} \\
&- \int a^{ij}(r)u_\alpha(|\mathbf{u}_\alpha(r)|^{q-2}u^l_\alpha(r))\partial_j u^l_\alpha(r)\,dx \\
&- \int \partial_i(|\mathbf{u}_\alpha(r)|^{q-2}u^l_\alpha(r))G^l_\alpha(r)\,dx \\
&+ \frac{1}{2}\int[(q-2)|\mathbf{u}_\alpha(r)|^{q-4}u^i_\alpha(r)u^j_\alpha(r) + |\mathbf{u}_\alpha(r)|^{q-2}\delta_{ij}]\tilde{b}^i(r)\cdot\tilde{b}^j(r)\,dx\} \\
&+ \frac{p}{2}(p-q)|\mathbf{u}_\alpha(r)|_q^{p-2q}|\int|\mathbf{u}_\alpha(r)|^{q-2}u^l_\alpha(r)\tilde{b}^l(r)\,dx|_Y^2,
\end{aligned}$$

and

$$g_\alpha(r) = p|\mathbf{u}_\alpha(r)|_q^{p-q}\int|\mathbf{u}_\alpha(r)|^{q-2}u^l_\alpha(r)\tilde{b}^l(r)\,dx.$$

Notice

$$\partial^\alpha(\mathbf{D}(\mathbf{u}(r),r)) \in \mathbb{H}_{-1,q}, \partial^\alpha(Q^k(\mathbf{u},r)) \in \mathbb{L}_q,$$

and, by our assumptions, there is a constant C so that

$$\begin{aligned}
|\partial^\alpha(\mathbf{D}(\mathbf{u}(r),r))|_{-1,q} &\le C|\mathbf{D}(\mathbf{u}(r),r)|_{s-1,q} \\
&\le C(|\mathbf{u}(r)|_{s,q} + |\mathbf{D}(\mathbf{0},r)|_{s-1,q}), \\
||\partial^\alpha Q^k(\mathbf{u},r)||_q \le C||Q^k(\mathbf{u},r)||_{s,q} &\le C(|\mathbf{u}(r)|_{s,q} + ||\mathbf{Q}(\mathbf{0},r)||_{s,q}),
\end{aligned} \tag{4.22}$$

$$|\int|\mathbf{u}_\alpha(r)|^{q-2}u^l_\alpha(r)\tilde{b}^l(r)\,dx|_Y \le C(|\mathbf{u}(r)|^q_{s,q} + |\mathbf{u}(r)|^{q-1}_{s,q}||\mathbf{Q}(\mathbf{0},r)||_{s,q}.$$

We have $h_\alpha(r) = p|\mathbf{u}_\alpha(r)|_q^{p-q}h^1_\alpha(r) + h^2_\alpha(r)$, where

$$\begin{aligned}
h^1_\alpha(r) = -\int a^{ij}(r)\partial_i(|\mathbf{u}_\alpha(r)|^{q-2}u^l_\alpha(r))\partial_j u^l_\alpha(r)\,dx + \frac{1}{2}\int[(q-2)|\mathbf{u}_\alpha(r)|^{q-4}u^i_\alpha(r)\tilde{u}^j \\
+ |\mathbf{u}_\alpha(r)|^{q-2}\delta_{ij}]\sigma^k(r)\partial_k u^i_\alpha(r)\cdot\sigma^l(r)\partial_l u^j_\alpha(r)\,dx.
\end{aligned}$$

Let

$$A^{ij}(r) = a^{ij}(r) - \frac{1}{2}\sigma^i(r)\cdot\sigma^j(r).$$

Then

$$h^1_\alpha(r) = -\int A^{ij}(r)[\partial_i u^l_\alpha(r)\partial_j u^l_\alpha(r)|\mathbf{u}_\alpha(r)|^{q-2}$$

$$+\frac{4(q-2)}{q^2}\partial_i(|\mathbf{u}_\alpha(r)|^{q/2})\partial_j(|\mathbf{u}_\alpha(r)|^{q/2})]\,dx \le -\delta\int|\mathbf{u}_\alpha(r)|^{q-2}|\nabla\mathbf{u}(r)|^2\,dx,$$

and for each $\varepsilon > 0$ there is a constant C_ε such that

$$|h^2_\alpha(r)| \le \varepsilon\int|\mathbf{u}_\alpha(r)|^{q-2}|\nabla\mathbf{u}(r)|^2\,dx + C_\varepsilon(|\mathbf{u}(r)|^q_{s,q} + |\mathbf{D}(\mathbf{0},r)|^p_{s-1,q}).$$

So, we obtain that

$$y(t) = \sum_{|\alpha|\le s} y_\alpha(t) = y(0) + \int_0^t h(r)\,dr + \int_0^t g(r)\cdot dW_r$$

and

$$h(r) = \sum_{|\alpha|\le s} h_\alpha(r) \le C(y(r) + f(r)), \tag{4.23}$$

$$|g(r)|_Y \le \sum_{|\alpha|\le s}|g_\alpha(r)|_Y \le C(y(r) + y(r)^{1-1/p}l(r)^{1/p}),$$

where

$$f(r) = |\mathbf{D}(\mathbf{0},r)|^p_{s-1,q}, l(r) = |\,|\mathbf{Q}(\mathbf{0},r)|\,|^p_{s,q}.$$

Let $v < t \le T, t-v \le 1/4$, and $\tilde\tau$ be a stopping time such that $\sup_{r\le\tilde\tau} y(r)$ is bounded and

$$\mathbf{E}\int_0^{\tilde\tau}(f(r) + l(r))\,dr < \infty.$$

Fix an arbitrary stopping time τ. Let $\bar\tau = \tau\wedge\tilde\tau$. Then by Burkhölder's inequality and (4.18)

$$\mathbf{E}\sup_{v\le r\le t} y(r\wedge\bar\tau) \le \mathbf{E}y(v\wedge\bar\tau) + C\mathbf{E}[\sup_{v\le r\le t} y(r\wedge\bar\tau)(t-v) + \int_{v\wedge\bar\tau}^{t\wedge\bar\tau} f(r)\,dr$$

$$+ (\int_{v\wedge\bar\tau}^{t\wedge\bar\tau}[y(r)^2 + y(r)^{2(1-1/p)}l(r)^{2/p})]\,dr)^{1/2}].$$

For each ε there is a constant C_ε independent of T such that

$$(\int_{v\wedge\bar\tau}^{t\wedge\bar\tau} y(r)^{2(1-1/p)} l(r)^{2/p}\, dr)^{1/2} \leq \sup_{v\leq r\leq t} y(r\wedge\bar\tau)^{1-1/p} (\int_{v\wedge\bar\tau}^{t\wedge\bar\tau} l(r)^{2/p}\, dr)^{1/2}$$

$$\leq \varepsilon \sup_{v\leq r\leq t} y(r\wedge\bar\tau) + C_\varepsilon (\int_{v\wedge\bar\tau}^{t\wedge\bar\tau} l(r)^{2/p}\, dr)^{p/2}$$

$$\leq \varepsilon \sup_{v\leq r\leq t} y(r\wedge\bar\tau) + C_\varepsilon \int_{v\wedge\bar\tau}^{t\wedge\bar\tau} l(r)\, dr.$$

So, there is a constant C such that

$$\mathbf{E} \sup_{v\leq r\leq t} y(r\wedge\bar\tau) \leq C\mathbf{E}[y(v\wedge\bar\tau) + \sup_{v\leq r\leq t} y(r\wedge\bar\tau)(t-v)^{1/2} + \int_{v\wedge\bar\tau}^{t\wedge\bar\tau} (f(r) + l(r))\, dr],$$

and we can find ε_0 such that for $|t-v| \leq \varepsilon_0$

$$\mathbf{E} \sup_{v\leq r\leq t} y(r\wedge\bar\tau) \leq C\mathbf{E}[y(v\wedge\bar\tau) + \int_{v\wedge\bar\tau}^{t\wedge\bar\tau} (f(r) + l(r))\, dr].$$

Now, the estimate easily follows. □

In the following two corollaries we combine Propositions 1 and 2, 3.

Corollary 3 *Let $s \in (-\infty, \infty), p \geq 2$. Assume that for each $\mathbf{v} \in \mathbb{H}_2^{s+1}$, $\mathbf{Q}(\mathbf{v}, t)$ is a predictable $\mathbb{H}_2^{s+1}$-valued process and $\mathbf{D}(\mathbf{v}, t)$ is a predictable $\mathbb{H}_2^s$-valued process. Let* ***A*, *A1***$(s+1, 2)$, ***A2***$(s+1, 2)$, ***A3***$(s, 2)$ *be satisfied, $|\mathbf{u}_0|_{s+1,2} < \infty$ with probability one, and for all $t > 0, \mathbf{v} \in \mathbb{H}_2^{s+1}$,*

$$|\,|\mathbf{Q}(\mathbf{v}, t)|\,|_{s+1,2} \leq |\,|\mathbf{Q}(\mathbf{0}, t)|\,|_{s+1,2} + C|\mathbf{v}|_{s+1,2}.$$

$$|\mathbf{D}(\mathbf{v}, t)|_{s,2} \leq |\mathbf{D}(\mathbf{0}, t)|_{s,2} + C|\mathbf{v}|_{s+1,2}.$$

Suppose also that,

$$\int_0^t (|\,|\mathbf{Q}(\mathbf{0}, r)|\,|_{s+1,2}^p + |\mathbf{D}(\mathbf{0}, r)|_{s,2}^p)\, dr < \infty$$

$\mathbf{P}$*-a.s. for all t. Then (4.1) has a unique continuous $\mathbb{H}_2^{s+1}$- solution. Moreover, for each $T > 0$, there is a constant C such that for each stopping time $\tau \leq T$ and set $A \in \mathcal{F}_0$,*

$$\mathbf{E}1_A \sup_{r\leq\tau} |\mathbf{u}(r)|_{s+1,2}^p \leq C\mathbf{E}1_A[|\mathbf{u}_0|_{s+1,2}^p + \int_0^\tau (|\mathbf{D}(\mathbf{0}, r)|_{s,2}^p + |\,|\mathbf{Q}(\mathbf{0}, r)|\,|_{s+1,2}^p)\, dr].$$

Proof Since all the assumptions of Proposition 1are satisfied, there is a unique $\mathbb{H}_p^{s+1}-$ solution $\mathbf{u}$ for which the estimate of Proposition 1 holds. Then applying again Proposition 2 to $s+1$ and the linear equation

$$\partial_t\xi(t,x) = \partial_i(a^{ij}(t,x)\partial_j\xi) + \mathbf{D}(\mathbf{u},t,x)+$$

$$[\sigma^k(t,x)\partial_k\xi(t,x) + \mathbf{Q}(\mathbf{u},t,x)]\cdot\dot{W},$$

$$\xi(0,x) = \mathbf{h}(x),$$

and using the fact that $\xi = \mathbf{u}$ we obtain the statement. □

Corollary 4 *Let $s \in \{0,1,\ldots\}, p \geq q \geq 2$. Assume that for each $\mathbf{v} \in \mathbb{H}_q^{s+1}$, $\mathbf{Q}(\mathbf{v},t)$ is a predictable $\mathbb{H}_q^{s+1}$-valued process and $\mathbf{D}(\mathbf{v},t)$ is a predictable $\mathbb{H}_q^{s}$-valued process. Let **A**, **A1**$(s+1,q)$, **A2**$(s+1,q)$, **A3**(s,q) be satisfied, $|\mathbf{u}_0|_{s+2-2/q,q} < \infty$ with probability one, and for all $t > 0, \mathbf{v} \in \mathbb{H}_q^{s+1}$,*

$$|\,|\mathbf{Q}(\mathbf{v},t)|\,|_{s+1,q} \leq |\mathbf{Q}(\mathbf{0},t)|\,|_{s+1,q} + C|\mathbf{v}|_{s+1,q}.$$

$$|\mathbf{D}(\mathbf{v},t)|_{s,q} \leq |\mathbf{D}(\mathbf{0},t)|_{s,q} + C|\mathbf{v}|_{s+1,q},.$$

Suppose also that,

$$\int_0^t (|\,|\mathbf{Q}(\mathbf{0},r)|\,|_{s+1,q}^p + |\mathbf{D}(\mathbf{0},r)|_{s,q}^p)\,dr < \infty$$

$\mathbf{P}$*-a.s. for all t. Then (4.1) has a unique continuous $\mathbb{H}_q^{s+1}$- solution. Moreover, for each $T > 0$, there is a constant N such that for each stopping time $\tau \leq T$ and set $A \in \mathcal{F}_0$,*

$$\mathbf{E}1_A \sup_{r\leq\tau} |\mathbf{u}(r)|_{s+1,q}^p \leq N\mathbf{E}1_A[|\mathbf{u}_0|_{s+1,q}^p + \int_0^\tau (|\mathbf{D}(\mathbf{0},r)|_{s,q}^p + |\,|\mathbf{Q}(\mathbf{0},r)|\,|_{s+1,q}^p)\,dr].$$

Proof Since all the assumptions of Proposition 1are satisfied, there is a unique $\mathbb{H}_q^{s+1}-$ solution $\mathbf{u}$ for which the estimate of Proposition 1 holds. Then applying again Proposition 3 to $s+1$ and the linear equation

$$\partial_t\xi(t,x) = \partial_i(a^{ij}(t,x)\partial_j\xi) + \mathbf{D}(\mathbf{u},t,x)+$$

$$[\sigma^k(t,x)\partial_k\xi(t,x) + \mathbf{Q}(\mathbf{u},t,x)]\cdot\dot{W},$$

$$\xi(0,x) = \mathbf{h}(x),$$

and using the fact that $\xi = \mathbf{u}$ we obtain the statement. □

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