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Wong-Zakai type convergence in infinite dimensions

Arnab Ganguly*

Abstract

The paper deals with convergence of solutions of a class of stochastic differential equations driven by infinite-dimensional semimartingales. The infinite-dimensional semimartingales considered in the paper are Hilbert-space valued. The theorems presented generalize the convergence result obtained by Wong and Zakai for stochastic differential equations driven by linear interpolations of a finite-dimensional Brownian motion. In particular, a general form of the correction factor is derived. Examples are given illustrating the use of the theorems to obtain other kinds of approximation results.

Keywords: Weak convergence; stochastic differential equation; Wong-Zakai, uniform tightness; infinite-dimensional semimartingales; Banach space-valued semimartingales; $H^{\#}$-semimartingales.
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1 Introduction

The subject of stochastic differential equations (SDEs) in infinite-dimensional spaces has gained substantial popularity since the publication of Itô's monograph [7] and Walsh's notes on stochastic partial differential equations [26]. The practical applications of infinite-dimensional stochastic analysis involve investigation of various problems in a variety of disciplines including neurophysiology, chemical reaction systems, infinite particle systems, turbulence etc.

The stability of stochastic integrals and stochastic differential equations is an important topic in stochastic analysis. More precisely, appropriate conditions on the driving sequence of semimartingales $\{Y_n\}$ are sought, such that $(X_n, Y_n) \Rightarrow (X, Y)$ will imply $X_{n-} \cdot Y_n \Rightarrow X_- \cdot Y$. Here and throughout the rest of the paper, '$\Rightarrow$' will denote convergence in distribution and $X_- \cdot Y \equiv \int X(s-)dY(s)$ is the stochastic integral of X with respect to the integrator Y. That it is not true automatically, is shown by Wong and Zakai in [27, 28]. Let W be a standard Brownian motion, and W_n a linear interpolation of W defined by

$$\frac{d}{dt}W_n(t) = n\left(W(\frac{k+1}{n}) - W(\frac{k}{n})\right), \quad \frac{k}{n} \le t < \frac{k+1}{n}.$$

*Brown University, Division of Applied Mathematics, USA. E-mail: arnab_ganguly@brown.edu
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Then

$$\int_0^t W_n(s)\, dW_n(s) \to \int_0^t W(s)\, dW(s) + t/2 = \int_0^t W(s) \circ dW(s),$$

where $\int W(s) \circ dW(s)$ is the Stratonovich integral. Moreover, if X_n satisfies

$$dX_n(t) = \sigma(X_n(t))dW_n(t) + b(X_n(t))dt, \tag{1.1}$$

then $\{X_n\}$ does not converge to the solution of the corresponding Itô SDE driven by W but goes to the solution of

$$dX(t) = \sigma(X(t))dW(t) + (b(X(t)) + \frac{1}{2}\sigma(X(t))\sigma'(X(t)))\, dt. \tag{1.2}$$

Generalization of the Wong-Zakai result to the multi-dimensional case has been done by Stroock and Varadhan in [22]. Further generalizations included replacement of the Brownian motion with general semimartingales. The observant reader will once again note that the above equation is just the Stratonovich SDE [21] written in Itô form. The connection shouldn't be surprising for the type of approximation considered in (1.1) as the Wong-Zakai correction factor and the Stratonovich correction factor (which appears when writing the Stratonovich integral in Itô form) both stem from the fact that Itô integral is defined as the limit of a Riemann-type sum with the integrand being evaluated at the leftmost point of each partition. The relationship between Stratonovich type integrals and Wong-Zakai type corrections is further investigated by Kurtz, Pardoux and Protter in [13] where they studied a broader class of Stratonovich type equations driven by general semimartingales. For continuous semimartingale differentials, Nakao and Yamato [17] proved the following result.

Theorem 1.1. *Let U be a continuous semimartingale. Suppose X_n satisfies*

$$dX_n(t) = \sigma(t, X_n(t), U_n(t))dU_n(t),$$

where the U_n are piecewise C^1 approximations of U. If U_n tends to U, then under suitable assumptions X_n goes to X, where X satisfies

$$dX(t) = \sigma(t, X(t), U(t))\, dU(t) + \frac{1}{2}(\sigma\, \partial_2\sigma\, + \partial_3\sigma)(t, X(t), U(t))d[U, U]_t.$$

Here $\partial_i\sigma$ denotes partial derivative of σ with respect to the i-th component.

Several extensions of the above theorem were made (see Marcus [14], Konecny [10], Protter [18]), where the requirement of continuous differentials was removed, and the coefficient σ was allowed to be more general. For semimartingales with jumps, most treatments consider the case of approximating differentials with jumps, as convergence is typically proved in uniform or Skorohod topology. This is mainly because in uniform or Skorohod topology the limit of continuous approximating differential has to be continuous. The case of continuous approximation of a general semimartingale has been considered in Kurtz, Pardoux and Protter [13]. Specifically, for a given semimartingale Z, the authors showed that the limit of a suitable sequence of SDEs driven by $Z^n \equiv n\int_{\cdot-\frac{1}{n}}^{\cdot} Z(s)ds$ is an appropriate Stratanovich SDE. However as Theorem 1.2 below and Theorems 5.1, 5.4 (in the infinite-dimensional case) show that the limit might not always be in Stratonovich (or Itô) form. The general form of the correction factor depends on the type of approximations considered.

In the infinite-dimensional case, generalizations are known for approximations of some stochastic evolution equations, where the driving Brownian motion is finite dimensional, but the state-space of the solution of the SDE is infinite dimensional (see e.g

[1, 2, 23]). Twardowska [24] considered the case where the driving Brownian motion is Hilbert space-valued.

Conditions like uniform tightness (UT) (Jakubowski, Meḿin and Pagès [8], also see Definition 3.2) and uniform controlled variation (UCV) (Kurtz and Protter [11]) were imposed on the driving semimartingale sequence $\{Y_n\}$ to ensure that $X_{n-} \cdot Y_n \Rightarrow X_- \cdot Y$ if $(X_n, Y_n) \Rightarrow (X, Y)$. Slominski [20] studied the limit of a sequence of SDEs driven by $\{Y_n\}$ under the UT condition while Kurtz and Protter [11] analyzed the limit for a broader class of SDEs under the UCV condition. Extensions of the notion of uniform tightness to a sequence of Hilbert space-valued semimartingales and the corresponding weak convergence theorems for stochastic integrals were proved in [9]. For martingale random measures, conditions for the desired convergence were given by Cho in [3, 4]. Kurtz and Protter [12] extended the notion of uniform tightness further to a sequence of $\mathbb{H}^{\#}$-semimartingales (semimartingales indexed by Banach space $\mathbb{H}$ satisfying certain properties) and proved limit theorems for both stochastic integrals and stochastic differential equations. These semimartingales form a broad class of infinite-dimensional semimartingales encompassing the class of most (semi)martingale random measures, Banach space-valued semimartingales, etc. Clearly, the approximations of the driving integrators discussed above are not UT.

In the finite-dimensional case, Kurtz and Protter [11] studied weak convergence of stochastic differential equations driven by a non-UT sequence of semimartingales. Their theorem, in particular, generalized the result obtained by Wong and Zakai. A simpler version of their theorem (Theorem 5.10, [11]) is stated below.

Theorem 1.2. *Let $\{U_n\}$ and $\{V_n\}$ be sequences of R-valued semimartingales, $b : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ be bounded with bounded first and second order derivatives. Suppose that X_n satisfies*

$$X_n(t) = X_n(0) + \int_0^t \sigma(X_n(s-))dU_n(s) + \int_0^t b(X_n(s-))dV_n(s).$$

Write $U_n = Y_n + Z_n$. Denote

$$\begin{aligned} H_n(t) &= \int_0^t Z_n(s-)dZ_n(s) \\ K_n(t) &= [Y_n, Z_n]_t. \end{aligned}$$

Assume that $\{Y_n\}, \{H_n\}$ and $\{V_n\}$ are UT, and

$$A_n \equiv (X_n(0), V_n, Y_n, Z_n, H_n, K_n) \Rightarrow (X_0, V, Y, 0, H, K) \equiv A$$

Then (A_n, X_n) is relatively compact and any limit point (A, X) satisfies

$$\begin{aligned} X(t) = X_0 &+ \int_0^t \sigma(X(s-))dY(s) + \int_0^t \sigma'(X(s-))\sigma(X(s-))d(H(s) - K(s)) \\ &+ \int_0^t b(X(s-))dV(s). \end{aligned}$$

Notice that in the original Wong-Zakai case $U_n(t) = W_n(t)$, $V_n(t) = t$, $Y_n(t) = W([nt+1]/n)$ and $Z_n(t) = W_n(t) - W([nt+1]/n$. It could easily be proved that $\{Y_n\}$ and $\{H_n\}$ satisfy the condition of Theorem 1.2 and $(H(t) - K(t)) = t/2$. Similarly, Theorem 1.1 can be derived from Theorem 1.2 by writing $U_n = Y_n + Z_n$ for suitable Y_n and Z_n (see Example 5.5 for a generalization).

The objective of the present paper is to study weak convergence of stochastic differential equations driven by infinite-dimensional semimartingales. The results obtained in this paper will be useful to investigate a broader class of approximation results. In particular, such approximation results are helpful in deriving continuous time models as limiting cases of discrete-time ones. We believe that our paper is a step towards a unified theory of weak convergence of infinite-dimensional stochastic differential equations.

The sequence of stochastic differential equations considered in this paper are driven by Hilbert space-valued semimartingales. However, the limiting semimartingale need not be Hilbert space-valued. The rest of the paper is structured as follows. In Section 2, we discuss briefly infinite-dimensional semimartingales focussing mainly on the concept of $\mathbb{H}^{\#}$-semimartingales and Banach space-valued semimartingales. In particular, it is shown that stochastic integrals with respect to Banach space-valued semimartingales are special cases of integrals with respect to appropriate $\mathbb{H}^{\#}$-semimartingales. The main reason for doing this is to pave the way for usage of results from [12] which are proven in the context of $\mathbb{H}^{\#}$-semimartingales. Section 3 is devoted to the review of the concept of uniform tightness and weak convergence results that serve as prerequisites for our proof. Section 4 contains technical lemmas that are required later. The main results are presented in Section 5. Theorem 5.1 treats the case when the SDE is driven by infinite-dimensional semimartingales, but the solutions are finite-dimensional, while Theorem 5.4 extends the result to the case when the solutions of the SDE are also infinite-dimensional. The section ends with illustrative examples. A few required facts about tensor product are collected in the Appendix.

2 Infinite-dimensional semimartingales

Infinite-dimensional stochastic analysis is an active research area and depending on the need, different types of infinite-dimensional semimartingales are used in modeling. A few popular notions of infinite-dimensional semimartingales include *orthogonal martingale random measure*[6], *worthy martingale random measures* [26], *Banach space-valued semimartingales* [16], *nuclear space-valued semimartingales* [25]. In [12], Kurtz and Protter introduced the notion of *standard* $\mathbb{H}^{\#}$-semimartingale. *Standard* $\mathbb{H}^{\#}$-semimartingales form a very general class of infinite-dimensional semimartingales which includes *Banach space valued-semimartingales, cylindrical Brownian motion* and most semimartingale random measures. In particular, they cover the two important cases: space-time Gaussian white noise and Poisson random measures. A few facts about $\mathbb{H}^{\#}$-semimartingales will be used in the present paper, and below we give a brief outline of $\mathbb{H}^{\#}$-semimartingales.

2.1 $\mathbb{H}^{\#}$-semimartingale

Let $\mathbb{H}$ be a separable Banach space.

Definition 2.1. *An* $\mathbb{R}$*-valued stochastic process* Y *indexed by* $\mathbb{H}\times[0,\infty)$ *is an* $\mathbb{H}^{\#}$**-semimartingale** *with respect to the filtration* $\{\mathcal{F}_t\}$ *if*

- *for each* $h\in\mathbb{H}$*,* $Y(h,\cdot)$ *is a cadlag* $\{\mathcal{F}_t\}$*-semimartingale, with* $Y(h,0)=0$*;*
- *for each* $t>0$*,* $h_1,\dots,h_m\in\mathbb{H}$ *and* $a_1,\dots,a_m\in\mathbb{R}$*, we have*

$$Y(\sum_{i=1}^m a_i h_i, t) = \sum_{i=1}^m a_i Y(h_i,t) \quad a.s.$$

As in almost all integration theory, the first step is to define the stochastic integral in a canonical way for simple functions and then extend it to a broader class of integrands.

Let Z be an $\mathbb{H}$-valued cadlag process of the form

$$Z(t) = \sum_{k=1}^{m} \xi_k(t) h_k, \quad h_1, \ldots, h_k \in \mathbb{H}, \tag{2.1}$$

where the ξ_k are $\{\mathcal{F}_t\}$-adapted real-valued cadlag processes.
The stochastic integral $Z_- \cdot Y$ is defined as

$$Z_- \cdot Y(t) = \sum_{k=1}^{m} \int_0^t \xi_k(s-) dY(h_k, s).$$

Note that the integral above is just a real-valued process. It is necessary to impose more conditions on the $\mathbb{H}^{\#}$-semimartingale Y to broaden the class of integrands Z.
Let $\mathcal{S}$ be the collection of all processes of the form (2.1). Define

$$\mathcal{H}_t = \left\{ \sup_{s \le t} |Z_- \cdot Y(s)| : Z \in \mathcal{S}, \sup_{s \le t} \|Z(s)\| \le 1 \right\}. \tag{2.2}$$

Definition 2.2. *An $\mathbb{H}^{\#}$-semimartingale Y is* **standard** *if for each $t > 0$, $\mathcal{H}_t$ is stochastically bounded, that is, for every $t > 0$ and $\epsilon > 0$, there exists $k(t, \epsilon)$ such that*

$$P\left[\sup_{s \le t} |Z_- \cdot Y(s)| \ge k(t, \epsilon) \right] \le \epsilon$$

for all $Z \in \mathcal{S}$ satisfying $\sup_{s \le t} \|Z(s)\| \le 1$.

The extension of the stochastic integral is then achieved by approximating the integrand X by processes of the form (2.1). More precisely,

Theorem 2.3. *Let Y be a standard $\mathbb{H}^{\#}$-semimartingale, and X an $\mathbb{H}$-valued adapted and cadlag process. Then for every $\epsilon > 0$, there exists a process X^ϵ such that $\|X(t) - X^\epsilon(t)\| < \epsilon$, and moreover*

$$X_- \cdot Y \equiv \lim_{\epsilon \to 0} X^\epsilon_- \cdot Y$$

exists in the sense that for each $\eta > 0, t > 0$,

$$\lim_{\epsilon \to 0} P\left[\sup_{s \le t} |X^\epsilon_- \cdot Y(s) - X_- \cdot Y(s)| > \eta \right] = 0.$$

$X_- \cdot Y$ is a cadlag process and is defined to be the stochastic integral of X with respect to Y

Example 2.4. Let (U, r) be a complete, separable metric space and μ a sigma finite measure on $(U, \mathcal{B}(U))$. Denote the Lebesgue measure on $[0, \infty)$ by λ, and let W be a space-time Gaussian white noise on $U \times [0, \infty)$ based on $\mu \otimes \lambda$, that is, W is a Gaussian process indexed by $\mathcal{B}(U) \times [0, \infty)$ with $E(W(A, t)) = 0$ and $E(W(A, t)W(B, s)) = \mu(A \cap B) \min\{t, s\}$. For $h \in L^2(\mu)$, define $W(h, t) = \int_{U \times [0,t)} h(x) W(dx, ds)$. The above integration is defined (see [26]), and it follows that W is an $\mathbb{H}^{\#}$-semimartingale with $\mathbb{H} = L^2(\mu)$. It is also easy to check that W is standard in the sense of Definition 2.2.

Example 2.5. Let U, r, μ and λ be as before. Let ξ be a Poisson random measure on $U \times [0, \infty)$ with mean measure $\mu \otimes \lambda$, that is, for each $\Gamma \in \mathcal{B}(U) \otimes \mathcal{B}([0, \infty))$, $\xi(\Gamma)$ is a Poisson random variable with mean $\mu \otimes \lambda(\Gamma)$, and for disjoint Γ_1 and Γ_2, $\xi(\Gamma_1)$ and $\xi(\Gamma_1)$ are independent. For $A \in \mathcal{B}(U)$, define $\widetilde{\xi}(A, t) = \xi(A \times [0, t]) - t\mu(A)$. For $h \in L^2(\mu)$, let $\widetilde{\xi}(h, t) = \int_{U \times [0,t)} h(x) \widetilde{\xi}(dx, ds)$ and for $h \in L^1(\mu)$, let $\xi(h, t) = \int_{U \times [0,t)} h(x) \xi(dx, ds)$. Then $\widetilde{\xi}$ is a standard $\mathbb{H}^{\#}$-martingale with $\mathbb{H} = L^2(\mu)$ and ξ is a standard $\mathbb{H}^{\#}$-semimartingale with $\mathbb{H} = L^1(\mu)$.

Remark 2.6. *In fact, it can be shown that most worthy martingale random measures or more generally semimartingale random measures are standard $\mathbb{H}^{\#}$-semimartingales for appropriate choices of indexing space $\mathbb{H}$ (see [12]).*

2.2 $(\mathbb{L}, \widehat{\mathbb{H}})^{\#}$-semimartingale and infinite-dimensional stochastic integrals

In the previous part, observe that the stochastic integrals with respect to infinite-dimensional standard $\mathbb{H}^{\#}$-semimartingales are real-valued. Function valued stochastic integrals are of interest in many areas of infinite-dimensional stochastic analysis, for example, stochastic partial differential equations. With that in mind, we want to study stochastic integrals taking values in some infinite-dimensional space. If Y is a standard $\mathbb{H}^{\#}$-semimartingale, we could put $H(x,t) = X(\cdot-,x)\cdot Y(t)$ where for each x in a Polish space E, $X(\cdot,x)$ is a cadlag process with values in $\mathbb{H}$. The above integral is defined, but the function properties of H are not immediately clear. Hence, a careful approach is needed for constructing infinite-dimensional stochastic integrals. In [12], Kurtz and Protter introduced the concept of $(\mathbb{L}, \widehat{\mathbb{H}})^{\#}$-semimartingale as a natural analogue of the $\mathbb{H}^{\#}$-semimartingale for developing infinite-dimensional stochastic integrals. Below, we give a brief outline of that theory.

Let (E, r_E) and (U, r_U) be two complete, separable metric spaces. Let $\mathbb{L}, \mathbb{H}$ be separable Banach spaces of $\mathbb{R}$-valued functions on E and U respectively. Note that for function spaces, the product $fg, f \in \mathbb{L}, g \in \mathbb{H}$ has the natural interpretation of pointwise product. Suppose that $\{f_i\}$ and $\{g_j\}$ are such that the finite linear combinations of the f_i and the finite linear combinations of the g_j are dense in $\mathbb{L}$ and $\mathbb{H}$ respectively.

Definition 2.7. *Let $\widehat{\mathbb{H}}$ be the completion of the linear space $\left\{\sum_{i=1}^{l}\sum_{j=1}^{m} a_{ij} f_i g_j : f_i \in \{f_i\}, g_j \in \{g_j\}\right\}$ with respect to some norm $\|\cdot\|_{\widehat{\mathbb{H}}}$.*

For example, if

$$\|\sum_{i=1}^{l}\sum_{j=1}^{m} a_{ij} f_i g_j\|_{\widehat{\mathbb{H}}} = \sup\left\{\sum_{i=1}^{l}\sum_{j=1}^{m} a_{ij}\langle\lambda, f_i\rangle\langle\eta, g_j\rangle : \lambda \in \mathbb{L}^*, \eta \in \mathbb{H}^*, \|\lambda\|_{\mathbb{L}^*} \le 1, \|\eta\|_{\mathbb{H}^*} \le 1\right\}$$

then $\widehat{\mathbb{H}}$ can be interpreted as a subspace of the space of bounded operators, $L(\mathbb{K}^*, \mathbb{L})$.

Let $\mathcal{S}_{\widehat{\mathbb{H}}}$ denote the space of all processes $X \in D_{\widehat{\mathbb{H}}}[0,\infty)$ of the form

$$X(t) = \sum_{ij} \xi_{ij}(t) f_i g_j, \tag{2.3}$$

where the ξ_{ij} are $\mathbb{R}$-valued, cadlag, adapted processes and only fintely many ξ_{ij} are non zero. For $X \in \mathcal{S}_{\widehat{\mathbb{H}}}$, define

$$X_- \cdot Y(t) = \sum_i f_i \sum_j \int_0^t \xi_{ij}(s-)\, dY(g_j, s).$$

Notice that $X_- \cdot Y \in D_{\mathbb{L}}[0,\infty)$.

Definition 2.8. *An $\mathbb{H}^{\#}$-semimartingale is a* **standard** $(\mathbb{L}, \widehat{\mathbb{H}})^{\#}$**-semimartingale** *if*

$$\mathcal{H}_t \equiv \left\{\sup_{s\le t}\|X_- \cdot Y(s)\|_{\mathbb{L}} : X \in \mathcal{S}_{\widehat{\mathbb{H}}}, \sup_{s\le t}\|X(s)\|_{\widehat{\mathbb{H}}} \le 1\right\}$$

is stochastically bounded for each $t > 0$.

As in Theorem 2.3, under the standardness assumption, the definition of $X_- \cdot Y$ can be extended to all cadlag $\widehat{\mathbb{H}}$-valued processes X, by approximating X by a sequence of processes of the form (2.3).

Remark 2.9. *The standardness condition in Definition 2.8 will follow if there exists a constant $C(t)$ such that*

$$E\left[\|X_- \cdot Y(t)\|_{\mathbb{L}}\right] \le C(t), \quad t > 0$$

for all $X \in \mathcal{S}_{\widehat{\mathbb{H}}}$ satisfying $\sup_{s \le t} \|X(s)\|_{\widehat{\mathbb{H}}} \le 1$.

Remark 2.10. *If $\mathbb{H}$ and $\mathbb{L}$ are general Banach spaces (rather than Banach spaces of functions), then $\widehat{\mathbb{H}}$ could be taken as the completion of $\mathbb{L} \otimes \mathbb{H}$ with respect to some norm, for example the Hilbert-Schmidt norm or the projective norm (see [19]).*

2.3 Banach space-valued semimartingales

Standard references for the materials in this section are [16, 15]. We start with the definition of martingales taking values in a separable Banach space $\mathbb{H}$. The definition is analogous to that of real-valued martingales.

Definition 2.11. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space. A stochastic process M taking values in $\mathbb{H}$ is an $\{\mathcal{F}_t\}$-**martingale** if*

- *M is $\{\mathcal{F}_t\}$-adapted;*
- *$E\|M_t\|_{\mathbb{H}} < \infty$, for all $t > 0$;*
- *for every $F \in \mathcal{F}_s$, $\int_F M_t \, dP = \int_F M_s \, dP$, where $t > s > 0$.*

The integration above is in the Bochner sense.

Remark 2.12. *In the above definition, the measurability of $\mathbb{H}$-valued process M is in the strong sense. However since $\mathbb{H}$ is separable, the notion of strong measurability of an $\mathbb{H}$-valued function f is same as that of its weak measurability; and the Bochner integral of f coincides with the Pettis integral of f provided that the scalar function $\|f\|_{\mathbb{H}}$ is integrable. Consequently, under the assumption $E\|M_t\|_{\mathbb{H}} < \infty$, M is an $\mathbb{H}$-valued martingale if and only if $\langle M(t), h^* \rangle_{\mathbb{H},\mathbb{H}^*}$ is a real-valued martingale for every $h^* \in \mathbb{H}^*$. Here, for every $h \in \mathbb{H}$ and $h^* \in \mathbb{H}^*$, $\langle h, h^* \rangle_{\mathbb{H},\mathbb{H}^*}$ is defined by*

$$\langle h, h^* \rangle_{\mathbb{H},\mathbb{H}^*} = h^*(h) = \langle h^*, h \rangle_{\mathbb{H}^*,\mathbb{H}}. \tag{2.4}$$

Just like the real-valued case, the notion of martingales can be generalized to that of local martingales. Below we define Banach space-valued semimartingales

Definition 2.13. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space. A stochastic process Y taking values in $\mathbb{H}$ is an $\{\mathcal{F}_t\}$-**semimartingale** if Y could be decomposed into*

$$Y = M + V,$$

where M is a local martingale, and V is a finite variation process on every bounded interval $[0, t] \subset [0, \infty)$.

Remark 2.14. *The local martingale M in the above decomposition can be taken as locally square integrable (see [15, Theorem 23.6] and [16, Section 9.16]). In fact, Métivier defined semimartingale when the local martingale part is locally square integrable.*

2.4 Integration with Banach space-valued semimartingales

Let X be an $\{\mathcal{F}_t\}$-adapted, cadlag process taking values in $\mathbb{H}^*$. Suppose that Y is an $\mathbb{H}$-valued $\{\mathcal{F}_t\}$-adapted semimartingale. Let $\sigma = \{t_i\}$ be a partition of $[0, \infty)$. Define

$$X^\sigma(s) = \sum_i X(t_i) 1_{[t_i, t_{i+1})}(s) \tag{2.5}$$

and the stochastic integral $X^\sigma_- \cdot Y(t)$ as

$$X^\sigma_- \cdot Y(t) = \sum_i \langle X(t_i), Y(t_{i+1} \vee t) - Y(t_i \vee t) \rangle_{\mathbb{H}^*, \mathbb{H}},$$

Notice that $X^\sigma_- \cdot Y$ is a real-valued process. The following theorem proves the existence of the stochastic integral

Theorem 2.15. *There exists an $\{\mathcal{F}_t\}$-adapted, real-valued cadlag process $X_- \cdot Y$ such that for all $T > 0$,*

$$\sup_{t \le T} |X^\sigma_- \cdot Y(t) - X_- \cdot Y(t)| \xrightarrow{P} 0, \quad \textit{as} \quad \|\sigma\| \to 0.$$

The following lemma (see [16, Section 10.9]) gives a bound for the stochastic integral.

Lemma 2.16. *Let Y be an $\{\mathcal{F}_t\}$-adapted semimartingale taking values in a Banach space $\mathbb{H}$ and X an $\{\mathcal{F}_t\}$-adapted, cadlag process taking values in $\mathbb{H}^*$. Then, there exists a nondecreasing, $\{\mathcal{F}_t\}$-adapted, real-valued cadlag process Q such that*

$$E[\sup_{t \le T} |X_- \cdot Y(s)|^2] \le E[\int_0^T \|X_{s-}\|^2_{\mathbb{H}} dQ_s] \tag{2.6}$$

Integration in the right side is in the Riemann-Stieltjes sense.

2.4.1 Banach space-valued semimartingale as standard $\mathbb{H}^\#$-semimartingale

Let Y be a semimartingale taking values in a Banach space $\mathbb{K}$. We will show that Y can be considered as an $\mathbb{H}^\#$-semimartingale, with $\mathbb{H} = \mathbb{K}^*$. Since $\mathbb{K}$ is isometrically embedded in $\mathbb{K}^{**}$, consider Y as an element of $\mathbb{K}^{**}$. Then notice that

- for each $h \in \mathbb{K}^*$, $Y(h, \cdot) \equiv \langle Y(t), h \rangle_{\mathbb{K}, \mathbb{K}^*}$ is a real-valued semimartingale;
- for $h_1, h_2 \in \mathbb{K}^*$, $Y(h_1 + h_2, \cdot) = Y(h_1, \cdot) + Y(h_2, \cdot)$.

This proves that Y is an $\mathbb{H}^\#$-semimartingale with $\mathbb{H} = \mathbb{K}^*$, and now (2.6) proves that Y is standard. It is obvious that the two definitions of stochastic integral (see Theorem 2.3 and Theorem 2.15) coincide.

Remark 2.17. *If $\mathbb{K} = \mathbb{L}^*$, for some Banach space $\mathbb{L}$, then Y can be considered as an $\mathbb{L}^\#$-semimartingale.*

2.4.2 Hilbert space-valued stochastic integrals

As before, let Y be a semimartingale taking values in a Banach space $\mathbb{K}$. Let $\mathbb{L}$ be a separable Hilbert space. Let X be an $\{\mathcal{F}_t\}$-adapted, cadlag process taking values in the operator space, $L(\mathbb{K}, \mathbb{L})$. Let $\sigma = \{t_i\}$ be a partition of $[0, \infty)$. Define

$$X^\sigma(s) = \sum_i X(t_i) 1_{[t_i, t_{i+1})}(s) \tag{2.7}$$

and the stochastic integral $X^\sigma_- \cdot Y(t)$ as

$$X^\sigma_- \cdot Y(t) = \sum_i X(t_i)(Y(t_{i+1} \wedge t) - Y(t_i \wedge t)).$$

Notice that $X^\sigma_- \cdot Y$ is an $\mathbb{L}$-valued process. The following theorem proves the existence of the stochastic integral.

Theorem 2.18. *There exists an $\{\mathcal{F}_t\}$-adapted, $\mathbb{L}$-valued cadlag process $X_- \cdot Y$, such that for all $T > 0$,*

$$\sup_{t\le T} \|X^\sigma_- \cdot Y(t) - X_- \cdot Y(t)\|_{\mathbb{L}} \xrightarrow{P} 0.$$

Similar to (2.6), we have:

Lemma 2.19. *Let Y be an $\{\mathcal{F}_t\}$-adapted semimartingale taking values in a Banach space $\mathbb{K}$. Then, there exists a nondecreasing, $\{\mathcal{F}_t\}$-adapted, real-valued cadlag process Q, such that for any Hilbert space $\mathbb{L}$*

$$E[\sup_{t\le T} \|X_- \cdot Y(s)\|^2_{\mathbb{L}}] \le E[\int_0^T \|X_{s-}\|^2_{op} dQ_s], \tag{2.8}$$

whenever X is an $\{\mathcal{F}_t\}$-adapted, cadlag $L(\mathbb{K}, \mathbb{L})$-valued process. Here $\|\cdot\|_{op}$ denotes the operator norm.

See [16, Section 10.9, Section 6.7])

Remark 2.20. *The above lemma might not be true if $\mathbb{L}$ is an arbitrary Banach space.*

Remark 2.21. *If Y is a $\mathbb{K}$-valued semimartingale, then (2.8) shows that for any Hilbert space $\mathbb{L}$, Y can be considered as a standard $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingale. Here $\widehat{\mathbb{H}}$ is the completion of the space $\mathbb{L} \otimes \mathbb{K}^*$ with respect to some norm which makes $\mathbb{L} \otimes \mathbb{K}^* \subset L(\mathbb{K}, \mathbb{L})$.*

Suppose that X and Y are two cadlag semimartingales taking values in $\mathbb{K}, \mathbb{K}^*$. Then both $X_- \cdot Y$ and $Y_- \cdot X$ are defined. We define the (scalar) covariation process $[X, Y]$ as

$$[X, Y]_t = \langle X(t), Y(t)\rangle_{\mathbb{K},\mathbb{K}^*} - \langle X(0), Y(0)\rangle_{\mathbb{K},\mathbb{K}^*} - X_- \cdot Y(t) - Y_- \cdot X(t). \tag{2.9}$$

It is easy to see that

$$[X, Y]_t = \lim_{\|\sigma\|\to 0} \sum_i \langle X(t_{i+1}) - X(t_i), Y(t_{i+1}) - Y(t_i)\rangle_{\mathbb{K},\mathbb{K}^*}$$

where $\sigma = \{t_i\}$ is a partition of $[0, t]$, and $\|\sigma\| = \sup(t_{i+1} - t_i)$ is the mesh of the partition σ.

2.5 Tensor stochastic integration

We briefly outline the theory of tensor stochastic integration. It will be used in the next chapter. The reader might want to look at Section A.1 before reading this part. We assume that Y is an adapted $\mathbb{K}$-valued semimartingale, where $\mathbb{K}$ is a separable Hilbert space with inner product denoted by $\langle\cdot, \cdot\rangle_{\mathbb{K}}$. Let X be a cadlag and adapted $\mathbb{K}$-valued process. The tensor stochastic integral $\int X_- \otimes dY$ is defined as

$$\int_0^t X(s-) \otimes dY(s) = \lim_{\|\sigma\|\to 0} \sum_i X(t_i) \otimes (Y(t_{i+1}) - Y(t_i)),$$

where $\sigma = \{t_i\}$ is a partition of $[0, t]$, and $\|\sigma\| = \sup(t_{i+1} - t_i)$ is the mesh of the partition σ.

Theorem 2.22. $\lim_{\|\sigma\|\to 0}\sum_i X(t_i)\otimes(Y(t_{i+1})-Y(t_i))$ *exists.*

Proof. Below, we give a quick proof which illustrates the fact that the tensor integration is an example of stochastic integration with respect to a standard $(\mathbb{L},\widehat{\mathbb{H}})^{\#}$-semimartingale, for appropriate $\mathbb{L}$ and $\widehat{\mathbb{H}}$. Take $\mathbb{L}=\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}$, the completion of the space $\mathbb{K}\otimes\mathbb{K}$ with respect to the Hilbert-Schmidt norm (see A.2). Recall that $\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}$ is a Hilbert space and can be identified with the space of Hilbert-Schmidt operators $HS(\mathbb{K},\mathbb{K})$. Let $\mathcal{H}=\mathbb{L}\otimes\mathbb{K}$, that is, $\mathcal{H}$ is the space of all elements of the form:

$$\sum_{i,j=1}^{I,J} c_{ij}\lambda_i\otimes k_j,\quad \lambda_i\in\mathbb{L},k_j\in\mathbb{K},c_{ij}\in\mathbb{R}.$$

Consider $\mathcal{H}$ as a subspace of $L(\mathbb{K},\mathbb{L})$, by defining the action of an element in $\mathcal{H}$ on $\mathbb{K}$ as

$$\sum_{i,j=1}^{I,J} c_{ij}\lambda_i\otimes k_j(k)=\sum_{i,j=1}^{I,J} c_{ij}\langle k_j,k\rangle_{\mathbb{K}}\lambda_i,\quad k\in\mathbb{K}.$$

Let $\widehat{\mathbb{H}}$ be the completion of the space $\mathcal{H}$ with respect to the operator norm. Suppose that $\{e_i\}$ forms an orthonormal basis of $\mathbb{K}$. For $h\in\mathbb{K}$, define

$$\widehat{h}=\sum_{i,j}\langle h,e_i\rangle_{\mathbb{K}}(e_i\otimes e_j)\otimes e_j$$

so that for any $g\in\mathbb{K}$,

$$\widehat{h}(g)=\sum_{i,j}\langle h,e_i\rangle_{\mathbb{K}}\langle g,e_j\rangle_{\mathbb{K}}e_i\otimes e_j.$$

Observe that

$$\widehat{h}(g)=h\otimes g.$$

It is now trivial to check that $\widehat{h}\in\widehat{\mathbb{H}}$, and $h\to\widehat{h}$ is an isometric isomorphism from $\mathbb{K}$ into $\widehat{\mathbb{H}}$. Consequently, h can be identified with $\widehat{h}$ and thought of as an element of $\widehat{\mathbb{H}}$. Therefore,

$$\sum_i X(t_i)\otimes(Y(t_{i+1})-Y(t_i))=\int_0^t X^{\sigma}(s-)\otimes dY(s)=\int_0^t \widehat{X}^{\sigma}(s-)dY(s).$$

The last quantity has a limit as $\|\sigma\|\to 0$, because Y is a standard $(\mathbb{L},\widehat{\mathbb{H}})^{\#}$-semimartingale, for any Hilbet space $\mathbb{L}$ (see Remark 2.21). □

Note that by the construction, $\int X_-\otimes dY\in\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}=HS(\mathbb{K},\mathbb{K})$. Since the tensor product is not usually symmetric, $\int X_-\otimes dY\neq\int dY\otimes X_-$. But as Lemma 2.23 shows, we have the following relation

$$(\int X_-\otimes dY)^*=\int dY\otimes X_-,$$

where $*$ denotes the operator adjoint.

Lemma 2.23. *Let X be an adapted, cadlag $\mathbb{K}$-valued process and Y an adapted $\mathbb{K}$-valued semimartingale.*

$$\begin{aligned}
\langle \int_0^t X(s-)\otimes dY(s)\phi_k, \psi_k\rangle_{\mathbb{K}} &= \langle \int_0^t X(s-)\otimes dY(s), \phi_k\otimes\psi_k\rangle_{\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}} \\
&= \int_0^t \langle X(s-),\phi_k\rangle_{\mathbb{K}}\ d\langle Y(s),\psi_k\rangle_{\mathbb{K}} \\
\langle \int_0^t dY(s)\otimes X(s-)\psi_k, \phi_k\rangle_{\mathbb{K}} &= \langle \int_0^t dY(s)\otimes X(s-), \psi_k\otimes\phi_k\rangle_{\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}} \\
&= \int_0^t \langle X(s-),\phi_k\rangle_{\mathbb{K}}\ d\langle Y(s),\psi_k\rangle_{\mathbb{K}}.
\end{aligned}$$

Proof. Let σ denote the partition $\{t_i\}$ of $[0,t]$, and denote X^σ by (2.5) Notice that

$$\begin{aligned}
\langle \int_0^t X_s^\sigma\otimes dY_s, \phi_k\otimes\psi_k\rangle_{\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}} &= \sum_i \langle (X(t_i))\otimes(Y(t_{i+1})-Y(t_i)), \phi_k\otimes\psi_k\rangle_{\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}} \\
&= \sum_i \langle X(t_i),\phi_k\rangle_{\mathbb{K}}\ \langle Y(t_{i+1})-Y(t_i),\psi_k\rangle_{\mathbb{K}} \\
&= \int_0^t \langle X_s^\sigma,\phi_k\rangle_{\mathbb{K}}\ d\langle Y_s,\psi_k\rangle_{\mathbb{K}}.
\end{aligned}$$

The theorem follows by taking limit as $\|\sigma\|\to 0$, and using the continuity of the inner product. The second part is similar. □

Define $Z=\int X_-\otimes dY$. Since $Z\in\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}=HS(\mathbb{K},\mathbb{K})$, by Remark 2.21, Z is a standard $(\mathbb{L},\widehat{\mathbb{H}})^{\#}$-semimartingale, where $\mathbb{L}$ is any Hilbert space and $\widehat{\mathbb{H}}$ is the completion of the space $\mathbb{L}\otimes(\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K})$ with respect to some norm such that $\widehat{\mathbb{H}}\subset L(\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K},\mathbb{L})$. Hence, if J is an $\widehat{\mathbb{H}}$-valued cadlag and adpated process, the stochastic integral $J_-\cdot Z$ is defined.

Recall that for any two Hilbert spaces $\mathbb{X},\mathbb{Y}$, $\mathbb{X}\widehat{\otimes}_{HS}\mathbb{Y}=HS(\mathbb{Y},\mathbb{X})\subset L(\mathbb{Y},\mathbb{X})$ (see A.2). In particular, for $u=\sum_{i=1}^m x_i\otimes y_i$ and $y\in\mathbb{Y}$, $u(y)=\sum_i x_i\langle y,y_i\rangle$. Note that $\|u\|_{op}\le\|u\|_{HS}$, where $\|\cdot\|_{op}$ denotes the operator norm. The following chain rule holds.

Theorem 2.24. *Suppose J is an $(\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K})$-valued cadlag and adapted process and $Z_t=\int_0^t X(s-)\otimes dY(s)$. Then*

$$\int_0^t J(s-)\ dZ(s)=\int_0^t J(s-)(X(s-))\ dY_s.$$

Proof. First, take J of the form

$$J(s)=\sum_{k=1}^n \xi_k(s)\phi_k\otimes\psi_k. \tag{2.10}$$

Then note that

$$\begin{aligned}
\int_0^t J(s-)\ dZ(s) &= \sum_{k=1}^n\int_0^t \xi_k(s-)d\langle Z(s),\phi_k\otimes\psi_k\rangle \\
&= \sum_{k=1}^n\int_0^t \xi_k(s-)\langle X(s-),\phi_k\rangle\ d\langle Y(s),\psi_k\rangle \quad \text{(by Lemma 2.23)} \\
&= \int_0^t\sum_{k=1}^n \xi_k(s-)\langle X(s-),\phi_k\rangle\psi_k\ dY(s) \\
&= \int_0^t J(s-)(X(s-))\ dY(s).
\end{aligned}$$

The third equality follows from the definition of the stochastic integral with respect to a standard $\mathbb{K}^{\#}$-semimartingale, and the last one by identifying $\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}$ with $L(\mathbb{K},\mathbb{K})$. Now, for any $\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}$-valued adapted process J, there is a sequence of $\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}$ valued adapted processes J_n of the form (2.10) such that $\sup_{s\leq t}\|J_n(s)-J(s)\|_{HS}\to 0$, which in turn implies $\sup_{s\leq t}\|J_n(s)-J(s)\|_{op}\to 0$. Letting $n\to\infty$ in

$$\int_0^t J_n(s-)\,dZ(s)=\int_0^t J_n(s-)(X(s-))\,dY(s),$$

we are done. □

Similar to (2.9), we define the tensor covariation as

$$[X,Y]_t^{\otimes}=X(t)\otimes Y(t)-X(0)\otimes Y(0)-\int_0^t X(s-)\otimes dY(s)-\int_0^t dX(s-)\otimes Y(s) \quad (2.11)$$

It is easy to see that

$$[X,Y]_t^{\otimes}=\lim_{\|\sigma\|\to 0}\sum_i (X(t_{i+1})-X(t_i))\otimes(Y(t_{i+1})-Y(t_i))$$

where $\sigma=\{t_i\}$ is a partition of $[0,t]$, and $\|\sigma\|=\sup(t_{i+1}-t_i)$ is the mesh of the partition σ.

Let B be a Banach space. For $\phi\in D_B[0,\infty)$, define the total variation of ϕ in the interval $[0,t]$ as

$$T_t(\phi)=\sup_\sigma\sum_i\|\phi(t_i)-\phi(t_{i-1})\|_B, \quad (2.12)$$

where as before, $\sigma=\{t_i\}$ is a partition of the interval $[0,t]$. We say ϕ is of locally finite variation (or sometimes simply finite variation) if $T_t(\phi)<\infty$, for all $t>0$.

Remark 2.25. *For any $\mathbb{K}$-valued semimartingale Y, $[Y,Y]^{\otimes}$ is an $\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}=HS(\mathbb{K},\mathbb{K})$-valued process. In fact, it can be shown that almost all paths of $[Y,Y]^{\otimes}$ take values in the space of nuclear operators $\mathcal{N}(\mathbb{K},\mathbb{K})$ and $trace([Y,Y]_t^{\otimes})=[Y,Y]_t$. Moreover, the total variation of paths of $[Y,Y]^{\otimes}$ in the nuclear norm (hence also in the Hilbert-Schmidt norm) satisfies $T_t([Y,Y]^{\otimes})\leq[Y,Y]_t$. (See [15, Theorem 26.11])*

3 Uniform tightness and weak convergence results

Since the state space of the $\mathbb{H}^{\#}$-semimartingales is not known, weak convergence of a sequence of $\mathbb{H}^{\#}$-semimartingales is defined in the following way.

Definition 3.1. *Let $\mathbb{L}$ and $\mathbb{H}$ be two separable Banach spaces. Let $\{Y_n\}$ be a sequence of $\{\mathcal{F}_t^n\}$-adapted $\mathbb{H}^{\#}$-semimartingales and $\{X_n\}$ be a sequence of cadlag, $\{\mathcal{F}_t^n\}$ adapted $\mathbb{L}$ valued processes. $(X_n,Y_n)\Rightarrow(X,Y)$ if for every finite collection of elements $\phi_1,\ldots\phi_d\in\mathbb{H}$,*

$$(X_n,Y_n(\phi_1,\cdot),\ldots,Y_n(\phi_d,\cdot))\Rightarrow(X,Y(\phi_1,\cdot),\ldots,Y(\phi_d,\cdot))$$

in $D_{\mathbb{L}\times\mathbb{R}^d}[0,\infty)$.

Let $\mathbb{L},\mathbb{K}$ be separable Banach spaces, and define $\widehat{\mathbb{H}}$ to be the completion of the space $\mathbb{L}\otimes\mathbb{K}$ with respect to some norm. Let $\{\mathcal{F}_t^n\}$ be a sequence of right continuous filtrations. Let $\mathcal{S}^n$ denote the space of all $\widehat{\mathbb{H}}$-valued processes Z, such that $\|Z(t)\|_{\widehat{\mathbb{H}}}\leq 1$ and is of the form

$$Z(t)=\sum_{i,j=1}^{I,J}\xi_{ij}(t)\lambda_i\otimes h_j,\quad \lambda_i\in\mathbb{L},h_j\in\mathbb{K}$$

where the ξ_{ij} are cadlag and $\{\mathcal{F}_t^n\}$-adapted $\mathbb{R}$-valued processes.

Definition 3.2. *A sequence of $\{\mathcal{F}_t^n\}$ adapted, standard $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingales $\{Y_n\}$ is uniformly tight (UT) if, for every $\delta > 0$ and $t > 0$, there exists a $M(t,\delta)$ such that*

$$\sup_{Z \in \mathcal{S}^n} P[\sup_{s \le t} \|Z_- \cdot Y_n(s)\|_{\mathbb{L}} > M(t,\delta)] \le \delta. \tag{3.1}$$

Remark 3.3. *Uniform tightness of the sequence $\{Y_n\}$ would follow if, for every $t > 0$, there exists a constant $C(t)$ (not depending on n), such that*

$$\sup_{Z \in \mathcal{S}^n} E[\sup_{s \le t} \|Z_- \cdot Y_n(s)\|_{\mathbb{L}}] \le C(t).$$

Theorem 3.4. *([12, Theorem 4.2]) For each $n = 1, 2, \ldots$, let Y_n be an $\{\mathcal{F}_t^n\}$-adapted, standard $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingale. Assume that the sequence $\{Y_n\}$ is UT. If $(X_n, Y_n) \Rightarrow (X, Y)$, then there is a filtration $\{\mathcal{F}_t\}$ such that Y is an $\{\mathcal{F}_t\}$-adapted, standard $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingale, X is $\{\mathcal{F}_t\}$-adapted and $(X_n, Y_n, X_{n-} \cdot Y_n) \Rightarrow (X, Y, X_- \cdot Y)$.*
If $(X_n, Y_n) \xrightarrow{P} (X, Y)$ in probability then $(X_n, Y_n, X_{n-} \cdot Y_n) \xrightarrow{P} (X, Y, X_- \cdot Y)$.

A similar theorem for stochastic differential equations has also been proved.

Theorem 3.5. *([12, Theorem 7.5]) Let $\mathbb{L} = \mathbb{R}^d$. For each $n = 1, 2, \ldots$, let Y_n be an $\{\mathcal{F}_t^n\}$-adapted, standard $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingale. Suppose that (U_n, X_n, Y_n) satisfies*

$$X_n = U_n + F_n(X_{n-}) \cdot Y_n,$$

where $F_n, F : \mathbb{R}^d \to \mathbb{K}^d$ are measurable functions satisfying

- *$F_n \to F$ uniformly over compact subsets of $\mathbb{R}^d$;*
- *F is continuous;*
- *$\sup_n \sup_x \|F_n(x)\|_{\mathbb{K}^d} < \infty$.*

If $(U_n, Y_n) \Rightarrow (U, Y)$ and $\{Y_n\}$ is UT, then $\{(U_n, X_n, Y_n)\}$ is relatively compact and any limit point (U, X, Y) satisfies

$$X = U + F(X_-) \cdot Y.$$

The corresponding theorem for general $\mathbb{L}$ is:

Theorem 3.6. *([12, Theorem 7.6]) For each $n = 1, 2, \ldots$, let Y_n be an $\{\mathcal{F}_t^n\}$-adapted, standard $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingale. Suppose that (U_n, X_n, Y_n) satisfies*

$$X_n = U_n + F_n(X_{n-}) \cdot Y_n,$$

where $F_n, F : \mathbb{L} \to \widehat{\mathbb{H}}$ are measurable functions satisfying

- *$F_n \to F$ uniformly over compact subsets of $\mathbb{L}$;*
- *F is continuous;*
- *$\sup_n \sup_x \|F_n(x)\|_{\widehat{\mathbb{H}}} < \infty$;*
- *for each $\delta > 0$, there exists a compact E_δ such that $\sup_{s \le t} \|x(s)\|_{\widehat{\mathbb{H}}} \le \delta$ implies that $F_n(x(t)) \in E_\delta$ for all n.*

If $(U_n, Y_n) \Rightarrow (U, Y)$ and $\{Y_n\}$ is UT, then $\{(U_n, X_n, Y_n)\}$ is relatively compact and any limit point (U, X, Y) satisfies

$$X = U + F(X_-) \cdot Y. \tag{3.2}$$

Remark 3.7. *Suppose that in addition to the conditions of Theorem 3.5 or Theorem 3.6, strong uniqueness holds for (3.2) for any versions of (U, Y) for which Y is an $\mathbb{H}^\#$ or $(\mathbb{L}, \widehat{\mathbb{H}})^\#$-semimartingale and that $(U_n, Y_n) \to (U, Y)$ in probability. Then $(U_n, Y_n, X_n) \to (U, Y, X)$ in probability.*

4 A few lemmas

Lemma 4.1. *Let* $\mathbb{H}$ *and* $\mathbb{K}$ *be two separable Hilbert spaces. Let* $\mathbb{L} = \mathbb{K}\widehat{\otimes}_{HS}\mathbb{H} = HS(\mathbb{H},\mathbb{K})$. *Then* $\mathbb{H}$ *is continuously embedded in* $L(\mathbb{L},\mathbb{K})$.

Proof. For $h \in \mathbb{H}$, define $\overline{h} \in L(\mathbb{L},\mathbb{K})$ by

$$\overline{h}(l) = l(h), \quad l \in \mathbb{L}.$$

Notice that $h \in \mathbb{H} \longrightarrow \overline{h} \in L(\mathbb{L},\mathbb{K})$ is an isomorphism and $\|\overline{h}\| \leq \|h\|_{\mathbb{H}}$.

□

Lemma 4.2. *Let* $\mathbb{H},\mathbb{K}$ *and* $\mathbb{L}$ *be as in Lemma 4.1. Let* $u \in L(\mathbb{K},\mathbb{L})$, $v \in HS(\mathbb{H},\mathbb{K})$. *Define* $\widetilde{uv} \in L(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H},\mathbb{K})$ *by*

$$\widetilde{uv}(h_1 \otimes h_2) = \overline{h}_1 uv(h_2),$$

where $\overline{h}_1$ *is as in the proof of the previous lemma. Then* $\widetilde{uv} \in HS(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}, K)$ *and*

$$\|\widetilde{uv}\|_{HS(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H},K)} = \|uv\|_{HS(\mathbb{H},\mathbb{L})}.$$

Proof. If $\{e_i\}$ is an orthonormal basis of $\mathbb{H}$, then $\{e_i \otimes e_j\}$ forms an orthonormal basis for $\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}$. Notice that

$$\widetilde{uv}(e_i \otimes e_j) = \overline{e}_i uv(e_j) = uv(e_j)(e_i).$$

It follows that

$$\sum_{i,j} \|\widetilde{uv}(e_i \otimes e_j)\|^2_{\mathbb{K}} = \sum_j \sum_i \|uv(e_j)(e_i)\|^2_{\mathbb{K}} = \sum_j \|uv(e_j)\|^2_{\mathbb{L}}$$
$$= \|uv\|^2_{HS(\mathbb{H},\mathbb{L})}$$

□

Notice that if $\mathbb{K} = \mathbb{R}$, then $\widetilde{uv} = u \otimes v$. The following lemma is a generalization of Lemma 2.24.

Lemma 4.3. *Let* H, V *and* U *be adapted cadlag processes taking values in* $\mathbb{H}$, $\mathbb{L} \equiv HS(\mathbb{H},\mathbb{K})$ *and* $L(\mathbb{K},\mathbb{L})$ *respectively. Let* Z *be an adapted* $\mathbb{H}$*-valued semimartingale. Then*

$$\int_0^t \overline{H(s-)}U(s-)V(s-)dZ(s) = \int_0^t U(\widetilde{s-)V}(s-)dR(s), \tag{4.1}$$

where $R(t) = \int_0^t H(s-) \otimes dZ(s)$. *Here the* $^{\sim}$ *and* $^{-}$ *mappings are as in Lemma 4.2 and the proof of Lemma 4.1 respectively.*

Proof. Notice that both sides take values in $\mathbb{K}$. For $u \in L(\mathbb{K},\mathbb{L})$, $v \in HS(\mathbb{H},\mathbb{K})$, define $\widehat{uv} \in L(\mathbb{H}, HS(\mathbb{H},\mathbb{K}))$ by

$$\widehat{uv}(h) = \overline{h}uv.$$

Note that $\widehat{uv} \neq uv$. It is easy to check that in fact, $\widehat{uv} \in HS(\mathbb{H}, HS(\mathbb{H},\mathbb{K}))$. Thus for any $\lambda \in L(\mathbb{K},\mathbb{R})$, $\lambda\widehat{uv} \in HS(\mathbb{H}, HS(\mathbb{H},R))$ and can be identified with $\lambda\widetilde{uv} \in HS(\mathbb{H},\mathbb{H})$. The proof now follows by applying λ on both sides of (4.1) and using Lemma 2.24 to verify their equality. □

Lemma 4.4. *Let* $\mathbb{H}$ *be a separable Hilbert space and* Y *an* $\mathbb{H}$*-valued adapted semimartingale. Suppose that* J *and* V *are cadlag, adapted processes taking values in* $\mathbb{H}$. *Define* $X = V_- \cdot Y$. *Note that* X *is a real- valued semimartingale. Let* $U(t) = \int_0^t J(s-)dX(s)$. *Then for any* $\mathbb{H}$*-valued adapted semimartingale* Z, *we have*

$$[U,Z] = \int J_- \otimes V_- \; d[Z,Y]^{\otimes}.$$

Proof. Let $\sigma = \{t_i\}$ be a partition of $[0,t]$. For a process H, define H^σ by

$$H^\sigma(s) = \sum_i H(t_i) 1_{[t_i, t_{i+1}]}(s).$$

Let

$$X_\sigma(u) \equiv \int_0^u V^\sigma(s-)\, dY_s = \sum_i \langle V(t_i \wedge u-), Y(t_{i+1} \wedge u) - Y(t_i \wedge u)\rangle_{\mathbb{H}} \quad 0 \le u \le t$$

and

$$U_\sigma(u) \equiv \int_0^u J^\sigma(s-) \otimes dX_\sigma(s) = \sum_i J(t_i \wedge u-) \otimes (X_\sigma(t_{i+1} \wedge u) - X_\sigma(t_i \wedge u)).$$

Denote

$$A = \int_0^t J(s-) \otimes V(s-) d[Z, Y]^{\otimes}(s)$$

and notice that

$$\begin{aligned}
A &= \lim_{\|\sigma\|\to 0} \sum_i \langle J(t_i) \otimes V(t_i), (Z(t_{i+1} - Z(t_i))) \otimes (Y(t_{i+1}) - Y(t_i))\rangle_{HS} \\
&= \lim_{\|\sigma\|\to 0} \sum_i \langle J(t_i), Z(t_{i+1} - Z(t_i))\rangle_{\mathbb{H}}\ \langle V(t_i), Y(t_{i+1}) - Y(t_i)\rangle_{\mathbb{H}} \\
&= \lim_{\|\sigma\|\to 0} \sum_i \langle J(t_i), Z(t_{i+1} - Z(t_i))\rangle_{\mathbb{H}}\ (V_-^\sigma \cdot Y(t_{i+1}) - V_-^\sigma \cdot Y(t_i)) \\
&= \lim_{\|\sigma\|\to 0} \sum_i \langle J(t_i), Z(t_{i+1} - Z(t_i))\rangle_{\mathbb{H}} (X_\sigma(t_{i+1}) - X_\sigma(t_i)) \\
&= \lim_{\|\sigma\|\to 0} \sum_i \langle J(t_i)(X_\sigma(t_{i+1}) - X_\sigma(t_i)), Z(t_{i+1} - Z(t_i))\rangle_{\mathbb{H}} \\
&= \lim_{\|\sigma\|\to 0} \sum_i \langle U_\sigma(t_{i+1}) - U_\sigma(t_i), Z(t_{i+1}) - Z(t_i)\rangle_{\mathbb{H}} = [U, Z]_t.
\end{aligned}$$

□

If X is an $\mathbb{L} \equiv HS(\mathbb{H}, \mathbb{K})$-valued semimartingale and Y is an $\mathbb{H}$-valued semimartingale, then by Theorem 2.18, the stochastic integral $X_- \cdot Y$ exists. Now, by Lemma 4.1, $\overline{Y}$ is an $L(\mathbb{L}, \mathbb{K})$-valued process, and consequently, $\overline{Y}_- \cdot X$ exists. Define the (generalized) quadratic variation process between X and Y as

$$[[X, Y]]_t = X(t)(Y(t)) - X(0)(Y(0)) - X_- \cdot Y(t) - \overline{Y}_- \cdot X(t). \tag{4.2}$$

$[[X, Y]]$ is a $\mathbb{K}$-valued process and

$$[[X, Y]]_t = \lim_{\|\sigma\|\to 0} \sum_i (X(t_{i+1}) - X(t_i))(Y(t_{i+1}) - Y(t_i))$$

where $\sigma = \{t_i\}$ is a partition of $[0,t]$, and $\|\sigma\| = \sup(t_{i+1} - t_i)$ is the mesh of the partition σ. The next result is a generalization of Lemma 4.4.

Lemma 4.5. *Let $\mathbb{H}$ be a separable Hilbert space and Y an $\mathbb{H}$-valued adapted semimartingale. Let $\mathbb{L} = \mathbb{K}\widehat{\otimes}_{HS}\mathbb{H} \equiv HS(\mathbb{H}, \mathbb{K})$. Suppose that J and V are cadlag, adapted processes taking values in $HS(\mathbb{K}, \mathbb{L})$ and $HS(\mathbb{H}, \mathbb{K})$ respectively. Define $X = V_- \cdot Y$. Note that X is a $\mathbb{K}$- valued semimartingale. Let $U(t) = \int_0^t J(s-) dX(s)$. Then for any $\mathbb{H}$-valued adapted semimartingale Z, we have*

$$[[U, Z]]_t = \int_0^t J(\widetilde{s-)V}(s-)\ d[Z, Y]^{\otimes}(s).$$

Proof. Similar to the previous lemma, we adopt the following notations: Let $\sigma = \{t_i\}$ be a partition of $[0, t]$. For a process H, define H^σ by

$$H^\sigma(s) = \sum_i H(t_i) 1_{[t_i, t_{i+1}]}(s).$$

Let

$$X_\sigma(u) \equiv \int_0^u V^\sigma(s-)\, dY_s = \sum_i V(t_i \wedge u-)(Y(t_{i+1} \wedge u) - Y(t_i \wedge u)) \quad 0 \le u \le t$$

and

$$U_\sigma(u) \equiv \int_0^u J^\sigma(s-) dX_\sigma(s) = \sum_i J(t_i \wedge u-)(X_\sigma(t_{i+1} \wedge u) - X_\sigma(t_i \wedge u)).$$

Denote

$$A = \int_0^t J(\widetilde{s-)V(}s-) d[Z, Y]^\otimes(s)$$

and notice that

$$\begin{aligned} A &= \lim_{\|\sigma\| \to 0} \sum_i J(\widetilde{t_i)V(}t_i)(Z(t_{i+1} - Z(t_i))) \otimes (Y(t_{i+1}) - Y(t_i)) \\ &= \lim_{\|\sigma\| \to 0} \sum_i \overline{(Z(t_{i+1} - Z(t_i))} J(t_i) V(t_i)(Y(t_{i+1}) - Y(t_i)) \\ &= \lim_{\|\sigma\| \to 0} \sum_i \overline{(Z(t_{i+1} - Z(t_i))} (J(t_i)(X_s(t_{i+1}) - X_s(t_i))) \\ &= \lim_{\|\sigma\| \to 0} \sum_i \overline{(Z(t_{i+1} - Z(t_i))} (U_\sigma(t_{i+1}) - U_\sigma(t_i)) \\ &= \lim_{\|\sigma\| \to 0} \sum_i (U_\sigma(t_{i+1}) - U_\sigma(t_i))(Z(t_{i+1}) - Z(t_i)) = [[U, Z]]_t. \end{aligned}$$

□

Recall that for a function ϕ mapping $[0, \infty)$ to a Banach space, the total variation $T_t(\phi)$ was defined in (2.12).

Theorem 4.6. *Let $\mathbb{H}$ be a separable Hilbert space. Suppose that $Y_n = M_n + A_n$ is an adapted $\mathbb{H}$-valued semimartingale, where $\{A_n\}$ is a sequence of $\mathbb{H}$-valued $\{\mathcal{F}_t^n\}$-adapted processes of locally finite variation and $\{M_n\}$ is a sequence of $\mathbb{H}$-valued $\{\mathcal{F}_t^n\}$-adapted local martingales . Then $\{Y_n\}$ is UT if for each $t > 0$, $\{T_t(A_n)\}$ is stochastically bounded (tight) and there exists a constant $C(t)$ such that $\mathsf{E}([M_n, M_n]_t) < C(t)$.*

Proof. It is enough to prove that $\{A_n\}$ and $\{M_n\}$ are UT.

For an $\{\mathcal{F}_t^n\}$-adapted, cadlag process J, we have

$$|\int_0^t J(s-)\, dA_n(s)| \le \int_0^t \|J(s-)\|\, dT_s(A_n).$$

Thus, if $\sup_{s \le t} \|J(s)\|_{\mathbb{H}} \le 1$, we have

$$P(\sup_{s \le t} |\int_0^s J(r-)\, dA_n(r)| > K) \le P(\int_0^t \|J(s-)\|\, dT_s(A_n) > K) \le P(T_t(A_n) > K)$$

which proves that $\{A_n\}$ is UT.

Next notice that

$$\begin{aligned}\mathsf{P}(\sup_{s\le t}|\int_0^s J(r-)dM_n(r)|>K) &\le \mathsf{E}(\sup_{s\le t}|\int_0^s J(r-)dM_n(r)|^2)/K^2\\ &\le 4\mathsf{E}(\int_0^t \|J(r-)\|^2 d[M_n,M_n]_r)/K^2\\ &\le \mathsf{E}([M_n,M_n]_t)/K^2 \le C(t)/K^2\end{aligned}$$

which proves that $\{M_n\}$ is UT.

□

5 Wong-Zakai type SDE

We are now ready to state our main results. Notice that from Section 2.5, the H_n and K_n defined below are $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}) = (\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^*$-valued semimartingales, hence standard $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^{\#}$-semimartingales. In fact, by Theorem 2.18, for any Hilbert space $\mathbb{X}$ and an adapted cadlag process ξ taking values in $L(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H},\mathbb{X})$, the stochastic integrals $\xi_-\cdot H_n$ and $\xi_-\cdot K_n$ exist. Therefore, more generally the H_n and K_n are $(\mathbb{X},\widehat{\mathcal{H}})^{\#}$-semimartingales (see Remark 2.21), where $\widehat{\mathcal{H}}$ can be taken to be the completion of the space $\mathbb{X}\otimes(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})$ with respect to some norm such that $\widehat{\mathcal{H}}\subset L(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H},\mathbb{X})$.

Theorem 5.1. *Let $\mathbb{H}$ be a separable Hilbert space. Let Y_n, Z_n be two cadlag and adapted $\mathbb{H}$-valued semimartingales and $f:\mathbb{R}\longrightarrow\mathbb{H}$ a twice continuously differentiable function with first and second-order derivatives denoted by Df and D^2f respectively. Define*

$$H_n(t)=\int_0^t Z_n(s-)\otimes dZ_n(s)\ ,\ K_n(t)=[Y_n,Z_n]_t^{\otimes}.$$

Suppose X_n satisfies

$$X_n(t)=X_n(0)+\int_0^t f(X_n(s-))\,dY_n(s)+\int_0^t f(X_n(s-))\,dZ_n(s). \tag{5.1}$$

Assume that $\{Y_n\}$ and $\{H_n\}$ are UT sequences, and for each $t>0$, $\{[Z_n,Z_n]_t\}$ is a tight sequence. Also assume that there exist an $\mathbb{H}^{\#}$-semimartingale Y and $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^{\#}$-semimartingales H,K such that

$$A_n:=(X_n(0),Y_n,Z_n,H_n,K_n)\Rightarrow(X(0),Y,0,H,K):=A,$$

in the following sense: for any $\{h_i,h_i'\}_{i=1}^m\subset\mathbb{H}$ and $\{u_i,u_i'\}_{i=1}^m\subset\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}$

$$\{(X_n(0),Y_n(h_i,\cdot),Z_n(h_i',\cdot),H_n(u_i,\cdot),K_n(u_i',\cdot))\}_{i=1}^m\Rightarrow\{(X(0),Y(h_i,\cdot),0,H(u_i,\cdot),K(u_i',\cdot))\}_{i=1}^m$$

in $D_{\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m}[0,\infty)$. Then $\{(A_n,X_n)\}$ is relatively compact, and any limit point (A,X) satisfies

$$X(t)=X(0)+\int_0^t f(X(s-))\,dY(s)+\int_0^t Df(X(s-))\otimes f(X(s-))\,d(H^*(s)-K(s)).$$

Remark 5.2. *Notice that*

$$\int dZ_n(s)\otimes Z_n(s-)=H_n^*,$$

where H_n^* denotes adjoint of H_n. Therefore, by the hypothesis

$$\int dZ_n(s)\otimes Z_n(s-)\Rightarrow H^*.$$

Remark 5.3. *For a function ϕ, let $\Delta\phi(s) = \phi(s) - \phi(s-)$. Notice that $\Delta H_n(s) = Z_n(s-) \otimes \Delta Z_n(s) \Rightarrow 0$. It follows that H is continuous. Similarly, K is continuous.*

Proof. By Remark 2.25,

$$T_t([Z_n, Z_n]^{\otimes}) \leq [Z_n, Z_n]_t.$$

Since for each $t > 0$, $[Z_n, Z_n]_t$ is tight by the assumption, it follows from Theorem 4.6 that $\{[Z_n, Z_n]^{\otimes}\}$ is UT. Note that by the integration by parts formula for tensor stochastic integral, we have

$$[Z_n, Z_n]^{\otimes}_t = Z_n(t) \otimes Z_n(t) - Z_n(0) \otimes Z_n(0) - \int_0^t Z_n(s-) \otimes dZ_n(s) - \int_0^t dZ_n(s) \otimes Z_n(s-).$$

It follows from the hypothesis that

$$[Z_n, Z_n]^{\otimes} \Rightarrow -(H + H^*).$$

Observe that $T_t([Y_n, Z_n]^{\otimes}) \leq [Y_n, Y_n]_t + [Z_n, Z_n]_t$, and since $\{[Y_n, Y_n]_t\}$ and $\{[Z_n, Z_n]_t\}$ are tight for each $t > 0$, it follows again from Theorem 4.6 that $\{K_n \equiv [Y_n, Z_n]^{\otimes}\}$ is UT.

By Itô's formula we have

$$f(X_n(t)) = f(X_n(0)) + \int_0^t Df(X_n(s-))\, dX_n(s) + R_n(t).$$

$$R_n(t) = \frac{1}{2}\int_0^t D^2 f(X_n(s-)) d[X_n, X_n]^c_s + \sum_{s \leq t}[\Delta f(X_n(\cdot))(s) - Df(X_n(s-))\Delta X_n(s)].$$

where $[X_n, X_n]^c_t = [X_n, X_n]_t - \sum_{s \leq t} \Delta X_n(s) \Delta X_n(s)$ is the continuous part of $[X_n, X_n]$. It follows that $\{R_n\}$ is a locally finite variation process. Notice that $T_t(R_n)$ is dominated by a linear combination of $[Z_n, Z_n]_t, [Y_n, Y_n]_t$, and since each of them is tight, we have $\{R_n\}$ to be UT.
Next, an application of the integration by parts formula (see (2.9)) gives

$$\int_0^t f(X_n(s-))\, dZ_n(s) = \langle f(X_n(s), Z_n(s)\rangle - \int_0^t Z_n(s-)\, df(X_n(s)) - [Z_n, f(X_n)]_t.$$

Now notice that

$$\int_0^t Z_n(s-)\, df(X_n(s)) = \int_0^t Z_n(s-) Df(X_n(s-))\, dX_n(s) + \int_0^t Z_n(s-) dR_n(s).$$

Notice that $Z_n(s) \in \mathbb{H}^* = L(\mathbb{H}, \mathbb{R})$ and $Df(X_n(s)) \in L(\mathbb{R}, \mathbb{H})$. Therefore $Z_n(s-)Df(X_n(s-))$ is well defined and $\in L(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.
Hence,

$$\begin{aligned}
\int_0^t Z_n(s-)\, df(X_n(s)) &= \int_0^t (Z_n(s-)Df(X_n(s-)))f(X_n(s-))dY_n(s) \\
&\quad + \int_0^t (Z_n(s-)Df(X_n(s-)))f(X_n(s-))dZ_n(s) + \int_0^t Z_n(s-)dR_n(s) \\
&= \int_0^t (Z_n(s-)Df(X_n(s-)))f(X_n(s-))dY_n(s) \\
&\quad + \int_0^t Df(X_n(s-)) \otimes f(X_n(s-))\, dH_n(s) + \int_0^t Z_n(s-)\, dR_n(s),
\end{aligned}$$

where $H_n(s) = \int_0^t Z_n(s-) \otimes dZ_n(s)$, and the equality of the middle terms in the above two lines follows by Lemma 2.24. Next by Lemma 4.4, we have

$$[Z_n, f(X_n)]_t = \int_0^t Df(X_n(s-)) \otimes f(X_n(s-)) \; d[Z_n, Y_n]_s^{\otimes} + \int_0^t Df(X_n(s-)) \otimes f(X_n(s-)) \; d[Z_n, Z_n]_s^{\otimes} + [Z_n, R_n]_t.$$

Putting things together, we see that

$$\int_0^t f(X_n(s-)) \, dZ_n(s) = V_n(t) - \int_0^t Df(X_n(s-)) \otimes f(X_n(s-)) \; d(H_n(s) + K_n(s) + [Z_n, Z_n]^{\otimes}(s),$$

where

$$V_n(t) = \langle f(X_n(s)), Z_n(s) \rangle - \int_0^t (Z_n(s-) Df(X_n(s-))) f(X_n(s-)) \; dY_n(s) - \int_0^t Z_n(s-) \; dR_n(s) - [Z_n, R_n]_t.$$

From the hypothesis, we get $V^n \Rightarrow 0$. Plugging things back in the original equation, we have,

$$X_n(t) = X_n(0) + V_n(t) + \int_0^t f(X_n(s-)) \; dY_n(s) - \int_0^t Df(X_n(s-)) \otimes f(X_n(s-)) \; d(H_n(s) + K_n(s) + [Z_n, Z_n]^{\otimes}(s)).$$

Take the indexing space $\mathbb{Y} = (\mathbb{H}, \mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})$ with the norm as $\|(h,u)\|_{\mathbb{Y}} = \|h\|_{\mathbb{H}} + \|u\|_{HS}$. Consider Y_n as an $\mathbb{Y}^{\#}$-semimartingale by defining

$$Y_n((h,u), \cdot) \equiv Y_n(h, \cdot).$$

Similarly, H_n, K_n and $[Z_n, Z_n]$ can be considered as $\mathbb{Y}^{\#}$-semimartingales. Since $\{Y_n\}, \{H_n\}, \{K_n\}$ and $\{[Z_n, Z_n]^{\otimes}\}$ are UT, Theorem 3.5 gives the desired result.

□

We next consider the case when the solutions of the stochastic differential equations of the form (5.1) are also infinite-dimensional. We follows the steps in the above proof, however, the difficulties lie in handling of infinite-dimensional Itô's lemma, infinite-dimensional covariation, chain rule, appropriate integration by parts etc. They are taken care of by suitable use of results from Section 4. Notice that as in Theorem 5.1, the H_n and K_n defined below are $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}) = (\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^*$-valued semimartingales.

Theorem 5.4. *Let $\mathbb{H}$ and $\mathbb{K}$ be separable Hilbert spaces. Let Y_n, Z_n be two cadlag and adapted $\mathbb{H}$-valued semimartingales and $f : \mathbb{K} \longrightarrow \mathbb{L} \equiv HS(\mathbb{H}, \mathbb{K})$ a twice continuously differentiable function with first and second-order derivatives denoted by Df and $D^2 f$ respectively. Notice that $Df : \mathbb{K} \to L(\mathbb{K}, \mathbb{L})$. Assume that for each $x \in \mathbb{K}$, $D^2 f(x)$ is an element of $L(\mathbb{K}\widehat{\otimes}_{HS}\mathbb{K}, \mathbb{L})$ and the mapping $x \to D^2 f(x)$ is uniformly continuous on any bounded subset of $\mathbb{K}$. Define*

$$H_n(t) = \int_0^t Z_n(s-) \otimes dZ_n(s) \; , \; K_n(t) = [Y_n, Z_n]_t^{\otimes}.$$

Suppose X_n satisfies

$$X_n(t) = X_n(0) + \int_0^t f(X_n(s-)) \; dY_n(s) + \int_0^t f(X_n(s-)) \; dZ_n(s). \tag{5.2}$$

Assume that $\{Y_n\}$ *and* $\{H_n\}$ *are uniformly tight sequences, and for each* $t > 0$*,* $\{[Z_n, Z_n]_t\}$ *is a tight sequence. Also assume that there exist an* $\mathbb{H}^{\#}$*-semimartingale* Y *and* $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^{\#}$*-semimartingales* H, K *such that*

$$A_n := (X_n(0), Y_n, Z_n, H_n, K_n) \Rightarrow (X(0), Y, 0, H, K) := A,$$

in the same sense as in Theorem 5.1. Then (A_n, X_n) *is relatively compact, and any limit point* (A, X) *satisfies*

$$X(t) = X(0) + \int_0^t f(X(s-))\ dY(s) + \int_0^t Df(X(\widetilde{s-))f}(X(s-))\ d(H^*(s) - K(s)),$$

where the mapping $\widetilde{\ }$ *was defined in Lemma 4.2.*

Proof. As before, $\int_0^{\cdot} dZ_n(s)\otimes Z_n(s-) \Rightarrow H^*$, $[Z_n, \cdot Z_n]^{\otimes} \Rightarrow -(H+H^*)$ and $\{[Z_n, Z_n]^{\otimes}\}$, $\{K_n \equiv [Y_n, Z_n]^{\otimes}\}$ are UT.

By the infinite-dimensional Itô's formula (Theorem A.2), we have

$$f(X_n(t)) = f(X_n(0)) + \int_0^t Df(X_n(s-))\ dX_n(s) + R_n(t). \tag{5.3}$$

where R_n is given by

$$R_n(t) = \frac{1}{2}\int_0^t D^2 f(X_n(s-))d[X_n, X_n]_s^{c,\otimes} + \sum_{s\le t}[\Delta f(X_n(\cdot))(s) - Df(X_n(s-))\Delta X_n(s)].$$

where $[X_n, X_n]_t^{c,\otimes} = [X_n, X_n]_t^{\otimes} - \sum_{s\le t}\Delta X_n(s)\otimes\Delta X_n(s)$ is the continuous part of $[X_n, X_n]^{\otimes}$. As before, $\{R_n\}$ is UT.
Next, an application of the integration by parts formula (see (4.2)) gives

$$\int_0^t f(X_n(s-))\ dZ_n(s) = f(X_n(s)(Z_n(s)) - \int_0^t \overline{Z_n(s-)}\ df(X_n(s)) - [[f(X_n), Z_n]]_t,$$

where the mapping $\overline{\ }$ is defined in the proof of Lemma 4.1.

Notice that by (5.3)

$$\int_0^t \overline{Z_n(s-)}\ df(X_n(s)) = \int_0^t \overline{Z_n(s-)}Df(X_n(s-))\ dX_n(s) + \int_0^t \overline{Z_n(s-)}dR_n(s).$$

Hence, by (5.2)

$$\begin{aligned}\int_0^t \overline{Z_n(s-)}\ df(X_n(s)) &= \int_0^t \overline{Z_n(s-)}Df(X_n(s-))f(X_n(s-))dY_n(s)\\ &\quad + \int_0^t \overline{Z_n(s-)}Df(X_n(s-))f(X_n(s-))dZ_n(s) + \int_0^t \overline{Z_n(s-)}dR_n(s)\\ &= \int_0^t \overline{Z_n(s-)}Df(X_n(s-))f(X_n(s-))dY_n(s)\\ &\quad + \int_0^t Df(X_n(\widetilde{s-))f}(X_n(s-))\ dH_n(s) + \int_0^t \overline{Z_n(s-)}\ dR_n(s),\end{aligned}$$

where $H_n(s) = \int_0^t Z_n(s-)\otimes dZ_n(s)$, and the equality of the middle terms in the above two lines follows by (4.1). Next by Lemma 4.5

$$\begin{aligned}[[f(X_n), Z_n]]_t &= \int_0^t Df(X_n(\widetilde{s-))f}(X_n(s-))\ d[Z_n, Y_n]_s^{\otimes}\\ &\quad + \int_0^t Df(X_n(\widetilde{s-))f}(X_n(s-))\ d[Z_n, Z_n]_s^{\otimes} + [[R_n, Z_n]]_t.\end{aligned}$$

Putting things together, we see that

$$\int_0^t f(X_n(s-))\,dZ_n(s) = V_n(t) - \int_0^t Df(X_n(\widetilde{s-))f}(X_n(s-))\,d(H_n(s)+K_n(s)+[Z_n,Z_n]^{\otimes}(s),$$

where

$$\begin{aligned} V_n(t) &= f(X_n(s))(Z_n(s)) - \int_0^t \overline{Z_n(s-)} Df(X_n(s-))f(X_n(s-))\,dY_n(s) \\ &\quad - \int_0^t \overline{Z_n(s-)}\,dR_n(s) - [[Z_n,R_n]]_t. \end{aligned}$$

From the hypothesis, we get $V^n \Rightarrow 0$. Plugging things back in the original equation, we have,

$$\begin{aligned} X_n(t) &= X_n(0) + V_n(t) + \int_0^t f(X_n(s-))\,dY_n(s) \\ &\quad - \int_0^t Df(X_n(\widetilde{s-))f}(X_n(s-))\,d(H_n(s) + K_n(s) + [Z_n,Z_n]^{\otimes}(s)) \end{aligned}$$

Take the indexing space $\mathbb{Y} = (\mathbb{H}, \mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})$ with the norm as $\|(h,u)\|_{\mathbb{Y}} = \|h\|_{\mathbb{H}} + \|u\|_{HS}$. Consider Y_n as an $\mathbb{Y}^{\#}$-semimartingale by defining

$$Y_n((h,u),\cdot) \equiv Y_n(h,\cdot).$$

Similarly H_n, K_n and $[Z_n, Z_n]$ can be considered as $\mathbb{Y}^{\#}$-semimartingales. Since $\{Y_n\},\{H_n\},\{K_n\}$ and $\{[Z_n,Z_n]^{\otimes}\}$ are UT, the desired result now follows from Theorem 3.6.

□

Example 5.5. Let U be an adapted semimartingale taking values in a Hilbert space $\mathbb{H}$. Let $\{G_n\}$ be a sequence of adapted $\mathbb{H}$-valued semimartingales with $G_n \Rightarrow U$. Suppose that $G_n = M_n + A_n$ is a decomposition of the semimartingale G_n into its local martingale and finite variation parts and that $\{M_n\}$ and $\{A_n\}$ satisfy the assumptions of Theorem 4.6. Note that this implies $\{G_n\}$ is UT. In many examples $G_n \equiv U$. As a first example, consider the stochastic differential equation

$$X_n(t) = X_n(0) + \int_0^t \sigma(s, X_n(s), U_n(s))dU_n(s), \tag{5.4}$$

where $\sigma : \mathbb{R}\times\mathbb{R}\times\mathbb{H} \longrightarrow \mathbb{H}$ is twice continuously differentiable and

$$U_n(t) = G_n(\frac{k}{n}) + n(t-\frac{k}{n})\left(G_n(\frac{k+1}{n}) - G_n(\frac{k}{n})\right), \quad \frac{k}{n} \le t < \frac{k+1}{n}.$$

Notice that the X_n are real-valued processes. Let $\partial_i\sigma$ denote the partial derivative of σ with respect to the i-th component. Notice that $\partial_1\sigma, \partial_2\sigma \in \mathbb{H}$ and $\partial_3\sigma \in L(\mathbb{H},\mathbb{H})$. Assume that $\partial_3\sigma \in HS(\mathbb{H},\mathbb{H})$.

As discussed, U, G_n and U_n can be considered as $\mathbb{H}^{\#}$-semimartingales. It is easy to see that $\{U_n\}$ is not UT. Let $U_n = Y_n + Z_n$, where $Y_n(t) = G_n(\frac{[nt]+1}{n})$ and $Z_n = U_n - Y_n$. We claim that $\{Y_n\}$ is UT. To see this, write $Y_n(t) \equiv \overline{M}_n(t) + \overline{A}_n(t) \equiv M_n(\frac{[nt]+1}{n}) + A_n(\frac{[nt]+1}{n})$. Note that $\{\overline{M}_n\}$ is a sequence of martingales with respect to the filtration $\mathcal{F}^n_t \equiv \mathcal{F}_{[nt]+1}$, with $E[\overline{M}_n, \overline{M}_n]_t \le E[M_n, M_n]_{t+1}$. Also, $T_t(\overline{A}_n) \le T_{t+1}(A_n)$. The assertion now follows by Theorem 4.6 and the assumptions on $\{M_n\}$ and $\{A_n\}$.

Next note that $\int_0^t Z_n(s-)\otimes dZ_n(s) \to -[U,U]_t^{\otimes}/2$. To see this, note that since $Z_n(s-)=0$, at the discontinuity points of Y_n

$$\begin{aligned}
\int_0^t Z_n(s-)\otimes dZ_n(s) &= \int_0^t Z_n(s-)\otimes dU_n(s) - \sum_{s\le t} Z_n(s-)\otimes \Delta Y_n(s) = \int_0^t Z_n(s-)\otimes dU_n(s)\\
&= \sum_k n\int_{k/n}^{(k+1)/n}\Big(n\,(s-k/n)\Big(G_n(\frac{k+1}{n})-G_n(\frac{k}{n})\Big)\\
&\quad + G_n(\frac{k}{n})-G_n(\frac{k+1}{n})\Big)\otimes\Big(G_n(\frac{k+1}{n})-G_n(\frac{k}{n})\Big)\,ds\\
&= n\sum_k \left(G_n\left((k+1)/n\right)-G_n(k/n)\right)\otimes\left(G_n\left((k+1)/n\right)-G_n(k/n)\right)\\
&\quad \int_{k/n}^{(k+1)/n} n(s-k/n)-1\ ds\\
&= -\frac{1}{2}\sum_k \left(G_n\left((k+1)/n\right)-G_n(k/n)\right)\otimes\left(G_n\left((k+1)/n\right)-G_n(k/n)\right)\\
&\Rightarrow -\frac{[U,U]_t^{\otimes}}{2}, \quad \text{since } \{G_n\} \text{ is UT.}
\end{aligned}$$

Also since,

$$[Z_n,Z_n]^{\otimes} = Z_n(t)\otimes Z_n(t) - \int_0^t Z_n(s-)\otimes dZ_n(s) - \int_0^t dZ_n(s-)\otimes Z_n(s),$$

it follows that

$$[Y_n,Z_n]^{\otimes} = -[Y_n,Y_n]^{\otimes} = -[Z_n,Z_n]^{\otimes} \Rightarrow [U,U]^{\otimes}$$

Moreover, $T_t(\int Z_{n-}\otimes dZ_n) \le n\sum_k \|\left(G_n\left((k+1)/n\right)-G_n(k/n)\right)\|^2 \int_{k/n}^{(k+1)/n} |n(s-k/n)-1|\ ds \to t\ [U,U]_t/2$. It follows that for each $t>0$, $\{T_t(\int Z_{n-}\otimes dZ_n)\}$ is tight, and hence the sequence $\{\int Z_{n-}\otimes dZ_n\}$ is UT.

We next derive the limiting stochastic differential equation for (5.4). Define

$$\widetilde{X}_n(t) = (t, X_n(t), U_n(t))^T,\ \widetilde{U}_n(t) = (t, U_n(t), U_n(t))^T,\ \widetilde{U}(t) = (t, U(t), U(t))^T$$

and $F:\mathbb{R}\times\mathbb{R}\times\mathbb{H}\to L(\mathbb{R}\times\mathbb{H}\times\mathbb{H},\mathbb{R}\times\mathbb{R}\times\mathbb{H})$ by

$$F(t,x,h) = \begin{pmatrix} 1 & 0 & 0\\ 0 & \sigma(t,x,h) & 0\\ 0 & 0 & 1\end{pmatrix}$$

In other words, the operator $F(t,x,h)$ is defined as

$$F(t,x,h)y = (y_1, \langle\sigma(t,x,h), y_2\rangle_{\mathbb{H}}, y_3)^T \in \mathbb{R}\times\mathbb{R}\times\mathbb{H},\quad y=(y_1,y_2,y_3)^T\in\mathbb{R}\times\mathbb{H}\times\mathbb{H}. \tag{5.5}$$

Note that (5.4) implies

$$\widetilde{X}_n(t) = \widetilde{X}_n(0) + F(\widetilde{X}_n)\cdot\widetilde{U}_n(t) = \widetilde{X}_n(0) + F(\widetilde{X}_n)\cdot\widetilde{Y}_n(t) + F(\widetilde{X}_n)\cdot\widetilde{Z}_n(t), \tag{5.6}$$

where $\widetilde{Y}_n(t) = (t, Y_n(t), Y_n(t))$ and $\widetilde{Z}_n(t) = (0, Z_n(t), Z_n(t))$. Now the previous discussion tells that the sequences of $\mathbb{R}\times\mathbb{H}\times\mathbb{H}$-valued processes $\{\widetilde{Y}_n\}$ and $HS(\mathbb{R}\times\mathbb{H}\times\mathbb{H},\mathbb{R}\times\mathbb{H}\times\mathbb{H})$-valued processes $\{\int\widetilde{Z}_n\otimes d\widetilde{Z}_n\}$ are UT, and

$$\widetilde{Y}_n\Rightarrow\widetilde{U},\ \int\widetilde{Z}_n\otimes d\widetilde{Z}_n \Rightarrow -\frac{[\widetilde{U},\widetilde{U}]_t^{\otimes}}{2} \equiv -\frac{1}{2}\begin{pmatrix} 0 & 0 & 0\\ 0 & [U,U]^{\otimes} & [U,U]^{\otimes}\\ 0 & [U,U]^{\otimes} & [U,U]^{\otimes}\end{pmatrix} \in HS(\mathbb{R}\times\mathbb{H}\times\mathbb{H},\mathbb{R}\times\mathbb{H}\times\mathbb{H})$$

and

$$[\widetilde{Y}_n, \widetilde{Z}_n]^{\otimes} = -[\widetilde{Y}_n, \widetilde{Y}_n]^{\otimes} = -[\widetilde{Z}_n, \widetilde{Z}_n]^{\otimes} \Rightarrow -\begin{pmatrix} 0 & 0 & 0 \\ 0 & [U,U]^{\otimes} & [U,U]^{\otimes} \\ 0 & [U,U]^{\otimes} & [U,U]^{\otimes} \end{pmatrix}.$$

Here, the elements of $HS(\mathbb{R}\times\mathbb{H}\times\mathbb{H}, \mathbb{R}\times\mathbb{H}\times\mathbb{H})$ are represented by matrix like structures of the form:

$$\Xi = \begin{pmatrix} \beta & h_{12} & h_{13} \\ h_{21} & \xi_{22} & \xi_{23} \\ h_{31} & \xi_{32} & \xi_{33} \end{pmatrix}, \quad \beta \in \mathbb{R},\ h_{ij} \in \mathbb{H},\ \xi_{ij} \in \mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}.$$

In other words, the Hilbert-Schmidt operator $\Xi \in HS(\mathbb{R} \times \mathbb{H} \times \mathbb{H}, \mathbb{R} \times \mathbb{H} \times \mathbb{H})$ is defined as

$$\Xi(y) = (\beta y_1 + \langle h_{12}, y_2\rangle_{\mathbb{H}} + \langle h_{13}, y_3\rangle_{\mathbb{H}}, y_1 h_{21} + \xi_{22}(y_2) + \xi_{23}(y_3), y_1 h_{31} + \xi_{32}(y_2) + \xi_{33}(y_3))$$

Observe that the derivative operator, $DF(t,x,h) \in L(\mathbb{R}\times\mathbb{R}\times\mathbb{H}, L(\mathbb{R}\times\mathbb{H}\times\mathbb{H}, \mathbb{R}\times\mathbb{R}\times\mathbb{H}))$ is given by

$$DF(t,x,h)(b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_1\partial_1\sigma + b_2\partial_2\sigma + \partial_3\sigma b_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b \in \mathbb{R} \times \mathbb{R} \times \mathbb{H}. \tag{5.7}$$

Now an application of Theorem 5.4 to (5.6) gives the limiting stochastic differential equation as

$$\widetilde{X}(t) = \widetilde{X}(0) + \int_0^t F(\widetilde{X}(s-))\, d\widetilde{U}(s) + \frac{1}{2}\int_0^t \widetilde{DF(\widetilde{X}(s-))F(\widetilde{X}(s-))}\, d[\widetilde{U}, \widetilde{U}]_t^{\otimes}, \tag{5.8}$$

where the mapping $\widetilde{\ }$ was defined in Lemma 4.2. Observe that in the present example (see Section A.4 for a proof),

$$\widetilde{DF(\widetilde{x})F(\widetilde{x})}(\Xi) = (0, \langle \partial_1\sigma, h_{21}\rangle_{\mathbb{H}} + \langle \partial_2\sigma\otimes\sigma, \xi_{22}\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}} + \langle \partial_3\sigma, \xi_{23}\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}}, 0)^T. \tag{5.9}$$

Therefore, considering the middle component of (5.8), it follows that $X_n \Rightarrow X$ where X satisfies

$$X(t) = X(0) + \int_0^t \sigma dU(s) + \frac{1}{2}\int_0^t (\partial_3\sigma + \partial_2\sigma\otimes\sigma) d[U,U]_s^{\otimes}$$

Remark 5.6. *Example 5.5 is a generalization of the results obtained by Nakao and Yamato [17] (see Theorem 1.1) and Konecny [10] to stochastic differential equations driven by infinite-dimensional semimartingales. One important example of U in Example 5.5 is an $\mathbb{H}$-valued Brownian motion W with covariance operator Q, where Q is nuclear (see [5]). In other words, for $h_1, h_2 \in \mathbb{H}$, $[W(h_1,\cdot), W(h_2,\cdot)]_t = \langle Qh_1, h_2\rangle_{\mathbb{H}} t$. The tensor quadratic variation of the process W is given by $[W,W]_t^{\otimes} = tQ$.*

Remark 5.7. *Using the same technique, Example 5.5 can easily be extended to the case where the solutions X_n are also infinite-dimensional. Also, the approximation of the semimartingale U by linear interpolation is just chosen for illustrative purpose. It can be easily extended to more general approximation techniques.*

Example 5.8. As a second example, we consider a space-time Gaussian white noise and its mollified version as its approximation. More precisely, let W be an $\{\mathcal{F}_t\}$-adapted space-time Gaussian white noise and $B_r(x) \subset \mathbb{R}^d$ denote the ball of radius r, centered

at x. Let $\rho : \mathbb{R}^d \to [0, \infty)$ and $\eta : \mathbb{R} \to [0, \infty)$ be smooth functions with $supp(\rho) \subset B_1(0)$, $supp(\eta) \subset (-1, 0)$, and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1, \int_{-\infty}^{\infty} \eta(s)\, ds = 1$.
Define $\rho_n(x) = n^d \eta(nx)$, and $\eta_n(s) = n\eta(nx)$. Notice that ρ_n is supported on $B_{1/n}(0) \subset \mathbb{R}^d$, and η_n is supported on $[-1/n, 0]$, and $\int_{B_{1/n}(0)} \rho_n(x)\, dx = 1$, $\int_{[-1/n,0]} \eta_n(s) = 1$.
Define

$$\dot{W}_n(x,t) = \int_{\mathbb{R}^d \times [0,\infty)} \rho_n(x-y)\eta_n(s-t) W(dy \times ds). \tag{5.10}$$

For $h \in L^2(\mathbb{R}^d)$, let

$$W_n(h,t) = \int_{\mathbb{R}^d \times [0,\infty)} h(x)\dot{W}_n(x,s) dx\, ds, \tag{5.11}$$

Notice that $\{W_n\}$ is a sequence of $\{\mathcal{F}_t\}$-adapted $\mathbb{H}^\#$-semimartingales, for $\mathbb{H} = L^2(\mathbb{R}^d)$, and $W_n \overset{P}{\to} W$ in the sense that, for any finite $h_1, \ldots, h_m \in L^2(\mathbb{R}^d)$

$$(W_n(h_1, \cdot), \ldots, W_n(h_m, \cdot)) \overset{P}{\to} (W(h_1, \cdot), \ldots, W_n(h, \cdot)).$$

Consider the following SDE

$$X_n(t) = X_n(0) + \int_{\mathbb{R}^d \times [0,t)} g(X_n(s), x)\dot{W}_n(x,s) dx\, ds.$$

Assume that

- $|g(\cdot, x)| \le \kappa(x)$ for some $\kappa \in L^1$ so that the integration in the right side is defined;
- $g(y, x) = Sf(y, \cdot)(x)$, where S is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$. For example, if $\int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma^2(x,u) dx\, du < \infty$, then S could be defined as

$$Sh(x) = \int_{\mathbb{R}^d} h(u)\gamma(x,u)\, du; \tag{5.12}$$

  (In other words, if S is defined by (5.12), then $g(y,x) = \int_{\mathbb{R}^d} f(y,u)\gamma(x,u)\, du$.)
- $\sup_u \int_{R^d} |f(u,x)|^2\, dx < \infty, \sup_u \int_{R^d} |\partial_1 f(u,x)|^2\, dx < \infty$ and $\sup_u \int_{R^d} |\partial_1^2 f(u,x)|^2\, dx < \infty$, where $\partial_1 f$ and $\partial_1^2 f$ denote the first and second order partial derivative of f with respect to the first co-ordinate.

The above assumptions imply that the mapping

$$u \in \mathbb{R}^d \to f(u, \cdot) \in L^2(\mathbb{R}^d)$$

is bounded in $L^2(\mathbb{R}^d)$ with bounded first and second-order (Frechet) derivative.
Thus X_n satisfies

$$X_n(t) = X_n(0) + Sf(X_n(\cdot), \cdot) \cdot W_n(t), \tag{5.13}$$

that is

$$X_n(t) = X_n(0) + \int_{\mathbb{R}^d \times [0,t)} Sf(X_n(s), \cdot)(x)\dot{W}_n(x,s) dx\, ds.$$

Observe that $\{W_n\}$ is not a UT sequence, as

$$\int_0^t W_n(h,s) dW_n(h,s) = \frac{1}{2} W_n(h,t)^2 \nRightarrow \int_0^t W(h,s) dW(h,s) = \frac{1}{2}(W(h,t)^2 - \|h\|^2 t).$$

We apply Theorem 5.1 to find the limit of $\{X_n\}$.
First, notice that the SDE (5.13) could be written as

$$X_n(t) = X_n(0) + f(X_n(\cdot), \cdot) \cdot \mathcal{W}_n(t), \tag{5.14}$$

where $\mathcal{W}_n$ is defined as

$$\mathcal{W}_n(h,t) = \int_{\mathbb{R}^d \times [0,t)} (Sh)(x)\dot{W}_n(x,s) dx\ ds, \tag{5.15}$$

It is easy to check that

$$E[\sup_{s \le t} |\mathcal{W}_n(h,s) - W(Sh,s)|^2] \to 0.$$

Observe that

$$\begin{aligned} \mathcal{W}_n(h,t) &= \int_{\mathbb{R}^d \times [0,t)} (Sh)(x)(\int_{\mathbb{R}^d \times [0,\infty)} \rho_n(x-y)\eta_n(r-s)W(dy \times dr))dx\ ds \\ &= \int_{\mathbb{R}^d \times [0,t)} (\int_{\mathbb{R}^d \times [0,t)} (Sh)(x)\rho_n(x-y)\eta_n(r-s)dx\ ds)W(dy \times dr) \\ &= \int_{\mathbb{R}^d \times [0,t)} (S_n Sh)(y)(\int_0^t (\eta_n(r-s)\ ds)W(dy \times dr) \end{aligned} \tag{5.16}$$

where the operator S_n is defined as

$$S_n h(x) = \int_{\mathbb{R}^d} h(y)\rho_n(x-y)\ dy.$$

Note that $\|S_n\|_{op} \le 1$. Write

$$\mathcal{W}_n(h,t) = Y_n(h,t) + Z_n(h,t).$$

where $Y_n(h,t) \equiv W(S_n Sh, t)$. Define

$$\widetilde{\mathcal{W}}_n(t) = \sum_j \mathcal{W}_n(e_j,t)e_j$$

Notice that the infinite sum above converges, as from (5.16)

$$\begin{aligned} \sup_{s \le t} E(\| \sum_{j=K}^M \mathcal{W}_n(e_j,t)e_j\|_2^2 &= \sup_{s \le t} \sum_{j=K}^M E(\mathcal{W}_n(e_j,s)^2) \\ &\le \sum_{j=K}^M \|S_n S e_j\|_2^2\ t \\ &\le \sum_{j=K}^M \|S e_j\|_2^2\ t, \quad \text{as } \|S_n\|_{op} \le 1 \\ &\to 0, \text{ as } K, M \to \infty, \quad \text{since } S \text{ is Hilbert-Schmidt.} \end{aligned}$$

It follows that $\widetilde{\mathcal{W}}_n \in D_{L^2(\mathbb{R}^d)}[0,\infty)$. Similarly,

$$\widetilde{Y}_n \equiv \sum_j Y_n(e_j,\cdot)e_j \in D_{L^2(\mathbb{R}^d)}[0,\infty).$$

Thus, $\widetilde{\mathcal{W}}_n$ and $\widetilde{Y}_n$ are versions of $\mathcal{W}_n$ and Y_n taking values in $L^2(\mathbb{R}^d)$ and it is easily checked that $H \cdot \mathcal{W}_n = H \cdot \widetilde{\mathcal{W}}_n$ and $H \cdot Y_n = H \cdot \widetilde{Y}_n$. With a slight abuse of notation, we will continue to use Y_n and $\mathcal{W}_n$ instead of $\widetilde{Y}_n$ and $\widetilde{\mathcal{W}}_n$, and consider them as $L^2(\mathbb{R}^d)$-valued semimartingales.

Put $K^n \equiv [Y_n, Z_n]^{\otimes} = -[Z_n, Z_n]^{\otimes} = -[Y_n, Y_n]^{\otimes}$, and notice that

$$\begin{aligned}[Y_n, Y_n]^{\otimes}_t &= \sum_{j,k} [Y_n(e_j, \cdot), Y_n(e_j, \cdot)]_t e_j \otimes e_k \\ &= t \sum_{j,k} \langle S_n S e_j, S_n S e_k \rangle e_j \otimes e_k = t(S_n S)^*(S_n S).\end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$. Recall that K_n is an $\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}$-valued (hence a standard $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^{\#}$) semimartingale. We verify $K_n(t) \equiv [Y_n, Z_n]^{\otimes}_t = -t(S_n S)^*(S_n S)$ converges to $-tS^*S$ in the sense of convergence of $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})^{\#}$-semimartingale, that is we need to verify for $u = \sum_{i=1}^{I} x_i \otimes y_i$, $\langle u, tK_n \rangle \to \langle u, tI \rangle$. This follows because

$$\begin{aligned}\langle u, tK_n \rangle &= \sum_i \langle x_i, tK_n y_i \rangle = -t \sum_i \langle x_i, (S_n S)^*(S_n S) y_i \rangle \\ &= -t \sum_i \langle S_n S x_i, S_n S y_i \rangle \longrightarrow -t \sum_i \langle S x_i, S y_i \rangle.\end{aligned}$$

The last equality is because $S_n S \to S$ in the strong operator topology.
Similarly, $H_n(t) \equiv \int_0^t Z_n(s) \otimes dZ_n(s) \to -\frac{t}{2} S^*S$.

Next, observe that $\{Y_n\}$ is a uniformly tight sequence, and $Y_n \overset{P}{\to} \mathcal{W}$ in the sense that, for any finite $h_1, \ldots, h_m \in L^2(\mathbb{R}^d)$

$$(Y_n(h_1, \cdot), \ldots, Y_n(h_m, \cdot)) \overset{P}{\to} (\mathcal{W}(h_1, \cdot), \ldots, \mathcal{W}(h, \cdot)),$$

where $\mathcal{W}(h, t) = W(Sh, t)$, that is, $\mathcal{W}$ is a space-time Gaussian white noise with

$$[\mathcal{W}(h, \cdot), \mathcal{W}(g, \cdot)]_t = t \langle Sh, Sg \rangle.$$

To apply Theorem 5.1, we only need to prove that the sequence $\{H_n \equiv \int_0^t Z_n(s-) \otimes dZ_n(s)\}$ is uniformly tight. For this purpose, we first compute $E\|Z_n(t)\|_2^2$. Notice that $\|Z_n(t)\|_2^2 = \sum_j \|Y_n(e_j, t) - \mathcal{W}_n(e_j, t)\|^2$. For any $h \in L^2(\mathbb{R}^d)$, we have using (5.16)

$$\begin{aligned}\mathcal{W}_n(h, t) - Y_n(h, t) &= \int_{\mathbb{R}^d \times [0,t)} (S_n Sh)(y) (\int_0^t (\eta_n(r-s)\, ds) W(dy \times dr) \\ &\quad - \int_{\mathbb{R}^d \times [0,t)} (S_n Sh)(y) W(dy \times dr) \\ &= \int_{\mathbb{R}^d \times [0,t)} (S_n Sh)(y) (\int_0^t \eta_n(r-s)\, ds - 1) W(dy \times dr) \\ &= \int_{\mathbb{R}^d \times [0,t-1/n)} (S_n Sh)(y) (\int_0^t \eta_n(r-s)\, ds - 1) W(dy \times dr) \\ &\quad + \int_{\mathbb{R}^d \times [t-1/n,t)} (S_n Sh)(y) (\int_0^t \eta_n(r-s)\, ds - 1) W(dy \times dr) \\ &= \int_{\mathbb{R}^d \times [t-1/n,t)} (S_n Sh)(y) (-\int_t^{r+1/n} \eta_n(r-s)\, ds) W(dy \times dr).\end{aligned}$$

The last equality is because $\int_0^t \eta_n(r-s)\, ds = 1,$ if $t \geq r + 1/n$. Thus,

$$E(\mathcal{W}_n(h, t) - Y_n(h, t))^2 = \int_{\mathbb{R}^d \times [t-1/n,t)} |S_n Sh(y)|^2 dy\, dr \leq \frac{1}{n} \|S_n Sh\|_2^2$$

It follows that

$$E\|Z_n(t)\|_2^2 \le \frac{1}{n}\sum_j \|S_nSe_j\|_2^2 = \frac{1}{n}\|S_nS\|_{HS} \le \frac{1}{n}\|S\|_{HS}. \tag{5.17}$$

Take G to be an $\{\mathcal{F}_t^n\}$-adapted $(\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H})$-valued cadlag process, and $\|G(s)\|_{HS} \le 1$. Notice that by Theorem 2.24

$$\begin{aligned} G_n \cdot H_n(t) &= \int_0^t G(s-)(Z_n(s-))\, dZ_n(s) \\ &= \int_0^t G(s-)(Z_n(s-))\, dY_n(s) - \int_0^t G(s-)(Z_n(s-))\, d\mathcal{W}_n(s) \\ &= A + B. \end{aligned}$$

We have,

$$\begin{aligned} E(B^2) &= \int_0^t \|S_nS[G(s)(Z_n(s))]\|_2^2\, ds \\ &\le \int_0^t \|S_nS\|_{op}\|G(s)\|_{op}\|Z_n(s)\|_2^2\, ds \le \int_0^t \|S_nS\|_{op}\|G(s)\|_{HS}\|Z_n(s)\|_2^2\, ds \\ &\le \|S\|_{op}\int_0^t \|Z_n(s)\|^2\, ds \le \frac{1}{n}\|S\|_{HS}\|S\|_{op}t \end{aligned}$$

and

$$\begin{aligned} A &= \int_0^t G(s-)(Z_n(s-))d\mathcal{W}_n(s) \\ &= \int_0^t SG(s-)(Z_n(s-))(x)\int_{\mathbb{R}^d\times[0,\infty)} \rho_n(x-y)\eta_n(r-s)W(dy\times dr)dxds \\ &= \int_0^t \int_{\mathbb{R}^d\times[0,\infty)} S_nSG(s-)(Z_n(s-))(y)\eta_n(r-s)W(dy\times dr)ds. \end{aligned}$$

Thus,

$$\begin{aligned} E(A^2) &\le \int_0^t E[\int_{\mathbb{R}^d\times[0,\infty)} S_nSG(s-)(Z_n(s-))(y)\eta_n(r-s)W(dy\times dr)]^2 ds \\ &= \int_0^t (\int_{\mathbb{R}^d\times[0,\infty)} E|S_nSG(s-)(Z_n(s-))(y)|^2|\eta_n(r-s)|^2\, dy\, dr)\, ds \\ &\le Cn\int_0^t (\int_{\mathbb{R}^d} E|S_nSG(s-)(Z_n(s-))(y)|^2 dy)\, ds, \quad C = \int_{-\infty}^{\infty}\eta^2(r)\, dr \\ &= Cn\int_0^t E\|S_nSG(s-)Z_n(s-)\|_2^2\, ds \\ &\le Cn\|S\|_{op}\int_0^t E\|Z_n(s-)\|_2^2\, ds \\ &\le Ct\|S\|_{op}\|S\|_{HS}, \quad \text{using (5.17)}. \end{aligned}$$

It follows that $\{H_n\}$ is uniformly tight.

Now if $X_n(0) \xrightarrow{P} X(0)$, then applying Theorem 5.1, we conclude $X_n \xrightarrow{P} X$, where X satisfies

$$\begin{aligned} X(t) = X(0) &+ \int_{\mathbb{R}^d\times[0,t)} Sf(X(s),\cdot)(x)W(dx\times ds) \\ &+ \int_0^t Df(X_{s-})\otimes f(X_{s-})\, d(-\frac{s}{2}SS^* + sS^*S). \end{aligned}$$

If $S = S^*$, which will be the case if S is defined by (5.12), then the above SDE could be written as

$$X(t) = X(0) + \int_{\mathbb{R}^d\times[0,t)} Sf(X(s),\cdot)(x)W(dx\times ds) + \frac{1}{2}\int_{\mathbb{R}^d\times[0,t)} Sf(X(s),\cdot)(x)\partial_1 Sf(X(s),\cdot)(x)\ dx\ ds.$$

Example 5.9. Let $\{\xi_n\}$ be a ϕ-irreducible Markov chain taking values in a separable metric space U. Let P denote the transition kernel of $\{\xi_n\}$. Assume that the chain is ergodic with unique stationary distribution π. Let $\mathbb{H} = L^2(U,\pi)$. Let $\{\widetilde{W}_n\}, \{\widetilde{Y}_n\}$ and $\{\widetilde{Z}_n\}$ be $\mathbb{H}^\#$-semimartingales defined by

$$\begin{aligned}\widetilde{W}_n(h,t) &\equiv \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}(Ph(\xi_k) - h(\xi_k))\\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}(Ph(\xi_{k-1}) - h(\xi_k)) + \frac{1}{\sqrt{n}}\left(Ph(\xi_{[nt]}) - h(\xi_0)\right)\\ &\equiv \widetilde{Y}_n(h,t) + \widetilde{Z}_n(h,t).\end{aligned}$$

Let S be a Hilbert-Schmidt operator from $\mathbb{H}$ to $\mathbb{H}$. Define $Y_n(h,t) \equiv \widetilde{Y}_n(Sh,t)$ and $Z_n(h,t) \equiv \widetilde{Z}_n(Sh,t)$. Let $\{e_k\}$ be an orthonormal basis of $\mathbb{H}$. With a slight abuse of notation, define

$$Y_n(t) = \sum_k Y_n(e_k,t)e_k, \quad Z_n(t) = \sum_k Z_n(e_k,t)e_k$$

Then Y_n and Z_n are $\mathbb{H}$-valued processes. To see this, first note that

$$\sup_{t\le T}\mathsf{E}[\|\sum_{j=K}^{M} Y_n(e_j,t)e_j\|_2^2] = \sup_{t\le T}\sum_{j=K}^{M}\mathsf{E}[\|Y_n(e_j,t)\|_2^2]. \tag{5.18}$$

Now observe that for any $h\in\mathbb{H}$,

$$\mathsf{E}[\|Y_n(h,t)\|_2^2] = \frac{1}{n}\sum_{j=1}^{[nt]}\mathsf{E}[PSh(\xi_{k-1}) - Sh(\xi_k)]^2 \le 2\frac{[nt]}{n}(\|PSh\|_2^2 + \|Sh\|_2^2) \le 4\frac{[nt]}{n}\|Sh\|_2^2.$$

It follows from (5.18) that

$$\sup_{t\le T}\mathsf{E}[\|\sum_{j=K}^{M} Y_n(e_j,t)e_j\|_2^2] \le 4\frac{[nT]}{n}\sum_{j=K}^{M}\|Se_j\|_2^2 \to 0, \text{ as } K,M\to\infty, \text{ since } S \text{ is Hilbert-Schmidt.}$$

Similarly, Z_n is an $\mathbb{H}$-valued process.
Consider a sequence of SDEs of the form (5.2) driven by $\{Y_n\}$ and $\{Z_n\}$. We show that $\{Y_n\}$ and $\{Z_n\}$ satisfy the assumptions of Theorem 5.4.

For each n, Y_n is a martingale, and by the martingale central limit theorem it follows that for any collection of $h_1,\dots,h_m\in\mathbb{H}$

$$(Y_n(h_1,\cdot),\dots,Y_n(h_m\cdot)) \Rightarrow W,$$

where W is an m-dimensional Gaussian process with covariance matrix, tC, and C is given by

$$C_{i,j} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(PSh_i(\xi_{k-1}) - Sh_i(\xi_k))(PSh_j(\xi_{k-1}) - Sh_j(\xi_k))$$
$$= \int \pi(dx)\int P(x,dy)(PSh_i(x) - Sh_i(y))(PSh_j(x) - Sh_j(y)).$$

Next, we prove that $\{Y_n\}$ is UT. By Theorem 4.6, it is enough to show that $\sup_n \mathsf{E}[Y_n, Y_n]_t < \infty$. To see this, observe that

$$[Y_n, Y_n]_t = \text{trace}([Y_n, Y_n]_t^{\otimes}) = \sum_k [Y_n(e_k,\cdot), Y_n(e_k,\cdot)]_t,$$

where $\{e_k\}$ is an orthonormal basis of $\mathbb{H}$.
Note that $[Y_n(h,\cdot), Y_n(g,\cdot)]_t = \frac{1}{n}\sum_{k=1}^{[nt]}(PSh(\xi_{k-1}) - Sh(\xi_k))(PSg(\xi_{k-1}) - Sg(\xi_k))$ and therefore,

$$\mathsf{E}[Y_n, Y_n]_t = \frac{[nt]}{n}\sum_k \int \pi(dx)\int P(x,dy)(PSe_k(x) - Se_k(y))^2$$
$$\le 4\frac{[nt]}{n}\sum_k \|Se_k\|_2^2.$$

Since S is Hilbert-Schmidt, it follows that $\sup_n \mathsf{E}[Y_n, Y_n]_t < \infty$.

Also, it is immediate that $Z_n \Rightarrow 0$, in the sense that for any collection of $h_1,\ldots,h_m \in \mathbb{H}$ $(Z_n(h_1,\cdot),\ldots,Z_n(h_m\cdot)) \Rightarrow 0$. Next, note that since $Z_n(h,t) = \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}(PSh(\xi_k) - PSh(\xi_{k-1})$

$$[Z_n(g,\cdot), Z_n(h,\cdot)]_t = \frac{1}{n}\sum_{k=1}^{[nt]}(PSg(\xi_k) - PSg(\xi_{k-1})(PSh(\xi_k) - PSh(\xi_{k-1})$$
$$\Rightarrow t\int \pi(dx)\int P(x,dy)(PSg(y) - PSg(x))(PSh(y) - PSh(x)).$$

It follows that for any $g_1, h_1, \ldots, g_m, h_m \in \mathbb{H}$,

$$([Z_n, Z_n]_t^{\otimes}(g_1\otimes h_1),\ldots,[Z_n, Z_n]_t^{\otimes}(g_m\otimes h_m)) = ([Z_n(g_1,\cdot), Z_n(h_1,\cdot)]_t,\ldots,[Z_n(g_m,\cdot), Z_n(h_m,\cdot)]_t)$$
$$\Rightarrow t\rho,$$

where $\rho = (\rho_i)_{i=1}^m$ and $\rho_i = \int \pi(dx)\int P(x,dy)(PSg_i(y) - PSg_i(x))(PSh_i(y) - PSh_i(x))$.
Also,

$$[Z_n(g,\cdot), Y_n(h,\cdot)]_t = \frac{1}{n}\sum_{k=1}^{[nt]}(PSg(\xi_k) - PSg(\xi_{k-1})(PSh(\xi_{k-1}) - Sh(\xi_k)$$
$$\Rightarrow t\int \pi(dx)\int P(x,dy)(PSg(y) - PSg(x))(PSh(x) - Sh(y)).$$

Therefore,

$$([Z_n, Y_n]_t^{\otimes}(g_1\otimes h_1),\ldots,[Z_n, Y_n]_t^{\otimes}(g_m\otimes h_m)) \Rightarrow t\rho',$$

where $\rho' = (\rho'_i)_{i=1}^m$ and $\rho'_i = \int \pi(dx)\int P(x,dy)(PSg_i(y) - PSg_i(x))(PSh_i(x) - Sh_i(y))$.
Similarly,

$$(\int_0^t Z_n(s-)\otimes dZ_n(s)(g_1\otimes h_1),\ldots,\int_0^t Z_n(s-)\otimes dZ_n(s)^{\otimes}(g_m\otimes h_m)) \Rightarrow t\rho'',$$

where $\rho'' = (\rho''_i)_{i=1}^m$ and $\rho''_i = \int \pi(dx) \int P(x,dy) PSg_i(x)(PSh_i(y) - Sh_i(x))$. Finally, we need to prove that $\{\int Z_n(s-) \otimes dZ_n(s)\}$ is UT. By Theorem 4.6, it is enough to show that for every $t > 0$, $\{T_t(\int Z_n(s-) \otimes dZ_n(s))\}$ is tight. Call $H_n = \int Z_n(s-) \otimes dZ_n(s)$. Recall that if $\{e_k\}$ is an orthonormal basis of $\mathbb{H}$, then $\{e_k \otimes e_l\}$ forms an orthonormal basis of $\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}$. Hence,

$$
\begin{aligned}
T_t(\int Z_n(s-) \otimes dZ_n(s)) &= \sup_{\{t_i\}} \sum_i \sqrt{\sum_k |H_n(e_k \otimes e_l, t_i) - H_n(e_k \otimes e_l, t_{i-1})|^2} \\
&= \frac{1}{n} \sum_{j=1}^{[nt]} \sqrt{\sum_{k,l} |PSe_k(\xi_{j-1})|^2 |PSe_l(\xi_j) - PSe_l(\xi_{j-1})|^2} \\
&\leq \sqrt{\sum_k \frac{1}{n} \sum_{j=1}^{[nt]} |PSe_k(\xi_{j-1})|^2} \sqrt{\sum_l \frac{1}{n} \sum_{j=1}^{[nt]} |PSe_l(\xi_j) - PSe_l(\xi_{j-1})|^2} \\
&\Rightarrow \sqrt{\sum_k \|PSe_k\|^2} \sqrt{\sum_l \int \pi(dx) \int P(x,dy) |PSe_l(x) - PSe_l(y)|^2} < \infty.
\end{aligned}
$$

It follows that $\{T_t(\int Z_n(s-) \otimes dZ_n(s))\}$ is tight.

Appendix

A.1 Tensor product

All the results in this section are from Ryan [19]. Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces. Let $B(\mathbb{X} \times \mathbb{Y}, \mathbb{Z})$ be the space of all bounded bilinear forms from $\mathbb{X} \times \mathbb{Y} \to \mathbb{Z}$, that is set of all bilinear forms A such that

$$\|A(x,y)\|_{\mathbb{Z}} \leq \gamma \|x\|_{\mathbb{X}} \|y\|_{\mathbb{Y}}, \text{ for some } \gamma > 0.$$

The smallest such constant γ is the norm of A, and will be denoted by $\|A\|$. If $\mathbb{Z} = \mathbb{R}$, then we will denote $B(\mathbb{X} \times \mathbb{Y}, \mathbb{Z})$ by $B(\mathbb{X} \times \mathbb{Y})$.

For a vector space V, let $V^\#$ denote the algebracic dual of V. The tensor product $\mathbb{X} \otimes \mathbb{Y}$ will be constructed as $B(\mathbb{X} \times \mathbb{Y})^\#$, by defining the action of $x \otimes y$ on $B(\mathbb{X} \times \mathbb{Y})^\#$ as

$$x \otimes y(A) = A(x,y), \quad x \in \mathbb{X}, y \in \mathbb{Y}.$$

Thus, a typical tensor $u \in \mathbb{X} \otimes \mathbb{Y}$, has the form

$$u = \sum_{i=1}^{I} x_i \otimes y_i. \tag{A.1}$$

Notice that by definition $u = 0$, if

$$\sum_{i=1}^{I} A(x_i, y_i) = 0, \quad \text{for all } A \in B(\mathbb{X} \times \mathbb{Y}).$$

The following theorem gives an easy criterion to check if $u = 0$.

Theorem A.1. *Let u be a tensor of the form (A.1). Then $u = 0$ if and only if*

$$\sum_{i=1}^{I} \phi(x_i)\psi(y_i) = 0, \quad \textit{for all } \phi \in \mathbb{X}^*, \psi \in \mathbb{Y}^*.$$

So far we have introduced tensor product $\mathbb{X} \otimes \mathbb{Y}$ as a vector space. Many choices of norm exist to complete the space $\mathbb{X} \otimes \mathbb{Y}$, e.g the projective norm, the nuclear norm etc. Here however, we focus on the case when $\mathbb{X}$ and $\mathbb{Y}$ are separable Hilbert spaces and the norm considered on $\mathbb{X} \otimes \mathbb{Y}$ is Hilbert-Schmidt.

A.2 Hilbert-Schmidt operator and tensor product

Let $\mathbb{X}$ and $\mathbb{Y}$ be two separable Hilbert spaces. Let $\{e_j\}$ be a complete orthonormal system of $\mathbb{X}$. $S \in L(\mathbb{X}, \mathbb{Y})$ is a Hilbert-Schmidt operator if

$$\sum_j \|Se_j\|_{\mathbb{K}}^2 < \infty.$$

The quantity in the left side does not depend on the orthonormal system $\{e_j\}$, and its square root is defined as the Hilbert-Schmidt norm $\|S\|_{HS}$. The space of all Hilbert-Schmidt operators is denoted by $HS(\mathbb{X}, \mathbb{Y})$. $HS(\mathbb{X}, \mathbb{Y})$ is a separable Hilbert space.

Let $h, h' \in \mathbb{X}$ and $k, k' \in \mathbb{Y}$. Define an inner product $\langle \cdot, \cdot \rangle_{HS}$ on $\mathbb{X} \otimes \mathbb{Y}$ by

$$\langle h \otimes k, h' \otimes k' \rangle_{HS} = \langle h, h' \rangle_{\mathbb{X}} \ \langle k, k, \rangle_{\mathbb{Y}}.$$

Let $\mathbb{X}\widehat{\otimes}_{HS}\mathbb{Y}$ denote the completion of the space with respect to the inner product $\langle \cdot, \cdot \rangle_{HS}$. Then $\mathbb{X}\widehat{\otimes}_{HS}\mathbb{Y}$ is isometrically isomorphic to $HS(\mathbb{Y}, \mathbb{X})$ and also $HS(\mathbb{X}, \mathbb{Y})$.
If $\{e_j\}$ and $\{f_j\}$ are complete orthonormal systems of $\mathbb{X}$ and $\mathbb{Y}$, then $\{e_j \otimes f_k\}_{j,k}$ forms a complete orthonormal system of $\mathbb{X}\widehat{\otimes}_{HS}\mathbb{Y}$. If $T \in HS(\mathbb{X}, \mathbb{Y})$, then T can be represented as

$$T = \sum_{j,k} \langle Te_j, f_k \rangle e_j \otimes f_k.$$

A.3 Infinite-dimensional Itô's lemma

Theorem A.2. *[15, Theorem 27.2] Let $\mathbb{X}$ and $\mathbb{Y}$ be two separable Hilbert spaces, Z an adapted $\mathbb{X}$-valued semimartingale and $\phi : \mathbb{X} \to \mathbb{Y}$ be a twice continuously differentiable function with first and second-order derivatives denoted by $D\phi$ and $D^2\phi$ respectively. Assume that for each $x \in \mathbb{X}$, $D^2\phi(x)$ is an element of $L(\mathbb{X}\widehat{\otimes}_{HS}\mathbb{X}, \mathbb{Y})$ and the mapping $x \to D^2\phi(x)$ is uniformly continuous on any bounded subset of $\mathbb{X}$. Then*

$$\begin{aligned}
\phi(Z_t) &= \phi(Z_0) + \int_0^t D\phi(Z(s-))\, dZ(s) + \frac{1}{2}\int_0^t D^2\phi(Z(s-))\, d[Z,Z]^{\otimes}(s) \\
&\quad + \sum_{s \le t} \left(\phi(Z(s)) - \phi(Z(s-)) - D\phi(Z(s-))\Delta Z(s) - \frac{1}{2} D^2\phi(Z(s-))\Delta Z(s) \otimes \Delta Z(s) \right) \\
&= \phi(Z_0) + \int_0^t D\phi(Z(s-))\, dZ(s) + \frac{1}{2}\int_0^t D^2\phi(Z(s-))\, d[Z,Z]_s^{c,\otimes} \\
&\quad + \sum_{s \le t} \left(\phi(Z(s)) - \phi(Z(s-)) - D\phi(Z(s-))\Delta Z(s) \right)
\end{aligned}$$

where $[Z,Z]_t^{c,\otimes} = [Z,Z]_t^{\otimes} - \sum_{s \le t} \Delta Z(s) \otimes \Delta Z(s)$.

A.4 Proof of (5.9)

Let $\{\gamma_k\}$ be an orthonormal basis of the Hilbert space $\mathbb{H}$. Then a basis for $\mathbb{R} \times \mathbb{H} \times \mathbb{H}$ is given by $\{e^1 = (1,0,0)^T, e_i^2 = (0, \gamma_i, 0)^T, e_i^3 = (0,0,\gamma_i)^T : i = 1, 2, \ldots\}$. Consequently, a basis for $HS(\mathbb{R} \times \mathbb{H} \times \mathbb{H}, \mathbb{R} \times \mathbb{H} \times \mathbb{H})$ is given by $\{e^1 \otimes e_i^k, e_i^k \otimes e^1, e_i^k \otimes e_j^l : k, l = 2, 3,\ i, j =$

$1, 2, \ldots\}$. Now an expansion of $\Xi \in HS(\mathbb{R}\times\mathbb{H}\times\mathbb{H}, \mathbb{R}\times\mathbb{H}\times\mathbb{H})$ gives

$$\begin{aligned}\Xi = {} & \beta e^1\otimes e^1 + \sum_i \langle h_{12}, \gamma_i\rangle_{\mathbb{H}} e^1\otimes e_i^2 + \sum_i \langle h_{13}, \gamma_i\rangle_{\mathbb{H}} e^1\otimes e_i^3 \\ & + \sum_i \langle h_{21}, \gamma_i\rangle_{\mathbb{H}} e_i^2\otimes e^1 + \sum_{i,j} \langle \xi_{22}, \gamma_i\otimes\gamma_j\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}} e_i^2\otimes e_j^2 + \sum_{i,j} \langle \xi_{23}, \gamma_i\otimes\gamma_j\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}} e_i^2\otimes e_j^3 \\ & + \sum_i \langle h_{31}, \gamma_i\rangle_{\mathbb{H}} e_i^3\otimes e^1 + \sum_{i,j} \langle \xi_{32}, \gamma_i\otimes\gamma_j\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}} e_i^3\otimes e_j^2 + \sum_{i,j} \langle \xi_{33}, \gamma_i\otimes\gamma_j\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}} e_i^3\otimes e_j^3.\end{aligned} \tag{A.2}$$

Observe that

$$F(\widetilde{x})(e^1) = (1,0,0)^T,\ F(\widetilde{x})(e_i^2) = (0, \langle\sigma,\gamma_i\rangle_{\mathbb{H}}, 0),\quad F(\widetilde{x})(e_i^3) = (0,0,\gamma_i)$$

By the definition of the mapping $\widetilde{\ }$ in Lemma 4.2, and using (5.5) and (5.7)

$$\begin{aligned}\widetilde{DF(\widetilde{x})F}(\widetilde{x})(e_i^2\otimes e_j^2) &= \left(\widetilde{DF(\widetilde{x})F}(\widetilde{x})(e_j^2)\right)(e_i^2) = \begin{pmatrix}0 & 0 & 0\\ 0 & \langle\sigma,\gamma_j\rangle_{\mathbb{H}}\partial_2\sigma & 0\\ 0 & 0 & 0\end{pmatrix} e_i^2 \\ &= (0, \langle\sigma,\gamma_j\rangle_{\mathbb{H}}\langle\partial_2\sigma,\gamma_i\rangle_{\mathbb{H}}, 0) = (0, \langle\partial_2\sigma\otimes\sigma, \gamma_i\otimes\gamma_j\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}}, 0).\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{DF(\widetilde{x})F}(\widetilde{x})(e_i^2\otimes e_j^3) &= \left(\widetilde{DF(\widetilde{x})F}(\widetilde{x})(e_j^3)\right)(e_i^2) = \begin{pmatrix}0 & 0 & 0\\ 0 & \partial_3\sigma\gamma_j & 0\\ 0 & 0 & 0\end{pmatrix} e_i^2 \\ &= (0, \langle\partial_3\sigma\gamma_j, \gamma_i\rangle_{\mathbb{H}}, 0)^T = (0, \langle\partial_3\sigma, \gamma_i\otimes\gamma_j\rangle_{\mathbb{H}\widehat{\otimes}_{HS}\mathbb{H}}, 0)^T.\end{aligned}$$

and

$$\widetilde{DF(\widetilde{x})F}(\widetilde{x})(e_i^2\otimes e^1) = \left(\widetilde{DF(\widetilde{x})F}(\widetilde{x})(e^1)\right)(e_i^2) = (0, \langle\partial_1\sigma,\gamma_i\rangle_{\mathbb{H}}, 0)^T.$$

It can easily be checked that other terms are $(0,0,0)^T$. (5.9) now follows from the expansion (A.2).

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