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WHERE DID THE BROWNIAN PARTICLE GO?

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Abstract Consider the radial projection onto the unit sphere of the path a d-dimensional Brownian motion W, started at the center of the sphere and run for unit time. Given the occupation measure μ of this projected path, what can be said about the terminal point $W(1)$, or about the range of the original path? In any dimension, for each Borel set $A \subseteq S^{d-1}$, the conditional probability that the projection of $W(1)$ is in A given $\mu(A)$ is just $\mu(A)$. Nevertheless, in dimension $d \geq 3$, both the range and the terminal point of W can be recovered with probability 1 from μ. In particular, for $d \geq 3$ the conditional law of the projection of $W(1)$ given μ is not μ. In dimension 2 we conjecture that the projection of $W(1)$ cannot be recovered almost surely from μ, and show that the conditional law of the projection of $W(1)$ given μ is not μ.

Keywords Brownian motion, conditional distribution of a path given its occupation measure, radial projection.

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1 Introduction

> *'This track, as you perceive, was made by a rider who was going from the direction of the school.'*
> *'Or towards it?'*
> *'No, no, my dear Watson. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school.'*
>
> From *The Adventure of the Priory School*, a Sherlock Holmes story by A. Conan Doyle.

The radial projection of a Brownian motion started at the origin and run for unit time in d dimensions defines a random occupation measure on the sphere S^{d-1}. Can we determine the endpoint of the Brownian path from this projected occupation measure? The problem of recovering data given a projection of the data is a common theme both inside and outside of probability theory. The title of this paper is adapted from a handout distributed by Peter Doyle, where the geometric problem of recovering from bicycle tracks the exit direction of the cyclist was posed.

An interesting feature of the present reconstruction problem is that the answer in low dimensions is different from the answer in dimensions $d \geq 3$. This would not be too surprising, except that the behavior in the one-dimensional case involves a conditioning identity which does not seem inherently one-dimensional. This identity concerns the conditional distribution of the endpoint given the occupation measure. One of the aims of this paper is to understand why this identity breaks down in higher dimensions, and what version of this identity might hold even when the occupation measure determines the endpoint and indeed determines the entire unprojected path. In high dimensions, recovery of the endpoint (and entire path), while intuitively plausible, is somewhat tricky because, as described in [17, page 275], the particle "comes in spinning". In particular, the range of the projected path is a.s. a dense subset of the sphere (see remark at the end of this introduction). Thus some quantitative criterion on accumulation of measure is required even to recover the set of occupied points on the sphere from the occupation measure.

Throughout the paper d is a positive integer, and $S^{d-1} \subseteq \mathbb{R}^d$ is the unit sphere. We often omit d in the notation for various spaces and mappings whose definition depends on d. Let $\pi : \mathbb{R}^d \to S^{d-1}$ be the spherical projection $\pi(x) = x/|x|$ for $x \neq 0$, with some arbitrary conventional value for $\pi(0)$. Let $(W_t, t \geq 0)$ denote a standard Brownian motion in $\mathbb{R}^d$ with $W_0 = 0$, which we take to be defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geq 0$ let $\Theta_t := \pi(W_t)$, and let $\Theta := (\Theta_t, 0 < t \leq 1)$. Let μ_Θ denote the random occupation measure of Θ on S^{d-1}, that is

$$\mu_\Theta(B) := \int_0^1 \mathbf{1}_{\Theta_t \in B}\, dt \tag{1.1}$$

for Borel subsets B of S^{d-1}. We may regard μ_Θ as a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the space $(\mathsf{prob}(S^{d-1}), \mathcal{F}_2)$ of Borel probability measure on S^{d-1} endowed with the σ-field generated by the measures of Borel sets.

The questions considered in this paper arose from the following identity: for each Borel subset B of S^{d-1}, we have

$$P(\Theta_1 \in B \,|\, \mu_\Theta(B)) = \mu_\Theta(B). \tag{1.2}$$

If $d = 1$ then $S^0 = \{-1, 1\}$, and $\mu_\Theta(\{1\})$ and $\mu_\Theta(\{-1\})$ are the times spent positive and negative respectively by a one-dimensional Brownian motion up to time 1. As observed in Pitman-Yor [28], formula (1.2) in this case can be read from Lévy's description [22] of the joint law of the arcsine distributed variable $\mu_\Theta(\{1\})$ and the Bernoulli(1/2) distributed indicator $\mathbf{1}_{W_1 > 0}$. The truth of (1.2) in higher dimensions is not so easily checked, due to the lack of explicit formulae for the distribution of $\mu_\Theta(B)$ even for the simplest subsets B of S^{d-1}; see for instance [4]. However, (1.2) can be deduced from the scaling property of Brownian motion, which implies that the process $(\Theta_t, t \geq 0)$ is 0-*self-similar*, meaning there is the equality in distribution

$$(\Theta_t, t \geq 0) \stackrel{d}{=} (\Theta_{ct}, t \geq 0)$$

for all $c > 0$. According to an identity of Pitman and Yor [30], recalled as Proposition 2.1 in Section 2, the identityi (1.2) holds for an arbitrary jointly measurable 0-self-similar process $(\Theta_t, t \geq 0)$ with values in an abstract measurable space, for any measurable subset B of that space.

Formula (1.2) led us to the following question, which we discuss further in Section 3:

Question 1.1 *For which processes $\Theta := (\Theta_t, 0 \leq t \leq 1)$, does the identity*

$$P(\Theta_1 \in B \,|\, \mu_\Theta) = \mu_\Theta(B) \tag{1.3}$$

hold for all measurable subsets B of the range space of Θ?

To clarify the difference between (1.2) and (1.3), $P(\Theta_1 \in B \,|\, \mu_\Theta(B))$ on the LHS of (1.3) is a conditional probability given the σ-field generated by the real random variable $\mu_\Theta(B)$, whereas $P(\Theta_1 \in B \,|\, \mu_\Theta)$ on the LHS of (1.3) is a conditional probability given the σ-field generated by the random measure μ_Θ, that is by all the random variables $\mu_\Theta(C)$ as C ranges over measurable subsets of the range space of Θ. For a general process Θ, formula (1.3) implies (1.2), but not conversely.

Now let Θ be the spherical projection of Brownian motion. If $d = 1$ then the σ-field generated by μ_Θ is identical to that generated by either $\mu_\Theta(\{1\})$ or by $\mu_\Theta(\{-1\}) = 1 - \mu_\Theta(\{-1\})$. So (1.3) is a consequence of (1.2) if $d = 1$. But (1.3) fails for $d \geq 2$. We show this for $d = 2$ in Section 4 by some explicit estimates involving the occupation times of quadrants. For $d \geq 3$ formula (1.3) fails even more dramatically. In Section 5 we show that if $d \geq 3$ then Θ_1 is a.s. equal to a measurable function of μ_Θ. Less formally, we say that Θ_1 *can be recovered from* μ_Θ. This brings us to the question of what features of the path of the original Brownian motion $W := (W_t, 0 \leq t \leq 1)$ can be recovered from μ_Θ. If $d = 3$ it is well known that the path W has self-intersections almost surely, so one can define a measure-preserving map T on the Brownian path space that reverses the direction of an appropriately selected closed loop in the path. Regarding $\mu_\Theta = \mu_\Theta(W)$ as a function on path space, we then have $\mu_\Theta(W) = \mu_\Theta(TW)$, hence $\mathbb{P}(W \in A \,|\, \mu_\Theta) = \mathbb{P}(W \in T^{-1}A \,|\, \mu_\Theta)$, from which it follows that W itself cannot be recovered from μ_Θ. However, for $d = 3$ it is possible to recover from μ_Θ both the random set

$$\mathsf{range}(W) := \{W_t : 0 \leq t \leq 1\}$$

and the final value W_1. We regard $\mathsf{range}(W)$ as a random variable with values in the space $\mathsf{cb\text{-}sets}$ of closed bounded subsets of $\mathbb{R}^d$, equipped with the Borel σ-field for the Hausdorff metric

$$d(S,T) := \max\{\sup_{x\in S} d(x,T)\,,\ \sup_{y\in T} d(y,S)\}$$

where $d(x,S) := \inf_{y\in S} |x-y|$. We now make a formal statement of the recovery result:

Theorem 1.2 *Fix the dimension $d \geq 3$. There exist measurable functions*

$$\psi : \mathsf{prob}(S^{d-1}) \to \mathsf{cb\text{-}sets} \ \text{ and } \ \varphi : \mathsf{cb\text{-}sets} \to \mathbb{R}^d,$$

such that there are almost sure equalities

$$\psi(\mu_\Theta) = \mathsf{range}(W) \tag{1.4}$$

and

$$\varphi \circ \psi(\mu_\Theta) = W_1. \tag{1.5}$$

For $d \geq 4$ it is a routine consequence of the almost sure parameterizability of a Brownian path by its quadratic variation that W can be recovered from $\mathsf{range}(W)$. So Theorem 1.2 implies that W can be recovered from μ_Θ in dimensions $d \geq 4$. The only part of the proof of Theorem 1.2 which involves probabilistic estimates for Brownian motion is Lemma 5.1, the remainder being mostly point-set topology. We remark that the topological arguments below also show that in dimension $d = 2$, the range of W and the endpoint W_1 can be recovered from the occupation measure of the planar path $W := (W_t, 0 \leq t \leq 1)$

As usual, the hardest (and most interesting) dimension is two. We conjecture that the high-dimensional behavior does not extend to two dimensions, that is,

Conjecture 1.3 *When $d = 2$, there is no map $\psi : \mathsf{prob}(S^{d-1}) \to \mathsf{cb\text{-}sets}$ such that almost surely*

$$\psi(\mu_\Theta) = \mathsf{range}(W).$$

When $d = 2$ it can be deduced from work of Bass and Khoshnevisan [3, Theorem 2.9] that μ_Θ almost surely has a continuous density, call it the *angular local time process*. The problem of describing the conditional law of W given μ_Θ for $d = 2$ is then analogous to the problem studied by Warren and Yor [36], who give an account of the randomness left in a one-dimensional Brownian motion after conditioning on its occupation measure up to a suitable random time. Aldous [1] and Knight [19] treat related questions involving the distribution of Brownian motion conditioned on its local time process. However, as far as we know there is no Ray-Knight type description available for the angular local time process, and this makes it difficult to settle the conjecture.

Remark. Let $\Theta_t := W_t/|W_t|$ be the radial projection of Brownian motion in $\mathbb{R}^d$. It is a classical fact that for any $\epsilon > 0$, the initial path segment $\{\Theta_t : 0 < t < \epsilon\}$ is dense in the unit sphere S^{d-1}. Since this fact motivates much of our work, we include an elementary explanation for it, which is valid in greater generality. It suffices to show that for any open set U on the sphere and any $\epsilon > 0$, the probability of the event $E(U,\epsilon)$ that $\{\Theta_t : 0 < t < \epsilon\}$ intersects U, equals one. By compactness, some finite number N_U of rotated copies of U cover the sphere, so by rotation invariance of Brownian motion, $\mathbb{P}[E(U,\epsilon)] \geq N_U^{-1}$. Therefore

$$\mathbb{P}\Big[\bigcap_{n=1}^{\infty} E(U,1/n)\Big] \geq N_U^{-1}\,,$$

whence by the Blumenthal zero-one law, this probability must be 1.

2 Identities for scalar self-similar processes

Recall that a real or vector-valued process $(X_t, t \geq 0)$ is called *β-self-similar* for a $\beta \in \mathbb{R}$ if for every $c > 0$

$$(X_{ct}, t \geq 0) \stackrel{d}{=} (c^{\beta} X_t, t \geq 0) \tag{2.1}$$

Such processes were studied by Lamperti [20, 21], who called them *semi-stable*. See [34] for a survey of the literature of these processes. The conditioning formula (1.2) for any 0-self-similar process $(\Theta_t, t \geq 0)$ is an immediate consequence of the following identity. To see the direct implication, take $X(t,\omega)$ to equal $\mathbf{1}_{(0,\infty)}(\omega(t))$.

Proposition 2.1 (Pitman and Yor [30]) *Let $(X_t, t \geq 0)$ be stochastic process with $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ jointly measurable. Let $\overline{X}_t := t^{-1}\int_0^t X_s\,ds$ and suppose that*

$$\begin{aligned}
(X_t, \overline{X}_t) \quad &\stackrel{d}{=} \quad (X_1, \overline{X}_1) && (2.2)\\
\textit{and} \quad \mathbb{E}|X_1| \quad &< \quad \infty. && (2.3)
\end{aligned}$$

Then for every $t > 0$,

$$\mathbb{E}(X_t \mid \overline{X}_t) = \overline{X}_t. \tag{2.4}$$

Proof: We simplify slightly the proof in [30]. Due to (2.2) it suffices to prove (2.4) for $t = 1$. It also suffices to prove this on the event $\{\overline{X}_1 \neq 0\}$, since this implies $\mathbb{E}X_1\mathbf{1}_{\overline{X}_1\neq 0} = \mathbb{E}\overline{X}_1\mathbf{1}_{\overline{X}_1\neq 0}$ and subtracting the relation $\mathbb{E}X_1 = \mathbb{E}\overline{X}_1$ (a consequence of (2.2)) yields $\mathbb{E}X_1\mathbf{1}_{\overline{X}_1=0} = 0$. This is equivalent to proving

$$\mathbb{E}[f(\overline{X}_1); \overline{X}_1 \neq 0] = \mathbb{E}\left[f(\overline{X}_1)\frac{X_1}{\overline{X}_1}; \overline{X}_1 \neq 0\right] \tag{2.5}$$

for a suitably large class of functions f. Let ν be the law of $\overline{X}_1$. Since $f(x)\mathbf{1}_{x\neq 0}$ for bounded measurable f may be approximated in $L^2(\nu)$ by bounded functions vanishing in a neighborhood of zero and having bounded continuous derivative, this class suffices. Fix such a function f and apply the chain rule for Lebesgue integrals (see, e.g., [32], Chapter 0, Prop. (4.6)), treating ω as fixed, to obtain

$$f\left(\int_0^1 X_t\,dt\right) = \int_0^1 f'\left(\int_0^t X_s\,ds\right) X_t\,dt.$$

Boundedness of f' allows the interchange of expectation with integration, so using (2.2) we get (2.5) from the following computation:

$$\begin{aligned}
\mathbb{E}[f(\overline{X}_1); \overline{X}_1 \neq 0] = \mathbb{E}f(\overline{X}_1) &= \int_0^1 \mathbb{E}\left[f'\left(\int_0^t X_s\, ds\right) X_t\right] dt \\
&= \int_0^1 \mathbb{E}\left[f'(t\overline{X}_1)X_1\right] dt \\
&= \mathbb{E}\left[\int_0^1 f'(t\overline{X}_1)X_1\, dt\right] \\
&= \mathbb{E}\left[f(\overline{X}_1)\frac{X_1}{\overline{X}_1}\right].
\end{aligned}$$

$\square$

For a different proof and variations of the identity see [29]. We see immediately that (2.2) holds for any 0-self-similar process X. We observe also:

Corollary 2.2 *Let (Y_t) be any β-self-similar vector-valued process. Let $X_t := \mathbf{1}_{\{Y_t \in \mathcal{C}\}}$ where $\mathcal{C}$ is any Borel set which is a cone, i.e., for $\lambda > 0$, $x \in \mathcal{C} \Leftrightarrow \lambda x \in \mathcal{C}$. Then (X_t) satisfies* (2.2)*, and hence*

$$\mathbb{P}(Y_1 \in \mathcal{C} \,|\, \overline{X}_1) = \overline{X}_1. \tag{2.6}$$

Applying Bayes' rule to (2.6) yields the following corollary.

Corollary 2.3 *Let $\{Y_t\}$ be any β-self-similar vector-valued process, and let $V_t = \int_0^t X_s\, ds$ with $X_t := \mathbf{1}_{\{Y_t \in \mathcal{C}\}}$ for a fixed positive cone $\mathcal{C}$. Then*

$$\mathbb{P}(V_t \in dv \,|\, Y_t \in \mathcal{C}) = \frac{v\mathbb{P}(V_t \in dv)}{t\mathbb{P}(Y_t \in \mathcal{C})}.$$

Corollary 2.4 *Under the hypotheses of the Corollary 2.3, suppose $\overline{X}_t$ has a* $\mathrm{beta}(a, b)$ *distribution. Then the conditional distribution of $\overline{X}_t$ given $Y_t \in C$ is* $\mathrm{beta}(a + 1, b)$ *and the conditional distribution of $\overline{X}_t$ given $Y_t \notin \mathcal{C}$ is* $\mathrm{beta}(a, b + 1)$*.*

Example 2.5 *Stable Lévy Processes.* Let $\{Y_t\}$ be a stable Lévy process that satisfies $\mathbb{P}(Y_t > 0) = p$ for all t. It is well known [23, 15] that the distribution of the total duration V_1 that $\{Y_t\}$ is positive up to time 1, is $\mathrm{beta}(p, 1-p)$. It follows that the conditional distributions of V_1 given the sign of Y_1 are respectively

$$(V_1 \,|\, Y_1 > 0) \overset{d}{=} \mathrm{beta}(1 + p, 1 - p) \tag{2.7}$$

$$(V_1 \,|\, Y_1 < 0) \overset{d}{=} \mathrm{beta}(p, 2 - p). \tag{2.8}$$

Example 2.6 *Perturbed Brownian Motions.* Let $Y_t := |B_t| - \mu \ell_t, t \geq 0$, where B is a standard one-dimensional Brownian motion started at 0, $\mu > 0$ and $(\ell_t, t \geq 0)$ is the local time process of B at zero. F. Petit [27] showed that $V_1^- := \int_0^1 ds 1_{(Y_s < 0)}$ has beta$\left(\frac{1}{2}, \frac{1}{2\mu}\right)$ distribution. Corollary 2.4 implies that the conditional distribution of V_1^- given $Y_1 < 0$ is beta$\left(\frac{3}{2}, \frac{1}{2\mu}\right)$ and that the conditional distribution of V_1^- given $Y_1 > 0$ is beta$\left(\frac{1}{2}, 1 + \frac{1}{2\mu}\right)$. These results have been stated and proved in [37, Th. 8.3] and in [8]. A more general class of beta laws has been obtained for the times spent in $\mathbb{R}_\pm$ by doubly perturbed Brownian motions, that is to say solutions of the stochastic equation

$$Y_t = B_t + \alpha \sup_{0 \leq s \leq t} Y_s + \beta \inf_{0 \leq s \leq t} Y_s.$$

See, e.g., Carmona-Petit-Yor [9], Perman-Werner [26] and Chaumont-Doney [10].

Example 2.7 *More about the Brownian case.* Formula (1.2) has some surprising consequences even in the simplest case when $d = 1$. Consider the function

$$f(t, a) := P(B_t > 0 | V_1 = a) \tag{2.9}$$

for $0 < t \leq 1$ and $0 \leq a \leq 1$, where B is a one-dimensional Brownian motion and $V_1 = \int_0^1 1(B_t > 0)dt$. Without attempting to compute $f(t, a)$ explicitly, which appears to be quite difficult, let us presume that f can be chosen to be continuous in (t, a). Then

$$\int_0^1 f(t, a)dt = a = f(1, a) \qquad (0 \leq a \leq 1) \tag{2.10}$$

where the first equality follows from (2.9) and the second equality is read from (1.2). On the other hand, it is easily seen that

$$f(0+, a) = \tfrac{1}{2} \qquad (0 < a < 1) \tag{2.11}$$

which implies that

$$\textit{for each } a > \tfrac{1}{2} \textit{ there exists } t \in (0, 1) \textit{ such that } f(t, a) > a \tag{2.12}$$

That is to say, given $V_1 = a > \frac{1}{2}$, there is some time $t < 1$ such that the BM is more likely to be positive at time t than it is at time 1.

3 Identities for self-similar processes in dimension $d \geq 2$

Say that a jointly measurable process $\Theta := (\Theta_t, 0 < t \leq 1)$ has the *sampling property* if

$$\mathbb{P}(\Theta_1 \in B \,|\, \mu_\Theta) = \mu_\Theta(B) \tag{3.1}$$

for all measurable subsets B of the range space of Θ. The results of this section consist of two examples where the sampling property does hold, and a characterization of the sampling property in terms of exchangeability.

Proposition 3.1 *Suppose that Θ takes values in a Borel space. Let $U_1, U_2, \cdots$ be a sequence of i.i.d. random variables with uniform distribution on $(0,1)$, independent of Θ. Then the following are equivalent:*

(i) (Θ_t) has the sampling property;

(ii) for each $n = 1, 2, 3, \cdots$,

$$(\Theta_1, \Theta_{U_2}, \Theta_{U_3}, \cdots, \Theta_{U_n}) \stackrel{d}{=} (\Theta_{U_1}, \Theta_{U_2}, \Theta_{U_3}, \cdots, \Theta_{U_n}). \tag{3.2}$$

PROOF: Clearly (ii) is equivalent to

$$\mathbb{P}(\Theta_1 \in B \,|\, \{\Theta_{U_j}\}_{j=2}^{\infty}) = \mathbb{P}(\Theta_{U_1} \in B \,|\, \{\Theta_{U_j}\}_{j=2}^{\infty}) \tag{3.3}$$

for all measurable subsets B of the range space of Θ. To connect this to (i), observe that $\{\Theta_{U_j}\}_{j=2}^{\infty}$ is a sequence of i.i.d. picks from μ_Θ. Hence this sequence is conditionally independent of Θ given μ_Θ. Therefore, (3.3) can be rewritten as

$$\mathbb{P}(\Theta_1 \in B \,|\, \mu_\Theta) = \mathbb{P}(\Theta_{U_1} \in B \,|\, \mu_\Theta) \tag{3.4}$$

for all measurable B, which is equivalent to (i). $\square$

The conditions (3.2) increase in strength as n increases. For $n = 2$, (3.2) is just

$$(\Theta_1, \Theta_{U_2}) \stackrel{d}{=} (\Theta_{U_1}, \Theta_{U_2}). \tag{3.5}$$

which immediately implies

$$(\Theta_1, \Theta_{U_2}) \stackrel{d}{=} (\Theta_{U_2}, \Theta_1). \tag{3.6}$$

Proposition 3.2 *If the distribution of (Θ_s, Θ_t) depends only on t/s then the conditions* (3.5) *and* (3.6) *are equivalent.*

PROOF: Construct U_1 and U_2 as follows. Let Y and Z be independent with Y uniform on $[0,1]$ and Z having density $2x$ on $[0,1]$. Let X be an independent ± 1 fair coin-flip and set (U_1, U_2) equal to (Z, YZ) if $X = 1$ and (YZ, Z) if $X = -1$. By construction, the law of $(\Theta_{U_1}, \Theta_{U_2})$ is one half the law of (Θ_Z, Θ_{YZ}) plus one half the law of (Θ_{YZ}, Θ_Z). By the assumption on Θ this is one half the law of (Θ_1, Θ_{U_2}) plus one half the law of (Θ_{U_2}, Θ_1). This and (3.6) imply (3.5). $\square$

We note that the spherical projection of Brownian motion in $\mathbb{R}^d$ satisfies (3.6) for all d. So this condition is not enough to imply the sampling property for a 0-self-similar process Θ. When Θ is not 0-self-similar it is easy to find cases where (3.6) holds but not (3.5).

Example 3.3 Let (X, Y) have a symmetric distribution and let $\Theta_t = X\mathbf{1}_{t<a} + Y\mathbf{1}_{t\geq a}$ for a fixed $a \in (0,1)$. It is easy to see that (3.6) holds. On the other hand, if $\mathbb{P}(X = Y) = 0$, then $\mathbb{P}(\Theta_1 = \Theta_{U_2}) = 1 - a$ while $\mathbb{P}(\Theta_{U_1} = \Theta_{U_2}) = a^2 + (1-a)^2$. Unless $a = \frac{1}{2}$, these two probabilities are not equal.

We now mention some interesting examples of 0-self-similar processes which do have the sampling property.

Example 3.4 *Walsh's Brownian motions.* Let B be a one-dimensional BM started at 0. Suppose that each excursion of B away from 0 is assigned a random angle in $[0, 2\pi)$ according to some arbitrary distribution, independently of all other excursions. Let Θ_t be the angle assigned to the excursion in progress at time t, with the convention $\Theta_t = 0$ if $B_t = 0$. So $(|B_t|, \Theta_t)$ is Walsh's singular Brownian motion in the plane [35, 2]. As shown in [28, Section 4], the process (Θ_t) is a 0-self-similar process with the sampling property, and the same is true of (Θ_t) defined similarly for a δ-dimensional Bessel process instead of $|B|$ for arbitrary $0 < \delta < 2$.

The proof of the sampling property of the angular part (Θ_t) of Walsh's Brownian motion is based on the following lemma, which is implicit in arguments of [28, Section 4] and [30, formula (24)].

Lemma 3.5 *Let Z be a random closed subset of $[0,1]$ with Lebesgue measure zero. For $0 \leq t \leq 1$ let $N_t - 1$ be the number of component intervals of the set $[0,t]\backslash Z$ whose length exceeds $t - G_t$, where $G_t = \sup\{s : s < t, s \in Z\}$. So N_t has values in $\{1, 2, \cdots, \infty\}$. Given Z, let (Θ_t) be a process constructed by assigning each complementary interval of Z an independent angle according to some arbitrary distribution on $[0, 2\pi)$, and letting $\Theta_t = 0$ if $t \in Z$. If (N_t) has the sampling property, then so does (Θ_t).*

According to [28, Theorem 1.2] and [30, formula (24)], for Z the zero set of a Brownian motion, or more generally the range of a stable(α) subordinator for $0 < \alpha < 1$, the process (N_t) has the sampling property, hence so does the angular part (Θ_t) of Walsh's Brownian motion whose radial part is a Bessel process of dimension δ for arbitrary $0 < \delta < 2$.

Example 3.6 *A Dirichlet Distribution.* Let Z be the set of points of a Poisson random measure on $(0, \infty)$ with intensity measure $\theta x^{-1}dx, x > 0$. Construct (Θ_t) from Z as in Lemma 3.5. So between each pair of points of the Poisson process, an independent angle is assigned, with some common distribution H of angles on $[0, 2\pi)$. It was shown in [30] that (N_t) derived from this Z has the sampling property, hence so does (Θ_t) derived from this Z. In this example μ_Θ is a Dirichlet random measure governed by θH as studied in [14, 18, 16, 33].

We close this section by rewriting Proposition 2.1 as a statement concerning stationary processes. Let (X_t) be a jointly measurable process and $Y_t = X_{e^t}$. The process (X_t) being 0-self-similar is equivalent to the process (Y_t) being stationary, so a change of variables turns Proposition 2.1 into:

Corollary 3.7 *(Pitman-Yor [29]) Fix $\lambda > 0$ and define $\overline{Y}_\lambda := \int_0^\infty \lambda e^{-\lambda t} Y_t \, dt$, where $\{Y_t : t \in \mathbb{R}\}$ is a stationary process and $\mathbb{E}|Y_0| < \infty$. Then*

$$\mathbb{E}(Y_0 \mid \overline{Y}_\lambda) = \overline{Y}_\lambda .$$

The following proposition provides a partial converse:

Proposition 3.8 *Let F be a distribution on $[0, \infty)$ and for a stationary process (Y_t) let $\overline{Y}_F$ denote $\int_0^\infty Y_t \, dF$. Assuming either F has a density or F is a lattice distribution, the identity $\mathbb{E}(Y_0 \mid \overline{Y}_F) = \overline{Y}_F$ holds for every such process $\{Y_t\}$ if and only if F has density $\lambda e^{-\lambda t}$ for some $\lambda \in (0, \infty)$ or $F = \delta_0$.*

Proof: Fix F and suppose that $\mathbb{E}(Y_0 \mid \overline{Y}_F) = \overline{Y}_F$ holds for all stationary $\{Y_t\}$ with $\mathbb{E}|Y_0| < \infty$. When also $\mathbb{E}|Y_0|^2 < \infty$, this implies $\mathbb{E} Y_0 \overline{Y}_F = \mathbb{E}(\overline{Y}_F)^2$. Let $r(t) = \mathbb{E} Y_0 Y_t$ and let ξ_1, ξ_2 be i.i.d. according to F. Comparing

$$\mathbb{E} Y_0 \overline{Y}_F = \mathbb{E} r(|\xi_1|)$$

with

$$\mathbb{E}(\overline{Y}_F)^2 = \mathbb{E} r(|\xi_1 - \xi_2|),$$

we find that

$$\mathbb{E} r(|\xi_1|) = \mathbb{E} r(|\xi_1 - \xi_2|).$$

Taking Y to be an Ornstein-Uhlenbeck process shows that this holds for $r(t) = e^{-\alpha t}$, so that $|\xi_1 - \xi_2|$ has the same Laplace transform, hence the same distribution, as $|\xi_1|$. Assuming that F is concentrated on $[0, \infty)$ and has a density, Puri and Rubin [31] showed that this condition implies F is an exponential. If F is a lattice distribution, they showed it must be δ_0 or $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_a$ or a times a geometric for some $a > 0$. It is easy to construct examples ruling out the nondegenerate discrete cases. $\square$

Changing back to $X_t := Y_{\log t}$, Proposition 3.8 yields:

Corollary 3.9 *Suppose F has a density f on $(0, 1)$. The identity*

$$\mathbb{E}(X_1 \mid \int_0^1 X_s \, dF) = \int_0^1 X_s \, dF$$

holds for all 0-self-similar processes (X_t) with $\mathbb{E}|X_1| < \infty$ if and only if $f(x) = \lambda x^{\lambda - 1}$ for some $\lambda > 0$.

4 Quadrants and the two-dimensional case

In this section we establish the following Proposition.

Proposition 4.1 *Let $d = 2$ and let Q_1, Q_2, Q_3, Q_4 be the four quadrants in the plane, in clockwise order. Let*

$$\mu(Q_i) := \int_0^1 \mathbf{1}_{W_t \in Q_i}\, dt$$

denote the time spent in Q_i up to time 1 by a planar Brownian motion W started at the origin. Then for each $k \leq 4$, the random variable

$$\mathbb{P}(W_1 \in Q_k \,|\, \mu(Q_i) : 1 \leq i \leq 4)$$

is not almost surely equal to $\mu(Q_k)$.

Fix the dimension $d = 2$ throughout, and denote by A_ϵ the event that $\mu(Q_2) \in [\epsilon, 2\epsilon]$, $\mu(Q_3) \in [\epsilon, 2\epsilon]$, and $\mu(Q_4) \in [\epsilon, 2\epsilon]$. Thus, if A_ϵ occurs, then the Brownian motion W spends only a small amount of time in Q_2, Q_3, and Q_4. The idea behind the proof is that if Brownian motion spends most of its time in Q_1, then it is very unlikely to be in Q_3 at time 1, since Q_1 and Q_3 do not share a common boundary. More precisely, we will show that there is a constant C for which

$$\mathbb{P}(W_1 \in Q_3 | A_\epsilon) \leq C\epsilon^2[\log(1/\epsilon)]^3 \tag{4.1}$$

for sufficiently small $\epsilon > 0$, which clearly implies Proposition 4.1. The estimate (4.1) follows immediately from the lower bound for $\mathbb{P}(A_\epsilon)$ and the upper bound for $\mathbb{P}(\{W_1 \in Q_3\} \cap A_\epsilon)$ given in Lemmas 4.3 and 4.4 below.

Lemma 4.2 *Let (B_t) be one-dimensional Brownian motion started from the origin. Then as $\delta \to 0$*

$$\delta^{-1}\mathbb{P}(\min_{t\in[0,1]} B_t \geq -\delta) \to \sqrt{\frac{2}{\pi}} \tag{4.2}$$

and

$$\delta^{-3}\mathbb{P}(\min_{t\in[0,1]} B_t \geq -\delta \text{ and } B_1 < 0) \to \frac{1}{\sqrt{2\pi}} \tag{4.3}$$

Proof: The first limit results from the fact that $\min_{t\in[0,1]} B_t$ has density $2\phi(x)$ on $(-\infty, 0]$ where ϕ is the standard normal density of B_1. The second follows easily from the reflection principle, which shows that the probability involved equals

$$\int_{-\delta}^{0} (\phi(x) - \phi(x - \delta))\, dx$$

.

□

Lemma 4.3 *There exists a constant $C_1 > 0$ such that*

$$\mathbb{P}(A_\epsilon) \geq C_1 \epsilon$$

for sufficiently small $\epsilon > 0$.

PROOF: Let D_ϵ be the set $(\sqrt{\epsilon}, \sqrt{\epsilon}) + Q_1$ and let S_ϵ be the event that

$$\{|\{t \in [0, 4\epsilon] : W_t \in Q_i\}| \geq \epsilon \text{ for } i = 2, 3, 4, \text{ and } W_{4\epsilon} \in D_\epsilon\}.$$

Let $C_2 = \mathbb{P}(S_\epsilon) > 0$. From the scaling properties of Brownian motion, we see that C_2 does not depend on ϵ. Let p_ϵ be the probability that $\min_{t \in [0, 1-4\epsilon]} B_t > -\sqrt{\epsilon}$. By the Markov property and independence of the coordinates of W, $\mathbb{P}(A_\epsilon) \geq C_2 p_\epsilon^2$. Lemma 4.2 tells us that $p_\epsilon \geq \sqrt{(2-\beta)\epsilon/\pi}$ for any $\beta > 0$ and sufficiently small $\epsilon(\beta)$. This proves the lemma with any $C_1 < 2C_2/\pi$. □

Lemma 4.4 *There exists a constant $C_3 < \infty$ such that*

$$\mathbb{P}(\{W_1 \in Q_3\} \cap A_\epsilon) \leq C_3 \epsilon^3 [\log(1/\epsilon)]^3$$

for sufficiently small $\epsilon > 0$.

PROOF: Choose $C_4 > 12$ and let $\delta = C_4 \sqrt{\epsilon \log(1/\epsilon)}$. Let $Q_1^\delta = \{(x, y) : x > -\delta, y > -\delta\}$. Also define $T_\delta = \min\{t : W_t \notin Q_1^\delta\}$. Let $R_1 = A_\epsilon \cap \{T_\delta \leq 1 - 6\epsilon\}$, $R_2 = A_\epsilon \cap \{1 - 6\epsilon < T_\delta \leq 1\}$, and $R_3 = \{W_1 \in Q_3\} \cap \{T_\delta > 1\}$. By splitting up the event $\{W_1 \in Q_3\} \cap A_\epsilon$ according to the value of T_δ, we see that if $\{W_1 \in Q_3\} \cap A_\epsilon$ occurs, then either R_1, R_2, or R_3 must occur. We will prove the lemma by establishing upper bounds on $\mathbb{P}(R_1)$, $\mathbb{P}(R_2)$, and $\mathbb{P}(R_3)$.

To bound $\mathbb{P}(R_3)$, apply (4.3) to the two independent coordinate processes, yielding for sufficiently small ϵ

$$\mathbb{P}(R_3) \leq \delta^6 = C_4^6 \epsilon^3 \log(1/\epsilon)^3.$$

A bound for $\mathbb{P}(R_2)$ follows from the observation that on A_ϵ, there must be some $t \in [1 - 6\epsilon, 1]$ for which $W_t \in Q_1$. Thus on R_2, one of the two coordinate processes has an oscillation of at least δ on the time interval $[1 - 6\epsilon, 1]$. This implies that one of the coordinate processes strays by at least $\delta/2$ from its starting value in the interval $[1 - 6\epsilon, 1]$, hence by the Markov property,

$$\begin{aligned} \mathbb{P}(R_2) &\leq 2\mathbb{P}(\max_{0 \leq t \leq 6\epsilon} |B_t| \geq \delta/2) \\ &\leq 8\mathbb{P}(B_{6\epsilon} \geq \delta/2), \text{ by the reflection principle,} \\ &\leq 8 \exp(-\delta^2/48\epsilon) = 8\epsilon^{C_4^2/48}. \end{aligned}$$

By choice of $C_4 > 12$, this is $o(\epsilon^3)$.

A bound on $\mathbb{P}(R_1)$ may be obtained in a similar way. Observe that on A_ϵ, there must be some $t \in [T_\delta, T_\delta + 6\epsilon]$ for which $W_t \in Q_1$. Thus one of the coordinates increases by at least δ from its starting value on the time interval $[T_\delta, T_\delta + 6\epsilon]$. The strong Markov property yields

$$\mathbb{P}(R_1) \leq 2\mathbb{P}(\max_{0 \leq t \leq 6\epsilon} B_t \geq \delta) \leq 4\mathbb{P}(B_{6\epsilon} \geq \delta).$$

As before, the choice of C_4 implies that $\mathbb{P}(R_1) = o(\epsilon^3)$ and summing the upper bounds on $\mathbb{P}(R_1)$, $\mathbb{P}(R_2)$ and $\mathbb{P}(R_3)$ proves the lemma. $\square$

Proof of Proposition 4.1: The inequality (4.1), and the theorem, follow directly from Lemmas 4.3 and 4.4: for sufficiently small $\epsilon > 0$

$$\mathbb{P}(W_1 \in Q_3 | A_\epsilon) = \frac{\mathbb{P}(\{W_1 \in Q_3\} \cap A_\epsilon)}{\mathbb{P}(A_\epsilon)} \leq \frac{C_3 \epsilon^3 \log(1/\epsilon)^3}{C_1 \epsilon} = C\epsilon^2 \log(1/\epsilon)^3.$$

$\square$

5 Recovery of the endpoint

In this section, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of continuous functions $\omega : [0,1] \to \mathbb{R}^d$, endowed with the Borel σ-field $\mathcal{F}$ (in the topology of uniform convergence) and Wiener measure $\mathbb{P}$ on paths from the origin. We write simply μ instead of μ_Θ for the occupation measure of the spherical projection $(\pi(\omega_t), 0 \leq t \leq 1)$. So μ is a measurable map from $(\Omega, \mathcal{F})$ to the space $(\mathsf{prob}(S^{d-1}), \mathcal{F}_2)$ of Borel probability measures on S^{d-1}. For a subinterval I of $[0,1]$, say $I = (a,b)$ or $I = [a,b]$, let ωI denote the range of the restriction of ω to I.

We will use some known topological facts about Brownian motion in dimensions $d \geq 3$:

(1) If I and J are disjoint open sub-intervals of $[0,1]$, then $\mathbb{P}$ almost surely the random set $\{\pi(\omega_t), t \in I\}$ does not contain $\{\pi(\omega_t), t \in J\}$.

(2) Almost every Brownian path $\omega : [0,1] \to \mathbb{R}^d$ has a sequence of cut-times $t_n \uparrow 1$, that is, $\omega(0, t_n) \cap \omega(t_n, 1) = \emptyset$.

(3) With probability 1, no cut-point is a double point. Formally, for $\mathbb{P}$-almost every ω, if $\omega[0,1] \setminus \{\omega(t)\}$ is not connected, then $\omega(s) \neq \omega(t)$ for $s \neq t$.

Fact (1) follows easily from Fubini's theorem. Fact (2) is proved in Theorem 2.2 of Burdzy [5] (see also [6]) and fact (3) is proved in Theorem 1.4 of Burdzy-Lawler [7].

The following lemma contains the probabilistic content of the argument and is proved at the end of this section. Facts (2) and (3) are true when $d = 2$ as well, which is all that is needed to establish the remark after Theorem 1.2.

Lemma 5.1 *Let D be a ball in the sphere $S^{d-1} \subseteq \mathbb{R}^d$. Then there is a measurable function $\rho_D : \mathsf{prob}(S^{d-1}) \to \mathbb{R}^+$ such that $\mathbb{P}$-almost surely,*

$$\rho_D(\mu(\omega)) = \sup\{|\omega(t)| : \pi(\omega(t)) \in D\}. \tag{5.1}$$

To construct ψ from Lemma 5.1 and fact (1), let $\mathcal{C}_j$ be a finite cover of S^{d-1} by balls of radius 2^{-j} and let $\mathsf{Cen}(D)$ denote the center of the ball D. The set of limit points of a sequence $\{S_n\}$ of elements in $\mathsf{cb\text{-}sets}$, defined by $\{x : \liminf_n d(x, S_n) = 0\}$, is called the Hausdorff limsup, denoted $\limsup_{n\to\infty} S_n$. Observe that if S_n are $\mathsf{cb\text{-}sets}$-valued random variables, then $\limsup_{n\to\infty} S_n$ is measurable as well.

Lemma 5.2 *Define measurable functions $A_j : \mathsf{prob}(S^{d-1}) \to \mathsf{cb\text{-}sets}$ to be the sets of vectors*

$$A_j(\mu) := \{\rho_D(\mu)\mathsf{Cen}(D) : D \in \mathcal{C}_j\}.$$

Then $\psi := \limsup_{j\to\infty} A_j$ satisfies (1.4):

$$\psi \circ \mu(\omega) = \mathsf{range}(\omega) \qquad \textit{for almost every } \omega.$$

Remark: In fact, from the proof we see that $\psi = \lim A_j$ almost surely when $\mu = \mu(\omega)$ and ω is chosen from $\mathbb{P}$.

Proof: It is easy to see that $\psi \circ \mu(\omega) \subseteq \mathsf{range}(\omega)$ for every ω: if $D \in \mathcal{C}_j$ then $\rho_D(\mu)(\omega)\mathsf{Cen}(D)$ is equal to $|\omega(t)|\mathsf{Cen}(D)$ for some t with $|\pi(\omega(t)) - \mathsf{Cen}(D)| \le 2^{-j}$, and is hence within $2^{-j}|\omega(t)|$ of the point $\omega(t) \in \mathsf{range}(\omega)$; since $\mathsf{range}(\omega)$ is closed and j is arbitrary, all limit points of sequences $\{x_j\}$ with $x_j \in A_j$ are in $\mathsf{range}(\omega)$.

To see that $\mathsf{range}(\omega) \subseteq \psi \circ \mu(\omega)$, fix $t \in (0,1)$ and consider $x = \omega(t) \in \mathsf{range}(\omega)$. For any $\varepsilon > 0$, choose a $\delta > 0$ such that $|\omega(s) - \omega(t)| \le \varepsilon$ when $|s - t| \le \delta$, and $|\omega(s)| < |x|$ for $0 \le s \le \delta$. By fact (1), the union $\pi(\omega[\delta, t-\delta]) \cup \pi(\omega[t+\delta, 1])$ does not cover $\pi(\omega[t-\delta, t+\delta])$. Thus we may choose an open ball D intersecting $\pi(\omega[t-\delta, t+\delta])$ such that $\pi(D)$ is disjoint from $\pi(\omega[\delta, t-\delta]) \cup \pi(\omega[t+\delta, 1])$. For any $D' \subseteq D$, it follows that $|\rho_{D'}(\mu) - |\omega(t)|| \le \varepsilon$. For sufficiently large j there is a ball $D' \in \mathcal{C}_j$ with $x \in D' \subseteq D$, which implies A_j contains a point $\rho_{D'}(\mu)\mathsf{Cen}(D')$ within $2^{-j}|x| + \varepsilon$ of x. Since ε and j are arbitrary, x is a limit point of the sets A_j. $\square$

The construction of φ from here uses two further non-probabilistic lemmas.

Definition 5.3 *Define the map $N_\delta : \mathsf{cb\text{-}sets} \to \{0, 1, 2, \ldots, \infty\}$ by setting $N_\delta(S)$ to be the number N of connected components of the closed set S that have diameter at least δ.*

Lemma 5.4 *For each δ the map N_δ is measurable.*

Proof: It suffices to show this when S is a subset of the unit ball. It will be convenient to have a nested sequence of sets $\mathsf{GRID}_1 \subseteq \mathsf{GRID}_2 \subseteq \cdots$ such that GRID_j is 2^{-j-1}-dense in the unit ball. (To construct this, inductively choose GRID_j to be a maximal set with no two points within distance 2^{-j-1}.) The sets BALLS_j defined to be the set of balls of radius 2^{-j} centered at points of GRID_j, form a sequence of covers of the unit ball such that each element of BALLS_{j+1} is contained in an element of BALLS_j.

For each j and each $S \in \mathsf{cb\text{-}sets}$ let

$$X_j(S) = \bigcup\{D \in \mathsf{BALLS}_j : D \cap S \neq \emptyset\}.$$

Let P_j be the set of connected components of $X_j(S)$ viewed as subsets of BALLS_j. In other words, $P_j(S) = \{\mathcal{C} \subseteq \mathsf{BALLS}_j : \bigcup \mathcal{C} \text{ is a component of } X_j(S)\}$. By the finiteness of BALLS_j, we see that each P_j is measurable. Since each $D \subseteq X_j(S)$ is contained in a ball $D' \in \mathsf{BALLS}_{j-1}$ also intersecting S, $X_j \subseteq X_{j-1}$ and hence each component of X_j is contained in a unique component of X_{j-1}. This defines a map $\mathsf{parent}_j : P_j \to P_{j-1}$ which is measurable since it depends only on P_j and P_{j-1}. Letting $P_{j,\delta}$ be the subset of P_j consisting of components of diameter at least δ, it is clear that parent_j maps $P_{j,\delta}$ to $P_{j-1,\delta}$ and that these are measurable.

Claim: $N_\delta(S)$ is the cardinality of the inverse limit of the system $\{P_{j,\delta}, \mathsf{parent}_j : j \geq 1\}$. Indeed, suppose that $\{x_j^{(i)}\}$ satisfy $x_j^{(i)} \in P_{j,\delta}$ and $\mathsf{parent}_j(x_j^{(i)}) = x_{j-1}^{(i)}$ for all j and $i = 1, 2$. Letting $\mathsf{set}(x_j^{(i)}) := \bigcup x_j^{(i)}$ denote the set of points in the component $x_j^{(i)}$, we see that $\bigcap_j(\mathsf{set}(x_j^{(i)}))$ are non-empty subsets of S and lie in different components unless $x_j^{(1)} = x_j^{(2)}$ for all j. Conversely, if x and y are points of S lying in different connected components, then S is contained in a disjoint union $X^\epsilon \cup Y^\epsilon$ for some sets X, Y with $x \in X$, $y \in Y$ (where Z^ϵ denotes the set of points within ϵ of the set Z). It follows that for each j there is an $x_j \in P_{j,\delta}$ with $x \in \bigcup x_j$, there is a $y_j \in P_{j,\delta}$ with $y \in \bigcup y_j$, and that for $2^{-j} < \epsilon$, $x_j \neq y_j$.

Finally, the cardinality of the inverse limit is easily seen to be measurable. Say $x_j \in P_{j,\delta}$ is a survivor if for each $k > j$ there is some $y_k \in P_{k,\delta}$ with $\bigcup y_k \subseteq \bigcup x_j$. The set of survivors is clearly measurable, and the cardinality of the inverse limit is the increasing limit of the number of survivors in the set $P_{j,\delta}$ as $j \to \infty$. $\square$

The endpoint $\omega(1)$ will be recovered from $\omega[0,1]$ as the only nonzero limit point of cutpoints, which is not a cutpoint itself. To justify measurability of this operation, the following definition and lemma are useful.

Definition 5.5 *Let $\text{cut}_\delta(S)$ denote the set of δ-cutpoints of S, that is, those $x \in S$ such that $S \setminus x$ has at least two components of diameter at least δ (note: if S is not connected this may be all of S). For each positive integer j and each $\delta > 0$, define the measurable function $A_{\delta,j} : \mathsf{cb\text{-}sets} \to \mathsf{cb\text{-}sets}$ by*

$$A_{\delta,j}(S) := \bigcup\{D' \in \mathsf{BALLS}_j : D' \cap S \neq \emptyset \text{ and } N_\delta(S \setminus D') \geq 2\}.$$

Let

$$A_\delta := \limsup_{j\to\infty} A_{\delta,j}.$$

Lemma 5.6 *Let $f : [0,1] \to \mathbb{R}^d$ be any continuous function and denote its range by S. Fix $\delta' > \delta > \delta'' > 0$. Then*

$$\mathrm{cut}_{\delta'}(S) \subseteq A_\delta \subseteq \mathrm{cut}_{\delta''}(S). \tag{5.2}$$

PROOF: Suppose first that x is a δ'-cutpoint of S. Let T and U be two components of $S \setminus x$ of diameter at least δ. If D is a ball of radius $\epsilon < (\delta' - \delta)/2$ containing x, then $S \setminus D$ will have at least two components of diameter at least $\delta' - 2\epsilon$. Thus $x \in A_{\delta,j}$ for $2^{-j} < \epsilon$, hence $x \in A_\delta$.

Suppose now that $x \in A_\delta$ and let $\{D_n\}$ be balls converging to x in the Hausdorff metric, such that each intersects S and has $N_\delta(S \setminus D_n) \geq 2$. Let $\{D_n'\}$ be balls with diameters going to zero such that $\bigcup_{j=n}^\infty D_j \subseteq D_n'$. Then $N_{\delta''}(S \setminus D_n') \geq 2$ when n is large enough so that the diameter of D_n' is at most $\delta - \delta''$.

> Claim: there are points $x_1, \ldots, x_k$ and an N_0 such that for $n \geq N_0$, each component of $S \setminus D_n'$ of diameter at least δ'' contains one of $x_1, \ldots, x_k$.
>
> Proof: Pick N_0 so that $D_n \subseteq \mathcal{B}(x, \delta''/2)$ when $n \geq N_0$. Pick $\epsilon > 0$ such that $|f(s) - f(t)| < \delta''/2$ when $|s - t| \leq \epsilon$. The open set $\{t : |f(t) - x| > \delta''/2\}$ decomposes into a countable set of intervals. At most $k := \lfloor 1/\epsilon \rfloor$ of these intervals $(u_j, v_j), j = 1, \ldots, k$ can have $v - u \geq \epsilon$, and these are the only ones containing times t with $|f(t) - x| \geq \delta''$. Since S is connected, every component G of $S \setminus D_n'$ intersects $\partial D_n'$, and if G has diameter at least δ'' then G must contain one of the k sojourns $f(u_j, v_j)$. Choose $x_j \in f(u_j, v_j)$.

Since $N_{\delta''}(S \setminus D_n') \geq 2$ for all $n \geq N_0$, there are $i < j \leq k$ such that infinitely many of the sets $S \setminus D_n$ have distinct components G_n and H_n of size at least δ'' containing x_i and x_j respectively. The increasing limits $\bigcup G_n$ and $\bigcup H_n$ must then be contained in distinct components of $S \setminus \{x\}$, showing that $x \in \mathrm{cut}_{\delta''}(S)$. $\square$

PROOF OF THEOREM 1.2 ASSUMING LEMMA 5.1: Clearly the sets A_δ increase as $\delta \to 0$. Define

$$\varphi(S) = (\limsup_{\delta\to 0} A_\delta) \setminus (\bigcup_\delta A_\delta \cup \{0\}).$$

We have shown that $\psi \circ \mu(\omega) = \omega[0,1]$ almost surely with respect to $\mathbb{P}$, and it follows from Lemma 5.6 that $\varphi(S) \cup \{0\}$ is the topological boundary of the set of cut-points of S. Fact (2) then implies that $\omega(1) \in \varphi(S)$. On the other hand, let $x = \omega(t)$ be any limit of cut-points, where $0 < t < 1$. Thus there are times $t_j \to t$ with $\omega(t_j)$ a cut-point. By fact (3), the sets $\omega(t_j, 1)$ and $\omega(0, t_j)$ are disjoint, and each of them is connected. For $t_j > t$, the set $\omega(t_j, 1)$ is disjoint from $\omega(0, t)$ so if $t_j \downarrow t$, then $\omega(t, 1)$ is disjoint from $\omega(0, t)$. Likewise if $t_j \uparrow t$ then $\omega(0, t)$

is disjoint from $\omega(t,1)$, hence t is a cut-time. This shows that $x \notin \varphi(S)$, so the only limits of cut-points that are not cutpoints are $\omega(0)$ and $\omega(1)$, which completes the proof. $\square$

To prove Lemma 5.1 we state several more lemmas. The cases $d = 3$ and $d \geq 4$ differ slightly in that the estimates required for two-dimensional balls ($d = 3$) include logarithmic terms. Since recovery of the endpoint in dimension $d \geq 4$ can be reduced to the three-dimensional case, and since the estimates for two-dimensional balls are strictly harder than for higher-dimensional balls, we assume for the remainder of the proof that $d = 3$. The formula for ρ_D in this case is given by:

$$\rho_D(\mu) := \left[\limsup_{D' \subseteq D, \mathrm{r}(D') \to 0} \frac{\mu(D')}{2\mathrm{r}(D')^2 \log^2 \mathrm{r}(D')} \right]^{1/2}. \tag{5.3}$$

We remark that when $d > 3$, the term $\log^2 \mathrm{r}(D')$ is replaced by $|\log \mathrm{r}(D')|$ and the constant 2 in the denominator changes as well; this is due to the different normalization needed for "thick points" in dimension 3 and higher, see [11].

We begin by quoting two results from Dembo, Peres, Rosen and Zeitouni [12].

Lemma 5.7 *([12], Theorem 1.2). Let $(W_t : t \geq 0)$ be a standard Brownian motion in $\mathbb{R}^2$. Let $\mathrm{r}(D)$ denote the radius of the ball D. Then for any fixed $A > 0$,*

$$\limsup_{\mathrm{r}(D) \to 0} \frac{\int_0^A \mathbf{1}_D(W_t)dt}{\mathrm{r}(D)^2 \log^2 \mathrm{r}(D)} = 2 \quad a.s. \tag{5.4}$$

$\square$

Lemma 5.8 *([12], Lemma 2.1). Let $Z_t = \int_0^t \mathbf{1}_D(W_t)\,dt$ be the occupation time of a standard two-dimensional Brownian motion up to time t in a ball D of radius r. Then for each $t > 0$ there is some $\lambda > 0$ not depending on r for which $\mathbb{E}e^{\lambda Z_t/(r^2|\log r|)} < \infty$. Consequently, $\mathbb{P}(Z_t > Ar^2 \log(1/r) < Ce^{-\gamma A}$ for some positive C and γ.*

Proof: Dembo et al prove the result when the Brownian motion is started at radius r (in their notation $r = r_1 = r_2$) and the time t is instead the time to hit a ball of fixed radius $r_3 = O(1)$. Accomodating these changes is trivial. $\square$

We now state three more lemmas which together imply Lemma 5.1.

Lemma 5.9 *Let $(W_t : t \geq 0)$ be a standard three-dimensional Brownian motion. For $0 \leq a < b \leq 1$, let $\mu_{a,b}$ be projected occupation measure in the time interval $[a,b]$, i.e., for $D \subseteq S^2$,*

$$\mu_{a,b}(D) := \int_a^b \mathbf{1}_D(\pi(W_t))\, dt.$$

Then for each ball $D \subseteq S^2$ and each $\epsilon > 0$, with probability 1,

$$\limsup_{D' \subseteq D, \mathrm{r}(D') \to 0} \frac{\mu_{\epsilon,1}(D')}{\mathrm{r}(D')^2 \log^2 \mathrm{r}(D')} \leq 2(\sup\{|W_t| : \pi(W_t) \in D\})^2. \tag{5.5}$$

Lemma 5.10 *In the notation of the previous lemma, there is a constant c_2 such that for each $\epsilon > 0$, with probability 1,*

$$\limsup_{D' \subseteq D, \mathrm{r}(D') \to 0} \frac{\mu_{0,\epsilon}(D')}{\mathrm{r}(D')^2 \log^2 \mathrm{r}(D')} \leq c_2(\sup\{|W_t| : t \in [0,\epsilon]\})^2. \tag{5.6}$$

Lemma 5.11 *For each $t \in (0,1)$, with probability 1,*

$$\limsup_{D \to \pi(W_t)} \frac{\mu(D)}{\mathrm{r}(D)^2 \log^2 \mathrm{r}(D)} \geq 2|W_t|^2.$$

To see why Lemma 5.1 follows from Lemmas 5.9 - 5.11, define ρ_D as in equation (5.3). Since the limsup may be taken over balls with rational centers and radii, ρ_D is measurable. Lemmas 5.9 and 5.10 together imply that with probability 1, for all $\epsilon > 0$,

$$\rho_D(\mu(\omega)) \leq \left[(\sup\{|W_t| : \pi(W_t) \in D\})^2 + (c_2/2)(\sup\{|W_t| : t \in [0,\epsilon]\})^2\right]^{1/2},$$

and sending ϵ to 0 shows that the LHS of (5.1) is less than or equal to the RHS. On the other hand, applying Lemma 5.11 for all rational t shows that with probability 1,

$$\rho_D(\mu(\omega)) \geq \sup\{|\omega(t)| : \pi(\omega(t)) \in \mathsf{interior}(D), t \text{ rational}\}$$

which yields the reverse inequality. It remains to prove Lemmas 5.9 - 5.11.

Proof of Lemma 5.9: Covering D with small balls, it suffices to assume $\mathrm{r}(D) < \delta$ and prove an upper bound of $(1 + o(1))$ times the RHS of (5.5) as $\delta \to 0$. Let $\beta : S^2 \to \mathbb{R}^2$ be a conformal map with Jacobian going to 1 near $\mathsf{Cen}(D)$. For example, take β to be stereographic projection from the antipode to $\mathsf{Cen}(D)$ to a plane (identified with $\mathbb{R}^2$) tangent to S^2 at $\mathsf{Cen}(D)$. The path $\{\pi(W_t) : t \geq \epsilon\}$ is a time-changed Brownian motion on S^2, and in particular, $\pi(W_{G(t)})$ is a Brownian motion started from $\pi(W_1)$, where $G(t)$ is defined by $\int_{G(t)}^1 |W_s|^{-2}\, ds = t$. Similarly, $(X_t := \beta(\pi(W_{G(H(t))})), t \in [0, M := H^{-1}(G^{-1}(\epsilon))])$ is a Brownian motion in $\mathbb{R}^2$, where M is random and $H(t)$ is another time change, with $|H'|$ going to 1 uniformly as $\mathrm{r}(D) \to 0$ and $\pi(W_{G(H(t))})$ is in D.

Let D' be any ball inside D. Let D'' be a ball containing $\beta(D')$ and observe that we can take $\mathrm{r}(D'')/\mathrm{r}(D') \to 1$ uniformly over $D' \subseteq D$ as $\mathrm{r}(D) \to 0$. When $\pi(W_s) \in D$, $G'(s) \geq \sup\{|W_t| : \pi(W_t) \in D\}^2$. Thus

$$\begin{aligned}
\mu_{\epsilon,1}(D') &= \left|\left\{t \in [\epsilon, 1] : \pi(W_t) \in D'\right\}\right| \\
&\leq \left|\left\{G(H(s)) : \beta(\pi(W_{G(H(s))})) \in D''\right\}\right| \\
&\leq \sup\{|W_t| : \pi(W_t) \in D\}^2 \sup H' \left|\left\{s : \beta(\pi(W_{G(H(s))})) \in D''\right\}\right| \\
&= \sup\{|W_t| : \pi(W_t) \in D\}^2 \sup H' \left|\left\{s : X_s \in D''\right\}\right| \\
&\leq (2 + o(1)) \sup\{|W_t| : \pi(W_t) \in D\}^2 r(D'')^2 \log^2(\frac{1}{r(D'')})
\end{aligned}$$

by Lemma 5.7 and the convergence of H' to 1. $\square$

Proof of Lemma 5.10: Let D be any ball in S^2 with center x. Let β_x be projection to the orthogonal complement of x in $\mathbb{R}^3$. If $\pi(W_t) \in D$ then $\beta_x(W_t) \in \mathcal{B}(0, s)$ for $s := \mathrm{r}(D) \sup\{|W_t| : t \in [0, 1]\}$. For fixed x, $\beta_x(W_t)$ is a standard Brownian motion, so an application of the Lemma (5.8) yields

$$\begin{aligned}
&\mathbb{P}\left(\frac{\mu_{0,\epsilon}(D)}{\mathrm{r}(D)^2 \log^2 \mathrm{r}(D)} \geq c(\sup\{|W_t| : t \in [0, 1]\})^2\right) \\
\leq\ &\mathbb{P}\left(\frac{\int_0^1 dt\, \mathbf{1}_{\mathcal{B}(0,s)}(\beta_x(W_t))}{s^2 |\log s| |\log r(D)| (\log(\mathrm{r}(D)) / \log s)} \geq c\right) \\
\leq\ &C \mathrm{r}(D)^{-\gamma c \log \mathrm{r}(D) / \log s}.
\end{aligned}$$

We may choose c_2 so that $c_2\gamma > 2$, and find classes $\mathcal{C}_r$ of balls of radius r so that for any $\epsilon > 0$, for sufficiently small r, any ball of radius $(1 - \epsilon)r$ is contained in some element of $\mathcal{C}_r$. One can arrange for $|\mathcal{C}_r| = O(1/r)^{c_2 c_0 - \delta}$, where $c_2 c_0 - \delta > 2$, ensuring that

$$\mathbb{P}\left(\exists D \in \mathcal{C}_r : \frac{\mu_{0,\epsilon}(D)}{\mathrm{r}(D)^2 \log^2 \mathrm{r}(D)} \geq c_2(\sup\{|W_t| : t \in [0, 1]\})^2\right) = o(r^\delta).$$

Summing over $r = (1 - \alpha)^n$ and using Borel-Cantelli shows that the limsup on the LHS of (5.6) is at most $(1 - \alpha)^{-2} c_2 \sup\{|W_t| : t \in [0, \epsilon]\}^2$, proving the lemma since α may be chosen arbitrarily small. $\square$

Proof of Lemma 5.11: Fix $t \in (0, 1)$. Define β, G and H as in the proof of Lemma 5.9, so that $(X_s := \beta(\pi(W_{G(H(s))})))$ is a planar Brownian motion. For any $\epsilon > 0$, Lemma 5.7 yields a random sequence of balls $D_n \to 0$ in $\mathbb{R}^2$ with

$$\int_0^M \frac{\mathbf{1}_{D_n}(X_s)\,ds}{\mathrm{r}(D_n)^2 \log^2 \mathrm{r}(D_n)} \to 2.$$

With probability 1, W_t is a single value, i.e., $W_t \neq W_s$ for $t \neq s$, in which case for n sufficiently large, $X_s \in D_n$ implies $|W_{G(H(s))}| \to |W_t|$ and $G(H(s)) \to t$. The sets $\beta^{-1}(D_n)$ are contained in balls D'_n with $\mathrm{r}(D'_n)/\mathrm{r}(D_n) \to 1$, so

$$\int_0^M \frac{\mathbf{1}_{D'_n}(\pi(W_{G(H(s))})\,ds}{\mathrm{r}(D'_n)^2 \log^2 \mathrm{r}(D'_n)} \to 2.$$

Changing variables reduces this integral to

$$\int_\epsilon^1 \frac{\mathbf{1}_{D'_n}(\pi(W_u))(G \circ H)'((G \circ H)^{-1}(u))\,du}{\mathrm{r}(D'_n)^2 \log^2 \mathrm{r}(D'_n)}$$

and since $(G \circ H)' = (1 + o(1))|W_t|^{-2}$ uniformly on an interval containing $H^{-1}(G^{-1}(t))$, we get

$$|W_t|^{-2} \frac{\mu_{\epsilon,1}(D'_n)}{\mathrm{r}(D'_n) \log^2 \mathrm{r}(D'_n)} \to 2,$$

proving the lemma. $\square$

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