Electronic Journal of Probability

Electron. J. Probab. **18** (2013), no. 70, 1–32.
ISSN: 1083-6489 DOI: 10.1214/EJP.v18-2797

On the spatial dynamics of the solution to the stochastic heat equation

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Abstract

We consider the solution of $\partial_t u = \partial_x^2 u + \partial_x\partial_t B$, $(x,t) \in \mathbb{R} \times (0,\infty)$, subject to the initial condition $u(x,0) = 0$, $x \in \mathbb{R}$, where B is a Brownian sheet. We show that u also satisfies $\partial_x^2 u + [\,(-\partial_t^2)^{1/2} + \sqrt{2}\partial_x(-\partial_t^2)^{1/4}\,]\,u^a = \partial_x\partial_t\tilde{B}$ in $\mathbb{R} \times (0,\infty)$ where u^a stands for the extension of $u(x,t)$ to $(x,t) \in \mathbb{R}^2$ which is antisymmetric in t and $\tilde{B}$ is another Brownian sheet. The new SPDE allows us to prove the strong Markov property of the pair $(u, \partial_x u)$ when seen as a process indexed by $x \geq x_0$, x_0 fixed, taking values in a state space of functions in t. The method of proof is based on enlargement of filtration and we discuss how our method could be applied to other quasi-linear SPDEs.

Keywords: stochastic partial differential equation; enlargement of filtration; Brownian sheet; Gaussian analysis.
AMS MSC 2010: Primary 60H15, Secondary 60H30.
Submitted to EJP on May 14, 2013, final version accepted on July 6, 2013.
Supersedes arXiv:1305.3325.

1 Introduction

When studying stochastic partial differential equations (SPDEs) one has to understand the behaviour of multi-parameter random fields $u(\mathbf{x})$, $\mathbf{x} \in \mathcal{Q}$, where $\mathcal{Q} \subseteq \mathbb{R}^d$ is a given domain. A first and of course important question deals with the possibility of a Markovian 'behaviour' of such a field. Since Lévy's sharp Markov property (see [13]) already fails in the case of multi-parameter Brownian sheets, the only comprehensive Markovian 'behaviour' one can hope for is the so-called germ Markov property–the reader is referred to the early papers [9, 14] and in particular to [10] for a good introduction to this concept.

There is an early paper by Y.A. Rozanov [18] on the Markovian 'behaviour' of SPDEs and then there are three influential papers [6, 7, 16] on the germ Markov property of solutions to SPDEs of type

$$\mathcal{L}u = \eta + f(u) \quad \text{in} \quad \mathcal{Q}$$

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where $\mathcal{L}$ is a linear partial differential operator of elliptic or parabolic type and η stands for a multi-parameter noise. The method applied in all three papers consisted in, first, establishing the germ Markov property for the solution of the linear equation

$$\mathcal{L}u = \eta \quad \text{in} \quad \mathcal{Q}$$

and, second, getting it for the drift-perturbed equation by a Kusuoka or Girsanov transform. It should be mentioned that [1] provides another useful method for the second step.

The main method for the first step is usually based on the paper [17] which was later improved by H. Künsch [12]. For example, the more recent paper [2] on the germ Markov property of the solution of a linear stochastic heat equation is still about checking the conditions stated in [17, 12] which can be demanding.

However, the purpose of our paper is to refine this first step in the following sense: study a more specific Markovian 'behaviour' of solutions of linear SPDEs. Our working example is the stochastic heat equation with additive space-time white noise. But all calculations are based on only two ingredients:

- a Green's function for $\quad \mathcal{L}u = \eta \quad \text{in} \quad \mathcal{Q}$;
- the covariance of a Gaussian noise η.

So, following our scheme of calculations but using a different Green's function or covariance would produce similar results with respect to other linear SPDEs. The explicit calculations are involved and will be different in other cases. That's why we have to restrict ourselves to the case of an important example in order to show how the method works in the very detail. However, at the end of this introduction, we list the tasks to be dealt with when treating another SPDE.

We now explain what we mean by a *Markovian 'behaviour' more specific than the germ Markov property*. A random field $u(\mathbf{x})$, $\mathbf{x} \in \mathcal{Q}$, satisfies the germ Markov property if $\sigma\{u(\mathbf{y}) : \mathbf{y} \in A\}$ and $\sigma\{u(\mathbf{y}) : \mathbf{y} \in A^c\}$ are conditionally independent given the germ-σ-algebra

$$germ(\partial A) \stackrel{\text{def}}{=} \bigcap_{\{\partial A \subset O : O \text{ open in } \mathcal{Q}\}} \sigma\{u(\mathbf{y}) : \mathbf{y} \in O\}$$

for any Borel set $A \in \mathcal{Q}$. This type of Markov property is 'directionless' with respect to the d-dimensional domain $\mathcal{Q}$. But often it is desirable to emphasise a direction in $\mathbb{R}^d$ and to study the behaviour of an SPDE along this direction. In the case of parabolic SPDEs, a natural direction to emphasise is the direction of 'time' we denote by t in what follows. Many solutions $u(\mathbf{x}, t)$ of parabolic SPDEs are even constructed as Markov processes $t \mapsto u(\cdot, t)$ taking values in a function space. Hence, along the direction of time, these solutions satisfy a sharp Markov property with an associated martingale problem.

But we want to be able to pick other directions with respect to the space-variable $\mathbf{x} = (x_1, \ldots, x_{d-1})$ of $u(\mathbf{x}, t)$, for example, the direction of x_1. Assume we would already know that $u(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \mathcal{Q}$, satisfies the germ Markov property. Then, the process $x_1 \mapsto u(x_1, \cdot)$ is at least Markovian relative to

$$germ\Big(\, (\{x_1\} \times \mathbb{R}^{d-1}) \cap \mathcal{Q} \, \Big).$$

But this does not give an associated martingale problem with martingales indexed by x_1 hence many useful probabilistic methods cannot be applied. So one wants to know if there is a σ-algebra included in $germ\big(\, (\{x_1\} \times \mathbb{R}^{d-1}) \cap \mathcal{Q} \, \big)$ so that $x_1 \mapsto u(x_1, \cdot)$ is still Markovian relative to this smaller σ-algebra but also satisfies an associated martingale problem. And, because we are dealing with SPDEs, it is very likely that a σ-algebra generated by certain partial derivatives of $u(\mathbf{x},t)$ with respect to x_1 would serve the purpose.

Our main result, Theorem 3.17, states that the above can be achieved in the case of the stochastic heat equation $(\partial_x^2 - \partial_t)u = -\partial_x\partial_t B$, $(x,t) \in \mathbb{R}\times(0,\infty)$, driven by a Brownian sheet B. We find a martingale problem for the pair of processes $(u(x,\cdot), \partial_x u(x,\cdot))$, $x \geq x_0$, which is given by an unbounded operator we explicitly calculate using the technique of enlargement of filtration.

The need for an enlargement of filtration in this context can be considered the key idea of this paper. The problem is explained in Section 3–the reader is referred to the second equation of (3.9). The observation is that, for a given test function h, there is no filtration such that $x \mapsto U(x,h')$ is adapted with respect to this filtration and $x \mapsto W_{x-x_0}(h)$ is a Wiener process with respect to this filtration. Enlargement of filtration solves this problem subject to a drift correction. But the new drift requires test functions which are less regular than the original test functions h. As a consequence one has to discuss the regularity of all involved processes very carefully.

We are then able to derive, by showing the uniqueness of the martingale problem, the strong Markov property of $u(x,\cdot)$, $x \geq x_0$, with respect to the natural filtration generated by $(u(x,\cdot), \partial_x u(x,\cdot))$, $x \geq x_0$.

As explained earlier, the same method could be used to find interesting martingale problems associated with other linear SPDEs or even drift-perturbed linear SPDEs, by applying Girsanov's transform for example, the latter being beyond the scope of this paper.

The organisation of the paper is as follows. In Section 2 we list notation which is crucial for the understanding of the paper. Section 3 is a combination of results and further explanations which fully describes our method and can be summarised by:

- choose a direction along which one wishes to study the solution u of a stochastic partial differential equation $\mathcal{L}u = \eta$ in $\mathcal{Q}$ subject to given boundary data;
- describe the dynamics of $\mathcal{L}u = \eta$ in $\mathcal{Q}$ along the chosen direction—see (3.3);
- find the regularity of all involved partial derivatives—see Proposition 3.3;
- find a correction ϱ of η such that $\mathcal{L}u$ and $\tilde{\eta} = \eta - \varrho$ are both adapted with respect to a filtration along the chosen direction and that the probability law of $\tilde{\eta}$ is the same as the law of η—see Proposition 3.6, Lemma 3.8, Remark 4.1(ii);
- describe ϱ as a functional of the solution u—see Proposition 3.9, Theorem 3.11;
- show uniqueness of the martingale problem along the chosen direction which is associated with the new equation $\mathcal{L}u - \varrho(u) = \tilde{\eta}$—see Theorem 3.17.

The results are finally proven in Section 4 on page 15.

2 Notation

We use the notation ∂_i for the operation of taking the ith partial derivative, $i = 1,\ldots,d$, of a function $f(x_1,\ldots,x_d)$ and we write ∂_i^m for iterating this operation m times, that is, $\partial_i^m = \partial_i\partial_i^{m-1}$ for $m \geq 1$ where ∂_i^0 is defined to be the identity map. But if the function f only depends on a space variable $x \in \mathbb{R}$ and a time variable $t \geq 0$ and there is no ambiguity about the nature of the involved variables then we will also write ∂_x and ∂_t for the corresponding partial derivatives.

The heat kernel

$$g(y,s\,;x,t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{4\pi(t-s)}}\,\exp\{\frac{-(x-y)^2}{4(t-s)}\}\,\mathbf{1}_{(s,\infty)}(t)$$

is considered a function

$$g : \Big[\, \mathbb{R} \times \mathbb{R} \,\Big] \times \Big[\, \mathbb{R} \times \mathbb{R} \,\Big] \to \mathbb{R}.$$

We write g as an inhomogeneous transition kernel in order to emphasise that the method works for more general PDE problems than the heat equation. However for some explicit calculations we are going to apply the time - homogeneous structure of g and then we also use the notation

$$g_y^{x_0}(t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \exp\{-\frac{(x_0-y)^2}{4t}\}\, \mathbf{1}_{(0,\infty)}(t), \quad t \in \mathbb{R},$$

for given $x_0, y \in \mathbb{R}$.

We use $f_1 * f_2$ to denote the convolution of functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ and write $\widehat{f_i}$ for their Laplace transform

$$\widehat{f_i}(\nu) \stackrel{\text{def}}{=} \int_0^\infty f_i(t) e^{-\nu t}\, \mathrm{d}t, \quad \nu > 0,\ i = 1, 2.$$

Note that

$$(f_1 \mathbf{1}_{(0,\infty)} \widehat{) * (} f_2 \mathbf{1}_{(0,\infty)}) \;=\; \widehat{f}_1\, \widehat{f}_2.$$

Furthermore, if l is a function on $(0,\infty)$ or $[0,\infty)$ then we denote by l^a its antisymmetric extension

$$l^a : \mathbb{R} \to \mathbb{R} \quad \text{such that } l^a(0) = 0 \text{ and } \quad l^a(t) = \begin{cases} \quad l(t) & : \quad t \in (0,\infty); \\ -l(-t) & : \quad t \in (-\infty, 0). \end{cases}$$

Note that $\|l^a\|_{L^2(\mathbb{R})} = \sqrt{2}\|l\|_{L^2([0,\infty))}$, $\|l^a\|_{L^1(\mathbb{R})} = 2\|l\|_{L^1([0,\infty))}$ and that

$$\begin{aligned} \|\frac{1}{\sqrt{|\cdot|}} * l^a\|_{L^p(\mathbb{R})} \quad &\le \quad \|\frac{\mathbf{1}_{[-1,1]}(\cdot)}{\sqrt{|\cdot|}} * l^a\|_{L^p(\mathbb{R})} + \|\frac{\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(\cdot)}{\sqrt{|\cdot|}} * l^a\|_{L^p(\mathbb{R})} \\ &\le \quad c_p(\|l^a\|_{L^2(\mathbb{R})} + \|l^a\|_{L^1(\mathbb{R})}) \end{aligned} \tag{2.1}$$

for each $p > 2$ by Young's inequality.

The following domains

$$\mathcal{Q}_+ \;=\; \mathbb{R} \times (0,\infty) \quad \text{and} \quad \mathcal{Q}_+^y \;=\; (y,\infty) \times (0,\infty),\ y \in \mathbb{R},$$

will be frequently used.

The symbol $\mathscr{D}$ is reserved for $C_c^\infty\big((0,\infty)\big)$ the space of smooth functions on $(0,\infty)$ with compact support and $\mathscr{D}^a \stackrel{\text{def}}{=} \{h^a : h \in \mathscr{D}\}$.

$\langle \cdot\, ; \cdot \rangle$ denotes the scalar product in $L^2([0,\infty))$ and, whenever the dual pairing between a topological vector space and its dual is an extension of the scalar product in $L^2([0,\infty))$, this dual pairing is also denoted by $\langle \cdot\, ; \cdot \rangle$.

3 Results

The stochastic partial differential equation of our interest formally reads

$$\partial_t u \;=\; \partial_x^2 u \;+\; \partial_x \partial_t B \quad \text{in } \mathcal{Q}_+ \quad \text{subject to} \quad \lim_{t\downarrow 0} u(\cdot, t) \;=\; 0 \tag{3.1}$$

where $B = \{B_{xt};\, x \in \mathbb{R}, t \ge 0\}$ is a Brownian sheet given on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that the transition kernel g introduced in Section 2 gives the Green's function associated with this Cauchy problem. It is well-known that the random field

$$U(x,t) \stackrel{\text{def}}{=} \iint_{\mathcal{Q}_+} B(\mathrm{d}y, \mathrm{d}s)\, g(y, s\,; x, t), \quad (x,t) \in \mathcal{Q}_+, \tag{3.2}$$

is the unique (weak) solution to (3.1) where the integral against $B(\mathrm{d}y, \mathrm{d}s)$ is understood as an Itô-type integral against a process indexed by two parameters. We always mean by U the version which can be continuously extended to the closure of $\mathcal{Q}_+$ – see [20] for a good account on the underlying theory of SPDEs.

Due to the parabolic nature of (3.1), the process $\{U(\cdot, t);\, t \geq 0\}$ taking values in the space of continuous functions is a strong Markov process with zero initial condition in the usual sense. But we are after the Markovian behaviour of $U(x, \cdot)$ as a process indexed by $x \geq x_0$. The method is to construct a (infinite dimensional) stochastic differential equation which is solved by the pair $(U(x, \cdot), \partial_1 U(x, \cdot))$ and then to prove that the solution of this stochastic differential equation is Markovian in the usual sense.

Remark 3.1. *It turns out that this Markov process is homogeneous and stationary in the case of (3.1), that is, in the case of zero initial condition. If the initial condition is not zero but a function b_0 then the underlying solution can be written as $U(x,t) + \int_{\mathbb{R}} b_0(y) g(y, 0\,; x, t)\, \mathrm{d}y$ if b_0 has enough regularity. Adding this deterministic integral gives an inhomogeneous Markov process instead without any extra proof.*

The initial idea is to rewrite (3.1) as

$$\left.\begin{array}{rcl} \partial_x u & = & v \\ \partial_x v & = & \partial_t u - \partial_x \partial_t B \end{array}\right\} \tag{3.3}$$

and to understand this system as an equivalent formulation of the Dirichlet problem

$$(\partial_x^2 - \partial_t)\, u \;=\; -\partial_x \partial_t B \quad \text{in} \quad \mathcal{Q}_+^{x_0} \;=\; (x_0, \infty) \times (0, \infty) \tag{3.4}$$

subject to a given continuous function on the boundary $\partial \mathcal{Q}_+^{x_0}$. So we are only interested in solutions (u, v) of (3.3) with respect to the domain $\mathcal{Q}_+^{x_0}$ where $u(x,t)$ can be extended to a continuous function on $\overline{\mathcal{Q}_+^{x_0}}$ satisfying

$$u(x_0, \cdot) \;=\; b^{x_0} \quad \text{and} \quad u(\cdot, 0) \;=\; b_0 \tag{3.5}$$

for given (maybe random) continuous functions

$$b^{x_0} : [0, \infty) \to \mathbb{R} \quad \text{and} \quad b_0 : [x_0, \infty) \to \mathbb{R} \quad \text{such that} \quad b^{x_0}(0) = b_0(x_0).$$

Remark 3.2. *(i) If a continuous function u on $\mathcal{Q}_+^{x_0}$ satisfies (3.4) in the weak sense of*

$$\iint_{\mathcal{Q}_+^{x_0}} u(x,t)\, (\partial_1^2 + \partial_2) f(x,t)\, \mathrm{d}x\mathrm{d}t \;\overset{\text{a.s.}}{=}\; -\iint_{\mathcal{Q}_+} B(\mathrm{d}x, \mathrm{d}t)\, f(x,t)$$

for all test functions $f \in C_c^\infty(\mathcal{Q}_+^{x_0})$ then the pair (u, v) where v is the generalized function given by $v(f) = -\iint_{\mathcal{Q}_+^{x_0}} u(x,t)\, \partial_1 f(x,t)\, \mathrm{d}x\mathrm{d}t$, $f \in C_c^\infty(\mathcal{Q}_+^{x_0})$, satisfies (3.3) in the corresponding weak sense and vice versa.

(ii) If b^{x_0} is an arbitrary continuous function on $[0, \infty)$ and b_0 is a continuous function on $[x_0, \infty)$ of polynomial growth such that $b^{x_0}(0) = b_0(x_0)$ then

$$\begin{aligned} u(x,t) \;&=\; \int_0^\infty b^{x_0}(s)\, 2\partial_1 g(x_0, s\,; x, t)\, \mathrm{d}s \\ &\quad + \int_{x_0}^\infty b_0(y)\, [g(y, 0\,; x, t) - g(2x_0 - y, 0\,; x, t)]\, \mathrm{d}y \\ &\quad + \iint_{\mathcal{Q}_+^{x_0}} B(\mathrm{d}y, \mathrm{d}s)\, [g(y, s\,; x, t) - g(2x_0 - y, s\,; x, t)] \end{aligned}$$

is the unique (weak) solution of (3.4),(3.5). Using the Green's function associated with this Dirichlet problem, the above existence/uniqueness result is standard – see [20] for example. The polynomial growth condition on b_0 is not optimal but sufficient for our purpose. Both, b^{x_0} and b_0, can of course be taken to be $\mathcal{F}$-measurable.

Let us return to the solution U of (3.1) given by (3.2). Of course, U is the unique weak solution of (3.4),(3.5) subject to $b^{x_0} = U(x_0, \cdot)$ and $b_0 = 0$ hence, by Remark 3.2, the pair $(U, \partial_1 U)$ solves (3.3) at least in the corresponding weak sense.

But this is not enough to justify why $(U(x,\cdot), \partial_1 U(x,\cdot))$ should be a Markov process indexed by $x \geq x_0$. First one needs a meaning of $(U(x,\cdot), \partial_1 U(x,\cdot))$ as a process indexed by $x \geq x_0$ which boils down to finding an appropriate state space, E, for the random variables $(U(x,\cdot), \partial_1 U(x,\cdot))$, $x \geq x_0$. Second, a well-posed martingale problem needs to be associated with the system (3.3).

To start with finding the right state space, observe that

$$\int_{\mathbb{R}} \int_0^\infty \left(\Big[\int_0^\infty g(y,s\,;x,t) h(t)\,\mathrm{d}t \Big]^2 + \Big[\int_0^\infty \partial_3 g(y,s\,;x,t) h(t)\,\mathrm{d}t \Big]^2 \right) \mathrm{d}s\mathrm{d}y \, < \, \infty$$

hence the stochastic integrals

$$U(x,h) \stackrel{\text{def}}{=} \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s) \Big[\int_0^\infty g(y,s\,;x,t) h(t)\,\mathrm{d}t \Big] \tag{3.6}$$

and

$$\partial_1 U(x,h) \stackrel{\text{def}}{=} \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s) \Big[\int_0^\infty \partial_3 g(y,s\,;x,t) h(t)\,\mathrm{d}t \Big] \tag{3.7}$$

are well-defined for all $x \geq x_0$ and $h \in \mathscr{D} = C_c^\infty\big((0,\infty)\big)$. Since the notation $\partial_1^0 U(x,h)$ and $\partial_1^1 U(x,h)$ can be used for $U(x,h)$ and $\partial_1 U(x,h)$, respectively, we have defined a centred Gaussian process $\partial_1^i U(x,h)$ indexed by $(i,x,h) \in \{0,1\} \times [x_0,\infty) \times \mathscr{D}$.

Proposition 3.3. *(i) The process $\{\partial_1^i U(x,h);\, (i,x,h) \in \{0,1\} \times [x_0,\infty) \times \mathscr{D}\}$ solves the system (3.3) in the sense of*

$$\begin{aligned} U(x,h) &\stackrel{\text{a.s.}}{=} U(x_0,h) + \int_{x_0}^x \partial_1 U(y,h)\,\mathrm{d}y \\ \partial_1 U(x,h) &\stackrel{\text{a.s.}}{=} \partial_1 U(x_0,h) - \int_{x_0}^x U(y,h')\,\mathrm{d}y - \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,(\mathbf{1}_{(x_0,x]} \otimes h)(y,s) \end{aligned}$$

for all $(x,h) \in [x_0,\infty) \times \mathscr{D}$.

(ii) For fixed $x \geq x_0$, the processes $\{U(x,h);\, h \in \mathscr{D}\}$ and $\{\partial_1 U(x,h);\, h \in \mathscr{D}\}$ are independent centred Gaussian processes with covariances

$$\mathbb{E}\, U(x,h_1) U(x,h_2) \, = \, \langle\, h_1\,; \frac{-\sqrt{|\cdot|}}{\sqrt{4\pi}} * h_2^a \,\rangle$$

and

$$\mathbb{E}\, \partial_1 U(x,h_1) \partial_1 U(x,h_2) \, = \, \langle\, h_1\,; \frac{1}{2\sqrt{4\pi|\cdot|}} * h_2^a \,\rangle$$

respectively.

(iii) For fixed $x \geq x_0$, the process $\{U(x,h);\, h \in \mathscr{D}\}$ has a version taking values in

$$E_1 \stackrel{\text{def}}{=} \{u \in (C_{0,\alpha})' : u \in C([0,\infty)) \text{ such that } u(0) = 0\}$$

for some $\alpha > 3/2$ where

$$C_{0,\alpha} \stackrel{\text{def}}{=} \{h \in C([0,\infty)) : h(t)(1+t)^\alpha \to 0,\, t \to \infty\},$$

is equipped with the norm $\|h\|_{0,\alpha} \stackrel{\text{def}}{=} \sup_{t\geq 0} |h(t)(1+t)^\alpha|$ and $(C_{0,\alpha})'$ denotes the topological dual of the Banach space $(C_{0,\alpha}\,, \|\cdot\|_{0,\alpha})$. Equip the space E_1 with the norm of $(C_{0,\alpha})'$.

(iv) *For fixed $x \geq x_0$, the process $\{\partial_1 U(x,h);\, h \in \mathscr{D}\}$ has a version taking values in*

$$E_2 \stackrel{\text{def}}{=} (H^a_{\mathfrak{w},\beta})' \quad \text{for some } \beta > 1/4$$

where

$$H^a_{\mathfrak{w},\beta} \,=\, \{l \in L^2([0,\infty)) : \mathfrak{w} l^a \in H_\beta\}.$$

Here H_β stands for the Sobolev space of functions $f \in L^2(\mathbb{R})$ whose Fourier transform f^F satisfies $\|(1+|\cdot|^2)^{\frac{\beta}{2}} f^F\|_{L^2(\mathbb{R})} < \infty$ and $\mathfrak{w}$ is a smooth weight function such that, for some $\varepsilon > 0$, $\mathfrak{w} \geq 1 + |\cdot|^{\frac{1}{2}+\varepsilon}$ but $\mathfrak{w} = 1 + |\cdot|^{\frac{1}{2}+\varepsilon}$ outside a neighbourhood of zero.

(v) *The family of random variables $\{(U(x,\cdot), \partial_1 U(x,\cdot));\, x \geq x_0\}$ is a stationary process taking values in $E = E_1 \times E_2$ which has an $\mathcal{F} \otimes \mathscr{B}([x_0,\infty))$ - measurable version such that*

$$\mathbb{E} \int_{x_0}^{x} \left(\|U(y,\cdot)\|^2_{E_1} + \|\partial_1 U(y,\cdot)\|^2_{E_2} \right) \mathrm{d}y \,<\, \infty \tag{3.8}$$

for all $x > x_0$. Furthermore, the process $\{U(x,\cdot);\, x \geq x_0\}$ taking values in E_1 has a version such that $(x,t) \mapsto U(x,t)$ is continuous on the closure of $\mathcal{Q}^{x_0}_+$.

Remark 3.4. (i) *The bound $3/2$ for the parameter α defining the state space E_1 is sharp in the following sense: for $\alpha \leq 3/2$ one cannot apply Lemma 3.5 below in the proof.*

(ii) *Denote by C_2 the covariance operator $C_2 h \stackrel{\text{def}}{=} \frac{1}{2\sqrt{4\pi|\cdot|}} * h^a$ associated with $\partial_1 U(x,\cdot)$ by item (ii) above. Of course, $C_2 h = const(-\partial_t^2)^{-\frac{1}{4}}\, h^a$ (see [19] for example), and the parameter β defining the space E_2 was chosen just big enough to ensure that $C_2^{1/2} : L^2([0,\infty)) \to E_2$ is a Hilbert-Schmidt operator which, by Sazonov's theorem, is needed for a meaningful state space of a Gaussian measure. The choice of a weighted (Sobolev) space is due to the 'infinite-volume' in t-direction. Finding the right space E_2 in the case of other SPDEs might be more complicated.*

The proof of the above proposition uses the following technical lemma. Recall that $B = \{B_{ys};\, y \in \mathbb{R}, s \geq 0\}$ is a Brownian sheet on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assign to B a family of σ-algebras

$$\mathcal{F}_A \,=\, \sigma(\{B_{ys} : y \in A, s > 0\}) \vee \mathcal{N}_{\mathbb{P}}, \quad A \subseteq \mathbb{R},$$

where $\mathcal{N}_{\mathbb{P}}$ is the collection of all null sets in $\mathcal{F}$. Note that this makes $\mathcal{F}_{(-\infty,x]}$, $x \in \mathbb{R}$, a right-continuous filtration.

Lemma 3.5 (special case of Theorem 2.6 in [20]). *Let $\phi \in L^1(I)$ where $I \subseteq \mathbb{R}$ is a measurable index set and let $f : \Omega \times \mathcal{Q}_+ \times I \to \mathbb{R}$ be an $\mathcal{F} \otimes \mathscr{B}(\mathcal{Q}_+) \otimes \mathscr{B}(I)$ - measurable function such that, for each $(y,\zeta) \in \mathbb{R} \times I$, the mapping $(\omega, s) \mapsto f(\omega, y, s, \zeta)$ is $\mathcal{F}_{(-\infty,y]} \otimes \mathscr{B}((0,\infty))$ - measurable.*

(i) *If $\mathbb{E} \int_{\mathbb{R}} \int_0^\infty [f(y,s,\zeta)]^2 \,\mathrm{d}s\mathrm{d}y < \infty$ for all $\zeta \in I$ then there is an $\mathcal{F} \otimes \mathscr{B}(I)$ - measurable version of the process $\{\iint_{\mathcal{Q}_+} B(\mathrm{d}y, \mathrm{d}s)\, f(y,s,\zeta);\, \zeta \in I\}$.*

(ii) If in addition

$$\mathbb{E}\int_{\mathbb{R}}\int_0^\infty\int_I [f(y,s,\zeta)]^2\,|\phi(\zeta)|\,\mathrm{d}\zeta\,\mathrm{d}s\mathrm{d}y\;<\;\infty$$

then the integrals below exist and satisfy

$$\int_I\Big[\iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,f(y,s,\zeta)\Big]\,\phi(\zeta)\,\mathrm{d}\zeta\stackrel{\text{a.s.}}{=}\iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,\Big[\int_I f(y,s,\zeta)\,\phi(\zeta)\,\mathrm{d}\zeta\Big].$$

Now we introduce the random variables

$$W_z(l)\stackrel{\text{def}}{=}\iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,(\mathbf{1}_{(x_0,x_0+z]}\otimes l)(y,s),\quad (z,l)\in[0,\infty)\times L^2([0,\infty)),$$

such that the equations of Proposition 3.3(i) can be rewritten as

$$\left.\begin{array}{rcl} U(x,h) & \stackrel{\text{a.s.}}{=} & U(x_0,h)\,+\int_{x_0}^x \partial_1 U(y,h)\,\mathrm{d}y \\[2ex] \partial_1 U(x,h) & \stackrel{\text{a.s.}}{=} & \partial_1 U(x_0,h)\,-\int_{x_0}^x U(y,h')\,\mathrm{d}y\,-\,W_{x-x_0}(h) \end{array}\right\}\tag{3.9}$$

for all $(x,h)\in[x_0,\infty)\times\mathscr{D}$. Note that, for fixed $l\in L^2([0,\infty))\setminus\{0\}$, a version of the process $\{W_z(l)/\|l\|_{L^2([0,\infty))};\,z\geq 0\}$ is a standard Wiener process with respect to the filtration $\mathcal{F}_{(-\infty,x_0+z]}$, $z\geq 0$, and $\{W_z(0)=0;\,z\geq 0\}$ is a version for $l=0$. These versions are used for all processes of type $\{aW_z(l);\,z\geq 0\}$ with fixed $(a,l)\in\mathbb{R}\times L^2([0,\infty))$ in what follows.

So, if the process $\{(U(x,\cdot),\partial_1 U(x,\cdot));\,x\geq x_0\}$ were $\mathcal{F}_{(-\infty,x]}$ - adapted then one could try to establish the Markov property of this process via the martingale problem corresponding to the stochastic differential equation (3.9). But, from (3.6) follows that

$$\mathbb{E}[U(x,h)\,|\,\mathcal{F}_{(-\infty,x]}]\stackrel{\text{a.s.}}{=}\iint_{\mathcal{Q}_+\setminus\mathcal{Q}_+^x} B(\mathrm{d}y,\mathrm{d}s)\,\Big[\int_0^\infty g(y,s\,;x,t)h(t)\,\mathrm{d}t\Big]\;\neq\;U(x,h)$$

for any $(x,h)\in[x_0,\infty)\times\mathscr{D}$ thus $\{(U(x,\cdot),\partial_1 U(x,\cdot));\,x\geq x_0\}$ cannot be $\mathcal{F}_{(-\infty,x]}$ - adapted.

The crucial observation is now that $\{(U(x,\cdot),\partial_1 U(x,\cdot));\,x\geq x_0\}$ is adapted with respect to the enlarged filtration

$$\tilde{\mathcal{F}}_x\stackrel{\text{def}}{=}\mathcal{F}_{(-\infty,x]}\vee\sigma(U(x_0,\cdot)),\quad x\geq x_0.$$

The intuition behind this is of course that a unique solution to (3.9) should be a functional of the initial data $U(x_0,\cdot)$, $\partial_1 U(x_0,\cdot)$ and the driving Wiener process. In our case this can easily be made precise by approximating the derivative h' in (3.9) by a bounded operator and showing that the $\tilde{\mathcal{F}}_x$ - adapted solutions of the approximating systems converge to the unique solution of (3.9). To do so, we would use the connection between (3.9) and (3.4) as explained in Remark 3.2. The wanted convergence can then be verified in a straight forward way using the Green's function given by Remark 3.2(ii).

As a consequence, $\{(U(x,\cdot),\partial_1 U(x,\cdot));\,x\geq x_0\}$ is at least adapted with respect to the filtration $\mathcal{F}_{(-\infty,x]}\vee\sigma(U(x_0,\cdot),\partial_1 U(x_0,\cdot))$ but, by Remark 4.2 on page 23, we know that $\partial_1 U(x_0,\cdot)$ is $\tilde{\mathcal{F}}_{x_0}$ - measurable so that $\{(U(x,\cdot),\partial_1 U(x,\cdot));\,x\geq x_0\}$ is indeed $\tilde{\mathcal{F}}_x$ - adapted. Note that, in the case of other SPDEs, it can easily happen that one has to enlarge $\mathcal{F}_{(-\infty,x]}$ by initial conditions with respect to several partial derivatives.

However, since $\{W_z(l);\,z\geq 0\}$ is not a martingale with respect to the bigger filtration $\tilde{\mathcal{F}}_{x_0+z}$, $z\geq 0$, the equation (3.9) cannot be associated with a martingale problem in a straight forward way, yet. One has to find a semimartingale decomposition of the process $\{W_z(l);\,z\geq 0\}$ with respect to $\tilde{\mathcal{F}}_{x_0+z}$, $z\geq 0$, and this problem is dealt with in the next proposition.

First we state a martingale representation theorem for Brownian sheet: if $\mathcal{L}$ is an $\mathcal{F}_{\mathbb{R}}$ - measurable random variable in $L^2(\Omega)$ then there exists an $\mathcal{F}_{(-\infty,y]}$ - adapted measurable process $(\dot\lambda_{y\,\cdot})_{y\in\mathbb{R}}$ in $L^2(\Omega\times\mathbb{R}\,;L^2([0,\infty))$ such that

$$\mathcal{L} \overset{\text{a.s.}}{=} \mathbb{E}\mathcal{L} + \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,\dot\lambda_{ys}\,.$$

This result which is more or less standard can easily be verified following the idea of the proof of Theorem 1.1.3 in [15].

As a consequence, any $\mathcal{F}_{\mathbb{R}}$ - measurable random variable L taking values in a measurable space E is associated with an additive stochastic kernel $\dot\lambda_{ys}(F)$ indexed by bounded measurable functions $F : E \to \mathbb{R}$ such that

$$F(L) \overset{\text{a.s.}}{=} \mathbb{E}F(L) + \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,\dot\lambda_{ys}(F)\,.$$

In what follows let E be a Souslin locally convex topological vector space and denote by E' its topological dual. Introduce

$$\mathfrak{F}C_b^\infty(D) \overset{\text{def}}{=} \Big\{ F : E \to \mathbb{R} \text{ such that } F(\phi) = f(h_1(\phi),\dots,h_n(\phi)) \text{ for } \\ f \in C_b^\infty(\mathbb{R}^n),\ h_i \in D,\ i = 1,\dots,n,\ n \in \mathbb{N} \Big\}$$

where $D \subseteq E'$ is supposed to separate the points of E. Then we have both $\sigma(L) = \sigma(\{F(L) : F \in \mathfrak{F}C_b^\infty(D)\})$ and $\{F(L) : F \in \mathfrak{F}C_b^\infty(D)\}$ is dense in $L^2(\Omega,\sigma(L),\mathbb{P})$ so that the kernel $\dot\lambda_{ys}(F)$ is fully described by $F \in \mathfrak{F}C_b^\infty(D)$.

Proposition 3.6. *Fix an $\mathcal{F}_{\mathbb{R}}$ - measurable random variable $L : \Omega \to E$ and $l \in L^2([0,\infty))$. Assume that there exists a measurable function*

$$\varrho_l : \Omega \times E \times [x_0,\infty) \to \mathbb{R}$$

such that

- *$\varrho_l(\phi,y)$ is $\mathcal{F}_{(-\infty,y]}$ - measurable for each $\phi \in E$ and $y \ge x_0$;*
- *$\varrho_l(L,y) \in L^1(\Omega)$ for almost every $y \ge x_0$;*
- *the mapping $y \mapsto \varrho_l(L,y)$ is in $L^1([x_0,x])$ almost surely for each $x > x_0$;*
- *for each $F \in \mathfrak{F}C_b^\infty(D)$ and almost every $y \ge x_0$ it holds that*

$$\int_0^\infty \dot\lambda_{ys}(F)\,l(s)\,\mathrm{d}s \overset{\text{a.s.}}{=} \mathbb{E}\left[F(L)\varrho_l(L,y)\,\Big|\,\mathcal{F}_{(-\infty,y]}\right]. \tag{3.10}$$

If

$$\tilde W_z(l) \overset{\text{def}}{=} W_z(l) - \int_{x_0}^{x_0+z} \varrho_l(L,y)\,\mathrm{d}y, \quad z \ge 0,$$

then, for $l \ne 0$, the process $\{\tilde W_z(l)/\|l\|_{L^2([0,\infty))};\ z \ge 0\}$ is a standard Wiener process with respect to the filtration $\mathcal{F}_{(-\infty,x_0+z]} \vee \sigma(L)$, $z \ge 0$. Moreover, if ϱ_l with the above properties exists for $l = l_1, l_2$ then $\varrho_{a_1l_1+a_2l_2}$ exists for each $a_1, a_2 \in \mathbb{R}$ and

$$\tilde W_z(a_1l_1 + a_2l_2) \overset{\text{a.s.}}{=} a_1\tilde W_z(l_1) + a_2\tilde W_z(l_2) \tag{3.11}$$

for each $z \ge 0$.

Remark 3.7. *(i) This proposition is a generalisation of Theorem 12.1 in [21] which deals with the semimartingale decomposition of a Wiener process* $\{W_t;\, t\geq 0\}$ *if its natural filtration* $\mathcal{F}_t^W$, $t\geq 0$, *is enlarged by the information given by an* $\mathcal{F}_\infty^W$*-measurable random variable. In our case, for fixed* $l\in L^2([0,\infty))\setminus\{0\}$, *the Wiener process* $\{W_z(l)/\|l\|_{L^2([0,\infty))};\, z\geq 0\}$ *is already a Wiener process with respect to a filtration larger than its natural filtration, that is* $\mathcal{F}_{(-\infty,x_0+z]}$, $z\geq 0$, *and this larger filtration is enlarged further. But we have both there is a martingale representation theorem with respect to* $\mathcal{F}_{(-\infty,x_0+z]}$, $z\geq 0$, *and* $\{W_z(l);\, z\geq 0\}$ *can be represented as a stochastic integral against the* $\mathcal{F}_{(-\infty,x_0+z]}$*-integrator which is the Brownian sheet. So the idea of proof is the same as in the proof of [21, Th.12.1] so that, in the Proof-Section, we will only deal with the following two elements of the proof: the part where the different type of martingale representation is used and the linearity (3.11).*

(ii) The proposition immediately implies that if ϱ_l *and* ϱ_l' *are two functions satisfying all properties stated in the above proposition then*

$$\mathbb{P}\left(\ \varrho_l(L,y)=\varrho_l'(L,y)\ \textit{for almost every}\ y\geq x_0\ \right)=1$$

because the process $\int_{x_0}^{x_0+z}(\varrho_l(L,y)-\varrho_l'(L,y))\,\mathrm{d}y$, $z\geq 0$, *is a continuous martingale and must vanish therefore.*

In our case, the role of L in the above proposition is played by $U(x_0,\cdot)$ hence, by Proposition 3.3(iii), the corresponding Souslin locally convex space is E_1. We choose $\mathscr{D}$ to be the subset of E_1' separating the points of E_1. The next lemma identifies a class of $l\in L^2([0,\infty))$ such that ϱ_l with the properties stated in Proposition 3.6 exists for $L=U(x_0,\cdot)$.

Lemma 3.8. *For an arbitrary but fixed* $\nu>0$ *set*

$$l_\nu \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi|\cdot|}}*(e^{-\nu\,\cdot})^a.$$

Then l_ν *is a bounded continuous function in* $L^2([0,\infty))$ *satisfying* $l_\nu(0)=0$ *and* ϱ_{l_ν} *with the properties stated in Proposition 3.6 exists for* $L=U(x_0,\cdot)$. *The function* ϱ_{l_ν} *can explicitly be given by*

$$\varrho_{l_\nu}(\phi,y)\;=\;\langle\phi-U(x_0,\cdot)_y\,;\frac{2\sqrt{\nu}e^{-\nu\,\cdot}}{e^{-\sqrt{\nu}(y-x_0)}}\rangle,\quad \phi\in E_1,\,y>x_0,$$

where

$$U(x_0,h)_y \overset{\text{def}}{=} \iint_{\mathcal{Q}_+\setminus\mathcal{Q}_+^y} B(\mathrm{d}y',\mathrm{d}s')\left[\int_0^\infty g(y',s'\,;x_0,t)h(t)\,\mathrm{d}t\right].$$

Furthermore

$$\varrho_{l_\nu}(U(x_0,\cdot),y)\overset{\text{a.s.}}{=}U(y,\sqrt{\nu}e^{-\nu\,\cdot})\,+\,\partial_1 U(y,e^{-\nu\,\cdot}),\quad y>x_0, \tag{3.12}$$

which does not depend on x_0 *anymore.*

The above lemma suggests that $\varrho_{l_\nu}(U(x_0,\cdot),y)$ can be written as a sum of operators acting on $U(y,\cdot)$ and $\partial_1 U(y,\cdot)$ respectively. In what follows, we will reveal the explicit nature of such operators.

For an absolutely continuous function $h:[0,\infty)\to\mathbb{R}$ satisfying $h'\in L^1([0,\infty))\cap L^2([0,\infty))$ define the functions $\mathfrak{A}_1 h$ and $\mathfrak{A}_2 h$ on $[0,\infty)$ by

$$\mathfrak{A}_1 h(t)\;=\int_t^\infty\frac{-h'(t')\,\mathrm{d}t'}{\sqrt{\pi(t'-t)}}\quad\text{and}\quad\mathfrak{A}_2 h(t)\;=\;\Big[\;\frac{sgn(\cdot)}{\sqrt{\pi|\cdot|}}*(h^a)'\;\Big](t)$$

respectively. Since $|\mathfrak{A}_1 h|\leq\Big[\;|\cdot|^{-\frac{1}{2}}*|(h^a)'|\;\Big]$, $\mathfrak{A}_1 h$ and $\mathfrak{A}_2 h$ are well-defined by (2.1).

Proposition 3.9. *(i) If $h \in \mathscr{D}$ then the function $(\mathfrak{A}_2 h)^a$ is a C^∞-function satisfying $\partial_t^k \mathfrak{A}_2 h \in C_{0,\alpha}$ for $0 \le \alpha < k+3/2$, $k = 0,1,2,\dots$, and $\mathfrak{A}_2 h \in H^a_{\mathfrak{w},\beta}$ for all $\beta \ge 0$ if the parameter $\varepsilon > 0$ used to determine the weight function $\mathfrak{w}$ in Proposition 3.3(iv) is less than $1/2$. Also, the function $\mathfrak{A}_1\mathfrak{A}_2 h$ is in $C_{0,\alpha}$ for $0 \le \alpha < 2$.*

(ii) For $h \in \mathscr{D}$, the process

$$\tilde{W}_z(h) \stackrel{\text{def}}{=} W_z(h) - \int_{x_0}^{x_0+z} [U(y, \mathfrak{A}_1\mathfrak{A}_2 h) + \partial_1 U(y, \mathfrak{A}_2 h)]\,\mathrm{d}y, \quad z \ge 0,$$

is well-defined and if $h \neq 0$ then $\{\tilde{W}_z(h)/\|h\|_{L^2([0,\infty))};\, z \ge 0\}$ is a standard Wiener process with respect to the filtration $\tilde{\mathcal{F}}_{x_0+z}$, $z \ge 0$. Moreover, it holds that

$$\tilde{W}_z(a_1h_1 + a_2h_2) \stackrel{\text{a.s.}}{=} a_1\tilde{W}_z(h_1) + a_2\tilde{W}_z(h_2) \tag{3.13}$$

for each $z \ge 0$, $a_1, a_2 \in \mathbb{R}$, $h_1, h_2 \in \mathscr{D}$.

(iii) If $h \in \mathscr{D}$ then

$$\mathfrak{A}_1\mathfrak{A}_2 h = -h' + (-\partial_t^2)^{\frac12}\, h^a \quad \textit{and} \quad \mathfrak{A}_2 h = \sqrt{2}\,(-\partial_t^2)^{\frac14}\, h^a$$

where, for $\beta \in \mathbb{R}$, the fractional Laplacian $(-\partial_t^2)^{\frac{\beta}{2}} f$ of $f \in C_c^\infty(\mathbb{R})$ is defined by its Fourier transform $((-\partial_t^2)^{\frac{\beta}{2}} f)^F = |\cdot|^\beta f^F$.

Remark 3.10. *The operators $\mathfrak{A}_1$ and $\mathfrak{A}_2$ were introduced to simplify the proof of item (ii) of the above proposition. Furthermore, if $h \in \mathscr{D}$ then $(-\partial_t^2)^{\frac12}\, h^a \in C_{0,\alpha}$ for $0 \le \alpha < 2$ by Proposition 3.9(i) because $(-\partial_t^2)^{\frac12}\, h^a = \mathfrak{A}_1\mathfrak{A}_2 h + h'$ and h' has compact support.*

So, in what follows, we will always assume that the parameter $\varepsilon > 0$ used to determine the weight function $\mathfrak{w}$ in Proposition 3.3(iv) is less than $1/2$. Then, recalling (3.9), Proposition 3.9 implies that $\{(U(x,\cdot), \partial_1 U(x,\cdot));\, x \ge x_0\}$ satisfies the equation

$$\left.\begin{aligned} U(x,h) &\stackrel{\text{a.s.}}{=} U(x_0,h) + \int_{x_0}^{x} \partial_1 U(y,h)\,\mathrm{d}y \\ \partial_1 U(x,h) &\stackrel{\text{a.s.}}{=} \partial_1 U(x_0,h) - \int_{x_0}^{x} [\,U(y, (-\partial_t^2)^{\frac12}\, h^a) + \partial_1 U(y, \sqrt{2}(-\partial_t^2)^{\frac14}\, h^a)\,]\,\mathrm{d}y \\ &\qquad - \tilde{W}_{x-x_0}(h) \end{aligned}\right\} \tag{3.14}$$

for all $(x,h) \in [x_0,\infty) \times \mathscr{D}$. Because both, $\{(U(x,\cdot), \partial_1 U(x,\cdot));\, x \ge x_0\}$ is $\tilde{\mathcal{F}}_x$-adapted and $\{\tilde{W}_z(h);\, z \ge 0\}$ is a martingale with respect to $\tilde{\mathcal{F}}_{x_0+z}$, this stochastic differential equation can eventually be associated with a martingale problem. But before we do so let us point out that (3.14), when seen as a family of stochastic differential equations indexed by x_0, gives raise to a new SPDE in $\mathcal{Q}_+$.

Theorem 3.11. *The unique weak solution U to (3.1) given by the continuous version of the right-hand side of (3.2) on page 4 satisfies*

$$\partial_x^2 U + [\,(-\partial_t^2)^{1/2} + \sqrt{2}\,\partial_x(-\partial_t^2)^{1/4}\,]\,U^a = \partial_x\partial_t\tilde{B}$$

in the sense of

$$U\left(\partial_x^2 f + [\,(-\partial_t^2)^{1/2} - \sqrt{2}\,\partial_x(-\partial_t^2)^{1/4}\,]\,f^a\right) = \partial_x\partial_t\tilde{B}(f) \quad \textit{for all } f \in C_c^\infty(\mathcal{Q}_+) \quad a.s.$$

where U^a, f^a stand for the extensions of $U(x,t), f(x,t)$ to $(x,t) \in \mathbb{R}^2$ which are antisymmetric in t and $\tilde{B} = \{\tilde{B}_{xt};\, x \in \mathbb{R}, t \ge 0\}$ is a Brownian sheet on $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 3.12. *In this theorem, U is considered a regular generalized function on $C_c^\infty(\mathcal{Q}_+)$, that is,*

$$U(f) \;=\; \int_{\mathbb{R}}\int_0^\infty U(x,t)\,f(x,t)\,\mathrm{d}t\mathrm{d}x \quad \textit{for} \quad f\in C_c^\infty(\mathcal{Q}_+).$$

However, the proof of Proposition 3.9(i) makes clear that, for fixed $x\in\mathbb{R}$ and $f\in C_c^\infty(\mathcal{Q}_+)$, the long-time behaviour of $(-\partial_t^2)^{1/4}f^a(x,\cdot)$ is not better than $\mathcal{O}(t^{-3/2})$, $t\to\infty$, in general while, by Proposition 3.3(iii), $U(x,\cdot)$ is only in $(C_{0,\alpha})'$ for $\alpha>3/2$. So the meaning of $U(\,\partial_x(-\partial_t^2)^{1/4}f^a)$ is based on an extension of the regular generalized function U which will be explained in the proof of the theorem.

Coming back to the martingale problem associated with (3.14), choose $D=\mathscr{D}\times\mathscr{D}$, which is a subset of the topological dual of $E=E_1\times E_2$, and denote by $\mathbf{A}$ the subset of $C_b(E)\times C(E)$ whose elements (F,G) are given by

$$F\in\mathfrak{F}C_b^\infty(D) \quad \text{such that} \quad F(\phi_1,\phi_2) \;=\; f(\langle\phi_1\,;h_1\rangle,\langle\phi_2\,;h_2\rangle,\dots,\langle\phi_1\,;h_{2n-1}\rangle,\langle\phi_2\,;h_{2n}\rangle)$$

for some $f\in C_b^\infty(\mathbb{R}^{2n})$, $h_i\in\mathscr{D}$, $i=1,2,\dots,2n$, and

$$\begin{aligned} G(\phi_1,\phi_2) \;&=\; \sum_{i\ odd}\partial_i f(\dots,\langle\phi_1\,;h_i\rangle,\dots)\,\langle\phi_2\,;h_i\rangle \\ &-\; \sum_{i\ even}\partial_i f(\dots,\langle\phi_2\,;h_i\rangle,\dots)\,[\,\langle\phi_1\,;(-\partial_t^2)^{\frac12}\,h_i^a\rangle+\langle\phi_2\,;\sqrt{2}(-\partial_t^2)^{\frac14}\,h_i^a\rangle\,] \\ &+\; \frac12\sum_{i,j\ even}\partial_i\partial_j f(\dots,\langle\phi_2\,;\begin{smallmatrix}h_i\\ or\\ h_j\end{smallmatrix}\rangle,\dots,\langle\phi_2\,;\begin{smallmatrix}h_j\\ or\\ h_i\end{smallmatrix}\rangle,\dots)\,\langle h_i\,;h_j\rangle. \end{aligned}$$

This definition of the subset $\mathbf{A}$ of course requires $(-\partial_t^2)^{\frac12}\,h^a\in E_1'$ and $(-\partial_t^2)^{\frac14}\,h^a\in E_2'$ for $h\in\mathscr{D}$ which follows from Proposition 3.9(i) and Remark 3.10.

Then, according to [8, Chapter 3], by a *solution of the martingale problem for* $\mathbf{A}$ *with respect to* $\mathscr{F}_z$ one would mean an $\mathscr{F}_z$ - progressively measurable process $R=(R_z)_{z\ge0}$ on a filtered probability space $(\Omega,\mathscr{F},\mathbb{P})$ taking values in E such that

$$F(R_z)-F(R_0)\;-\int_0^z G(R_y)\,\mathrm{d}y,\quad z\ge0,$$

is a martingale with respect to the filtration $\mathscr{F}_z$, $z\ge0$, for all $(F,G)\in\mathbf{A}$. When an initial condition μ is specified, it is also said that the process R is a *solution of the martingale problem for* $(\mathbf{A},\mu)$.

Next we check whether $\{(U(x_0+z,\cdot),\partial_1U(x_0+z,\cdot));\,z\ge0\}$ is a solution of the martingale problem for our set $\mathbf{A}$ with respect to $\tilde{\mathcal{F}}_{x_0+z}$.

First, the process $\{(U(x_0+z,\cdot),\partial_1U(x_0+z,\cdot));\,z\ge0\}$ has an $\tilde{\mathcal{F}}_{x_0+z}$ - progressively measurable version because it is $\tilde{\mathcal{F}}_{x_0+z}$ - adapted and, by Proposition 3.3(v), has an $\mathcal{F}\otimes\mathscr{B}([x_0,\infty))$ - measurable version taking values in the space $E=E_1\times E_2$. This can be verified the same way the analogous statement for real-valued adapted measurable processes was verified in [4]. Notice that, by construction, the filtration $\tilde{\mathcal{F}}_x$ inherits right-continuity from the filtration $\mathcal{F}_{(-\infty,x]}$ defined on page 7.

Second, knowing that $\{(U(x,\cdot),\partial_1U(x,\cdot));\,x\ge x_0\}$ satisfies (3.8) and (3.14), an easy application of Itô's formula to $R_z=(U(x_0+z,\cdot),\partial_1U(x_0+z,\cdot))$ yields that

$$F(R_z)-F(R_0)\;-\int_0^z G(R_y)\,\mathrm{d}y,\quad z\ge0,$$

is indeed a martingale with respect to $\tilde{\mathcal{F}}_{x_0+z}$ for all $(F,G)\in\mathbf{A}$.

Now we hope that the well-posedness-condition

$$\left.\begin{array}{l}\text{for each probability measure } \mu \text{ on } (E,\mathscr{B}(E)),\text{ any two solutions } R,R' \\ \text{of the martingale problem for } (\mathbf{A},\mu) \text{ with respect to } \mathscr{F}_z,\ \mathscr{F}'_z,\text{ have the} \\ \text{same one-dimensional distributions, that is, for each } z>0, \\ \qquad \mathbb{P}(\{R_z\in\Gamma\}) \,=\, \mathbb{P}'(\{R'_z\in\Gamma\}),\quad \Gamma\in\mathscr{B}(E),\end{array}\right\} \tag{wp}$$

as stated in Theorem 4.2 in Chapter 4 of [8] is enough to ensure that a solution $(R_z)_{z\geq 0}$ of the martingale problem is strong Markov in the sense of:

Definition 3.13. *Let E be a separable metric space and let μ be a probability measure on $(E,\mathscr{B}(E))$. An $\mathscr{F}_z$ - progressively measurable process $(R_z)_{z\geq 0}$ on a filtered probability space $(\Omega,\mathscr{F},\mathbb{P})$ taking values in E is said to be a* strong Markov process with initial condition *μ if*

(i) $\mathbb{P}(\{R_0\in\Gamma\}) \,=\, \mu(\Gamma)$ *for all* $\Gamma\in\mathscr{B}(E)$*;*

(ii) for any $\mathscr{F}_z$ - stopping time $\xi\geq 0$, $y\geq 0$ and $\Gamma\in\mathscr{B}(E)$,

$$\mathbb{P}[\{R_{\xi+y}\in\Gamma\}\,|\,\mathscr{F}_\xi\,] \overset{\text{a.s.}}{=} \mathbb{P}[\{R_{\xi+y}\in\Gamma\}\,|\,\sigma(R_\xi)] \quad \textit{on } \{\xi<\infty\}.$$

To show the strong Markov property of $R_z=(U(x_0+z,\cdot),\partial_1 U(x_0+z,\cdot))$ with initial condition $\mathbb{P}\circ(U(x_0,\cdot),\partial_1 U(x_0,\cdot))^{-1}$ we want to apply Theorem 4.2(b) in [8]. But the conclusion of this theorem is stated under the extra conditions that $\mathbf{A}\subseteq C_b(E)\times C_b(E)$ and that $(R_z)_{z\geq 0}$ has a right-continuous version taking values in E.

Remark 3.14. *(i) Our set $\mathbf{A}$ defining the martingale problem is not a subset of $C_b(E)\times C_b(E)$ but of $C_b(E)\times C(E)$ only. However, in the general situation of [8, Thm.4.2(b)], the boundedness of F and G is the natural condition to ensure that $|F(R_z)-F(R_0)\,-\int_0^z G(R_y)\,\mathrm{d}y|$ has finite expectation for each $z\geq 0$. In our specific situation, if $R_z=(U(x_0+z,\cdot),\partial_1 U(x_0+z,\cdot))$ then*

$$\mathbb{E}\left|F(R_z)-F(R_0)\,-\int_0^z G(R_y)\,\mathrm{d}y\right| \,<\, \infty$$

for given $(F,G)\in\mathbf{A}$ because of (3.8) and $F\in\mathfrak{F}C_b^\infty(D)$. It turns out that part (b) of Theorem 4.2 in [8] remains valid when adding a condition of type (3.8) to the definition of the martingale problem–see Definition 3.15(iii) and Remark 3.16(ii) below.

(ii) Taking another look at the proof of Theorem 4.2(b) in [8] reveals that the right-continuous version of the solution is only needed for

$$F(R_z)-F(R_0)\,-\int_0^z G(R_y)\,\mathrm{d}y,\quad z\geq 0,$$

to be a right-continuous martingale when $(F,G)\in\mathbf{A}$ in order to be able to apply Doob's optional sampling theorem. So, it is already enough to require right-continuity of $z\mapsto F(U(x_0+z,\cdot),\partial_1 U(x_0+z,\cdot))$ for all $F\in\mathfrak{F}C_b^\infty(D)$ to make the theorem work in our case.

It is easy to realize that there is a version of the process $\{(U(x,\cdot),\partial_1 U(x,\cdot));\,x\geq x_0\}$ such that $z\mapsto F(U(x_0+z,\cdot),\partial_1 U(x_0+z,\cdot))$ is continuous for all $F\in\mathfrak{F}C_b^\infty(D)$. First, it is well-known (see [20] for example) that the process $\{W_z(l);\,(z,l)\in[0,\infty)\times L^2([0,\infty))\}$ defined on page 8 has a version such that $\{W_z(\cdot);\,z\geq 0\}$ is a continuous $\mathscr{D}'$ - valued process. Second, using the above $\mathscr{D}'$-valued version of $\{W_z(\cdot);\, z \geq 0\}$ and the $\mathcal{F} \otimes \mathscr{B}([x_0,\infty))$-measurable version of $\{(U(x,\cdot),\partial_1 U(x,\cdot));\, x \geq x_0\}$ as stated in Proposition 3.3(v), one can construct from (3.9) a version of $\{(U(x,\cdot),\partial_1 U(x,\cdot));\, x \geq x_0\}$ such that

$$\mathbb{P}\left(\, x \mapsto U(x,h) \;\&\; x \mapsto \partial_1 U(x,h) \text{ are continuous for all } h \in \mathscr{D} \right) \,=\, 1.$$

So, by Thm.4.2(b) in [8] and Remark 3.14, the strong Markov property of our process $\{(U(x_0+z,\cdot),\partial_1 U(x_0+z,\cdot));\, z \geq 0\}$ becomes a direct implication of the well-posedness-condition (wp). But, for showing the uniqueness wanted in (wp), we need to work with a more restrictive martingale problem than Ethier/Kurtz in [8]. Recall the set $\mathbf{A} \subseteq C_b(E) \times C(E)$ introduced on page 12.

Definition 3.15. *An $\mathscr{F}_z$-progressively measurable process $\{(u_z,v_z);\, z \geq 0\}$ on a filtered probability space $(\Omega,\mathscr{F},\mathbb{P})$ taking values in $E = E_1 \times E_2$ is called a* solution of the martingale problem for $\mathbf{A}$ with respect to $\mathscr{F}_z$ *iff*

(i) the mappings $z \mapsto u_z(h)$ and $z \mapsto v_z(h)$ are continuous for all $h \in \mathscr{D}$;

(ii) the map $(z,t) \mapsto u_z(t)$ is continuous on the closure of $\mathcal{Q}_+^0$;

(iii) $\mathbb{E}\int_0^z (\|u_y\|_{E_1} + \|v_y\|_{E_2})\,\mathrm{d}y \,<\, \infty$ for all $z > 0$;

(iv) $\{F(u_z,v_z) - F(u_0,v_0) - \int_0^z G(u_y,v_y)\,\mathrm{d}y;\, z \geq 0\}$ is a martingale with respect to the filtration $\mathscr{F}_z$, $z \geq 0$, for all $(F,G) \in \mathbf{A}$.

Remark 3.16. *(i) We also use the phrase 'solution of the martingale problem for $(\mathbf{A},\mu)$' when a specific initial condition μ is emphasised as in (wp).*

(ii) We claim that Thm.4.2(b) in [8] remains valid with respect to our more restrictive definition of the martingale problem when being applied to show the strong Markov property of $\{(U(x_0+z,\cdot),\, \partial_1 U(x_0+z,\cdot));\, z \geq 0\}$. A quick glance at the proof of this theorem shows that one only has to pay attention to the property (iii) of our Definition 3.15 and this will be done in the next item of this remark.

(iii) Adapting the proof of Thm.4.2(b) in [8] to our setup, fix a finite $\tilde{\mathcal{F}}_{x_0+z}$-stopping time ξ, choose $\Xi \in \tilde{\mathcal{F}}_{x_0+\xi}$ such that $\mathbb{P}(\Xi) > 0$ and introduce

$$u_y \overset{\text{def}}{=} U(x_0+\xi+y,\cdot), \quad v_y \overset{\text{def}}{=} \partial_1 U(x_0+\xi+y,\cdot) \quad \text{for all } y \geq 0$$

and

$$\mathbb{P}_1(\Gamma) \overset{\text{def}}{=} \frac{\mathbb{E}\mathbf{1}_\Xi \mathbb{P}[\Gamma|\tilde{\mathcal{F}}_{x_0+\xi}]}{\mathbb{P}(\Xi)}, \quad \mathbb{P}_2(\Gamma) \overset{\text{def}}{=} \frac{\mathbb{E}\mathbf{1}_\Xi \mathbb{P}[\Gamma|\sigma(U(x_0+\xi,\cdot),\partial_1 U(x_0+\xi,\cdot))]}{\mathbb{P}(\Xi)}$$

for all $\Gamma \in \mathcal{F}$. The task is to show the property in Definition 3.15(iii) if $\mathbb{E}$ is replaced by the expectation operators $\mathbb{E}_1$ and $\mathbb{E}_2$ given by the measures $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively. But, for fixed $z > 0$, we obtain that

$$\mathbb{E}_i \int_0^z (\|u_y\|_{E_1} + \|v_y\|_{E_2})\,\mathrm{d}y \,\leq\, \frac{1}{\mathbb{P}(\Xi)}\,\mathbb{E}\int_{x_0}^{x_0+\xi+z} (\|U(y,\cdot)\|_{E_1} + \|\partial_1 U(y,\cdot)\|_{E_2})\,\mathrm{d}y$$

for $i = 1,2$ where, by Proposition 3.3(v), the last term is finite if the stopping time ξ is bounded. And it is sufficient to check the strong Markov property for bounded stopping times only–we refer to Problem 2.6.9 in [11] for example.

After this preparation, the key part of the proof of the below theorem consists in verifying the well-posedness-condition (wp) on page 13 for our martingale problem. Recall the spaces E_1, E_2 defined in Proposition 3.3 and assume that the parameter $\varepsilon > 0$ used to define E_2 is less than $1/2$.

Theorem 3.17. *The $\tilde{\mathcal{F}}_{x_0+z}$-progressively measurable version of the process $\{(U(x_0+z,\cdot),\, \partial_1 U(x_0+z,\cdot));\, z\ge 0\}$ taking values in $E_1\times E_2$ is a stationary homogeneous strong Markov process which is associated with the martingale problem of Definition 3.15 via a pathwise unique stochastic differential equation in $E_1\times E_2$ which can be formally written as*

$$\begin{aligned} \mathrm{d}u_z &= v_z\,\mathrm{d}z \\ \mathrm{d}v_z &= -\Big[\,(-\partial_t^2)^{\frac12}u_z^a+\sqrt{2}\,(-\partial_t^2)^{\frac14}v_z^a\,\Big]\,\mathrm{d}z\,-\,\mathrm{d}\mathscr{W}_z \end{aligned}$$

where $\{\mathscr{W}_z;\, z\ge 0\}$ stands for a $\mathscr{D}'$-valued Wiener process.

Corollary 3.18. *(i) The unique weak solution $U(x,t)$ to (3.1), when seen as a process $U(x,\cdot)$ indexed by $x\ge x_0$ taking values in E_1, satisfies*

$$\mathbb{P}\Big[\{U(\xi+y,\cdot)\in\Gamma\}\Big|\,\tilde{\mathcal{F}}_\xi\Big]\overset{\text{a.s.}}{=}\mathbb{P}\Big[\{U(\xi+y,\cdot)\in\Gamma\}\Big|\,\sigma(U(\xi,\cdot),\partial_1U(\xi,\cdot))\Big]$$

for any finite $\tilde{\mathcal{F}}_x$-stopping time $\xi\ge x_0$ and any $y\ge 0,\,\Gamma\in\mathscr{B}(E_1)$. This remains valid when the filtration $\tilde{\mathcal{F}}_x$, $x\ge x_0$, is replaced by the filtration generated by the process $\{(U(x,\cdot),\partial_1U(x,\cdot));\, x\ge x_0\}$ augmented by the $\mathbb{P}$-null sets in $\mathcal{F}$.

(ii) For any $x\in\mathbb{R}$, the σ-algebras $\sigma\{U(y,t): y<x, t>0\}$ and $\sigma\{U(y,t): y>x, t>0\}$ are conditionally independent given $\sigma(U(x,\cdot),\partial_1U(x,\cdot))$ where

$$\sigma(U(x,\cdot),\partial_1U(x,\cdot))\ \subsetneq\ \textit{germ}\Big(\,\{x\}\times(0,\infty)\,\Big)\ \overset{\textit{def}}{=}\bigcap_{\substack{O\ \textit{open in}\ \mathcal{Q}_+\\ \{x\}\times(0,\infty)\subseteq O}}\sigma\{U(y,t):(y,t)\in O\}.$$

4 Proofs of:

Proof of ***Proposition 3.3***. For (i) fix $(x,h)\in[x_0,\infty)\times\mathscr{D}$ and notice that

$$\int_{x_0}^{x}\partial_1U(y,h)\,\mathrm{d}y\;=\;\int_{\mathbb{R}}\Big(\iint_{\mathcal{Q}_+}B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_0^\infty\partial_3g(y',s';y,t)h(t)\,\mathrm{d}t\Big]\Big)\,\mathbf{1}_{(x_0,x]}(y)\,\mathrm{d}y$$

by the definition of $\partial_1U(y,h)$. The integral on the right-hand side a.s. equals

$$\iint_{\mathcal{Q}_+}B(\mathrm{d}y',\mathrm{d}s')\int_{\mathbb{R}}\Big[\int_0^\infty\partial_3g(y',s';y,t)h(t)\mathrm{d}t\Big]\;\mathbf{1}_{(x_0,x]}(y)\,\mathrm{d}y \tag{4.1}$$

by applying Lemma 3.5 with respect to the bounded function $\phi=\mathbf{1}_{(x_0,x]}$. Here the condition of Lemma 3.5(ii) is easily satisfied because the covariance of $\partial_1U(y,h)$ given in Proposition 3.3(ii) does not depend on y. Then the equation for $U(x,h)$ follows from (4.1) by Fubini's theorem with respect to $\mathrm{d}t\mathrm{d}y$ which can be applied for every $(y',s')\in\mathcal{Q}_+$ because

$$\int_{x_0}^{x}\int_0^\infty|\partial_3g(y',s';y,t)h(t)|\,\mathrm{d}t\mathrm{d}y\;<\;\infty.$$

In order to show the equation for $\partial_1 U(x,h)$ we first calculate:

$$
\begin{aligned}
& \partial_1 U(x,h) - \partial_1 U(x_0,h) \\
= \quad & \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_0^\infty \partial_3 g(y',s'\,;x,t)h(t)\,\mathrm{d}t\Big] - \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_0^\infty \partial_3 g(y',s'\,;x_0,t)h(t)\,\mathrm{d}t\Big] \\
\stackrel{\text{a.s.}}{=} \quad & \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_{s'}^\infty \Big[\int_{x_0}^x \partial_3^2 g(y',s'\,;y,t)\,\mathrm{d}y\Big]\,h(t)\,\mathrm{d}t\Big] \\
= \quad & \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_{s'}^\infty \Big[\int_{x_0}^x \partial_4 g(y',s'\,;y,t)\,\mathrm{d}y\Big]\,h(t)\,\mathrm{d}t\Big] \\
= \quad & \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_{s'}^\infty \partial_t\Big[\int_{x_0}^x g(y',s'\,;y,t)\,\mathrm{d}y\Big]\,h(t)\,\mathrm{d}t\Big] \\
\stackrel{\text{a.s.}}{=} - & \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_{s'}^\infty \Big[\int_{x_0}^x g(y',s';y,t)\,\mathrm{d}y\Big]\,h'(t)\,\mathrm{d}t\Big] - \iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\lim_{t\downarrow s'}\Big[\int_{x_0}^x g(y',s';y,t)\,\mathrm{d}y\Big]\,h(s').
\end{aligned}
$$

Again applying Fubini's Theorem and our stochastic Fubini Lemma 3.5, one sees that

$$
\iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_{s'}^\infty \Big[\int_{x_0}^x g(y',s'\,;y,t)\,\mathrm{d}y\Big]\,h'(t)\,\mathrm{d}t\Big] \;=\; \int_{x_0}^x U(y,h')\,\mathrm{d}y
$$

hence the equation for $\partial_1 U(x,h)$ follows since

$$
\iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\lim_{t\downarrow s'}\Big[\int_{x_0}^x g(y',s'\,;y,t)\,\mathrm{d}y\Big]\,h(s') \;\stackrel{\text{a.s.}}{=}\; \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,(\mathbf{1}_{(x_0,x]}\otimes h)(y,s)
$$

holds true by the strong continuity of the heat semigroup in $L^2(\mathbb{R})$.

For proving item (ii) of the proposition fix $x \geq x_0$ and $h_1,h_2 \in \mathscr{D}$. Then

$$
\begin{aligned}
& \mathbb{E}\,U(x,h_1)U(x,h_2) \\
= \quad & \mathbb{E}\iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,\Big[\int_0^\infty g(y,s\,;x,t)h_1(t)\,\mathrm{d}t\Big]\iint_{\mathcal{Q}_+} B(\mathrm{d}y',\mathrm{d}s')\,\Big[\int_0^\infty g(y',s'\,;x,t')h_2(t')\,\mathrm{d}t'\Big] \\
= \quad & \int_{\mathbb{R}}\int_0^\infty \Big[\int_0^\infty g(y,s\,;x,t)h_1(t)\,\mathrm{d}t\Big]\,\Big[\int_0^\infty g(y,s\,;x,t')h_2(t')\,\mathrm{d}t'\Big]\,\mathrm{d}s\mathrm{d}y \\
= \quad & \int_0^\infty h_1(t)\int_0^\infty \Big[\int_{\mathbb{R}}\int_0^\infty g(y,s\,;x,t)g(y,s\,;x,t')\,\mathrm{d}s\mathrm{d}y\Big]\,h_2(t')\,\mathrm{d}t'\,\mathrm{d}t \\
= \quad & \langle\, h_1\,;\frac{-\sqrt{|\cdot|}}{\sqrt{4\pi}} * h_2^a\,\rangle
\end{aligned}
$$

because

$$
\frac{1}{\sqrt{4\pi}}\,(\sqrt{t+t'}-\sqrt{|t-t'|}) \;=\; \int_{\mathbb{R}}\int_0^\infty g(y,s\,;x,t)g(y,s\,;x,t')\,\mathrm{d}s\mathrm{d}y.
$$

We only mention that, by the well-known properties of the Green's function g, the integrability conditions needed for the above calculation are satisfied in the case of test functions h_1,h_2 with compact support.

The covariance of the process $\{\partial_1 U(x,h);\, h\in\mathscr{D}\}$ can be verified by a similar calculation since

$$
\frac{1}{2\sqrt{4\pi}}\,(\frac{1}{\sqrt{|t-t'|}}-\frac{1}{\sqrt{t+t'}}) \;=\; \int_{\mathbb{R}}\int_0^\infty \partial_3 g(y,s\,;x,t)\partial_3 g(y,s\,;x,t')\,\mathrm{d}s\mathrm{d}y
$$

and

$$\int_{\mathbb{R}}\int_0^\infty g(y,s\,;x,t)\partial_3 g(y,s\,;x,t')\,\mathrm{d}s\mathrm{d}y \;=\; 0$$

for all $t, t' \geq 0$ gives the independence of the two processes.

We now show part (iii) of the proposition. Fix $x \geq x_0$ and $\alpha > 3/2$. If $h \in C_{0,\alpha}$ then

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^\infty\int_0^\infty g(y,s\,;x,t)^2\,|h(t)|\,\mathrm{d}t\,\mathrm{d}s\mathrm{d}y \\
\leq\quad & \int_{\mathbb{R}}\int_0^\infty\int_s^\infty \frac{1}{4\pi(t-s)}\exp\{\frac{-(x-y)^2}{2(t-s)}\}(1+t)^{-\alpha}\,\mathrm{d}t\,\mathrm{d}s\mathrm{d}y\cdot\|h\|_{0,\alpha} \\
=\quad & \frac{\|h\|_{0,\alpha}}{2\sqrt{2\pi}}\int_0^\infty\int_s^\infty (t-s)^{-\frac{1}{2}}(1+t)^{-\alpha}\,\mathrm{d}t\mathrm{d}s \;=\; \frac{\|h\|_{0,\alpha}}{\sqrt{2\pi}}\int_0^\infty \sqrt{t}\,(1+t)^{-\alpha}\,\mathrm{d}t \;<\;\infty
\end{aligned}$$

because $\alpha > 3/2$. Hence, by Lemma 3.5, the process $\{U(x,h);\, h\in\mathscr{D}\}$ can be extended to $h \in C_{0,\alpha}$ and

$$U(x,h) \;=\; \int_0^\infty U(x,t)\,h(t)\,\mathrm{d}t \quad \text{a.s. for all } h\in C_{0,\alpha}. \tag{4.2}$$

But the above calculation also shows that

$$\mathbb{E}\int_0^\infty U(x,t)^2\;(1+t)^{-\alpha}\mathrm{d}t \;<\;\infty \tag{4.3}$$

thus

$$\int_0^\infty |U(x,t)|\;(1+t)^{-\alpha}\mathrm{d}t \;<\;\infty \quad \text{a.s.}$$

which implies

$$|\int_0^\infty U(x,t)\,h(t)\,\mathrm{d}t| \;\leq\; \|h\|_{0,\alpha}\int_0^\infty |U(x,t)|\;(1+t)^{-\alpha}\mathrm{d}t, \quad \forall\, h\in C_{0,\alpha},\ \text{a.s.} \tag{4.4}$$

As a consequence, there is a version of the process $\{U(x,h);\, h\in\mathscr{D}\}$ taking values in $(C_{0,\alpha})'$. Since U given by (3.2) is continuous in $(x,t)\in\mathcal{Q}_+$ such that $\lim_{t\downarrow 0}U(x,t)=0$ for all $x\geq x_0$, this version takes values in E_1 even.

Next we prove item (iv) of Proposition 3.3. Note that the Sobolev space H_β can be identified with $(Id-\partial_t^2)^{-\beta/2}L^2(\mathbb{R})$ in the sense of generalized functions. Fix $\beta > 1/4$ and define the operator $Kh = (Id-\partial_t^2)^{-\beta/2}(\mathfrak{w}^{-1}h^a)$, $h\in\mathscr{D}$. Of course, K^{-1} exists and it holds that $K^{-1}h = \mathfrak{w}[(Id-\partial_t^2)^{\beta/2}h^a]$, $h\in\mathscr{D}$. Hence, if v is a linear form with domain of definition which contains $\{Ke_i\}_{i=1}^\infty$ where $\{e_i\}_{i=1}^\infty\subseteq\mathscr{D}$ is an orthonormal basis of $L^2([0,\infty))$ then

$$\sum_{i=1}^\infty |v(Ke_i)|^2 \;<\;\infty \quad \text{gives} \quad v \;=\; \sum_{i=1}^\infty v(Ke_i)K^{-1}e_i \;\in\; (H^a_{\mathfrak{w},\beta})'.$$

Fix $x\geq x_0$ and choose an orthonormal basis $\{e_i\}_{i=1}^\infty\subseteq\mathscr{D}$ of $L^2([0,\infty))$. The above implies that if the linear form $\partial_1 U(x,\cdot)$ can be extended to the linear hull of $\mathscr{D}\cup\{Ke_i\}_{i=1}^\infty$ and if

$$\mathbb{E}\sum_{i=1}^\infty |\partial_1 U(x,Ke_i)|^2 \;<\;\infty \tag{4.5}$$

then

$$\omega \;\mapsto\; \mathbf{1}_{\{\sum_{i=1}^\infty |\partial_1 U(x,Ke_i)|^2<\infty\}}(\omega)\sum_{i=1}^\infty \partial_1 U(\omega,x,Ke_i)K^{-1}e_i \tag{4.6}$$

defines a version of $\partial_1 U(x,\cdot)$ taking values in $(H^a_{\mathfrak{w},\beta})'$.

In order to show (4.5) recall from Remark 3.4(ii) that $C_2 h = const(-\partial_t^2)^{-\frac{1}{4}}\, h^a$. First, applying Proposition 3.3(ii), we have that[1]

$$\begin{aligned}
\mathbb{E}\,|\partial_1 U(x,Ke_i)|^2 &= \langle (Id-\partial_t^2)^{-\beta/2}(\mathfrak{w}^{-1}e_i^a)\,;C_2(Id-\partial_t^2)^{-\beta/2}(\mathfrak{w}^{-1}e_i^a)\rangle \\
&= \frac{const}{2}\,\|(-\partial_t^2)^{-\frac{1}{8}}(Id-\partial_t^2)^{-\beta/2}(\mathfrak{w}^{-1}e_i^a)\|^2_{L^2(\mathbb{R})} \\
&= \frac{const}{2\cdot 2\pi}\,\Big\|\,|\cdot|^{-1/4}(1+|\cdot|^2)^{-\beta/2}(\mathfrak{w}^{-1}e_i^a)^F\,\Big\|^2_{L^2(\mathbb{R})} \\
&\le \frac{const}{2\cdot 2\pi}\,\Big\|\,|\cdot|^{-1/4}(\mathfrak{w}^{-1}e_i^a)^F\,\Big\|^2_{L^2(\mathbb{R})} \le \langle\,|\mathfrak{w}^{-1}e_i|\,;C_2|\mathfrak{w}^{-1}e_i|\,\rangle < \infty
\end{aligned}$$

for each single i since $|\mathfrak{w}^{-1}e_i| \le |e_i|$ and $e_i \in \mathscr{D}$. As a consequence, $\partial_1 U(x,\cdot)$ can be extended to the linear hull of $\mathscr{D}\cup\{Ke_i\}_{i=1}^\infty$ and the left-hand side of (4.5) makes sense.

Taking into account the calculations of the last paragraph, condition (4.5) becomes equivalent to

$$\sum_{i=1}^\infty \Big\|\,|\cdot|^{-1/4}(1+|\cdot|^2)^{-\beta/2}(\mathfrak{w}^{-1}e_i^a)^F\,\Big\|^2_{L^2(\mathbb{R})} < \infty$$

where

$$(\mathfrak{w}^{-1}e_i^a)^F = \mathbf{i}\int_{\mathbb{R}} \sin(-\tau t)\,\mathfrak{w}^{-1}(t)e_i^a(t)\,\mathrm{d}t = -2\mathbf{i}\int_0^\infty \sin(\tau t)\,\mathfrak{w}^{-1}(t)e_i(t)\,\mathrm{d}t.$$

Thus

$$\begin{aligned}
&\sum_{i=1}^\infty \Big\|\,|\cdot|^{-1/4}(1+|\cdot|^2)^{-\beta/2}(\mathfrak{w}^{-1}e_i^a)^F\,\Big\|^2_{L^2(\mathbb{R})} \\
&= \sum_{i=1}^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{|\tau|}\,(1+|\tau|^2)^\beta}\int_0^\infty \sin(\tau t)\,\mathfrak{w}^{-1}(t)e_i(t)\,\mathrm{d}t\int_0^\infty \sin(\tau t')\,\mathfrak{w}^{-1}(t')e_i(t')\,\mathrm{d}t'\,\mathrm{d}\tau \\
&= \int_0^\infty \sum_{i=1}^\infty \Big\{\int_0^\infty \Big[\int_{\mathbb{R}} \frac{\sin(\tau t)\,\mathfrak{w}^{-1}(t)\sin(\tau t')\,\mathfrak{w}^{-1}(t')}{\sqrt{|\tau|}\,(1+|\tau|^2)^\beta}\,\mathrm{d}\tau\Big]\,e_i(t')\,\mathrm{d}t'\Big\}\,e_i(t)\,\mathrm{d}t \\
&= \int_0^\infty\int_{\mathbb{R}} \frac{\sin(\tau t)^2\,\mathfrak{w}^{-1}(t)^2}{\sqrt{|\tau|}\,(1+|\tau|^2)^\beta}\,\mathrm{d}\tau\mathrm{d}t \le \int_{\mathbb{R}} \frac{\mathrm{d}\tau}{\sqrt{|\tau|}\,(1+|\tau|^2)^\beta}\int_0^\infty \frac{\mathrm{d}t}{(1+|t|^{\frac{1}{2}+\varepsilon})^2} < \infty
\end{aligned}$$

since $\beta > 1/4$.

We finally justify item (v) of Proposition 3.3. First, the stationarity follows from item (ii) because the covariances do not depend on $x \ge x_0$. Second, $(\omega,x,t)\mapsto U(\omega,x,t)$ is clearly $\mathcal{F}\otimes\mathscr{B}([x_0,\infty))\otimes\mathscr{B}([0,\infty))$ - measurable leading to an $\mathcal{F}\otimes\mathscr{B}([x_0,\infty))$ - measurable version of $x\mapsto U(x,\cdot)\in E_1$ and the version of $(\omega,x)\mapsto \partial_1 U(\omega,x,\cdot)$ given by (4.6) is also $\mathcal{F}\otimes\mathscr{B}([x_0,\infty))$ - measurable. Third, using stationarity, (3.8) already follows from

$$\mathbb{E}\|U(x,\cdot)\|^2_{E_1} < \infty \quad\text{and}\quad \mathbb{E}\|\partial_1 U(x,\cdot)\|^2_{E_2} < \infty$$

for an arbitrary but fixed $x \ge x_0$ where the first expectation is finite because of (4.3), (4.4) and the second expectation is equal to the left-hand side of (4.5) which was shown to be finite above. Finally, the existence of a continuous version on the closure of $\mathcal{Q}_+^0$ of the solution $(x,t)\mapsto U(x,t)$ as given by (3.2) is standard–see [20]. □

[1] See Section 2 for $\|l^a\|^2_{L^2(\mathbb{R})} = 2\langle l\,;l\rangle$.

Proof of ***Proposition 3.6***. Recalling Remark 3.7(i), we only deal with the following two issues and refer to Theorem 12.1 in [21] otherwise.

First, after several steps, one has to identify

$$\int_{x_0}^{x_0+\,\cdot} \mathbb{E}\left[F(L)\varrho_l(L,y) \Big| \mathcal{F}_{(-\infty,y]}\right] \mathrm{d}y$$

with the covariation between the martingales

$$\mathbb{E}\left[F(L) \Big| \mathcal{F}_{(-\infty,x_0+\,\cdot\,]}\right] \quad \text{and} \quad \iint_{\mathcal{Q}_+} B(\mathrm{d}y,\mathrm{d}s)\,(\mathbf{1}_{(x_0,x_0+\,\cdot\,]}\otimes l)(y,s).$$

But, by the assumptions on ϱ_l made in the proposition, it holds that

$$\int_{x_0}^{x_0+z} \mathbb{E}\left[F(L)\varrho_l(L,y) \Big| \mathcal{F}_{(-\infty,y]}\right] \mathrm{d}y \;=\; \int_{x_0}^{x_0+z}\int_0^\infty \dot{\lambda}_{ys}(F)\, l(s)\,\mathrm{d}s\,\mathrm{d}y, \quad z\geq 0, \text{ a.s.,}$$

where, by the martingale representation of $F(L)$, the above right-hand side is the wanted covariation.

Second, if ϱ_l exists for $l_1, l_2 \in L^2([0,\infty))$ and if $a_1, a_2 \in \mathbb{R}$ then

$$\int_0^\infty \dot{\lambda}_{ys}(F)\,(a_1 l_1 + a_2 l_2)(s)\,\mathrm{d}s \overset{\text{a.s.}}{=} \mathbb{E}\left[F(L)(a_1\varrho_{l_1}(L,y) + a_2\varrho_{l_2}(L,y)) \Big| \mathcal{F}_{(-\infty,y]}\right].$$

But $(\omega,\phi,y)\mapsto a_1\varrho_{l_1}(\omega,\phi,y)+a_2\varrho_{l_2}(\omega,\phi,y)$ also satisfies the other properties of ϱ_l stated in the proposition hence it can be taken to be $\varrho_{a_1 l_1 + a_2 l_2}$. Then the linearity (3.11) follows from the uniqueness of ϱ_l explained in Remark 3.7(ii). Note that (3.11) is not required for the argument given in Remark 3.7(ii). □

Proof of ***Lemma 3.8***. Fix $\nu > 0$ and observe that, by change of variable ($t' = tr$), the test function l_ν can be represented as

$$l_\nu(t) \;=\; \frac{\sqrt{t}}{\sqrt{4\pi}} \int_0^\infty \Big(\frac{1}{\sqrt{|1-r|}} - \frac{1}{\sqrt{1+r}}\Big)\, e^{-\nu tr}\,\mathrm{d}r, \quad t\geq 0. \tag{4.7}$$

Thus, since the function $r\mapsto \big|\, |1-r|^{-1/2} - (1+r)^{-1/2}\,\big|^p$ on $[0,\infty)$ is integrable for all $1\leq p<2$, Lebesgue's dominated convergence theorem implies both the continuity of l_ν and $l_\nu(t)\to 0$ if t tends to zero. Moreover, $l_\nu \in L^2([0,\infty))\cap L^\infty([0,\infty))$ follows from

$$\begin{aligned}
&\int_0^\infty \Big(\frac{1}{\sqrt{|1-r|}} - \frac{1}{\sqrt{1+r}}\Big)\, e^{-\nu tr}\,\mathrm{d}r \\
\leq\;& e^{-\nu t}\int_0^1 \Big(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{2-r}}\Big)\, e^{\nu tr}\,\mathrm{d}r \;+\; \int_0^\infty \frac{1}{\sqrt{r}}\, e^{-\nu t(r+1)}\,\mathrm{d}r \\
\leq\;& e^{-\nu t}\Big[\sqrt{2}\,e^{\nu t/2} + \underbrace{\int_{1/2}^1 \Big(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{2-r}}\Big)\, e^{\nu tr}\,\mathrm{d}r}_{-(\sqrt{2}-\sqrt{\frac{2}{3}})\,\frac{e^{\nu t/2}}{\nu t} + \frac{1}{2}\int_{1/2}^1 (r^{-3/2}+(2-r)^{-3/2})\,\frac{e^{\nu tr}}{\nu t}\,\mathrm{d}r}\Big] + e^{-\nu t}\sqrt{\pi/(\nu t)}
\end{aligned}$$

since

$$\int_{1/2}^1 (r^{-3/2}+(2-r)^{-3/2})\,\frac{e^{\nu tr}}{\nu t}\,\mathrm{d}r \;\leq\; (2^{3/2}+1)\,[\,e^{\nu t} - e^{\nu t/2}\,]\,\frac{1}{(\nu t)^2}$$

which, in the end, yields $l_\nu(t) = \mathcal{O}(t^{-3/2})$ for $t\to\infty$ by (4.7).

The next step is to identify $\varrho_{l_\nu}(\phi, y)$ as given in the lemma so, in particular, we have to show (3.10). Fix $y > x_0$ and $F \in \mathfrak{F}C_b^\infty(\mathscr{D})$ given by

$$F(\phi) \,=\, f(\langle \phi\,; h_1\rangle, \dots, \langle \phi\,; h_n\rangle), \quad \phi \in E_1,$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $h_i \in \mathscr{D}$, $i = 1, \dots, n$, for some $n \geq 1$.

Since $U(x_0, \cdot)$ is a stochastic integral against the Brownian sheet B with deterministic integrand, the Malliavin derivative $D_{ys}F(U(x_0, \cdot))$ exists and can explicitly be given by

$$D_{ys}F(U(x_0, \cdot)) \,= \sum_{i=1}^n \partial_i f(\dots, \langle U(x_0, \cdot)\,; h_i\rangle, \dots) \int_0^\infty g(y, s\,; x_0, t) h_i(t)\,\mathrm{d}t.$$

Furthermore, by Clark-Ocone's formula, we have the identity

$$\dot{\lambda}_{ys}(F) \,=\, \mathbb{E}\left[D_{ys}F(U(x_0, \cdot)) \,\middle|\, \mathcal{F}_{(-\infty, y]} \right], \quad s \geq 0, \text{ a.s.}$$

Therefore we obtain that

$$\begin{aligned} & \int_0^\infty \dot{\lambda}_{ys}(F)\, l_\nu(s)\,\mathrm{d}s \\ \overset{\text{a.s.}}{=} & \sum_{i=1}^n \mathbb{E}\left[\partial_i f(\dots, \langle U(x_0, \cdot)\,; h_i\rangle, \dots) \,\middle|\, \mathcal{F}_{(-\infty, y]} \right] \int_0^\infty [\int_0^\infty g(y, s\,; x_0, t) l_\nu(s)\,\mathrm{d}s\,]\, h_i(t)\,\mathrm{d}t \\ = & \sum_{i=1}^n \mathbb{E}\left[\partial_i f(\dots, \langle U(x_0, \cdot)\,; h_i\rangle, \dots) \,\middle|\, \mathcal{F}_{(-\infty, y]} \right] \langle g_y^{x_0} * l_\nu^0\,; h_i\rangle \end{aligned}$$

where $l_\nu^0 \overset{\text{def}}{=} l_\nu \mathbf{1}_{(0,\infty)}$ is treated as a function on $\mathbb{R}$ and $g_y^{x_0}$ was defined in Section 2.

Since $U(x_0, \cdot) - U(x_0, \cdot)_y$ and $\mathcal{F}_{(-\infty, y]}$ are independent, the last sum of conditional expectations simplifies to

$$\int_{E_1} \sum_{i=1}^n \partial_i f(\dots, \langle U(x_0, \cdot)_y\,; h_i\rangle + \langle \phi^y\,; h_i\rangle, \dots) \langle g_y^{x_0} * l_\nu^0\,; h_i\rangle\, \mu^y(\mathrm{d}\phi^y)$$

where μ^y denotes the image measure of $U(x_0, \cdot) - U(x_0, \cdot)_y$ on E_1 equipped with the Borel-σ-algebra. Remark that, similar to the proof of Proposition 3.3(iii), there are versions of $U(x_0, \cdot)_y$ and $U(x_0, \cdot) - U(x_0, \cdot)_y$ taking values in E_1.

Now introduce the function

$$F_y(\omega, \phi^y) \,\overset{\text{def}}{=}\, F(U(x_0, \cdot)_y(\omega) + \phi^y)$$

and observe that

$$\sum_{i=1}^n \partial_i f(\dots, \langle U(x_0, \cdot)_y\,; h_i\rangle + \langle \phi^y\,; h_i\rangle, \dots) \langle g_y^{x_0} * l_\nu^0\,; h_i\rangle \,=\, \frac{\partial F_y(\phi^y)}{\partial (g_y^{x_0} * l_\nu^0)}$$

where the right-hand side is the Gâteaux derivative into the direction $g_y^{x_0} * l_\nu^0$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}r}\, F_y\left(\phi^y + r(g_y^{x_0} * l_\nu^0) \right)\Big|_{r=0}.$$

Note that this requires $g_y^{x_0} * l_\nu^0 \in E_1$ which can easily be verified using the explicit structure of $g_y^{x_0}$ because the function l_ν^0 is bounded.

Having found that

$$\int_0^\infty \dot{\lambda}_{ys}(F)\, l_\nu(s)\,\mathrm{d}s \,\overset{\text{a.s.}}{=}\, \int_{E_1} \frac{\partial F_y(\phi^y)}{\partial (g_y^{x_0} * l_\nu^0)}\, \mu^y(\mathrm{d}\phi^y) \tag{4.8}$$

we now want to justify that

$$\int_{E_1} \frac{\partial F_y(\phi^y)}{\partial(g_y^{x_0} * l_\nu^0)}\, \mu^y(\mathrm{d}\phi^y) \overset{\text{a.s.}}{=} \mathbb{E}\left[F(U(x_0,\cdot))\langle U(x_0,\cdot) - U(x_0,\cdot)_y\,;\, \frac{2\sqrt{\nu}e^{-\nu\,\cdot}}{e^{-\sqrt{\nu}(y-x_0)}}\rangle \,\middle|\, \mathcal{F}_{(-\infty,y]}\right].$$

But

$$\int_{E_1} \frac{\partial F_y(\phi^y)}{\partial(g_y^{x_0} * l_\nu^0)}\, \mu^y(\mathrm{d}\phi^y) = \int_{E_1} F_y(\phi^y)\langle \phi^y\,; C_y^{-1}(g_y^{x_0} * l_\nu^0)\rangle\, \mu^y(\mathrm{d}\phi^y) \tag{4.9}$$

if the direction $g_y^{x_0} * l_\nu^0$ is in the Cameron-Martin space H_y of the Gaussian measure μ^y with covariance $C_y : E_1' \to E_1''$.

Remark 4.1. *(i) We refer to [3] being a good reference for the theory of Gaussian measures on infinite-dimensional spaces. The covariance $C_y : E_1' \to E_1''$ can be extended to the reproducing kernel Hilbert space H_y' of μ^y and C_y acts on H_y' as an isomorphism between H_y' and the Cameron-Martin space H_y, $H_y \subseteq E_1 \subseteq E_1''$. So, checking if $g_y^{x_0} * l_\nu^0 \in H_y$ can be done by finding a solution $m_\nu^y \in H_y'$ of the equation*

$$g_y^{x_0} * l_\nu^0 = C_y\, m_\nu^y \tag{4.10}$$

and this will be the next step of the proof.

(ii) It is clear that we have chosen l_ν in a way that (4.10) can be solved in H_y'. When applying our method with respect to other linear SPDEs with additive Gaussian noise then one has to study an equation of similar type, i.e.

$$g_y^{x_0} * l = C_y\, m_l^y,$$

where $g_y^{x_0}$ comes from the Green's function associated with the SPDE and C_y is the covariance of some Gaussian measure. The task is then to identify the 'good' test functions l for which such an equation can be solved.

We will show that

$$g_y^{x_0} * l_\nu^0 = C_y\, e^{-\nu\,\cdot} / \widehat{g_y^{x_0}}(\nu)$$

which also implies that the direction $g_y^{x_0} * l_\nu^0$ must be in H_y because $e^{-\nu\,\cdot} \in E_1' \subseteq H_y'$ and $C_y : H_y' \to H_y$ is an isomorphism.

Let's show the claimed equality. As C_y is the covariance of the image measure of the random variable $U(x_0,\cdot) - U(x_0,\cdot)_y$ taking values in E_1 given by

$$U(x_0,t) - U(x_0,t)_y = \iint_{\mathcal{Q}_+^y} B(\mathrm{d}y',\mathrm{d}s')\, g(y',s'\,;x_0,t), \quad t \geq 0,$$

we obtain that

$$\begin{aligned} C_y\, e^{-\nu\,\cdot}(t) &= \int_0^\infty \Big[\int_y^\infty\!\!\int_0^\infty g(y',s'\,;x_0,t) g(y',s'\,;x_0,t')\, \mathrm{d}s'\mathrm{d}y' \Big]\, e^{-\nu t'}\, \mathrm{d}t' \\ &= \int_y^\infty \Big(g_{y'}^{x_0} * \{ \int_0^\infty g(y',\cdot\,;x_0,t') e^{-\nu t'}\, \mathrm{d}t'\, \mathbf{1}_{(0,\infty)}(\cdot) \} \Big)(t)\, \mathrm{d}y' \end{aligned}$$

for all $t \geq 0$. Thus

$$\begin{aligned} \widehat{C_y\, e^{-\nu\,\cdot}}(\tilde{\nu}) &= \int_y^\infty \widehat{g_{y'}^{x_0}}(\tilde{\nu}) \underbrace{\int_0^\infty \Big(g_{y'}^{x_0} * [e^{-\tilde{\nu}\,\cdot}\, \mathbf{1}_{(0,\infty)}(\cdot)] \Big)(t')\, e^{-\nu t'}\, \mathrm{d}t'}_{\widehat{g_{y'}^{x_0}}(\nu)(\tilde{\nu}+\nu)^{-1}}\, \mathrm{d}y' \\ &= \int_y^\infty \frac{e^{-(y'-x_0)(\sqrt{\tilde{\nu}}+\sqrt{\nu})}}{4\sqrt{\tilde{\nu}\nu}(\tilde{\nu}+\nu)}\, \mathrm{d}y' = \widehat{g_y^{x_0}}(\tilde{\nu})\widehat{g_y^{x_0}}(\nu)\, \frac{1}{(\sqrt{\tilde{\nu}}+\sqrt{\nu})(\tilde{\nu}+\nu)} \end{aligned}$$

such that
$$\widehat{C_y\, e^{-\nu\,\cdot}}(\tilde\nu)/\widehat{g^{x_0}_y}(\nu) \;=\; \widehat{g^{x_0}_y}(\tilde\nu)\,\frac{1}{(\sqrt{\tilde\nu}+\sqrt{\nu})(\tilde\nu+\nu)} \;=\; \widehat{g^{x_0}_y * l^0_\nu}(\tilde\nu)$$
for all $\tilde\nu > 0$ proving
$$C_y\, e^{-\nu\,\cdot}/\,\widehat{g^{x_0}_y}(\nu) \;=\; g^{x_0}_y * l^0_\nu$$
in the end.

The above allows us to use
$$e^{-\nu\,\cdot}/\,\widehat{g^{x_0}_y}(\nu) \;=\; \frac{2\sqrt{\nu}e^{-\nu\,\cdot}}{e^{-\sqrt{\nu}(y-x_0)}} \quad \text{for} \quad C_y^{-1}(g^{x_0}_y * l^0_\nu)$$
on the right-hand side of (4.9) leading to
$$\begin{aligned}\int_{E_1}\frac{\partial F_y(\phi^y)}{\partial(g^{x_0}_y * l^0_\nu)}\,\mu^y(\mathrm{d}\phi^y) \quad &= \quad \int_{E_1} F_y(\phi^y)\langle \phi^y\,;\frac{2\sqrt{\nu}e^{-\nu\,\cdot}}{e^{-\sqrt{\nu}(y-x_0)}}\rangle\,\mu^y(\mathrm{d}\phi^y)\\ &\overset{\text{a.s.}}{=} \quad \mathbb{E}\left[F(U(x_0,\cdot))\langle U(x_0,\cdot)-U(x_0,\cdot)_y\,;\frac{2\sqrt{\nu}e^{-\nu\,\cdot}}{e^{-\sqrt{\nu}(y-x_0)}}\rangle\,\middle|\,\mathcal{F}_{(-\infty,y]}\right].\end{aligned}$$
Because of (4.8), this justifies that $\varrho_{l_\nu}(\phi,y)$ as given in Lemma 3.8 satisfies (3.10). It also satisfies the measurability conditions stated in Proposition 3.6 and it only remains to show that $\varrho_{l_\nu}(U(x_0,\cdot),y)\in L^1(\Omega)$ for almost every $y\ge x_0$ and that $y\mapsto\varrho_{l_\nu}(U(x_0,\cdot),y)$ is in $L^1([x_0,x])$ almost surely for each $x\ge x_0$. But this follows from
$$\begin{aligned}&\mathbb{E}\int_{x_0}^{x}|\varrho_{l_\nu}(U(x_0,\cdot),y)|\,\mathrm{d}y\\ =\;& \mathbb{E}\int_{x_0}^{x}|\iint_{\mathcal{Q}_+^y}B(\mathrm{d}y',\mathrm{d}s')\,[\int_0^\infty g(y',s'\,;x_0,t)\,\frac{2\sqrt{\nu}e^{-\nu t}}{e^{-\sqrt{\nu}(y-x_0)}}\,\mathrm{d}t\,]\,|\,\mathrm{d}y\\ \le\;& \int_{x_0}^{x}\sqrt{\int_{\mathbb{R}}^{\infty}\!\!\int_0^\infty[\int_0^\infty g(y',s'\,;x_0,t)\,\frac{2\sqrt{\nu}e^{-\nu t}}{e^{-\sqrt{\nu}(y-x_0)}}\,\mathrm{d}t\,]^2\,\mathrm{d}s'\mathrm{d}y'}\;\mathrm{d}y\\ =\;& 2\sqrt{\nu}\int_{x_0}^{x}e^{\sqrt{\nu}(y-x_0)}\,\mathrm{d}y\,\sqrt{\langle e^{-\nu\,\cdot}\,;\frac{-\sqrt{|\cdot|}}{\sqrt{4\pi}}*(e^{-\nu\,\cdot})^a\rangle}\;<\;\infty\end{aligned}$$
where the equality in the last line is obtained by manipulations similar to the lines of proof of Proposition 3.3(ii).

We finally prove (3.12). On the one hand, for fixed $y\ge x_0$, we have that
$$\begin{aligned}\varrho_{l_\nu}(U(x_0,\cdot),y) \quad &\overset{\text{a.s.}}{=} \quad \iint_{\mathcal{Q}_+^y}B(\mathrm{d}y',\mathrm{d}s')\int_0^\infty g(y',s'\,;x_0,t)\,\frac{2\sqrt{\nu}e^{-\nu t}}{e^{-\sqrt{\nu}(y-x_0)}}\,\mathrm{d}t\\ &= \quad \iint_{\mathcal{Q}_+^y}B(\mathrm{d}y',\mathrm{d}s')\,2\sqrt{\nu}e^{\sqrt{\nu}(y-x_0)}\underbrace{\int_{s'}^\infty g^{x_0}_{y'}(t-s')\,e^{-\nu t}\,\mathrm{d}t}_{e^{-\nu s'}\widehat{g^{x_0}_{y'}}(\nu)}\\ &= \quad \iint_{\mathcal{Q}_+^y}B(\mathrm{d}y',\mathrm{d}s')\,e^{-\nu s'}e^{-\sqrt{\nu}(y'-y)}\end{aligned}$$
using again that $\widehat{g^{x_0}_{y'}}(\nu)=e^{-\sqrt{\nu}|y'-x_0|}/(2\sqrt{\nu})$ for $y'\in\mathbb{R}$. On the other hand, it also holds that
$$\begin{aligned}&U(y,\sqrt{\nu}e^{-\nu\,\cdot})\;+\;\partial_1 U(y,e^{-\nu\,\cdot})\\ =\;& \sqrt{\nu}\iint_{\mathcal{Q}_+}B(\mathrm{d}y',\mathrm{d}s')\int_{s'}^\infty g(y',s'\,;y,t)\,e^{-\nu t}\,\mathrm{d}t\\ +\;& \iint_{\mathcal{Q}_+}B(\mathrm{d}y',\mathrm{d}s')\int_{s'}^\infty \partial_3 g(y',s'\,;y,t)\,e^{-\nu t}\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\nu} \iint_{\mathcal{Q}_+} B(\mathrm{d}y', \mathrm{d}s')\, e^{-\nu s'} e^{-\sqrt{\nu}|y'-y|}/(2\sqrt{\nu}) \\
&+ \iint_{\mathcal{Q}_+} B(\mathrm{d}y', \mathrm{d}s') \underbrace{\int_{s'}^{\infty} \frac{-2(y-y')}{4(t-s')\sqrt{4\pi(t-s')}} \exp\{\frac{-(y-y')^2}{4(t-s')}\}\, e^{-\nu t}\,\mathrm{d}t}_{\substack{= -\frac{1}{2} e^{-\nu s'} e^{-\sqrt{\nu}|y'-y|} \mathbf{1}_{(-\infty,y]}(y') \\ +\frac{1}{2} e^{-\nu s'} e^{-\sqrt{\nu}|y'-y|} \mathbf{1}_{(y,\infty)}(y')}} \\
&\overset{\text{a.s.}}{=} \iint_{\mathcal{Q}_+^y} B(\mathrm{d}y', \mathrm{d}s')\, e^{-\nu s'} e^{-\sqrt{\nu}(y'-y)}.
\end{aligned}
$$

□

Remark 4.2. *The last part of the above proof also shows that*

$$
\partial_1 U(y, e^{-\nu\,\cdot}) \overset{\text{a.s.}}{=} U(y, \sqrt{\nu} e^{-\nu\,\cdot}) - \iint_{\mathcal{Q}_+ \setminus \mathcal{Q}_+^y} B(\mathrm{d}y', \mathrm{d}s')\, e^{-\nu s'} e^{-\sqrt{\nu}|y'-y|}
$$

for all $y \geq x_0$.

Proof of ***Proposition 3.9****.* Fix $h \in \mathscr{D}$. Then h^a is infinitely often differentiable but with compact support in $\mathbb{R}$. So, the function $(\mathfrak{A}_2 h)^a$ defined by convolution is a C^∞-function.

Next, choose an upper bound c_h for the support of h and fix $t > c_h$. Then

$$
\begin{aligned}
|\mathfrak{A}_2\, h(t)| &= \Big|\int_{-c_h}^{c_h} \frac{(h^a)'(t')\,\mathrm{d}t'}{\sqrt{\pi(t-t')}}\Big| = \frac{1}{2}\Big|\int_{-c_h}^{c_h} \frac{(h^a)(t')\,\mathrm{d}t'}{\sqrt{\pi}(t-t')^{3/2}}\Big| \\
&\leq \sup_{t' \geq 0} |h(t')|\, c_h (t-c_h)^{-3/2} = \mathcal{O}(t^{-3/2}), \quad t \to \infty,
\end{aligned}
$$

which yields $\mathfrak{A}_2\, h \in C_{0,\alpha}$ for $0 \leq \alpha < 3/2$. Since $\partial_t^k \mathfrak{A}_2\, h = [sgn(\cdot)(\sqrt{\pi}|\cdot|)^{-\frac{1}{2}}] * \partial_t^{k+1}(h^a)$, the claim that $\partial_t^k \mathfrak{A}_2\, h \in C_{0,\alpha}$ for $0 \leq \alpha < k + 3/2$, $k = 1, 2, \ldots$, can be shown exactly the same way only using $\int_{-c_h}^{c_h} (t-t')^{-k-3/2}\,\mathrm{d}t' \leq 2c_h(t-c_h)^{-k-3/2}$ instead.

For $\mathfrak{A}_2\, h \in H^a_{\mathfrak{w},\beta}$ recall that $\mathfrak{w}$ is a smooth weight function such that, for some $\varepsilon > 0$, $\mathfrak{w} \geq 1 + |\cdot|^{\frac{1}{2}+\varepsilon}$ but $\mathfrak{w} = 1 + |\cdot|^{\frac{1}{2}+\varepsilon}$ outside a neighbourhood of zero. Hence, $\partial_t^k \mathfrak{A}_2\, h \in C_{0,\alpha}$ for $0 \leq \alpha < k + 3/2$ implies $\partial_t^k [\mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a] \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ if $\varepsilon < k + 1/2$ for $k = 0, 1, 2, \ldots$ Note that $\mathfrak{A}_2\, h \in H^a_{\mathfrak{w},0}$ if and only if $\mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a \in L^2(\mathbb{R})$ hence, assuming $\varepsilon < 1/2$, it remains to discuss the case $\beta > 0$. Denote by $\lceil \beta \rceil$ the smallest integer larger than β. Then

$$
\begin{aligned}
&\|(1 + |\cdot|)^{\beta/2}\, (\mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a)^F\|^2_{L^2(\mathbb{R})} \\
= \;& \int_{\mathbb{R}} (1+\tau^2)^\beta\, |(\mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a)^F(\tau)|^2\,\mathrm{d}\tau \leq 2^{\lceil \beta \rceil - 1} \int_{\mathbb{R}} (1 + \tau^{2\lceil \beta \rceil})\, |(\mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a)^F(\tau)|^2\,\mathrm{d}\tau \\
= \;& 2^{\lceil \beta \rceil - 1} 2\pi \left\| \mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a \right\|^2_{L^2(\mathbb{R})} + 2^{\lceil \beta \rceil - 1} 2\pi \left\| \partial_t^{\lceil \beta \rceil} [\mathfrak{w} \cdot (\mathfrak{A}_2\, h)^a] \right\|^2_{L^2(\mathbb{R})} < \infty
\end{aligned}
$$

proving $\mathfrak{A}_2\, h \in H^a_{\mathfrak{w},\beta}$. Note that $\varepsilon < 1/2$ is needed for the finiteness of the first summand in the last line, again.

Using the large-t-behaviour of $(\mathfrak{A}_2\, h)'$ found above, $\mathfrak{A}_1 \mathfrak{A}_2 h$ is well-defined and, for

$\frac{1}{2} < \alpha' < \frac{5}{2}$, we obtain that

$$\begin{aligned}
|\mathfrak{A}_1\mathfrak{A}_2 h(t)| &\le \int_t^\infty \frac{|(\mathfrak{A}_2 h)'(t')|\,\mathrm{d}t'}{\sqrt{t'-t}} \;\le\; \|(\mathfrak{A}_2\, h)'\|_{0,\alpha'} \int_0^\infty \frac{\mathrm{d}t'}{(t+t')^{\alpha'}\sqrt{t'}} \\
&\overset{(t'\,=\,tr)}{=} \|(\mathfrak{A}_2\, h)'\|_{0,\alpha'} \int_0^\infty \frac{t\,\mathrm{d}r}{(t+tr)^{\alpha'}\sqrt{tr}} \\
&= t^{-(\alpha'-\frac{1}{2})}\,\|(\mathfrak{A}_2\, h)'\|_{0,\alpha'} \int_0^\infty \frac{\mathrm{d}r}{(1+r)^{\alpha'}\sqrt{r}} \qquad (4.11)\\
&= \mathcal{O}(t^{-(\alpha'-\frac{1}{2})}), \quad t\to\infty,
\end{aligned}$$

proving $\mathfrak{A}_1\mathfrak{A}_2 h \in C_{0,\alpha}$ for $0 \le \alpha < 2$.

We continue with the proof of item(ii) of Proposition 3.9. First fix $h \in \mathscr{D}$ and observe that, by (2.1) and $\mathfrak{A}_2 h \in C_{0,\alpha}$ for $0 \le \alpha < 3/2$, the convolution $(4\pi|\cdot|)^{-1/2} * (\mathfrak{A}_2 h)^a$ is well-defined and

$$\frac{1}{\sqrt{4\pi|\cdot|}} * (\mathfrak{A}_2 h)^a \;=\; \frac{1}{\sqrt{4\pi|\cdot|}} * \Big(\, \frac{sgn(\cdot)}{\sqrt{\pi|\cdot|}} * (h^a)' \,\Big) \;=\; h. \tag{4.12}$$

This is easiest seen by taking the Fourier transform of $\frac{1}{\sqrt{4\pi|\cdot|}} * \frac{sgn(\cdot)}{\sqrt{\pi|\cdot|}}$ which is equal to the (principal value) tempered distribution $\tau \mapsto \frac{1}{\mathrm{i}\tau}$.

The next step is based on the following classical result on the extension of the Stone-Weierstrass theorem for weighted topologies:

Lemma 4.3 (Corollary 3.7 in [5])**.** *The linear hull of* $\{e^{-\nu\,\cdot} : \nu > 0\}$ *is dense in the Banach space* $(C_{0,\alpha}\,, \|\cdot\|_{0,\alpha})$ *for each* $\alpha \ge 0$*.*

So, for fixed $\alpha_0 \in (2, 5/2)$, we can choose $\tilde{\mathbf{e}}_n \in \boldsymbol{Lin}\{e^{-\nu\,\cdot} : \nu > 0\}$, $n = 1, 2, \ldots$, such that $\tilde{\mathbf{e}}_n \to (\mathfrak{A}_2 h)'$ in C_{0,α_0} if $n \to \infty$. Here, $\alpha_0 < 5/2$ is required for $(\mathfrak{A}_2 h)' \in C_{0,\alpha_0}$ and the reason for $\alpha_0 > 2$ will become clear later.

Define

$$\mathbf{e}_n(t) \;=\; -\int_t^\infty \tilde{\mathbf{e}}_n(t')\,\mathrm{d}t', \quad n = 1, 2, \ldots,$$

and observe that, for all $q \ge 1$,

$$\begin{aligned}
&\int_0^\infty [\,\mathbf{e}_n(t) - \mathfrak{A}_2 h(t)\,]^q\,\mathrm{d}t \\
=\;& \int_0^\infty [\int_t^\infty \Big(\, \tilde{\mathbf{e}}_n(t') - (\mathfrak{A}_2 h)'(t') \,\Big)\,\mathrm{d}t'\,]^q\,\mathrm{d}t \\
\le\;& \|\tilde{\mathbf{e}}_n - (\mathfrak{A}_2 h)'\|_{0,\alpha_0}^q \underbrace{\int_0^\infty [\int_t^\infty (1+t')^{-\alpha_0}\,\mathrm{d}t'\,]^q\,\mathrm{d}t}_{\text{finite because } \alpha_0\,>\,2} \;\to\; 0, \quad n\to\infty,
\end{aligned}$$

hence, for all $q \ge 1$,

$$\mathbf{e}_n \overset{L^q([0,\infty))}{\longrightarrow} \mathfrak{A}_2 h \quad \text{and} \quad \mathbf{e}_n' \overset{C_{0,\alpha_0}}{\longrightarrow} (\mathfrak{A}_2 h)' \quad \text{if} \quad n\to\infty. \tag{4.13}$$

Furthermore, for each n, since $\mathbf{e}_n \in \boldsymbol{Lin}\{e^{-\nu\,\cdot} : \nu > 0\}$, we know that $(4\pi|\cdot|)^{-1/2} * \mathbf{e}_n^a$ is a bounded continuous function in $L^2([0,\infty))$ by Lemma 3.8. But, as being shown in the next paragraph, even

$$\frac{1}{\sqrt{4\pi|\cdot|}} * \mathbf{e}_n^a \overset{L^2([0,\infty))}{\longrightarrow} h, \quad n\to\infty, \tag{4.14}$$

holds true.

In fact, using (4.12), we can write

$$\begin{aligned}
\Big|\,(\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a-h)(t)\,\Big| &= \Big|\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi|t-t'|}}\,[\,\mathbf{e}_n(t)-\mathfrak{A}_2h(t)\,]^a(t')\,\mathrm{d}t'\,\Big| \\
= \quad & \frac{1}{\sqrt{4\pi}}\Big|\int_0^\infty(\frac{1}{\sqrt{|t-t'|}}-\frac{1}{\sqrt{t+t'}})\,[\int_{t'}^\infty\Big(\,\tilde{\mathbf{e}}_n(s)-(\mathfrak{A}_2h)'(s)\,\Big)\,\mathrm{d}s\,]\,\mathrm{d}t'\,\Big| \\
\le \quad & \frac{1}{\sqrt{4\pi}}\int_0^\infty(\frac{1}{\sqrt{|t-t'|}}-\frac{1}{\sqrt{t+t'}})\,[\int_{t'}^\infty(1+s)^{-\alpha_0}\,\mathrm{d}s\,]\,\mathrm{d}t'\,\|\tilde{\mathbf{e}}_n-(\mathfrak{A}_2h)'\|_{0,\alpha_0} \\
\overset{(t'\,=\,tr)}{=} \quad & (\alpha_0-1)\;\underbrace{\frac{\sqrt{t}}{\sqrt{4\pi}}\int_0^\infty(\frac{1}{\sqrt{|1-r|}}-\frac{1}{\sqrt{1+r}})\,(1+tr)^{-\alpha_0+1}\,\mathrm{d}r}\;\;\|\tilde{\mathbf{e}}_n-(\mathfrak{A}_2h)'\|_{0,\alpha_0}
\end{aligned}$$

where, if $2<\alpha_0<3$, then the underbraced term is $\mathcal{O}(t^{-\alpha_0+\frac{3}{2}})$, $t\to\infty$, by a calculation similar to how the right-hand side of (4.7) was shown to be $\mathcal{O}(t^{-3/2})$, $t\to\infty$, in the proof of Lemma 3.8. Therefore, when choosing $\alpha_0\in(2,5/2)$ as we do, this large-t-behaviour of $[(4\pi|\cdot|)^{-1/2}*\mathbf{e}_n^a-h](t)$ can be assumed proving (4.14) since, for $p>2$,

$$\begin{aligned}
&\Big\|\,\Big[\,\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a-h\,\Big]\,\mathbf{1}_{[0,1]}\,\Big\|_{L^2([0,\infty))} = \Big\|\,\Big[\,\frac{1}{\sqrt{4\pi|\cdot|}}*(\mathbf{e}_n-\mathfrak{A}_2h)^a\,\Big]\,\mathbf{1}_{[0,1]}\,\Big\|_{L^2([0,\infty))} \\
\le\; &\Big\|\,\frac{1}{\sqrt{4\pi|\cdot|}}*(\mathbf{e}_n-\mathfrak{A}_2h)^a\,\Big\|_{L^p(\mathbb{R})} \overset{(2.1)}{\le} c_p\,\Big(\,\|(\mathbf{e}_n-\mathfrak{A}_2h)^a\|_{L^2(\mathbb{R})}+\|(\mathbf{e}_n-\mathfrak{A}_2h)^a\|_{L^1(\mathbb{R})}\,\Big)
\end{aligned}$$

the right-hand side of which converges to zero by (4.13).

Now realize that $\sqrt{\nu}e^{-\nu\,\cdot}=\mathfrak{A}_1e^{-\nu\,\cdot}$ for all $\nu>0$. Thus, applying Lemma 3.8 again, the functions $\varrho_{\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a}$ exist and

$$\varrho_{\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a}(U(x_0,\cdot),y)\overset{\text{a.s.}}{=}U(y,\mathfrak{A}_1\mathbf{e}_n)+\partial_1U(y,\mathbf{e}_n),\quad y>x_0,$$

for all $n=1,2,\ldots$ Using the versions found in Proposition 3.3(v), it follows that the process

$$\tilde{W}_z(\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a)\overset{\text{def}}{=}W_z(\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a)-\int_{x_0}^{x_0+z}[\,U(y,\mathfrak{A}_1\mathbf{e}_n)+\partial_1U(y,\mathbf{e}_n)\,]\,\mathrm{d}y,\;z\ge0,$$

has all properties of a process $\{\tilde{W}_z(l);\,z\ge0\}$ given in Proposition 3.6 when replacing l by $\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a$, $n=1,2,\ldots$

If we can now verify that, for each $z\ge0$, the three sequences of random variables

$$W_z(\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a),\;\int_{x_0}^{x_0+z}U(y,\mathfrak{A}_1\mathbf{e}_n)\,\mathrm{d}y,\;\int_{x_0}^{x_0+z}\partial_1U(y,\mathbf{e}_n)\,\mathrm{d}y,\quad n=1,2,\ldots,\tag{4.15}$$

converge to

$$W_z(h),\;\int_{x_0}^{x_0+z}U(y,\mathfrak{A}_1\mathfrak{A}_2h)\,\mathrm{d}y,\;\int_{x_0}^{x_0+z}\partial_1U(y,\mathfrak{A}_2h)\,\mathrm{d}y$$

in $L^2(\Omega)$ when $n\to\infty$, respectively, then, for each $z\ge0$, the sequence of random variables $\tilde{W}_z(\frac{1}{\sqrt{4\pi|\cdot|}}*\mathbf{e}_n^a)$, $n=1,2,\ldots$, converges in $L^2(\Omega)$ to $\tilde{W}_z(h)$ as defined in item (ii) of Proposition 3.9. But $\{\tilde{W}_z(h);\,z\ge0\}$ is a continuous process so, for $h\ne0$, the process $\{\tilde{W}_z(h)/\|h\|_{L^2([0,\infty))};\,z\ge0\}$ is a standard Wiener process with respect to the filtration $\tilde{\mathcal{F}}_z,\, z \geq 0$ by simply checking the corresponding martingale problem. The linearity (3.13) is an easy consequence of the properties of the summands defining $\tilde{W}_z(h),\, z \geq 0$.

It remains to prove the convergence of the three sequences in (4.15). Fix $z \geq 0$. First, the convergence

$$W_z(\frac{1}{\sqrt{4\pi|\cdot|}} * \mathbf{e}_n^a) \stackrel{L^2(\Omega)}{\longrightarrow} W_z(h), \quad n \to \infty,$$

follows from (4.14) using the definition on page 8 of $W_z(l)$ for $l \in L^2([0,\infty))$ as a stochastic integral.

Second, applying Proposition 3.3(ii), we obtain that

$$\begin{aligned}
&\mathbb{E}\,[\int_{x_0}^{x_0+z} U(y,\mathfrak{A}_1\mathbf{e}_n - \mathfrak{A}_1\mathfrak{A}_2 h)\,\mathrm{d}y\,]^2 \\
\leq\;& z\int_{x_0}^{x_0+z} \mathbb{E}\,U(y,\mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h))^2\,\mathrm{d}y \;=\; z^2\,\langle \mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h)\,;\frac{-\sqrt{|\cdot|}}{\sqrt{4\pi}} * [\mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h)]^a\rangle \\
\leq\;& z^2\,\underbrace{\langle (1+|\cdot|)^{-\alpha}\,;\frac{-\sqrt{|\cdot|}}{\sqrt{4\pi}} * [(1+|\cdot|)^{-\alpha}]^a\rangle}_{\text{finite for all } \alpha\,>\,5/4}\,\|\mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h)\|_{0,\alpha}^2.
\end{aligned}$$

So we need $\|\mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h)\|_{0,\alpha} \to 0,\, n \to \infty$, for some $\alpha > 5/4$. Fix $\alpha > 5/4$. By Proposition 3.9(i), this α should be less than 2 to ensure that $\mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h) \in C_{0,\alpha}$. Then

$$\begin{aligned}
\|\mathfrak{A}_1(\mathbf{e}_n - \mathfrak{A}_2 h)\|_{0,\alpha} &= \sup_{t\geq 0}|(1+t)^\alpha \int_t^\infty \frac{-[\mathbf{e}_n'(t') - (\mathfrak{A}_2 h)'(t')]\,\mathrm{d}t'}{\sqrt{\pi(t'-t)}}| \\
&\leq \|\mathbf{e}_n' - (\mathfrak{A}_2 h)'\|_{0,\alpha_0}\sup_{t\geq 0}|(1+t)^\alpha \int_0^\infty \frac{\mathrm{d}t'}{(1+t'+t)^{\alpha_0}\sqrt{t'}}|
\end{aligned}$$

where the last supremum is finite for $\alpha < \alpha_0 - 1/2$ by manipulations similar to how (4.11) was derived. Recall that we can choose $\alpha \in (5/4, 2)$ and $\alpha_0 \in (2, 5/2)$ so that the convergence of the second sequence in (4.15) follows from (4.13).

Third, applying Proposition 3.3(ii) once more, we obtain that

$$\mathbb{E}\,[\int_{x_0}^{x_0+z} \partial_1 U(y,\mathbf{e}_n - \mathfrak{A}_2 h)\,\mathrm{d}y\,]^2 \;=\; z^2\,\langle \mathbf{e}_n - \mathfrak{A}_2 h\,;\frac{1}{2\sqrt{4\pi|\cdot|}} * (\mathbf{e}_n - \mathfrak{A}_2 h)^a\rangle$$

where the right-hand side converges to zero by (4.12),(4.13) and (4.14) which completes the discussion of the convergence of the sequences in (4.15).

We finally show item (iii) of Proposition 3.9. Fix $h \in \mathscr{D}$. First, since $((4\pi|\cdot|)^{-1/2})^F = 1/\sqrt{2|\cdot|}$, the equality $\mathfrak{A}_2 h = \sqrt{2}\,(-\partial_t^2)^{\frac{1}{4}}\,h^a$ follows from (4.12) by taking Fourier transforms. Second, observe that

$$\mathfrak{A}_1 h \;=\; \frac{-\mathbf{1}_{(-\infty,0)}}{\sqrt{\pi|\cdot|}} * (h^a)' \;=\; \frac{1}{2}\Big(\,\frac{sgn(\cdot)}{\sqrt{\pi|\cdot|}} - \frac{1}{\sqrt{\pi|\cdot|}}\,\Big) * (h^a)'$$

hence, using the regularity of $\mathfrak{A}_2 h$ as stated in Proposition 3.9(i), the wanted equality for $\mathfrak{A}_1\mathfrak{A}_2 h$ follows from

$$\begin{aligned}
\mathfrak{A}_1\mathfrak{A}_2 h &= \frac{1}{2}\mathfrak{A}_2^2 h \,-\, \frac{1}{2}\frac{1}{\sqrt{\pi|\cdot|}} * \big(\,(\mathfrak{A}_2 h)^a\,\big)' \\
&= \frac{1}{2}\mathfrak{A}_2^2 h \,-\, \partial_t\,[\,\frac{1}{\sqrt{4\pi|\cdot|}} * (\mathfrak{A}_2 h)^a\,] \\
&= \frac{1}{2}\mathfrak{A}_2^2 h \,-\, h'
\end{aligned}$$

where, for the last line above, we again applied (4.12). □

Proof of ***Theorem 3.11***. In this proof one has to differ between the regular generalized function U on $C_c^\infty(\mathcal{Q}_+)$ given by the continuous version of the right-hand side of (3.2) and the version of the process $\{(U(x,\cdot),\partial U(x,\cdot));\, x\ge x_0\}$ provided by Proposition 3.3(v).

First fix $x_0\in\mathbb{R}$ and $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathfrak{D}$. Then, using (4.2), we have that $U(-f'\otimes h)\stackrel{\text{a.s.}}{=}-\int_{\mathbb{R}} f'(x)\,U(x,h)\,\mathrm{d}x$ hence, by partial integration, the first equation of (3.14) yields $U(-f'\otimes h)\stackrel{\text{a.s.}}{=}\int_{\mathbb{R}} f(x)\,\partial_1 U(x,h)\,\mathrm{d}x$. As a consequence, $U(-f'\otimes h)=\int_{\mathbb{R}} f(x)\,\partial_1 U(x,h)\,\mathrm{d}x$ for all (f,h) from a countable dense subset of $C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$ almost surely. Of course, the map $C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}\to\mathbb{R}:(f,h)\mapsto U(-f'\otimes h)$ is continuous. Nevertheless, by (3.8), it holds that

$$\mathbb{P}\Big(\int_{x_0}^{x}\|\partial_1 U(y,\cdot)\|_{E_2}\,\mathrm{d}y<\infty\quad\text{for all } x\ge x_0\Big)=1$$

hence, since $\mathscr{D}$ is continuously embedded in E_2', the map $C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}\to\mathbb{R}:(f,h)\mapsto\int_{\mathbb{R}} f(x)\,\partial_1 U(x,h)\,\mathrm{d}x$ is continuous almost surely leading to

$$-U(f'\otimes h)=\int_{\mathbb{R}} f(x)\,\partial_1 U(x,h)\,\mathrm{d}x \tag{4.16}$$

for all $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$ almost surely.

Next, when integrating both sides of the second equation of (3.14) by $-f'$, we obtain that

$$\left.\begin{aligned} U(f''\otimes h)\;&\stackrel{\text{a.s.}}{=}\;-U(f\otimes(-\partial_t^2)^{1/2}h^a)-\sqrt{2}\int_{\mathbb{R}} f(x)\,\partial_1 U(x,(-\partial_t^2)^{1/4}h^a)\,\mathrm{d}x\\ &\qquad+\int_{\mathbb{R}} f'(x)\tilde W_{x-x_0}(h)\,\mathrm{d}x\end{aligned}\right\} \tag{4.17}$$

where we have used (4.2), the first equation of (3.14) and partial integration. Here, by Proposition 3.3(iii) and Proposition 3.9(i), the first summand on the right-hand side has the meaning of $-\int_{\mathbb{R}}\int_0^\infty U(x,t)\,f(x)[(-\partial_t^2)^{1/2}h^a](t)\,\mathrm{d}t\mathrm{d}x$. But if we want to write $\sqrt{2}\,U(f'\otimes(-\partial_t^2)^{1/4}h^a)$ for the second summand then, by Remark 3.12, the meaning of U needs to be extended.

Recall that a measurable version of the process $\{\partial U(x,\cdot);\, x\ge x_0\}$ taking values in E_2 was chosen at the beginning of the proof. Furthermore, let $\Omega_0\subseteq\Omega$ be such that, on Ω_0, both holds true: $\int_{x_0}^{x}\|\partial_1 U(y,\cdot)\|_{E_2}\,\mathrm{d}y<\infty$ for all $x\ge x_0$ and (4.16) is satisfied for all $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$. Then, since Proposition 3.9(i) gives $(-\partial_t^2)^{1/4}(\mathscr{D}^a)\subseteq E_2'$, the map $C_c^\infty\big((x_0,\infty)\big)\to\mathscr{D}':f\mapsto\int_{\mathbb{R}} f(x)\,\partial_1 U(x,(-\partial_t^2)^{1/4}[\cdot]^a)\,\mathrm{d}x$ is continuous on Ω_0 so that, for each $\omega\in\Omega_0$, the map

$$C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}\to\mathbb{R}:(f,h)\mapsto\int_{\mathbb{R}} f(x)\,\partial_1 U(\omega,x,(-\partial_t^2)^{1/4}h^a)\,\mathrm{d}x$$

extends to a generalized function on $C_c^\infty(\mathcal{Q}_+^{x_0})$ by Schwartz' kernel theorem. Also, if χ_N is a symmetric C^∞-function on $\mathbb{R}$ such that $\chi_N\equiv 1$ on $[-N,N]$ and $\operatorname{supp}\chi_N\subseteq(-N-1,N+1)$, $N=1,2,\dots$, then $\chi_N(-\partial_t^2)^{1/4}h^a\to(-\partial_t^2)^{1/4}h^a$, $N\to\infty$, in E_2' thus

$$-\lim_{N\uparrow\infty}U(f'\otimes[\chi_N(-\partial_t^2)^{1/4}h^a])=\int_{\mathbb{R}} f(x)\,\partial_1 U(x,(-\partial_t^2)^{1/4}h^a)\,\mathrm{d}x$$

for all $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$ on Ω_0. As a consequence, since the tensor product $C_c^\infty\big((x_0,\infty)\big)\otimes\mathscr{D}$ is dense in $C_c^\infty(\mathcal{Q}_+^{x_0})$, the generalized function given on Ω_0 by the above right-hand side must be equal to

$$-\lim_{N\uparrow\infty}(\chi_N U)(\partial_1(-\partial_2^2)^{1/4}f^a), \quad f\in C_c^\infty(\mathcal{Q}_+^{x_0}), \quad \text{on } \Omega_0, \tag{4.18}$$

where $\chi_N U$ is short for $\chi_N(t)\,U(x,t)$, $(x,t)\in\mathcal{Q}_+$. Setting $\partial_x(-\partial_t^2)^{1/4}\,U^a$ to be the above limit on Ω_0 and zero otherwise therefore defines a meaningful generalized function on $C_c^\infty(\mathcal{Q}_+^{x_0})$.

Now realize that Ω_0 can be chosen to be of probability one. Repeating the above construction for $x_0 > x_1 > \ldots$ where $x_k\to-\infty$, $k\to\infty$, gives meaningful definitions of $\partial_x(-\partial_t^2)^{1/4}\,U^a$ based on subsets Ω_k of measure one, $k=0,1,2,\ldots$ Of course, $\mathbb{P}(\bigcap_{k\geq 0}\Omega_k)=1$ and the definition of $\partial_x(-\partial_t^2)^{1/4}\,U^a$ based on $\bigcap_{k\geq 0}\Omega_k$ is consistent in the following sense: if $x_{k+1}<x_k$ and $\operatorname{supp} f\subseteq\mathcal{Q}_+^{x_k}$ then the limit defining $\partial_x(-\partial_t^2)^{1/4}\,U^a$ remains the same regardless of whether it was constructed with respect to x_{k+1} or x_k. In this sense, $\partial_x(-\partial_t^2)^{1/4}\,U^a$ can be considered a meaningful generalized function on $C_c^\infty(\mathcal{Q}_+)$ which is independent of the choice of x_0.

Using this definition of $\partial_x(-\partial_t^2)^{1/4}\,U^a$, equation (4.17) becomes

$$\begin{aligned}&\partial_x^2\,U(f\otimes h)+(-\partial_t^2)^{1/2}\,U^a(f\otimes h)+\sqrt{2}\,\partial_x(-\partial_t^2)^{1/4}\,U^a(f\otimes h)\\ \overset{\text{a.s.}}{=}\;&\int_{\mathbb{R}} f'(x)\tilde{W}_{x-x_0}(h)\,\mathrm{d}x\end{aligned}$$

for all $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$. Assume for a moment that we can show that one could construct a Brownian sheet $\tilde{B}$ on $(\Omega,\mathcal{F},\mathbb{P})$ such that

$$\int_{\mathbb{R}} f'(x)\tilde{W}_{x-x_0}(h)\,\mathrm{d}x \overset{\text{a.s.}}{=} \int_{\mathbb{R}}\int_0^\infty \tilde{B}(x,t)f'(x)h'(t)\,\mathrm{d}t\mathrm{d}x$$

for all $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$. Then, by continuity, the last equation can be extended to

$$\partial_x^2\,U(f)+(-\partial_t^2)^{1/2}\,U^a(f)+\sqrt{2}\,\partial_x(-\partial_t^2)^{1/4}\,U^a(f) \;=\; \partial_x\partial_t\tilde{B}(f) \tag{4.19}$$

for all $f\in C_c^\infty(\mathcal{Q}_+^{x_0})$ almost surely. Observe that the above left-hand side does not depend on the choice of x_0 hence, if the same Brownian sheet $\tilde{B}$ can be used for all x_0, then (4.19) can easily be extended to hold for all $f\in C_c^\infty(\mathcal{Q}_+)$ almost surely proving the theorem.

It remains to construct the Brownian sheet $\tilde{B}$. By Proposition 3.9, in particular using (3.13), the process $\int_{\mathbb{R}} f'(x)\tilde{W}_{x-x_0}(h)\,\mathrm{d}x$ indexed by $(f,h)\in C_c^\infty\big((x_0,\infty)\big)\times\mathscr{D}$ is a centred Gaussian process with covariance

$$\int_{\mathbb{R}}\int_0^\infty f_1(x)h_1(t)f_2(x)h_2(t)\,\mathrm{d}t\mathrm{d}x$$

which can be represented by

$$\eta(f,h)\overset{\text{def}}{=}\Big(\partial_t\,U+(-\partial_t^2)^{1/2}\,U^a+\sqrt{2}\,\partial_x(-\partial_t^2)^{1/4}\,U^a\Big)(f\otimes h)-\iint_{\mathcal{Q}_+}B(\mathrm{d}x,\mathrm{d}x)(f\otimes h)(x,t)$$

independently of x_0. Thus, the process η extends to a centred Gaussian process indexed by $L^2(\mathbb{R})\times L^2([0,\infty))$ and the continuous version of the field $\{\eta(sgn(x)\mathbf{1}_{[x\wedge 0,x\vee 0]},\mathbf{1}_{[0,t]});\ (x,t)\in\mathcal{Q}_+\}$ gives the wanted Brownian sheet $\tilde{B}$. □

Proof of ***Theorem 3.17***. According to the sequence of arguments laid down between Theorem 3.11 and Theorem 3.17 in the Results-Section, there is a version of the process $\{(U(x_0+z,\cdot),\ \partial_1U(x_0+z,\cdot));\ z\geq 0\}$ which satisfies all conditions of Definition 3.15.

Hence, by Thm.4.2(b) in [8] and Remark 3.14, if (wp) on page 13 holds true for the martingale problem of Definition 3.15 then $\{(U(x_0+z,\cdot),\ \partial_1 U(x_0+z,\cdot));\ z\geq 0\}$ is a strong Markov process in the sense of Definition 3.13. Moreover, it is stationary by Proposition 3.3(v).

The condition (wp) will be shown in two steps. First, for an arbitrary solution $\{(u_z,v_z);\ z\geq 0\}$ to the martingale problem of Definition 3.15, we prove that

$$\left.\begin{aligned}\langle u_z\,;h\rangle &\overset{\text{a.s.}}{=} \langle u_0\,;h\rangle + \int_0^z \langle v_y\,;h\rangle\,\mathrm{d}y\\ \langle v_z\,;h\rangle &\overset{\text{a.s.}}{=} \langle v_0\,;h\rangle - \int_0^z \Big[\langle u_y\,;(-\partial_t^2)^{\frac{1}{2}}h^a\rangle + \langle v_y\,;\sqrt{2}\,(-\partial_t^2)^{\frac{1}{4}}h^a\rangle\Big]\,\mathrm{d}y - \mathscr{W}_z(h)\end{aligned}\right\}\quad (4.20)$$

for all $(z,h)\in[0,\infty)\times\mathscr{D}$ where $\{\mathscr{W}_z;\ z\geq 0\}$ is a $\mathscr{D}'$-valued Wiener process with respect to the filtration $\mathscr{F}_z$ and, second, we verify that the above stochastic integral equation has a pathwise unique solution satisfying the conditions (i),(ii),(iii) of Definition 3.15. This indeed shows (wp) because pathwise uniqueness of stochastic differential equations implies weak uniqueness and weak uniqueness is sufficient for the uniqueness of the one-dimensional marginal distributions.

The first step is fairly standard and we only sketch the key idea. Also, the filtration to be considered for all martingales and Wiener processes mentioned below is $\mathscr{F}_z,\ z\geq 0$.

Define $F_N\in\mathfrak{F}C_b^\infty(D)$ by $h\in\mathscr{D}$ and $f_N\in C_b^\infty(\mathbb{R})$ satisfying $f_N(x)=x$ for $x\in[-N,N]$ and $\sup_{x\in\mathbb{R}}|f_N(x)|\leq N+1,\ N=1,2,\ldots$ Then, using both Definition 3.15(iv) with respect to F_N and stopping times $\inf\{z\geq 0:|\langle u_z\,;h\rangle|+|\langle v_z\,;h\rangle|\geq N\}$, the process $\{\langle u_z\,;h\rangle-\langle u_0\,;h\rangle-\int_0^z\langle v_y\,;h\rangle\,\mathrm{d}y;\ z\geq 0\}$ can be shown to be a continuous local martingale with quadratic variation zero which proves the first equation of (4.20).

In a similar way one shows that, for each $h\in\mathscr{D}$, the process $\{\langle v_z\,;h\rangle-\langle v_0\,;h\rangle+\int_0^z\big[\,\langle u_y\,;(-\partial_t^2)^{\frac{1}{2}}h^a\rangle+\langle v_y\,;\sqrt{2}\,(-\partial_t^2)^{\frac{1}{4}}h^a\rangle\,\big]\,\mathrm{d}y;\ z\geq 0\}$ is a continuous local martingale with quadratic variation $z\|h\|^2_{L^2([0,\infty))}$. Here one also needs that the stochastic integral of an adapted continuous process against a continuous local martingale always exists.

As a consequence, by the martingale characterisation of the standard Wiener process, for each $h\in\mathscr{D}$, there is a continuous process $\{\mathscr{W}_z(h);\ z\geq 0\}$ on $(\Omega,\mathcal{F},\mathbb{P})$ such that the 2nd equation of (4.20) is satisfied for all $z\geq 0$ almost surely and $\{\mathscr{W}_z(h)/\|h\|_{L^2([0,\infty))}$; $z\geq 0\}$ is a standard Wiener Process if $h\neq 0$. Of course, $\mathscr{W}_z(h)$ inherits the linearity $\mathscr{W}_z(a_1h_1+a_2h_2)\overset{\text{a.s.}}{=}a_1\mathscr{W}_z(h_1)+a_2\mathscr{W}_z(h_2)$ for each $z\geq 0,\ a_1,a_2\in\mathbb{R},\ h_1,h_2\in\mathscr{D}$ from the process $\{(u_z,v_z);\ z\geq 0\}$ taking values in $E_1\times E_2$. Hence, by standard theory – see [20] for example, there is a version of the centred Gaussian process $\mathscr{W}_z(h)$ indexed by $(z,h)\in[0,\infty)\times\mathscr{D}$ such that both the map $\mathscr{D}\to\mathbb{R}:h\mapsto\mathscr{W}_z(h)$ is an element of $\mathscr{D}'$ for each $z\geq 0$ and the map $[0,\infty)\to\mathscr{D}':z\mapsto\mathscr{W}_z$ is continuous. This means that $\{\mathscr{W}_z;\ z\geq 0\}$ can indeed be considered a $\mathscr{D}'$-valued Wiener process and the first step of proving (wp) is done.

It remains to show the pathwise uniqueness of the system of equations (4.20). So assume that two $\mathscr{F}_z$-progressively measurable processes $\{(u_z^1,v_z^1);\ z\geq 0\}$ and $\{(u_z^2,v_z^2)$; $z\geq 0\}$ on $(\Omega,\mathcal{F},\mathbb{P})$ taking values in $E_1\times E_2$ satisfy:

- $u_0^1=u_0^2$ and $v_0^1=v_0^2$;
- the equation (4.20) for all $(z,h)\in[0,\infty)\times\mathscr{D}$ driven by the same $\mathscr{D}'$-valued Wiener process $\{\mathscr{W}_z;\ z\geq 0\}$ with respect to the filtration $\mathscr{F}_z$ given on $(\Omega,\mathcal{F},\mathbb{P})$;
- the conditions (i),(ii),(iii) of Definition 3.15.

Set $u \overset{\text{def}}{=} u^1 - u^2$ and $v \overset{\text{def}}{=} v^1 - v^2$ and realize that

$$\begin{aligned}\langle u_z\,; h\rangle &\overset{\text{a.s.}}{=} \int_0^z \langle v_y\,; h\rangle\,\mathrm{d}y\\ \langle v_z\,; h\rangle &\overset{\text{a.s.}}{=} -\int_0^z \Big[\langle u_y\,; (-\partial_t^2)^{\frac{1}{2}} h^a\rangle + \langle v_y\,; \sqrt{2}\,(-\partial_t^2)^{\frac{1}{4}} h^a\rangle\Big]\,\mathrm{d}y\end{aligned}$$

for all $(z,h) \in [0,\infty) \times \mathscr{D}$. We want to show that $u \equiv 0$ almost surely.

First, by the same principles applied in the proof of Theorem 3.11, one can justify that

$$u\left(\,\partial_z^2 f_{|_{\mathcal{Q}_+^0}} + (-\partial_t^2)^{\frac{1}{2}}[\,f_{|_{\mathcal{Q}_+^0}}]^a - \sqrt{2}\,\partial_z(-\partial_t^2)^{\frac{1}{4}}[\,f_{|_{\mathcal{Q}_+^0}}]^a\,\right) = 0 \tag{4.21}$$

for all $f \in C_c^\infty(\mathcal{Q}_+)$ almost surely where, in this context, u stands for the regular generalized function given by $u_z(t)$, $(z,t) \in \mathcal{Q}_+^0$, and $f_{|_{\mathcal{Q}_+^0}}$ denotes the restriction of f to $\mathcal{Q}_+^0$. Notice that, because $f_{|_{\mathcal{Q}_+^0}}$ does not have compact support in $\mathcal{Q}_+^0$ for general $f \in C_c^\infty(\mathcal{Q}_+)$, one also has to approximate $f_{|_{\mathcal{Q}_+^0}}$ by functions from $C_c^\infty(\mathcal{Q}_+^0)$ when showing (4.21).

Second, since the map $(z,t) \mapsto u_z(t)$ is continuous on the closure of $\mathcal{Q}_+^0$ and zero on the boundary of $\mathcal{Q}_+^0$,

$$\underline{u}(z,t) \overset{\text{def}}{=} \begin{cases} 0 & : \quad z<0,\, t\in\mathbb{R}\\ u(z,t) & : \quad z\geq 0,\, t\geq 0\\ -u(z,-t) & : \quad z\geq 0,\, t<0\end{cases}$$

defines a continuous function on $\mathbb{R}^2$. Furthermore, for $f \in C_c^\infty(\mathcal{Q}_+)$, we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \underline{u}(z,t)\, f^a(z,t)\,\mathrm{d}t\mathrm{d}z \;=\; \int_0^\infty 2\int_0^\infty u(z,t)\, f_{|_{\mathcal{Q}_+^0}}(z,t)\,\mathrm{d}t\mathrm{d}z$$

because $\underline{u}$ is also antisymmetric in t. Hence (4.21) implies

$$\underline{u}\left(\,\partial_z^2 f^a + (-\partial_t^2)^{\frac{1}{2}} f^a - \sqrt{2}\,\partial_z(-\partial_t^2)^{\frac{1}{4}} f^a\,\right) = 0$$

for all $f \in C_c^\infty(\mathcal{Q}_+)$ almost surely.

Note that all arguments leading to the above equality remain valid if the corresponding functions f are complex-valued and all our test function spaces are supposed to be complex-valued for the rest of this proof.

Therefore, because $\underline{u}$ is a continuous linear form on the space $\mathscr{S}(\mathbb{R}^2)$ of rapidly decreasing functions and $\{f^a : f \in C_c^\infty(\mathcal{Q}_+)\}$ is dense in $\mathscr{S}^a(\mathbb{R}^2)$, we even obtain that

$$\underline{u}\left(\,\partial_z^2 f + (-\partial_t^2)^{\frac{1}{2}} f - \sqrt{2}\,\partial_z(-\partial_t^2)^{\frac{1}{4}} f\,\right) = 0$$

for all $f \in \mathscr{S}^a(\mathbb{R}^2)$ almost surely. But each $f \in \mathscr{S}(\mathbb{R}^2)$ can be split in a unique way into a sum of two functions in $\mathscr{S}(\mathbb{R}^2)$, one being symmetric and the other being antisymmetric in the second argument, and $\underline{u}$ maps the symmetric one to zero. Moreover, when a fractional Laplacian with respect to the second argument is applied to a function in $\mathscr{S}(\mathbb{R}^2)$ which is symmetric in the second argument, then the outcome is still symmetric in the second argument. So, the equality

$$\underline{u}\left(\,\partial_z^2 f + (-\partial_t^2)^{\frac{1}{2}} f - \sqrt{2}\,\partial_z(-\partial_t^2)^{\frac{1}{4}} f\,\right) = 0$$

must eventually hold for all $f \in \mathscr{S}(\mathbb{R}^2)$ almost surely and we can perform the calculation

$$\begin{aligned} 0 &= \underline{u}\Big(\partial_z^2 f + (-\partial_t^2)^{\frac{1}{2}} f - \sqrt{2}\,\partial_z(-\partial_t^2)^{\frac{1}{4}} f \Big) \\ &= \underline{u}^F \Big((\partial_z^2 f)^{F^{-1}} + ((-\partial_t^2)^{\frac{1}{2}} f)^{F^{-1}} - \sqrt{2}\,(\partial_z(-\partial_t^2)^{\frac{1}{4}} f)^{F^{-1}} \Big) \\ &= \underline{u}^F \Big(-\zeta^2 f^{F^{-1}} + |\tau| f^{F^{-1}} + \sqrt{2}\,\mathbf{i}\zeta\sqrt{|\tau|} f^{F^{-1}} \Big) \end{aligned}$$

leading to the well-defined equality

$$\Big(-\zeta^2 + |\tau| + \sqrt{2}\,\mathbf{i}\zeta\sqrt{|\tau|} \Big)\,\underline{u}^F \;\equiv\; 0 \quad \text{a.s.}$$

Observe that the only solution to the equation $\zeta^2 - |\tau| - \sqrt{2}\,\mathbf{i}\zeta\sqrt{|\tau|} = 0$ is $\zeta = \tau = 0$ hence the tempered distribution $\underline{u}^F$ must almost surely coincide with a series expansion of type

$$\sum_{\gamma_1=0}^{\infty}\sum_{\gamma_2=0}^{\infty} c_{\gamma_1,\gamma_2}\,\partial_1^{\gamma_1}\partial_2^{\gamma_2}\,\delta_{(0,0)}$$

where $\delta_{(0,0)}$ denotes Dirac's delta-function with respect to $(0,0) \in \mathbb{R}^2$ and c_{γ_1,γ_2} are complex-valued coefficients. Taking the inverse Fourier transform gives

$$\underline{u}(z,t) \;=\; \sum_{\gamma_1=0}^{\infty}\sum_{\gamma_2=0}^{\infty} \frac{(-\mathbf{i})^{\gamma_1+\gamma_2}}{(2\pi)^2}\,c_{\gamma_1,\gamma_2}\,z^{\gamma_1}t^{\gamma_2} \;=\; \sum_{\gamma_1=0}^{\infty}\sum_{\gamma_2=0}^{\infty} \underline{c}_{\gamma_1,\gamma_2}\,z^{\gamma_1}t^{\gamma_2}$$

for all $(z,t) \in \mathbb{R}^2$ almost surely with real-valued coefficients $\underline{c}_{\gamma_1,\gamma_2}$ because $\underline{u}$ is real-valued. But this proves $\underline{u} \equiv 0$ almost surely, hence the pathwise uniqueness of (4.20), because, on the one hand, the power series $\sum_{\gamma_1=0}^{\infty}\sum_{\gamma_2=0}^{\infty}\underline{c}_{\gamma_1,\gamma_2}\,z^{\gamma_1}t^{\gamma_2}$ cannot depend on z as $\underline{u}(z,t) = 0$ whenever $z < 0$ and, on the other hand, if $\underline{u}(z,t) = \sum_{\gamma_2=0}^{\infty}\underline{c}_{\gamma_2}\,t^{\gamma_2}$ then all coefficients $\underline{c}_{\gamma_2}$ must vanish since $\underline{u}(0,t) = 0$ for all $t \in \mathbb{R}$. □

*Proof of **Corollary 3.18**.* One only has to justify why, for fixed $x \in \mathbb{R}$, the σ-algebra $germ\,\big(\,\{x\} \times (0,\infty)\,\big)$ strictly contains $\sigma(U(x,\cdot), \partial_1 U(x,\cdot))$. But, the former includes information on all (possibly generalized) derivatives $\partial_1^m U(x,\cdot)$, $m = 0,1,2,\dots$, while the latter does not. □

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Acknowledgments. The authors would like to thank Roger Tribe for many fruitful discussions.