Estimating the covariance of random matrices

Pierre Youssef*

Abstract

We extend to the matrix setting a recent result of Srivastava-Vershynin [24] about estimating the covariance matrix of a random vector. The result can be interpreted as a quantified version of the law of large numbers for positive semi-definite matrices which verify some regularity assumption. Beside giving examples, we discuss the notion of log-concave matrices and give estimates on the smallest and largest eigenvalues of a sum of such matrices.

Keywords: Covariance matrix ; log-concave matrix.
AMS MSC 2010: 60B20.
Submitted to EJP on January 28, 2013, final version accepted on December 19, 2013.

1 Introduction

In recent years, interest in matrix valued random variables gained momentum. Many of the results dealing with real random variables and random vectors were extended to cover random matrices. Concentration inequalities like Bernstein, Hoeffding and others were obtained in the non-commutative setting ([5],[25], [17]). The methods used were mostly combination of methods from the real/vector case and some matrix inequalities like the Golden-Thompson inequality (see [8]).

Estimating the covariance matrix of a random vector has gained a lot of interest recently. Given a random vector X in $\mathbb{R}^n$, the question is to estimate $\Sigma = \mathbb{E}XX^t$. A natural way to do this is to take $X_1, .., X_N$ independent copies of X and try to approximate Σ with the sample covariance matrix $\Sigma_N = \frac{1}{N}\sum_i X_i X_i^t$. The challenging problem is to find the minimal number of samples needed to estimate Σ. It is known using a result of Rudelson (see [22]) that for general distributions supported on the sphere of radius $\sqrt{n}$, it suffices to take $cn\log(n)$ samples. But for many distributions, a number proportional to n is sufficient. Using standard arguments, one can verify this for gaussian vectors. It was conjectured by Kannan- Lovasz- Simonovits [14] that the same result holds for log-concave distributions. This problem was solved by Adamczak et al ([3], [4]). Recently, Srivatava-Vershynin proved in [24] covariance estimate with a number of samples proportional to n, for a larger class of distributions covering the log-concave

---
*University of Alberta, Canada. E-mail: pyoussef@ualberta.ca

case. The method used was different from previous work on this field and the main idea was to randomize the sparsification theorem of Batson-Spielman-Srivastava [7].

Our aim in this paper is to adapt the work of Srivastava-Vershynin to the matrix setting replacing the vector X in the problem of the covariance matrix by an $n \times m$ random matrix A and try to estimate $\mathbb{E}AA^t$ by the same techniques. This will be possible since in the deterministic setting, the sparsification theorem of Batson-Spielman-Srivastava [7] has been extended to a matrix setting by De Carli Silva-Harvey-Sato [10] who precisely proved the following:

Theorem 1.1. *Let $B_1, \ldots, B_m$ be positive semi-definite matrices of size $n \times n$ and arbitrary rank. Set $B := \sum_i B_i$. For any $\varepsilon \in (0, 1)$, there is a deterministic algorithm to construct a vector $y \in \mathbb{R}^m$ with $O(n/\varepsilon^2)$ nonzero entries such that $y \geq 0$ and*

$$B \preceq \sum_i y_i B_i \preceq (1+\varepsilon)B.$$

For an $n \times n$ matrix A, denote by $\|A\|$ the operator norm of A seen as an operator on l_2^n. The main idea is to randomize the previous result using the techniques of Srivastava-Vershynin [24]. Our problem can be formulated as follows:

Take B a positive semi-definite random matrix of size $n \times n$. How many independent copies of B are needed to approximate $\mathbb{E}B$ i.e taking $B_1, .., B_N$ independent copies of B, what is the minimal number of samples needed to make $\left\| \frac{1}{N} \sum_i B_i - \mathbb{E}B \right\|$ very small.

One can view this as a matrix analogue to the covariance estimate of a random vector by taking for B the matrix AA^t where A is an $n \times m$ random matrix. Moreover, this problem implies an averaging approximation of covariance matrices of many random vectors. Indeed, let $X_1, .., X_m$ be random vectors in $\mathbb{R}^n$ and take A' the $n \times m$ matrix which has $X_1, .., X_m$ as columns. Denote $B = \frac{1}{m} A'A'^t$; it is clear that $B = \frac{1}{m} \sum_{j \leqslant m} X_j X_j^t$. Therefore, when approximating $\mathbb{E}B$ we are approximating the average of the covariance matrices of the random vectors $(X_j)_{j \leqslant m}$.

With some regularity, we will be able to take a number of independent copies proportional to the dimension n. However, in the general case this is no longer true. In fact, take B uniformly distributed on $\{ne_ie_i^t\}_{i \leqslant n}$ where e_j denotes the canonical basis of $\mathbb{R}^n$. It is easy to verify that $\mathbb{E}B = I_n$ and when taking $B_1, .., B_N$ independent copies of B the matrix $\frac{1}{N} \sum_i B_i$ is diagonal and its diagonal coefficients are distributed as

$$\frac{n}{N}(p_1, .., p_n),$$

where p_i denotes the number of times $e_ie_i^t$ is chosen. This problem is well- studied and it is known (see [15]) that we must take $N \geqslant cn \log(n)$. This example is essentially due to Aubrun [6]. More generally, if B is a positive semi-definite matrix such that $\mathbb{E}B = I_n$ and $\mathrm{Tr}(B) \leqslant n$ almost surely, then by Rudelson's inequality in the non-commutative setting (see [18]) it is sufficient to take $cn \log(n)$ samples.

The method will work properly for a class of matrices satisfying a matrix strong regularity assumption which we denote by (MSR) and can be viewed as an analog to the property (SR) defined in [24].

Definition 1.2 (Property (MSR))**.**
Let B be an $n \times n$ positive semi-definite random matrix such that $\mathbb{E}B = I_n$. We will say that B satisfies (MSR) if for some $c, \eta > 0$ we have :

$$\mathbb{P}(\|PBP\| \geqslant t) \leqslant \frac{c}{t^{1+\eta}} \quad \forall t \geqslant c \cdot rank(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^n.$$

In the rest of the paper, c will always denote the parameter appearing in this definition while C will be a universal constant which may change from line to line. Also, $c(\eta)$ will denote a constant depending on c and η which may also change from line to line.

The main result of this paper is the following:

Theorem 1.3. *Let B be an $n \times n$ positive semi-definite random matrix verifying $\mathbb{E}B = I_n$ and (MSR) for some $\eta > 0$. Then for every $\varepsilon \in (0, 1)$, taking $N = C_1{}^1 \frac{n}{\varepsilon^{2 + \frac{2}{\eta}}}$ we have*

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^N B_i - I_n \right\| \leqslant \varepsilon \quad \text{where } B_1, .., B_N \text{ are independent copies of } B.$$

If X is an isotropic random vector of $\mathbb{R}^n$, put $B = XX^t$ then $\|PBP\| = \|PX\|_2^2$. Therefore if X verifies the property (SR) appearing in [24], then B verifies property (MSR). So applying Theorem 1.3 to $B = XX^t$, we recover the covariance estimation as stated in [24].

In order to apply our result, beside some examples, we investigate the notion of log-concave matrices in relation with the definition of log-concave vectors. Moreover remarking some strong concentration inequalities satisfied by these matrices we are able, using the ideas developed in the proof of the main theorem, to have some results with high probability rather than only in expectation as is the case in the main result. This will be discussed in the last section of the paper.

The paper is organized as follows: in section 2, we discuss Property (MSR) and give some examples, in section 3 we show how to prove Theorem 1.3 using two other results (Theorem 3.1, Theorem 3.3) which we prove respectively in sections 4 and 5 using again two other results (Theorem 4.1, Theorem 5.1) whose proofs are given respectively in sections 6 and 7. In section 8, we discuss the notion of log-concave matrices and prove some related results.

2　Property (MSR) **and examples**

A random vector X in $\mathbb{R}^l$ is called isotropic if its covariance matrix is the identity i.e $\mathbb{E}XX^t = Id$. In [24], an isotropic random vector X in $\mathbb{R}^l$ was said to satisfy (SR) if for some $c, \eta > 0$,

$$\mathbb{P} \left(\|PX\|_2^2 \geqslant t \right) \leqslant \frac{c}{t^{1+\eta}}, \quad \forall t \geqslant c \cdot rank(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^l.$$

Since $\|PXX^tP\| = \|PX\|_2^2$, then clearly $B = XX^t$ satisfies (MSR) if and only if X satisfies (SR). Therefore if X verifies the property (SR), applying Theorem 1.3 to $B = XX^t$, we recover the covariance estimate as stated in [24].

Let us note that (MSR) implies moment assumptions on the quadratic forms $\langle Bx, x \rangle$. To see this, first note that if $x \in \mathbb{S}^{n-1}$ then $\langle Bx, x \rangle = \|P_x B P_x\|$, where P_x is the orthogonal projection on $\mathrm{span}(x)$. Now, by integration of tails we have for $1 < q < 1 + \eta$,

$$\mathbb{E}\langle Bx, x \rangle^q \leqslant C(c, q, \eta).$$

Moreover, property (MSR) implies regularity assumption on the eigenvalues of the matrix B. Indeed, for any orthogonal projection of rank k one can write

$$\|PBP\| \geqslant \min_{\substack{F \subset \mathbb{R}^n \\ \dim F = k}} \max_{x \in F} \langle Bx, x \rangle = \lambda_{n-k+1}(B),$$

$^1 C_1 = (64c)^{1+\frac{2}{\eta}} (1 + \frac{1}{\eta})^{\frac{2}{\eta}} \vee 64(4c)^{\frac{1}{\eta}} (32 + \frac{32}{\eta})^{1+\frac{3}{\eta}} \vee 256(2c)^{\frac{3}{2}+\frac{2}{\eta}} (16 + \frac{16}{\eta})^{\frac{4}{\eta}}$

where the last equality is given by the Courant-Fischer minimax formula (see [9]). Therefore, property (MSR) implies the following: for some $c, \eta > 0$,

$$\mathbb{P}\left(\lambda_{n-k+1}(B) \geqslant t\right) \leqslant \frac{c}{t^{1+\eta}}, \quad \forall t \geqslant c \cdot k \text{ and } \forall k \leqslant n.$$

We may now discuss some examples for applications of the main result. Let us first replace (MSR) with a stronger, but easier to manipulate, property which we denote by $(MSR*)$. If B is an $n \times n$ positive semidefinite random matrix such that $\mathbb{E}B = I_n$, we will say that B satisfies $(MSR*)$ if for some $c, \eta > 0$:

$$\mathbb{P}(\mathrm{Tr}(PB) \geqslant t) \leqslant \frac{c}{t^{1+\eta}} \quad \forall t \geqslant c \cdot rank(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^n.$$

Note that since $\|PBP\| \leqslant \mathrm{Tr}(PBP) = \mathrm{Tr}(PB)$, then $(MSR*)$ is clearly stronger than (MSR).

2.1 $(2+\varepsilon)$-moments for the spectrum

As we mentioned before, (MSR), one can see that it implies regularity assumptions on the eigenvalues of B. Putting some independence in the spectral decomposition of B, we will only need to use the regularity of the eigenvalues. To be more precise, we have the following:

Proposition 2.1. *Let $B = UDU^t$ be the spectral decomposition of an $n \times n$ symmetric positive semidefinite random matrix, where U an orthogonal matrix and D is a diagonal matrix whose entries are denoted by $(\alpha_j)_{j \leqslant n}$. Suppose that U and D are independent and that $(\alpha_j)_{j \leqslant n}$ are independent and satisfy the following:*

$$\forall i \leqslant n, \ \mathbb{E}\alpha_i = 1 \quad and \quad (\mathbb{E}\alpha_i^p)^{\frac{1}{p}} \leqslant C,$$

for some $p > 2$. Then B satisfies $(MSR*)$ with $\eta = \frac{p}{2} - 1$.

Proof. First note that since U and D are independent and $\mathbb{E}\alpha_i = 1$, then $\mathbb{E}B = I_n$.

Now, $(MSR*)$ is a rotationally invariant property. Therefore we can assume without loss of generality that $U = I_n$ and thus that $B = D$. Let $k > 0$ and P be an orthogonal projection of rank k on $\mathbb{R}^n$ and denote by $(p_{ij})_{i,j \leqslant n}$ the entries of P. Note that $\mathrm{Tr}(PB) = \sum_{i \leqslant n} p_{ii}\alpha_i$, and now using Markov's inequality we have for $t > k$,

$$
\begin{aligned}
\mathbb{P}\left\{\mathrm{Tr}(PB) \geqslant t\right\} &= \mathbb{P}\left\{\sum_{i \leqslant n} p_{ii}(\alpha_i - 1) \geqslant t - k\right\} \\
&\leqslant \mathbb{P}\left\{\left|\sum_{i \leqslant n} p_{ii}(\alpha_i - 1)\right| \geqslant t - k\right\} \\
&\leqslant \frac{1}{(t-k)^p}\mathbb{E}\left|\sum_{i \leqslant n} p_{ii}(\alpha_i - 1)\right|^p.
\end{aligned}
$$

Using Rosenthal's inequality (see [21]) we get

$$\mathbb{E}\left|\sum_{i \leqslant n} p_{ii}(\alpha_i - 1)\right|^p \leqslant C(p) \max\left\{\sum_{i \leqslant n} p_{ii}^p \mathbb{E}|\alpha_i - 1|^p, \left(\sum_{i \leqslant n} p_{ii}^2 \mathbb{E}|\alpha_i - 1|^2\right)^{\frac{p}{2}}\right\}.$$

Taking in account that $p_{ii} \leqslant 1$, which implies that for any $l \geqslant 1$, $\sum_i p_{ii}^l \leqslant k$, we deduce that

$$\mathbb{E}\left|\sum_{i \leqslant n} p_{ii}(\alpha_i - 1)\right|^p \leqslant C(p) k^{\frac{p}{2}}.$$

Instead of Rosenthal's inequality, we could have used a symmetrization argument alongside Khintchine's inequality to get the estimate above.

One can easily conclude that B satisfies $(MSR*)$ with $\eta = \frac{p}{2} - 1$. $\qquad\square$

Applying Theorem 1.3, we can deduce the following proposition:

Proposition 2.2. *Let $B = UDU^t$ be the spectral decomposition of an $n \times n$ symmetric positive semidefinite random matrix, where U an orthogonal matrix and D is a diagonal matrix whose entries are denoted by $(\alpha_j)_{j \leqslant n}$. Suppose that U and D are independent and that $(\alpha_j)_{j \leqslant n}$ are independent and satisfy the following:*

$$\forall i \leqslant n, \ \mathbb{E}\alpha_i = 1 \quad and \quad (\mathbb{E}\alpha_i^p)^{\frac{1}{p}} \leqslant C,$$

for some $p > 2$. Let $\varepsilon \in (0, 1)$, then taking $N = C(p)\dfrac{n}{\varepsilon^{\frac{2p}{p-2}}}$ we have

$$\mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^N B_i - I_n\right\| \leqslant \varepsilon,$$

where $B_1, .., B_N$ are independent copies of B.

2.2 From (SR) to (MSR)

We will show how to jump from property (SR) dealing with vectors to the property $(MSR*)$ dealing with matrices.

Proposition 2.3. *Let A be an $n \times m$ random matrix and denote by $(C_i)_{i \leqslant m}$ its columns. Suppose that $A'^t = \sqrt{m}(C_1^t, .., C_m^t)$ is an isotropic random vector in $\mathbb{R}^{nm}$ which satisfies property (SR). Then $B = AA^t$ verifies $\mathbb{E}B = I_n$ and Property $(MSR*)$.*

Proof. For $l \leqslant nm$, one can write $l = (j-1)n + i$ with $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, so that the coordinates of A' are given by $a'_l = \sqrt{m}a_{i,j}$, and since A' is isotropic we get $\mathbb{E}a_{i,j}a_{r,s} = \frac{1}{m}\delta_{(i,j),(r,s)}$.

The terms of B are given by $b_{i,j} = \displaystyle\sum_{s=1}^m a_{i,s}a_{j,s}$. We deduce that $\mathbb{E}b_{i,j} = \delta_{i,j}$ and therefore $\mathbb{E}B = I_n$.

Let P be an orthogonal projection of $\mathbb{R}^n$ and put $P' = I_m \otimes P$ i.e. P' is an $nm \times nm$ matrix of the form

$$P' = \begin{pmatrix} P & 0 & \ldots & 0 \\ 0 & P & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & P \end{pmatrix}$$

Clearly we have $\|P'A'\|_2^2 = m\mathrm{Tr}(PB)$ and $rank(P') = m \cdot rank(P)$.

Let $t \geqslant c \cdot rank(P)$ then $mt \geqslant c \cdot rank(P')$ and by property (SR) we have

$$\mathbb{P}\left(\|P'A'\|_2^2 \geqslant mt\right) \leqslant \frac{c}{(mt)^{1+\eta}}.$$

This means that

$$\mathbb{P}\left(\mathrm{Tr}(PB) \geqslant t\right) \leqslant \frac{c}{(mt)^{1+\eta}},$$

and therefore B satisfies $(MSR*)$. $\qquad\square$

3 Proof of Theorem 1.3

Let us first introduce some notations which will be used in the rest of the paper. The set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}_n$. For $X \in \mathbb{S}_n$, the notation $\lambda(X)$ always refers to the eigenvalues of X. For $X, Y \in \mathbb{S}_n$, $X \succeq Y$ means that $X - Y$ is positive semidefinite. The vector space $\mathbb{S}_n$ can be endowed with the trace inner product $\langle \cdot, \cdot \rangle$ defined by $\langle X, Y \rangle := \mathrm{Tr}(XY)$.

Let us now introduce a regularity assumption on the moments which we denote by (MWR):

$$\exists p > 1 \ \text{ such that } \ \mathbb{E}\langle Bx, x \rangle^p \leqslant C_p \quad \forall x \in S^{n-1}.$$

Note that by a simple integration of tails, (MSR) (with P a rank one projection) implies (MWR) with $p < 1 + \eta$.

The proof of Theorem 1.3 is based on two theorems dealing with the smallest and largest eigenvalues of $\frac{1}{N}\sum_{i=1}^{N} B_i$.

Theorem 3.1. *Let B_i be $n \times n$ independent positive semidefinite random matrices verifying $\mathbb{E}B_i = I_n$ and (MWR) .*
Let $\varepsilon \in (0, 1)$, then for

$$N \geqslant 16\left(16C_p\right)^{\frac{1}{p-1}} \frac{n}{\varepsilon^{\frac{2p-1}{p-1}}},$$

we get

$$\mathbb{E}\lambda_{min}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \geqslant 1 - \varepsilon.$$

Remark 3.2. *The proof yields a more general estimate; precisely if $h = \frac{n}{N}$ then*

$$\mathbb{E}\lambda_{min}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \geqslant 1 - C(p)\max\left\{h^{\frac{p-1}{2p-1}}, h\right\}.$$

Theorem 3.3. *Let B_i be $n \times n$ independent positive semidefinite random matrices verifying $\mathbb{E}B_i = I_n$ and (MSR).*
For $\varepsilon \in (0, 1)$ and $N \geqslant C_2{}^2 \frac{n}{\varepsilon^{2+\frac{2}{\eta}}}$ we have

$$\mathbb{E}\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \leqslant 1 + \varepsilon.$$

Remark 3.4. *The proof yields a more general estimate; Precisely if $h = \frac{n}{N}$ then*

$$\mathbb{E}\lambda_{max}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \leqslant 1 + c(\eta)\max\left\{h^{\frac{\eta}{2+2\eta}}, h\right\}.$$

---

[2]$C_2 = 16c^{\frac{1}{\eta}}(32 + \frac{32}{\eta})^{1+\frac{3}{\eta}} \vee 16\sqrt{2}(4c)^{\frac{3}{2}+\frac{2}{\eta}}(8 + \frac{8}{\eta})^{\frac{4}{\eta}}$

Combining this with the previous remark, for any $B_1, ..., B_N$ $n \times n$ independent positive semidefinite random matrices verifying $\mathbb{E}B_i = I_n$ and (MSR), we have

$$1 - c(\eta) \max \left\{ h^{\frac{\eta}{2+2\eta}}, h \right\} \leqslant \mathbb{E}\lambda_{min} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) \leqslant \mathbb{E}\lambda_{max} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) \leqslant 1 + c(\eta) \max \left\{ h^{\frac{\eta}{2+2\eta}}, h \right\}$$

We will give the proof of these two theorems in sections 4 and 5 respectively. We need also a simple lemma:

Lemma 3.5. *Let $1 < r \leqslant 2$ and $Z_1, ..., Z_N$ be independent positive random variables with $\mathbb{E}Z_i = 1$ and satisfying $(\mathbb{E}Z_i^r)^{\frac{1}{r}} \leqslant M$ Then*

$$\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^{N} Z_i - 1 \right| \leqslant \frac{2M}{N^{\frac{r-1}{r}}}.$$

Proof. Let $(\varepsilon_i)_{i \leqslant N}$ independent ± 1 Bernoulli variables. By symmetrization and Jensen's inequality we can write

$$\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^{N} Z_i - 1 \right| \leqslant \frac{2}{N} \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i Z_i \right| \leqslant \frac{2}{N} \mathbb{E} \left(\sum_{i=1}^{N} Z_i^2 \right)^{\frac{1}{2}}$$

$$\leqslant \frac{2}{N} \mathbb{E} \left(\sum_{i=1}^{N} Z_i^r \right)^{\frac{1}{r}} \leqslant \frac{2}{N} \left(\sum_{i=1}^{N} \mathbb{E}Z_i^r \right)^{\frac{1}{r}} \leqslant \frac{2M}{N^{\frac{r-1}{r}}}.$$

$\square$

Proof of Theorem 1.3. Take $N \geqslant c(\eta) \frac{n}{\varepsilon^{2+\frac{2}{\eta}}}$ satisfying conditions of Theorem 3.1 (with $p = 1 + \frac{\eta}{2}$) and Theorem 3.3. Note that by the triangle inequality

$$\left\| \frac{1}{N} \sum_{i=1}^{N} B_i - I_n \right\| \leqslant \left\| \frac{1}{N} \sum_{i=1}^{N} B_i - \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) I_n \right\| + \left\| \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) I_n - I_n \right\|$$

$$:= \alpha + \beta.$$

Observe that

$$\alpha = \max \left| \lambda \left(\frac{1}{N} \sum_{i=1}^{N} B_i - \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) I_n \right) \right|$$

$$= \max \left[\lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) - \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right), \frac{1}{n} \text{Tr} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) - \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) \right].$$

Since the two terms in the max are non-negative, then one can bound the max by the sum of the two terms. More precisely, we get $\alpha \leqslant \lambda_{\max} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right) - \lambda_{\min} \left(\frac{1}{N} \sum_{i=1}^{N} B_i \right)$ and by Theorem 3.1 and Theorem 3.3 we deduce that $\mathbb{E}\alpha \leqslant 2\varepsilon$.

Note that

$$\beta = \left| \frac{1}{N} \sum_{i=1}^{N} \frac{\text{Tr}(B_i)}{n} - 1 \right| = \left| \frac{1}{N} \sum_{i=1}^{N} Z_i - 1 \right|,$$

where $Z_i = \frac{\mathrm{Tr}(B_i)}{n}$. Since B_i satisfies (MWR), then taking $r = \min(2, 1 + \frac{\eta}{2})$ we have

$$\forall i \leqslant N, \quad (\mathbb{E} Z_i^r)^{\frac{1}{r}} \leqslant \frac{1}{n} \sum_{j=1}^{n} (\mathbb{E} \langle B_i e_j, e_j \rangle^r)^{\frac{1}{r}} \leqslant c(\eta).$$

Therefore Z_i satisfy the conditions of Lemma 3.5 and we deduce that $\mathbb{E}\beta \leqslant \varepsilon$ by the choice of N.

As a conclusion

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^{N} B_i - I_n \right\| \leqslant \mathbb{E}\alpha + \mathbb{E}\beta \leqslant 3\varepsilon.$$

$\square$

4 Proof of Theorem 3.1

Given A an $n \times n$ positive semi-definite matrix such that all eigenvalues of A are greater than a lower barrier $l_A = l$ i.e $A \succ l.I_n$, define the corresponding potential function to be

$$\phi_l(A) = \mathrm{Tr}(A - l \cdot I_n)^{-1}.$$

The proof of Theorem 3.1 is based on the following result which will be proved in section 6:

Theorem 4.1. *Let $A \succ l \cdot I_n$ and $\phi_l(A) \leqslant \phi$, B a positive semi-definite random matrix satisfying $\mathbb{E}B = I_n$ and Property (MWR) with some $p > 1$.*
Let $\varepsilon \in (0, 1)$, if

$$\phi \leqslant \frac{1}{4 \left(8 C_p\right)^{\frac{1}{p-1}}} \varepsilon^{\frac{p}{p-1}}, \tag{4.1}$$

then there exist l' a random variable such that

$$A + B \succ l' \cdot I_n, \quad \phi_{l'}(A + B) \leqslant \phi_l(A) \ \text{ and } \ \mathbb{E}l' \geqslant l + 1 - \varepsilon.$$

Proof of Theorem 3.1. Let ϕ satisfying condition (4.1) of Theorem 4.1. We start with $A_0 = 0$ and $l_0 = -\frac{n}{\phi}$ so that $\phi_{l_0}(A_0) = -\frac{n}{l_0} = \phi$.
Applying Theorem 4.1, one can find l_1 such that

$$A_1 = A_0 + B_1 \succ l_1 \cdot I_n, \quad \phi_{l_1}(A_1) \leqslant \phi_{l_0}(A_0) = \phi$$

and

$$\mathbb{E}l_1 \geqslant l_0 + 1 - \varepsilon$$

Now apply Theorem 4.1 conditionally on A_1 to find l_2 such that

$$A_2 = A_1 + B_2 \succ l_2 \cdot I_n, \quad \phi_{l_2}(A_2) \leqslant \phi_{l_1}(A_1)$$

and

$$\mathbb{E}_{B_2} l_2 \geqslant l_1 + 1 - \varepsilon.$$

Using Fubini's Theorem we have

$$\mathbb{E}l_2 \geqslant \mathbb{E}l_1 + 1 - \varepsilon \geqslant l_0 + 2(1 - \varepsilon)$$

After N steps, we get $\mathbb{E}\lambda_{\min}(A_N) \geqslant \mathbb{E}l_N \geqslant l_0 + N(1-\varepsilon)$. Therefore,

$$\mathbb{E}\lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \geqslant 1 - \varepsilon - \frac{n}{N\phi}.$$

Taking $N = \frac{n}{\varepsilon\phi}$, we get $\mathbb{E}\lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \geqslant 1 - 2\varepsilon$.

$\square$

5   Proof of Theorem 3.3

Given A an $n \times n$ positive semi-definite matrix such that all eigenvalues of A are less than an upper barrier $u_A = u$ i.e. $A \prec u \cdot I_n$, define the corresponding potential function to be

$$\psi_u(A) = \mathrm{Tr}\,(u \cdot I_n - A)^{-1}.$$

The proof of Theorem 3.3 is based on the following result which will be proved in section 7:

Theorem 5.1. *Let $A \prec u \cdot I_n$ and $\psi_u(A) \leqslant \psi$, B a positive semi-definite random matrix satisfying $\mathbb{E}B = I_n$ and Property (MSR).*
Let $\varepsilon \in (0,1)$, if

$$\psi \leqslant C_3 \cdot {}^3\varepsilon^{1+\frac{2}{\eta}} \tag{5.1}$$

there exists u' a random variable such that

$$A + B \prec u' \cdot I_n, \quad \psi_{u'}(A+B) \leqslant \psi_u(A) \;\; and \;\; \mathbb{E}u' \leqslant u + 1 + \varepsilon.$$

Proof of Theorem 3.3. Let ψ satisfying the condition of Theorem 5.1. We start with $A_0 = 0$, $u_0 = \frac{n}{\psi}$ so that $\psi_{u_0}(A_0) = \psi$.
Applying Theorem 5.1, one can find u_1 such that

$$A_1 = A_0 + B_1 \prec u_1 \cdot I_n, \quad \psi_{u_1}(A_1) \leqslant \psi_{u_0}(A_0) \;\; and \;\; \mathbb{E}u_1 \leqslant u_0 + 1 + \varepsilon.$$

Now apply Theorem 5.1 conditionally on A_1 to find u_2 such that

$$A_2 = A_1 + B_2 \prec u_2 \cdot I_n, \quad \psi_{u_2}(A_2) \leqslant \psi_{u_1}(A_1) \;\; and \;\; \mathbb{E}_{B_2}u_2 \leqslant u_1 + 1 + \varepsilon.$$

Using Fubini's Theorem we have

$$\mathbb{E}u_2 \leqslant \mathbb{E}u_1 + 1 + \varepsilon \leqslant u_0 + 2(1 + \varepsilon).$$

After N steps we get $\mathbb{E}\lambda_{\max}\left(\sum_{i=1}^{N} B_i\right) \leqslant \mathbb{E}u_N \leqslant u_0 + N(1+\varepsilon)$.

Taking $N \geqslant \frac{n}{\varepsilon\psi} = C_3^{-1}\frac{n}{\varepsilon^{2+\frac{2}{\eta}}}$, we deduce that

$$\mathbb{E}\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \leqslant 1 + 2\varepsilon.$$

$\square$

${}^3 C_3 = \left[8(2c)^{\frac{1}{\eta}}(16 + \frac{16}{\eta})^{1+\frac{3}{\eta}}\right]^{-1} \wedge \left[16(2c)^{\frac{3}{2}+\frac{2}{\eta}}(8 + \frac{8}{\eta})^{\frac{4}{\eta}}\right]^{-1}$

6 Proof of Theorem 4.1

6.1 Notations

We work under the assumptions of Theorem 4.1. We are looking for a random variable l' of the form $l + \delta$ where δ is a positive random variable playing the role of the shift.

If in addition $A \succ (l + \delta) \cdot I_n$, we will note

$$L_\delta = A - (l + \delta) \cdot I_n$$

so that

$$\mathrm{Tr}\left(B^{\frac{1}{2}}(A - (l + \delta) \cdot I_n)^{-1}B^{\frac{1}{2}}\right) = \left\langle L_\delta^{-1}, B\right\rangle.$$

$\lambda_1, .., \lambda_n$ will denote the eigenvalues of A and $v_1, .., v_n$ the corresponding normalized eigenvectors. Note that $(v_i)_{i\leqslant n}$ are also the eigenvectors of L_δ^{-1} corresponding to the eigenvalues $\frac{1}{\lambda_i - (l+\delta)}$.

6.2 Finding the shift

To find sufficient conditions for such δ to exist, we need a matrix extension of Lemma 3.4 in [7] which, up to a minor change, is essentially contained in Lemma 20 in [10] and we formulate it here in Lemma 6.2. This method uses the Sherman-Morrison-Woodbury formula:

Lemma 6.1. *Let E be an $n \times n$ invertible matrix, C a $k \times k$ invertible matrix, U an $n \times k$ matrix and V a $k \times n$ matrix. Then we have:*

$$(E + UCV)^{-1} = E^{-1} - E^{-1}U(C^{-1} + VE^{-1}U)^{-1}VE^{-1}$$

Lemma 6.2. *Let A as above satisfying $A \succ l \cdot I_n$. Suppose that one can find $\delta > 0$ verifying $\delta \leqslant \frac{1}{\|L_0^{-1}\|}$ and*

$$\frac{\left\langle L_\delta^{-2}, B\right\rangle}{\phi_{l+\delta}(A) - \phi_l(A)} - \left\|B^{\frac{1}{2}}L_\delta^{-1}B^{\frac{1}{2}}\right\| \geqslant 1$$

Then

$$\lambda_{\min}(A + B) > l + \delta \quad and \quad \phi_{l+\delta}(A + B) \leqslant \phi_l(A).$$

Proof. First note that $\frac{1}{\|L_0^{-1}\|} = \lambda_{\min}(A) - l$, so the first condition on δ implies that $\lambda_{\min}(A) \geqslant l + \delta$.

Now using Sherman-Morrison-Woodbury formula with $E = L_\delta$, $U = V = B^{\frac{1}{2}}$, $C = I_n$ we get :

$$\phi_{l+\delta}(A + B) = \mathrm{Tr}\,(L_\delta + B)^{-1}$$

$$= \phi_{l+\delta}(A) - \mathrm{Tr}\left(L_\delta^{-1}B^{\frac{1}{2}}\left(I_n + B^{\frac{1}{2}}L_\delta^{-1}B^{\frac{1}{2}}\right)^{-1}B^{\frac{1}{2}}L_\delta^{-1}\right)$$

$$\leqslant \phi_{l+\delta}(A) - \frac{\left\langle L_\delta^{-2}, B\right\rangle}{1 + \left\|B^{\frac{1}{2}}L_\delta^{-1}B^{\frac{1}{2}}\right\|}$$

Rearranging the hypothesis, we get $\phi_{l+\delta}(A+B) \leqslant \phi_l(A)$.

$\square$

Since $\|L_0^{-1}\| \leqslant \mathrm{Tr}\left(L_0^{-1}\right) = \phi_l(A)$ and $\left\|B^{\frac{1}{2}}L_\delta^{-1}B^{\frac{1}{2}}\right\| \leqslant \left\langle L_\delta^{-1}, B\right\rangle$ then in order to satisfy conditions of Lemma 6.2, we may search for δ satisfying:

$$\delta \leqslant \frac{1}{\phi_l(A)} \quad \text{and} \quad \frac{\left\langle L_\delta^{-2}, B\right\rangle}{\phi_{l+\delta}(A) - \phi_l(A)} - \left\langle L_\delta^{-1}, B\right\rangle \geqslant 1 \tag{6.1}$$

For $t \leqslant \frac{1}{\phi}$, let us note :

$$q_1(t, B) = \left\langle L_t^{-1}, B\right\rangle = \mathrm{Tr}\left(B(A - (l+t) \cdot I_n)^{-1}\right)$$

and

$$q_2(t, B) = \frac{\left\langle L_t^{-2}, B\right\rangle}{\mathrm{Tr}(L_t^{-2})} = \frac{\mathrm{Tr}\left(B(A - (l+t) \cdot I_n)^{-2}\right)}{\mathrm{Tr}\left(A - (l+t) \cdot I_n\right)^{-2}}$$

We have already seen in Lemma 6.2 that if $t \leqslant \frac{1}{\phi} \leqslant \frac{1}{\|L_0^{-1}\|}$ then $A \succ (l+t) \cdot I_n$ so the definitions above make sense. Since we have

$$\begin{aligned}
\phi_{l+\delta}(A) - \phi_l(A) &= \mathrm{Tr}(A - (l+\delta) \cdot I_n)^{-1} - \mathrm{Tr}(A - l \cdot I_n)^{-1} \\
&= \delta \mathrm{Tr}((A - (l+\delta) \cdot I_n)^{-1}(A - l \cdot I_n)^{-1}) \\
&\leqslant \delta \mathrm{Tr}(A - (l+\delta) \cdot I_n)^{-2},
\end{aligned}$$

then in order to have (6.1), it will be sufficient to choose δ satisfying $\delta \leqslant \frac{1}{\phi}$ and

$$\frac{1}{\delta}q_2(\delta, B) - q_1(\delta, B) \geqslant 1 \tag{6.2}$$

Note that q_1 and q_2 can be expressed as follows:

$$q_1(t, B) = \sum_{i=1}^n \frac{\langle Bv_i, v_i\rangle}{\lambda_i - l - t} \quad \text{and} \quad q_2(t, B) = \frac{\sum_i \frac{\langle Bv_i, v_i\rangle}{(\lambda_i - l - t)^2}}{\sum_i (\lambda_i - l - t)^{-2}}.$$

Since $\phi_l(A) = \sum_{i=1}^n (\lambda_i - l)^{-1} \leqslant \phi$, then $(\lambda_i - l) \cdot \phi \geqslant 1$ for all i. Thus

$$(1 - t \cdot \phi)(\lambda_i - l) = \lambda_i - l - t \cdot (\lambda_i - l) \cdot \phi \leqslant \lambda_i - l - t \leqslant \lambda_i - l$$

and therefore

$$q_1(t, B) \leqslant (1 - t \cdot \phi)^{-1} q_1(0, B)$$

and

$$(1 - t \cdot \phi)^2 q_2(0, B) \leqslant q_2(t, B) \leqslant (1 - t \cdot \phi)^{-2} q_2(0, B).$$

Lemma 6.3. *Let $s \in (0, 1)$ and take $\delta = (1-s)^3 q_2(0, B)\mathbf{1}_{\{q_1(0,B)\leqslant s\}}\mathbf{1}_{\{q_2(0,B)\leqslant \frac{s}{\phi}\}}$. Then $A + B \succ (l+\delta) \cdot I_n$ and $\phi_{l+\delta}(A+B) \leqslant \phi_l(A)$.*

Proof. As stated before in (6.2), it is sufficient to check that $\delta \leqslant \frac{1}{\phi}$ and $\frac{1}{\delta}q_2(\delta, B) - q_1(\delta, B) \geqslant 1$.

If $q_1(0, B) \geqslant s$ or $q_2(0, B) \geqslant \frac{s}{\phi}$ then $\delta = 0$ and there is nothing to prove since $\phi_l(A+B) \leqslant \phi_l(A)$.

In the other case i.e. $q_1(0, B) \leqslant s$ and $q_2(0, B) \leqslant \frac{s}{\phi}$, we have $\delta = (1-s)^3 q_2(0, B)$. So $\delta \leqslant (1-s)^3 \frac{s}{\phi} \leqslant \frac{1}{\phi}$ and

$$\frac{1}{\delta}q_2(\delta, B) - q_1(\delta, B) = \frac{1}{(1-s)^3 q_2(0, B)}q_2(\delta, B) - q_1(\delta, B)$$

$$\geqslant \frac{1}{(1-s)^3 q_2(0, B)}(1-\delta\phi)^2 q_2(0, B) - (1-\delta\phi)^{-1}q_1(0, B)$$

$$\geqslant \frac{(1-s)^2}{(1-s)^3} - \frac{s}{(1-s)} = 1.$$

$\square$

6.3 Estimating the random shift

Now that we have found δ, we will estimate $\mathbb{E}\delta$ using the property (MWR). We will start by stating some basic facts about q_1 and q_2.

Proposition 6.4. *Let as above* $A \succ l \cdot I_n$ *and* $\phi_l(A) \leqslant \phi$, B *satisfying* (MWR) *and* $\mathbb{E}B = I_n$. *Then we have the following :*

1. $\mathbb{E}q_1(0, B) = \phi_l(A) \leqslant \phi$ *and* $\mathbb{E}q_1(0, B)^p \leqslant C_p\phi^p$.
2. $\mathbb{E}q_2(0, B) = 1$ *and* $\mathbb{E}q_2(0, B)^p \leqslant C_p$.
3. $\mathbb{P}(q_1(0, B) \geqslant u) \leqslant C_p(\frac{\phi}{u})^p$ *and* $\mathbb{P}(q_2(0, B) \geqslant u) \leqslant \frac{C_p}{u^p}$.

Proof. Since $\mathbb{E}B = I_n$ then $\mathbb{E}q_1(0, B) = \phi_l(A)$ and $\mathbb{E}q_2(0, B) = 1$.
Now using the triangle inequality and Property (MWR) we get :

$$(\mathbb{E}q_1(0, B)^p)^{\frac{1}{p}} = \left[\mathbb{E}\left(\sum_{i=1}^n \frac{\langle Bv_i, v_i\rangle}{\lambda_i - l}\right)^p\right]^{\frac{1}{p}} \leqslant \sum_{i=1}^n \frac{(\mathbb{E}\langle Bv_i, v_i\rangle^p)^{\frac{1}{p}}}{\lambda_i - l} \leqslant \sum_{i=1}^n \frac{C_p^{\frac{1}{p}}}{\lambda_i - l} \leqslant C_p^{\frac{1}{p}}\phi$$

With the same argument we prove that $\mathbb{E}q_2(0, B)^p \leqslant C_p$. The third part of the proposition follows by Markov's inequality. $\square$

Lemma 6.5. *If* δ *is as in Lemma 6.3. Then*

$$\mathbb{E}\delta \geqslant (1-s)^3\left[1 - 2C_p\left(\frac{\phi}{s}\right)^{p-1}\right].$$

Proof. Using the above proposition and Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we get :

$$\mathbb{E}\delta = \mathbb{E}(1-s)^3 q_2(0, B)\mathbf{1}_{\{q_1(0,B)\leqslant s\}}\mathbf{1}_{\{q_2(0,B)\leqslant \frac{s}{\phi}\}}$$

$$= (1-s^3)\left[\mathbb{E}q_2(0, B) - \mathbb{E}q_2(0, B)\mathbf{1}_{\{q_1(0,B)>s \text{ or } q_2(0,B)>\frac{s}{\phi}\}}\right]$$

$$\geqslant (1-s)^3\left[1 - (\mathbb{E}q_2(0, B)^p)^{\frac{1}{p}}.\left(\mathbb{P}\left\{q_1(0, B) > s \text{ or } q_2(0, B) > \frac{s}{\phi}\right\}\right)^{\frac{1}{q}}\right]$$

$$\geqslant (1-s)^3\left[1 - C_p^{\frac{1}{p}}\left(C_p\left(\frac{\phi}{s}\right)^p + C_p\left(\frac{\phi}{s}\right)^p\right)^{\frac{1}{q}}\right]$$

$$\geqslant (1-s)^3\left[1 - 2C_p\left(\frac{\phi}{s}\right)^{p-1}\right].$$

$\square$

Now it remains to make good choice of s and ϕ in order to finish the prove Theorem 4.1. Take $l' = l + \delta$, the choice of δ being as before with $s = \frac{\varepsilon}{4}$.

As we have seen, we get $A + B \succ l' \cdot I_n$ and $\phi_{l'}(A + B) \leqslant \phi_l(A)$. Moreover,

$$\mathbb{E}l' = l + \mathbb{E}\delta \geqslant l + (1 - s)^3 \left[1 - 2C_p \left(\frac{\phi}{s} \right)^{p-1} \right] \geqslant 1 - \varepsilon,$$

by the assumption on ϕ. This ends the proof of Theorem 4.1.

7 Proof of Theorem 5.1

7.1 Notations

We work under the assumptions of Theorem 4.1. We are looking for a random variable u' of the form $u + \Delta$ where Δ is a positive random variable playing the role of the shift.

We will denote

$$U_t = (u + t) \cdot I_n - A,$$

so that

$$\mathrm{Tr}\left(B^{\frac{1}{2}} ((u + t) \cdot I_n - A)^{-1} B^{\frac{1}{2}} \right) = \left\langle U_t^{-1}, B \right\rangle.$$

As before, $\lambda_1, .., \lambda_n$ will denote the eigenvalues of A and $v_1, .., v_n$ the corresponding normalized eigenvectors. $(v_i)_{i \leqslant n}$ are also the eigenvectors of U_t^{-1} corresponding to the eigenvalues $\frac{1}{u+t-\lambda_i}$.

7.2 Finding the shift

To find sufficient conditions for such Δ to exist, we need a matrix extension of Lemma 3.3 in [7] which, up to a minor change, is essentially contained in Lemma 19 in [10]. For the sake of completeness, we include the proof.

Lemma 7.1. *Let A as above satisfying $A \prec u \cdot I_n$. Suppose that one can find $\Delta > 0$ verifying*

$$\frac{\left\langle U_\Delta^{-2}, B \right\rangle}{\psi_u(A) - \psi_{u+\Delta}(A)} + \left\| B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}} \right\| \leqslant 1. \tag{7.1}$$

Then

$$A + B \prec (u + \Delta) \cdot I_n \quad and \quad \psi_{u+\Delta}(A + B) \leqslant \psi_u(A).$$

Proof. Since $\left\langle U_\Delta^{-2}, B \right\rangle$ and $\psi_u(A) - \psi_{u+\Delta}(A)$ are positive, then by (7.1) we have

$$\left\| B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}} \right\| < 1 \quad and \quad \frac{\left\langle U_\Delta^{-2}, B \right\rangle}{1 - \left\| B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}} \right\|} \leqslant \psi_u(A) - \psi_{u+\Delta}(A).$$

First note that $\left\| B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}} \right\| = \left\| U_\Delta^{-\frac{1}{2}} B U_\Delta^{-\frac{1}{2}} \right\| < 1$, so $U_\Delta^{-\frac{1}{2}} B U_\Delta^{-\frac{1}{2}} \prec I_n$. Therefore we get $B \prec U_\Delta$ which means that $A + B \prec (u + \Delta) \cdot I_n$.

Now using the Sherman-Morrison-Woodbury (see Lemma 6.1) with $E = U_\Delta, U = V = B^{\frac{1}{2}}, C = I_n$ we get :

$$\psi_{u+\Delta}(A+B) = \mathrm{Tr}\,(U_\Delta - B)^{-1}$$

$$= \psi_{u+\Delta}(A) + \mathrm{Tr}\left(U_\Delta^{-1} B^{\frac{1}{2}} \left(I_n - B^{\frac{1}{2}} U_\Delta^{-1} B^{\frac{1}{2}}\right)^{-1} B^{\frac{1}{2}} U_\Delta^{-1}\right)$$

$$\leqslant \psi_{u+\Delta}(A) + \frac{\langle U_\Delta^{-2}, B\rangle}{1 - \left\|U_\Delta^{-\frac{1}{2}} B U_\Delta^{-\frac{1}{2}}\right\|} \leqslant \psi_u(A)$$

$\square$

We may now find Δ satisfying (7.1). Let us note :

$$Q_1(t, B) = \left\|B^{\frac{1}{2}} U_t^{-1} B^{\frac{1}{2}}\right\| = \left\|B^{\frac{1}{2}}\left((u+t)\cdot I_n - A\right)^{-1} B^{\frac{1}{2}}\right\|$$

and

$$Q_2(t, B) = \frac{\langle U_t^{-2}, B\rangle}{\psi_u(A) - \psi_{u+t}(A)} = \frac{\mathrm{Tr}\left(B\left((u+t)\cdot I_n - A\right)^{-2}\right)}{\psi_u(A) - \psi_{u+t}(A)}.$$

Since Q_1 and Q_2 are both decreasing in t, we work with each separately. Note that by (5.1), we have $4c \cdot \psi \leqslant 1/2$. Fix $\theta \in (4c \cdot \psi, 1/2)$ and define Δ_1, Δ_2 as follows :

$$\Delta_1 \text{ the smallest positive number such that } Q_1(\Delta_1, B) \leqslant \theta$$

and

$$\Delta_2 \text{ the smallest positive number such that } Q_2(\Delta_2, B) \leqslant 1 - \theta.$$

Now take $\Delta = \Delta_1 + \Delta_2$, then $Q_1(\Delta, B) + Q_2(\Delta, B) \leqslant \theta + 1 - \theta = 1$. So this choice of Δ satisfies (7.1) and it remains now to estimate Δ_1 and Δ_2 separately.

7.3 Estimating Δ_1

We may write $Q_1(\Delta_1, B) = \left\|\sum_{i=1}^{n} \frac{B^{\frac{1}{2}} v_i v_i^t B^{\frac{1}{2}}}{u + \Delta_1 - \lambda_i}\right\|.$

Put $\xi_i = B^{\frac{1}{2}} v_i v_i^t B^{\frac{1}{2}}$, $\mu_i = \psi(u - \lambda_i)$ and $\mu = \psi\Delta_1$. Denote P_S the orthogonal projection on $\mathrm{span}\left(\{v_i\}_{i\in S}\right)$, clearly $rank(P_S) = |S|$. Then we have :

$$\begin{cases} \mathbb{E}\|\xi_i\| = \mathbb{E}\langle Bv_i, v_i\rangle = 1. \\[2ex] \mathbb{P}\left(\left\|\sum_{i\in S}\xi_i\right\| \geqslant t\right) = \mathbb{P}\left(\|P_S B P_S\| \geqslant t\right) \leqslant \frac{c}{t^{1+\eta}} \quad \forall t \geqslant c\,|S|. \\[2ex] \sum_{i=1}^{n} \frac{1}{\mu_i} = \frac{\psi_u(A)}{\psi} \leqslant 1. \\[2ex] \mu \text{ is the smallest positive number such that } \sum_{i=1}^{n} \frac{\xi_i}{\mu_i + \mu} \preceq \frac{\theta}{\psi} I_n. \end{cases}$$

We will need an analog of Lemma 3.5 appearing in [24]. We extend this lemma to a matrix setting:

Lemma 7.2. *Suppose $\{\xi_i\}_{i \leqslant n}$ are symmetric positive semi-definite random matrices with $\mathbb{E}\|\xi_i\| = 1$ and*

$$\mathbb{P}\left(\left\|\sum_{i \in S} \xi_i\right\| \geqslant t\right) \leqslant \frac{c}{t^{1+\eta}} \quad provided \quad t > c|S| = c \sum_{i \in S} \mathbb{E}\|\xi_i\|,$$

for all subsets $S \subset [n]$ and some constants $c, \eta > 0$. Consider positive numbers μ_i such that

$$\sum_{i=1}^n \frac{1}{\mu_i} \leqslant 1.$$

Let μ be the minimal positive number such that

$$\sum_{i=1}^n \frac{\xi_i}{\mu_i + \mu} \preceq K \cdot I_n,$$

for some $K \geqslant C = 4c$. Then $\mathbb{E}\mu \leqslant \frac{c(\eta)}{K^{1+\eta}}$.

Proof. For any $j \geqslant 0$, denote

$$I_j = \left\{i / \ 2^j \leqslant \mu_i < 2^{j+1}\right\}$$

and let $n_j = |I_j|$. Note that

$$\sum_{j=0}^{+\infty} \frac{n_j}{2^{j+1}} \leqslant \sum_{j=0}^{+\infty} \sum_{i \in I_j} \frac{1}{\mu_i} = \sum_{i \leqslant n} \frac{1}{\mu_i} \leqslant 1.$$

Define μ' as the minimal positive number such that

$$\forall j \geqslant 0, \qquad \frac{1}{2^j + \mu'} \left\|\sum_{i \in I_j} \xi_i\right\| \leqslant \varepsilon_j,$$

where $\varepsilon_j = \frac{K}{2} \frac{n_j}{2^{j+1}} \vee \frac{K}{2a} 2^{-j\frac{\eta}{2+2\eta}}$ and $a = \sum_j 2^{-j\frac{\eta}{2+2\eta}}$. First note that $\mu \leqslant \mu'$; indeed,

$$\left\|\sum_{i=1}^n \frac{\xi_i}{\mu_i + \mu'}\right\| = \left\|\sum_{j=0}^{+\infty} \sum_{i \in I_j} \frac{\xi_i}{\mu_i + \mu'}\right\| \leqslant \left\|\sum_{j=0}^{+\infty} \frac{1}{2^j + \mu'} \sum_{i \in I_j} \xi_i\right\|$$

$$\leqslant \sum_{j=0}^{+\infty} \frac{1}{2^j + \mu'} \left\|\sum_{i \in I_j} \xi_i\right\|$$

$$\leqslant \sum_{j=0}^{\infty} \varepsilon_j$$

$$\leqslant \frac{K}{2} + \frac{K}{2} = K,$$

and since μ is the minimal positive number satisfying the inequality above, then $\mu \leqslant \mu'$.

We may now estimate $\mathbb{E}\mu'$; to this aim, we need to look at $\mathbb{P}\{\mu' \geqslant t\}$. For $t \geqslant 0$,

$$
\begin{aligned}
\mathbb{P}\{\mu' \geqslant t\} &= \mathbb{P}\left\{\exists j \geqslant 0 \Big/ \frac{1}{2^j + t}\left\|\sum_{i \in I_j} \xi_i\right\| \geqslant \varepsilon_j\right\} \\
&\leqslant \sum_{j=0}^{+\infty} \mathbb{P}\left\{\frac{1}{2^j + t}\left\|\sum_{i \in I_j} \xi_i\right\| \geqslant \varepsilon_j\right\} \\
&= \sum_{j=0}^{+\infty} \mathbb{P}\left\{\left\|\sum_{i \in I_j} \xi_i\right\| \geqslant \varepsilon_j(2^j + t)\right\} \\
&\leqslant \sum_{j=0}^{+\infty} \frac{c}{[\varepsilon_j(2^j + t)]^{1+\eta}},
\end{aligned}
$$

where the last inequality comes from the fact that $\varepsilon_j(2^j + t) \geqslant \frac{K}{4}n_j \geqslant c|I_j|$ and by applying the hypothesis satisfied by the ξ_i. Now since $\varepsilon_j \geqslant \frac{K}{2a}2^{-j\frac{\eta}{2+2\eta}}$, we have

$$
\mathbb{P}\{\mu' \geqslant t\} \leqslant \left(\frac{2a}{K}\right)^{1+\eta} \sum_{j=0}^{+\infty} \frac{c}{2^{-j\frac{\eta}{2}}(2^j + t)^{1+\eta}} \leqslant \frac{c(\eta)}{K^{1+\eta}}\sum_{j=0}^{+\infty}\frac{1}{(2^j + t)^{1+\frac{\eta}{2}}}.
$$

Now by integration we get,

$$
\mathbb{E}\mu' = \int_0^{+\infty}\mathbb{P}\{\mu' \geqslant t\}\,dt \leqslant \frac{c(\eta)}{K^{1+\eta}}\sum_{j=0}^{+\infty}\int_0^{+\infty}\frac{1}{(2^j + t)^{1+\frac{\eta}{2}}}\,dt \leqslant \frac{c(\eta)}{K^{1+\eta}}.
$$

$\square$

Applying Lemma 7.2 we get $\mathbb{E}\mu \leqslant c(\eta)\left(\frac{\psi}{\theta}\right)^{1+\eta}$, so that

$$
\mathbb{E}\Delta_1 \leqslant c(\eta)\frac{\psi^\eta}{\theta^{1+\eta}}. \tag{7.2}
$$

7.4 Estimating Δ_2

Since
$$
\psi_u(A) - \psi_{u+t}(A) = t \cdot \mathrm{Tr}\left((u \cdot I_n - A)^{-1}\left((u+t)\cdot I_n - A\right)^{-1}\right),
$$
we can write

$$
Q_2(t, B) = \frac{\sum_i \frac{\langle Bv_i, v_i\rangle}{(u+t-\lambda_i)^2}}{t\sum_i(u+t-\lambda_i)^{-1}(u-\lambda_i)^{-1}} \leqslant \frac{1}{t}\frac{\sum_i \frac{\langle Bv_i, v_i\rangle}{(u+t-\lambda_i)(u-\lambda_i)}}{\sum_i(u+t-\lambda_i)^{-1}(u-\lambda_i)^{-1}} \tag{7.3}
$$
$$
:= \frac{1}{t}P_2(t, B).
$$

First note that $P_2(t, B)$ can be written as $\sum_i \alpha_i(t)\langle Bv_i, v_i\rangle$ with $\sum_i \alpha_i(t) = 1$. Having this in mind, one can easily check that $\mathbb{E}P_2(t, B) = 1$ and

$$
\mathbb{E}P_2(t, B)^{1+\frac{3\eta}{4}} \leqslant c(\eta), \tag{7.4}
$$

where for the last inequality, we used the fact that B satisfies (MWR) with $p = 1 + \frac{3\eta}{4}$. In order to estimate Δ_2, we will divide it into two parts as follows:

$$
\Delta_2 = \Delta_2 \mathbf{1}_{\{P_2(0, B)\leqslant \frac{\theta}{4\psi}\}} + \Delta_2 \mathbf{1}_{\{P_2(0, B)> \frac{\theta}{4\psi}\}} := H_1 + H_2.
$$

Let us start by estimating $\mathbb{E}H_1$. Suppose that $P_2(0, B) \leqslant \frac{\theta}{4\psi}$ and denote

$$x = (1 + 4\theta)P_2(0, B).$$

Since $\psi_u(A) \leqslant \psi$, we have $(u - \lambda_i).\psi \geqslant 1 \; \forall i$ and therefore $u + x - \lambda_i \leqslant (1 + x\psi)(u - \lambda_i)$. This implies that

$$P_2(x, B) \leqslant (1 + x\psi)P_2(0, B).$$

Now write

$$Q_2(x, B) \leqslant \frac{1}{x}P_2(x, B) \leqslant \frac{1 + x\psi}{x}P_2(0, B) \leqslant \frac{1 + (1 + 4\theta)\frac{\theta}{4}}{1 + 4\theta} \leqslant 1 - \theta,$$

which means that

$$\Delta_2 \mathbf{1}_{\left\{P_2(0,B) \leqslant \frac{\theta}{4\psi}\right\}} \leqslant (1 + 4\theta)P_2(0, B),$$

and therefore

$$\mathbb{E}H_1 = \mathbb{E}\Delta_2 \mathbf{1}_{\left\{P_2(0,B) \leqslant \frac{\theta}{4\psi}\right\}} \leqslant 1 + 4\theta. \qquad (7.5)$$

Now it remains to estimate $\mathbb{E}H_2$. For that we need to prove a moment estimate for Δ_2. First observe that using (7.4) we have

$$\mathbb{P}\{\Delta_2 > t\} = \mathbb{P}\{Q_2(t, B) > 1 - \theta\} \leqslant \mathbb{P}\{P_2(t, B) > t.(1 - \theta)\} \leqslant \frac{c(\eta)}{t^{1 + \frac{3\eta}{4}}}.$$

By integration, this implies

$$\mathbb{E}\Delta_2^{1 + \frac{\eta}{2}} = \int_0^\infty \mathbb{P}\{\Delta_2 > t\}(1 + \frac{\eta}{2})t^{\frac{\eta}{2}}\,dt \leqslant \int_0^1 (1 + \frac{\eta}{2})t^{\frac{\eta}{2}}\,dt + \int_1^\infty \frac{c(\eta)}{t^{1 + \frac{\eta}{4}}} \leqslant c(\eta).$$

Let $p' = 1 + \frac{\eta}{2}$, applying Hölder's inequality with $\frac{1}{p'} + \frac{1}{q'} = 1$ we have :

$$\mathbb{E}H_2 = \mathbb{E}\Delta_2 \mathbf{1}_{\left\{P_2(0,B) > \frac{\theta}{4\psi}\right\}} \leqslant \left(\mathbb{E}\Delta_2^{p'}\right)^{\frac{1}{p'}} \left(\mathbb{P}\left\{P_2(0, B) > \frac{\theta}{4\psi}\right\}\right)^{\frac{1}{q'}} \qquad (7.6)$$

$$\leqslant c(\eta)\left(\left(\frac{\psi}{\theta}\right)^{1 + \frac{\eta}{2}} \mathbb{E}P_2(0, B)^{1 + \frac{\eta}{2}}\right)^{\frac{1}{q'}}$$

$$\leqslant c(\eta)\left(\frac{\psi}{\theta}\right)^{\frac{\eta}{2}}.$$

Looking at (7.5) and (7.6) we have

$$\mathbb{E}\Delta_2 \leqslant 1 + 4\theta + c(\eta)\left(\frac{\psi}{\theta}\right)^{\frac{\eta}{2}}.$$

Putting the estimates of Δ_1 and Δ_2 together we deduce

$$\mathbb{E}\Delta \leqslant 1 + 4\theta + c(\eta)\left(\frac{\psi}{\theta}\right)^{\frac{\eta}{2}} + c(\eta)\frac{\psi^\eta}{\theta^{1+\eta}}.$$

We are now ready to finish the proof. Take $u' = u + \Delta$ with $\theta = \frac{\varepsilon}{8}$. Then taking $\psi = c(\eta)\varepsilon^{1 + \frac{2}{\eta}}$ with the constant depending on c and η properly chosen, we get $\mathbb{E}\Delta \leqslant 1 + \varepsilon$.

8 Isotropic log-concave matrices

A natural way to define a log-concave matrix is to ask that it has a log-concave distribution. We will define the isotropic condition as follows:

Definition 8.1. *Let A be an $n \times m$ random matrix and denote by $(C_i)_{i \leqslant m}$ its columns. We will say that A is an isotropic log-concave matrix if $A'^t = \sqrt{m}(C_1^t, .., C_m^t)$ is an isotropic log-concave random vector in $\mathbb{R}^{nm}$.*

Remark 8.2. *Let $(a_{i,j})$ the entries of A. Saying that A' is isotropic means that*

$$\mathbb{E} a_{i,j} a_{k,l} = \frac{1}{m} \delta_{(i,j),(k,l)}$$

This implies that for any $n \times m$ matrix M we have

$$\mathbb{E} \langle A, M \rangle A = \mathbb{E} \mathrm{Tr} \left(A^t M \right) A = \frac{1}{m} M.$$

One can view this as an analogue to the isotropic condition in the vector case: in fact if $A = X$ is a vector (i.e an $n \times 1$ matrix), the above condition would be

$$\mathbb{E} \langle X, y \rangle X = y \quad \text{for all } y \in \mathbb{R}^n,$$

which means that X is isotropic in $\mathbb{R}^n$.

In [19] and [20], Paouris established large deviation inequality and small ball probability estimate satisfied by an isotropic log-concave vector. Moreover, Guédon-Milman obtained in [12] what is known as thin-shell estimate for isotropic log-concave vector. We will derive analogue properties for isotropic log-concave matrices using the results above.

Proposition 8.3. *Let A be an $n \times m$ isotropic log-concave matrix and denote $B = AA^t$. Then for every orthogonal projection P on $\mathbb{R}^n$ we have the following large deviation estimate for $\mathrm{Tr}(PB)$*

$$\mathbb{P} \left\{ \mathrm{Tr}(PB) \geqslant c_1 t \right\} \leqslant \exp \left(-\sqrt{t.m} \right) \quad \forall t \geqslant rank(P) \tag{8.1}$$

and a small ball probability estimate

$$\mathbb{P} \left\{ \mathrm{Tr}(PB) \leqslant c_2 \varepsilon . rank(P) \right\} \leqslant \varepsilon^{c_2 \sqrt{m.rank(P)}} \quad \forall \varepsilon \leqslant 1. \tag{8.2}$$

Moreover, we also have a thin-shell estimate

$$\mathbb{P} \left\{ |\mathrm{Tr}(PB) - rank(P)| \geqslant t.\mathrm{rank}(P) \right\} \leqslant C \exp \left(-ct^3 \sqrt{m.\mathrm{rank}(P)} \right) \quad \forall t \leqslant 1. \tag{8.3}$$

Proof. Let P be an orthogonal projection on $\mathbb{R}^n$ and denote $P' = I_m \otimes P$. As we have seen before $\mathrm{Tr}(PB) = \|PA\|_{\mathrm{HS}}^2 = \frac{1}{m} \|P'A'\|_2^2$ and $rank(P') = m.rank(P)$. Since $P'A'$ is an isotropic log-concave vector, then using the large deviation inequality [19] satisfied by $P'A'$, we have

$$\mathbb{P} \left\{ \|P'A'\|_2^2 \geqslant c_1 u \right\} \leqslant \exp \left(-\sqrt{u} \right) \quad \forall u \geqslant rank(P').$$

Let $t \geqslant rank(P)$ and write $u = t.m$. Since $u \geqslant m.rank(P) = rank(P')$ we have

$$\mathbb{P} \left\{ m.Tr(PB) \geqslant c_1 t.m \right\} \leqslant \exp \left(-\sqrt{t.m} \right)$$

which gives the large deviation estimate stated above.

For the small ball probability estimate, we apply Paouris result [20] to $P'A'$:

$$\mathbb{P}\left\{\|P'A'\|_2^2 \leqslant c_2\varepsilon.rank(P')\right\} \leqslant \varepsilon^{c_2\sqrt{rank(P')}} \quad \forall\varepsilon \leqslant 1.$$

Writing this in terms of B and P, we easily get the conclusion. Using the thin-shell estimate obtained in [12] and following the same procedure as above, we get the last part of the proposition. □

Remark 8.4. *Recently, it was shown in [11] that s-concave random vectors satisfy a thin-shell concentration similar to the log-concave one. Therefore, all results of this section extend to the case of s-concave random matrices.*

In [24], it was shown that an isotropic log-concave vector satisfies (SR) and we showed in Proposition 2.3 how to pass from (SR) to $(MSR*)$. Therefore, we may apply Theorem 1.3 to log-concave matrices and get the following:

Proposition 8.5. *Let A be an $n \times m$ isotropic log-concave matrix. Then $B = AA^t$ satisfies (MSR). Moreover $\forall\varepsilon \in (0,1)$, taking $N > c(\varepsilon)n$ independent copies of B we have*

$$\mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^{N} B_i - I_n\right\| \leqslant \varepsilon.$$

Proof. Note first that since A is isotropic in the sense of definition 8.1, then $B = AA^t$ satisfies $\mathbb{E}B = I_n$.

By proposition 8.3, B satisfies

$$\mathbb{P}\left(\mathrm{Tr}(PB) \geqslant c_1 t\right) \leqslant \exp(-\sqrt{tm}) \quad \forall t \geqslant rank(P) \text{ and } \forall P \text{ orthogonal projection of } \mathbb{R}^n.$$

and therefore $(MSR*)$. Applying Theorem 1.3 we deduce the result. □

8.1 Eigenvalues of the empirical sum of a log-concave matrix

The concentration inequalities satisfied by log-concave matrices will allow us obtain some results with high probability rather than in expectation as was the case before. Precisely, we can prove the following :

Theorem 8.6. *Let n, m and N some fixed integers. Let A be an $n \times m$ isotropic log-concave matrix and denote $B = AA^t$. For any $\varepsilon \in (0,1)$, if $m \geqslant \frac{C}{\varepsilon^6}\left[\log(CnN)\right]^2$, then with probability $\geqslant 1 - \exp(-c\varepsilon^3\sqrt{m})$ we have*

$$\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N} B_i\right) \leqslant 1 + \varepsilon + \frac{6n}{\varepsilon N},$$

where $(B_i)_{i\leqslant N}$ are independent copies of B.

Proof. The proof of Theorem 8.6 follows the same ideas developed in the previous sections. Let $\varepsilon \in (0,1)$, we only need the following property satisfied by our matrix $B = AA^t$:

$$\mathbb{P}\left(\langle Bx, x\rangle \geqslant 1 + \frac{\varepsilon}{2}\right) \leqslant \exp(-c\varepsilon^3\sqrt{m}) \quad \forall x \in \mathbb{S}^{n-1}.$$

This is obtained by applying (8.3) for rank 1 projections and looking only at the large deviation part.

Define Δ, ψ and α as follows:

$$\Delta = 1 + \varepsilon; \quad \psi = \frac{\varepsilon}{6} \quad \text{and} \quad \alpha = \frac{1 + \frac{\varepsilon}{2}}{1 + \varepsilon}.$$

Recall some notations

$$A_0 = 0, \ A_1 = B_1, \ A_2 = A_1 + B_1, \ .., \ A_N = A_{N-1} + B_N = \sum_{i=1}^{N} B_i.$$

Denote

$$u_0 = \frac{n}{\psi}, \ u_1 = u_0 + \Delta, \ u_2 = u_1 + \Delta, \ .., \ u_N = u_{N-1} + \Delta = (1 + \varepsilon)N + \frac{6n}{\varepsilon}.$$

Define

$$\psi_{u_i}(A_i) = \text{Tr}\,(u_i.I_n - A_i)^{-1},$$

the corresponding potential function when $A_i \prec u_i.I_n$. Denote by $\Im_i$ the event

$$\Im_i := ``A_i \prec u_i.I_n \quad \text{and} \quad \psi_{u_i}(A_i) \leqslant \psi".$$

Clearly $\mathbb{P}(\Im_0) = 1$. Suppose now that $\Im_i$ is satisfied; as we have seen in Lemma 7.1, the following condition is sufficient for the occurrence of the event $\Im_{i+1}$:

$$Q_2(\Delta, B_{i+1}) + Q_1(\Delta, B_{i+1}) \leqslant 1.$$

Note that

$$Q_2(\Delta, B_{i+1}) \leqslant \frac{1}{\Delta} P_2(\Delta, B_{i+1}),$$

where P_2 is defined in (7.3) but with A_i instead of A. Now denoting λ_j the eigenvalues of A_i and v_j the corresponding eigenvectors, taking the probability with respect to B_{i+1} one can write

$$\mathbb{P}_{B_{i+1}}\{Q_2(\Delta, B_{i+1}) + Q_1(\Delta, B_{i+1}) > 1\} \leqslant \mathbb{P}_{B_{i+1}}\left\{\frac{1}{\Delta} P_2(\Delta, B_{i+1}) + Q_1(\Delta, B_{i+1}) > 1\right\}$$

$$\leqslant \mathbb{P}_{B_{i+1}}\left\{\frac{1}{\Delta} P_2(\Delta, B_{i+1}) > \alpha\right\} + \mathbb{P}_{B_{i+1}}\{Q_1(\Delta, B_{i+1}) > 1 - \alpha\}$$

$$\leqslant \mathbb{P}_{B_{i+1}}\left\{\sum_{j=1}^{n} \frac{\langle B_{i+1}v_j, v_j\rangle}{(u_{i+1} - \lambda_j)(u_i - \lambda_j)} > \alpha\Delta \sum_{j=1}^{n} \frac{1}{(u_{i+1} - \lambda_j)(u_i - \lambda_j)}\right\}$$

$$+ \mathbb{P}_{B_{i+1}}\left\{\sum_{j=1}^{n} \frac{\langle B_{i+1}v_j, v_j\rangle}{u_{i+1} - \lambda_j} > 1 - \alpha\right\}$$

$$\leqslant \mathbb{P}_{B_{i+1}}\left\{\exists j \leqslant n \mid \langle B_{i+1}v_j, v_j\rangle > 1 + \frac{\varepsilon}{2}\right\} + \mathbb{P}_{B_{i+1}}\left\{\exists j \leqslant n \mid \langle B_{i+1}v_j, v_j\rangle > \frac{1 - \alpha}{\psi}\right\}$$

$$\leqslant 2Cn.\exp(-c\varepsilon^3\sqrt{m}).$$

Keeping in mind that B_i are independent, we have shown that

$$\mathbb{P}(\Im_{i+1}|\Im_i) \geqslant 1 - 2Cn.\exp(-c\varepsilon^3\sqrt{m}).$$

Moreover, we have

$$\mathbb{P}\left\{\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N}B_i\right)\leqslant 1+\varepsilon+\frac{6n}{\varepsilon N}\right\}\geqslant \mathbb{P}\left(\Im_N\right)$$

$$\geqslant \mathbb{P}\left(\Im_N|\Im_{N-1}\right)\mathbb{P}\left(\Im_{N-1}|\Im_{N-2}\right)\cdots\mathbb{P}\left(\Im_0\right)$$

$$\geqslant 1-2CNn.\exp(-c\varepsilon^3\sqrt{m}).$$

Therefore, Theorem 8.6 follows by the choice of m. $\qquad\square$

Remark 8.7. *Note that in the previous proof, we only used the large deviation inequality given by the thin-shell estimate (8.3). If one uses the deviation inequality given by (8.1), then by the same proof it can be proved that*

$$\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N}B_i\right)\leqslant C(1+\frac{n}{N}),$$

with high probability and with similar condition on m. The advantage of using thin-shell is that we can get an estimate close to 1.

By the same techniques, we also get an estimate of the smallest eigenvalue.

Theorem 8.8. *Let n,m and N some fixed integers. Let A be an $n\times m$ isotropic log-concave matrix and denote $B=AA^t$. For any $\varepsilon\in(0,1)$, if $m\geqslant\frac{C}{\varepsilon^6}\left[\log(CnN)\right]^2$, then with probability $\geqslant 1-\exp(-c\varepsilon^3\sqrt{m})$ we have*

$$\lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N}B_i\right)\geqslant 1-\varepsilon-\frac{3n}{\varepsilon N},$$

where $(B_i)_{i\leqslant N}$ are independent copies of B.

Proof. Here we will use the lower and the upper estimate given by thin-shell (8.3). Applying (8.3) for rank 1 projections, we have:

$$\mathbb{P}\left(1-\frac{\varepsilon}{2}\leqslant\langle Bx,x\rangle\leqslant 1+\frac{\varepsilon}{2}\right)\geqslant 1-C\exp(-c\varepsilon^3\sqrt{m})\quad\forall x\in\mathbb{S}^{n-1}.$$

Define δ, ϕ and α as follows:

$$\delta=1-\varepsilon;\quad\phi=\frac{\varepsilon}{3}\quad\text{and}\quad\alpha=\frac{1-\frac{\varepsilon}{2}}{1-\varepsilon}.$$

Recall some notations

$$A_0=0,\ A_1=B_1,\ A_2=A_1+B_1,\ ..,\ A_N=A_{N-1}+B_N=\sum_{i=1}^{N}B_i.$$

Denote

$$l_0=-\frac{n}{\phi},\ l_1=l_0+\delta,\ l_2=l_1+\delta,\ ..,\ l_N=l_{N-1}+\delta=N(1-\varepsilon)-\frac{3n}{\varepsilon}.$$

Define

$$\phi_{l_i}(A_i)=\operatorname{Tr}\left(A_i-l_i.I_n\right)^{-1},$$

the corresponding potential function when $A_i\succeq l_i.I_n$. Note also that $\delta\leqslant\frac{1}{\phi}$.
Denote by $\Im_i$ the event

$$\Im_i:=\text{“}A_i\succeq l_i.I_n\quad\text{and}\quad\phi_{l_i}(A_i)\leqslant\phi\text{”}.$$

Clearly $\mathbb{P}\left(\Im_0\right) = 1$. Suppose now that $\Im_i$ is satisfied, following what was done after Lemma 6.2, condition (6.2) is sufficient for the occurrence of the event $\Im_{i+1}$:

$$\frac{1}{\delta}q_2(\delta, B_{i+1}) - q_1(\delta, B_{i+1}) \geqslant 1$$

Denoting λ_j the eigenvalues of A_i and v_j the corresponding eigenvectors, taking the probability with respect to B_{i+1} one can write :

$$\mathbb{P}_{B_{i+1}}\left\{\frac{1}{\delta}q_2(\delta, B_{i+1}) - q_1(\delta, B_{i+1}) < 1\right\} \leqslant$$

$$\leqslant \mathbb{P}_{B_{i+1}}\left\{\frac{1}{\delta}q_2(\delta, B_{i+1}) < \alpha\right\} + \mathbb{P}_{B_{i+1}}\left\{q_1(\delta, B_{i+1}) > \alpha - 1\right\}$$

$$\leqslant \mathbb{P}_{B_{i+1}}\left\{\sum_{j=1}^{n}\frac{\langle B_{i+1}v_j, v_j\rangle}{(\lambda_j - l_{i+1})^2} < \alpha\delta\sum_{j=1}^{n}\frac{1}{(\lambda_j - l_{i+1})^2}\right\} + \mathbb{P}_{B_{i+1}}\left\{\sum_{j=1}^{n}\frac{\langle B_{i+1}v_j, v_j\rangle}{\lambda_j - l_{i+1}} > \alpha - 1\right\}$$

$$\leqslant \mathbb{P}_{B_{i+1}}\left\{\exists j \leqslant n \mid \langle B_{i+1}v_j, v_j\rangle < 1 - \frac{\varepsilon}{2}\right\} + \mathbb{P}_{B_{i+1}}\left\{\exists j \leqslant n \mid \langle B_{i+1}v_j, v_j\rangle > \frac{\alpha - 1}{\phi}\right\}$$

$$\leqslant 2Cn.\exp(-c\varepsilon^3\sqrt{m}).$$

Keeping in mind that B_i are independent, we have shown that

$$\mathbb{P}\left(\Im_{i+1}|\Im_i\right) \geqslant 1 - 2Cn.\exp(-c\varepsilon^3\sqrt{m}).$$

Moreover, we have

$$\mathbb{P}\left\{\lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N}B_i\right) \geqslant 1 - \varepsilon - \frac{3n}{\varepsilon N}\right\} \geqslant \mathbb{P}\left(\Im_N\right)$$

$$\geqslant \mathbb{P}\left(\Im_N|\Im_{N-1}\right)\mathbb{P}\left(\Im_{N-1}|\Im_{N-2}\right)\cdots\mathbb{P}\left(\Im_0\right)$$

$$\geqslant 1 - 2CNn.\exp(-c\varepsilon^3\sqrt{m}).$$

Therefore, Theorem 8.8 follows by the choice of m.

$\square$

Remark 8.9. *Note that in the previous proof, we used the large deviation inequality alongside the small ball probability estimate given by thin-shell (8.3). If one uses the deviation inequality given by (8.1) alongside the small ball probability estimate given by (8.2), then by the same proof it can be proved that*

$$\lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N}B_i\right) \geqslant -c + C\frac{n}{N},$$

with high probability and with similar condition on m. The advantage of using thin-shell is that we can get an estimate close to 1.

Combining the two previous results, we will be able to obtain, with high probability, a similar result to Proposition 8.5 for log-concave matrices:

Theorem 8.10. *Let A be an $n \times m$ isotropic log-concave matrix. For any $\varepsilon \in (0, 1)$, if $m \geqslant \frac{C}{\varepsilon^6}\left[\log(\frac{Cn}{\varepsilon})\right]^2$, taking*

$$N \geqslant \frac{96n}{\varepsilon^2}$$

copies $A_1, ..., A_N$ of A, then with probability $\geqslant 1 - \exp(-c\varepsilon^3 \sqrt{m})$ we have

$$\left\| \frac{1}{N} \sum_{i=1}^{N} A_i A_i^t - I_n \right\| \leqslant \varepsilon.$$

Proof. Let $\varepsilon \in (0, 1)$, $N \geqslant \frac{6n}{\varepsilon^2}$ and suppose that m satisfies the assumption of the theorem. Note that

$$\mathbb{P}\left\{ \left\| \frac{1}{N} \sum_{i=1}^{N} A_i A_i^t - I_n \right\| > 4\varepsilon \right\} \leqslant \mathbb{P}\left\{ \lambda_{\max}\left(\frac{1}{N} \sum_{i=1}^{N} A_i A_i^t \right) > 1 + 2\varepsilon \right\}$$
$$+ \mathbb{P}\left\{ \lambda_{\min}\left(\frac{1}{N} \sum_{i=1}^{N} A_i A_i^t \right) < 1 - 2\varepsilon \right\}$$

and therefore it is sufficient to apply Theorem 8.6 and Theorem 8.8. $\qquad\square$

8.2 Concrete examples of isotropic log-concave matrices

For $x \in \mathbb{R}^k$, we denote by x^* the vector with components $|x_i|$ arranged in nonincreasing order. Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$, we say that f is absolutely symmetric if $f(x) = f(x^*)$ for all $x \in \mathbb{R}^k$. (For example, $\| \cdot \|_p$ is absolutely symmetric).
Define F a function on $\mathbb{M}_{n,m}$ by $F(A) = f(s_1(A), .., s_k(A))$ for $A \in \mathbb{M}_{n,m}$ and $k = \min(n, m)$. It was shown by Lewis [16] that f is absolutely symmetric if and only if F is unitary invariant. Moreover, f is convex if and only if F is convex.
Let A be an $n \times m$ random matrix whose density with respect to Lebesgue measure is given by $G(A) = \exp(-f(s_1(A), .., s_k(A)))$, where f is an absolutely symmetric convex function. By the remark above, G is log-concave. This covers the case of random matrices with density of the form $\exp(-\sum_i V(s_i(A)))$, where V is an increasing convex function on $\mathbb{R}^+$. When $V(x) = x^2$, this would be the gaussian unitary ensemble GUE.

Proposition 8.11. *Let A be an $n \times m$ random matrix whose density with respect to Lebesgue is given by*

$$G(A) = \exp(-f(s_1(A), ..., s_k(A))),$$

where f is an absolutely symmetric convex function and $k = \min(n, m)$. Suppose that $\mathbb{E}\|A\|_{HS}^2 = n$, then A is an isotropic log-concave matrix, and $\sqrt{\frac{n}{m}} A^t$ is an $m \times n$ isotropic log-concave matrix.

Proof. Since f is an absolutely symmetric convex function, then G is log-concave as we have seen above. It remains to prove the isotropic condition.

Let $(a_{i,j})$ be the entries of A. Fix (i, j) and (k, l) two different indices. Note $D_j = diag(1, .., -1, .., 1)$ the $m \times m$ diagonal matrix where the -1 is on the jth term. Let $E_{(i,k)}$ be the $n \times n$ matrix obtained by swapping the ith and k^{th} rows in the identity matrix. Note also $F_{(j,l)}$ the $m \times m$ matrix obtained by swapping the jth and l^{th} rows in the identity matrix.

It is easy to see that $\alpha := AD_j$ change the jth column of A to its opposite and keep the rest unchanged. Note that AD_j has the same singular values as A.

Similarly, $\beta := E_{(i,k)} A F_{(j,l)}$ permute $a_{i,j}$ with $a_{k,l}$ and keep the other terms unchanged. Note also that $E_{(i,k)} A F_{(j,l)}$ has the same singular values as A.

Finally note that these α and β have a Jacobian equal to 1, and since f is absolutely symmetric, these transformations, which preserve the singular values, don't affect the density.

If $j \neq l$, by a change of variables $M = AD_j$ the density is invariant and we have

$$\int a_{i,j} a_{k,l} G(A) dA = - \int a_{i,j} a_{k,l} G(A) dA.$$

Doing the change of variables $M = D_i A$ when $i \neq k$, we can conclude that

$$\mathbb{E} a_{i,j} a_{k,l} = 0 \quad \text{if } (i,j) \neq (k,l).$$

Now by a change of variables $M = E_{(i,j)} A F_{(k,l)}$ the density is invariant and we have

$$\int a_{i,j}^2 G(A) dA = \int a_{k,l}^2 G(A) dA.$$

This implies that

$$\int a_{i,j}^2 G(A) dA = \frac{1}{nm} \sum_{k \leqslant n, l \leqslant m} \int a_{k,l}^2 G(A) dA = \frac{1}{nm} \int \|A\|_{\mathrm{HS}}^2 G(A) dA = \frac{1}{m}.$$

As a conclusion, we have shown that

$$\mathbb{E} a_{i,j} a_{k,l} = \frac{1}{m} \delta_{(i,j),(k,l)},$$

which means that A is isotropic. $\qquad\square$

Remark 8.12. *The condition $\mathbb{E}\|A\|_{HS}^2 = n$ can be obtained by a good normalization of the function f. Indeed, suppose that*

$$\frac{1}{n} \int \|A\|_{\mathrm{HS}}^2 G(A) dA = c.$$

Define $\hat{f}(x) = f(\sqrt{c}x) - nm\log(\sqrt{c})$ and $\hat{G}(A) = \exp\left(-\hat{f}(s_1(A), .., s_k(A))\right).$

Note that $\hat{G}$ is a probability density. Indeed, by the change of variables $M = \sqrt{c}A$ we have

$$\int \hat{G}(A) dA = \int \exp\left(-f(\sqrt{c}s_1(A), .., \sqrt{c}s_k(A))\right) (\sqrt{c})^{nm} dA$$
$$= \int \exp\left(-f(s_1(M), .., s_k(M))\right) dM = 1.$$

Note also that $\hat{G}$ satisfies the isotropic condition. Indeed, by the same change of variables we can write

$$\frac{1}{n} \int \|A\|_{\mathrm{HS}}^2 \hat{G}(A) dA = \frac{1}{cn} \int \|M\|_{\mathrm{HS}}^2 G(M) dM = 1.$$

Applying Theorem 8.6 and Theorem 8.8 for A and A^t we get:

Proposition 8.13. *Let A be an $n \times m$ random matrix whose density with respect to Lebesgue is given by*

$$G(A) = \exp\left(-f(s_1(A), ..., s_k(A))\right),$$

where f is an absolutely symmetric convex function, properly normalized as above and $k = \min(n, m)$.

Suppose that $m \geqslant \frac{C}{\varepsilon^6} \left[\log(\frac{Cn}{\varepsilon})\right]^2$ and $n \geqslant \frac{C}{\varepsilon^6} \left[\log(\frac{Cm}{\varepsilon})\right]^2$, taking $N = \frac{96 \max(n,m)}{\varepsilon^2}$ then with probability $\geqslant 1 - \exp(-c\varepsilon^3 \sqrt{k})$ we have

$$1 - \varepsilon \leqslant \lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N} A_i A_i^t\right) \leqslant \lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N} A_i A_i^t\right) \leqslant 1 + \varepsilon$$

and

$$(1-\varepsilon)\frac{n}{m} \leqslant \lambda_{\min}\left(\frac{1}{N}\sum_{i=1}^{N} A_i^t A_i\right) \leqslant \lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N} A_i^t A_i\right) \leqslant (1+\varepsilon)\frac{n}{m}.$$

References

[1] R. Adamczak, O. Guédon, R. Latała, K. Oleszkiewicz, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Moment estimates for convex measures. *Electronic Journal of Probability*, 17(101):1–19, 2012. MR-3005719

[2] R. Adamczak, A. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Tail estimates for norms of sums of log-concave random vectors. arXiv:1107.4070v1, July 2011.

[3] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles. *J. Amer. Math. Soc.*, 23(2):535–561, 2010. MR-2601042

[4] R. Adamczak, A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann. Sharp bounds on the rate of convergence of the empirical covariance matrix. *C. R. Math. Acad. Sci. Paris*, 349(3-4):195–200, 2011. MR-2769907

[5] R. Ahlswede and A. Winter. Addendum to: "Strong converse for identification via quantum channels" [IEEE Trans. Inform. Theory **48** (2002), no. 3, 569–579; MR1889969 (2003d:94069)]. *IEEE Trans. Inform. Theory*, 49(1):346, 2003. MR-1889969

[6] G. Aubrun. Sampling convex bodies: a random matrix approach. *Proc. Amer. Math. Soc.*, 135(5):1293–1303 (electronic), 2007. MR-2276637

[7] J. D. Batson, D. A. Spielman, and N. Srivastava. Twice-Ramanujan sparsifiers. In *STOC'09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 255–262. ACM, New York, 2009. MR-2780071

[8] R. Bhatia. *Matrix analysis*, volume 169 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. MR-1477662

[9] D. Chafaï, O. Guédon, G. Lecué, and A. Pajor. *Interactions between compressed sensing random matrices and high dimensional geometry*, volume 37 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2012. MR-3113826

[10] M. De Carli Silva, N. Harvey, and c. Sato. Sparse Sums of Positive Semidefinite Matrices. arXiv:1107.0088v2, July 2011.

[11] M. Fradelizi, O. Guédon, and A. Pajor. Spherical thin-shell concentration for convex measures. Available at arXiv:1306.6794.

[12] O. Guédon and E. Milman. Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures. *Geom. Funct. Anal.*, 21(5):1043–1068, 2011. MR-2846382

[13] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963. MR-0144363

[14] R. Kannan, L. Lovász, and M. Simonovits. Random walks and an $O^*(n^5)$ volume algorithm for convex bodies. *Random Structures Algorithms*, 11(1):1–50, 1997. MR-1608200

[15] V. F. Kolchin, B. A. Sevast′yanov, and V. P. Chistyakov. *Random allocations*. V. H. Winston & Sons, Washington, D.C., 1978. Translated from the Russian, Translation edited by A. V. Balakrishnan, Scripta Series in Mathematics. MR-0471016

[16] A. S. Lewis. The convex analysis of unitarily invariant matrix functions. *J. Convex Anal.*, 2(1-2):173–183, 1995. MR-1363368

[17] L. Mackey, M. Jordan, R. Chen, B. Farrell, and J. Tropp. Matrix concentration inequalities via the method of exchangeable pairs. Available at arXiv:1201.6002.

[18] R. I. Oliveira. Sums of random Hermitian matrices and an inequality by Rudelson. *Electron. Commun. Probab.*, 15:203–212, 2010. MR-2653725

[19] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006. MR-2276533

[20] G. Paouris. Small ball probability estimates for log-concave measures. *Trans. Amer. Math. Soc.*, 364(1):287–308, 2012. MR-2833584

[21] H. P. Rosenthal. On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.*, 8:273–303, 1970. MR-0271721

[22] M. Rudelson. Random vectors in the isotropic position. *J. Funct. Anal.*, 164(1):60–72, 1999. MR-1694526

[23] N. Srivastava. *Spectral Sparsification and Restricted Invertibility*. PhD thesis, Yale University, Mar. 2010. MR-2941475

[24] N. Srivastava and R. Vershynin. Covariance estimation for distributions with $2 + \varepsilon$ moments. *Ann. Probab.*, 41(5):3081–3111, 2013. MR-3127875

[25] J. A. Tropp. User-friendly tail bounds for sums of random matrices. *Found. Comput. Math.*, 12(4):389–434, 2012. MR-2946459