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On the expectation of the norm of random matrices with non-identically distributed entries

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Abstract

Let $X_{i,j}$, $i,j=1,...,n$, be independent, not necessarily identically distributed random variables with finite first moments. We show that the norm of the random matrix $(X_{i,j})_{i,j=1}^n$ is up to a logarithmic factor of the order of

$$\mathbb{E}\max_{i=1,\dots,n}\left\|(X_{i,j})_{j=1}^n\right\|_2+\mathbb{E}\max_{i=1,\dots,n}\left\|(X_{i,j})_{j=1}^n\right\|_2.$$

This extends (and improves in most cases) the previous results of Seginer and Latała.

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1 Introduction and Notation

We study the order of magnitude of the expectation of the largest singular value, i.e. the norm of random matrices with independent entries

$$\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\right),$$

where $a_{i,j}\in\mathbb{R}$, $i,j=1,\dots,n$, $g_{i,j}$, $i,j=1,\dots,n$, are standard Gaussian random variables and $\|\ \|_{2\to2}$ the operator norm on ℓ_2^n. There are two cases with a complete answer. Chevet [2] showed for matrices satisfying $a_{i,j}=a_ib_j$ that the expectation is proportional to

$$\|a\|_2\|b\|_\infty+\|a\|_\infty\|b\|_2,$$

where $\|a\|_2$ denotes the Euclidean norm of $a=(a_1,\dots,a_n)$ and $\|a\|_\infty=\max_{1\le i\le n}|a_i|$.

For diagonal matrices with diagonal elements $d_1,\dots,d_n$ the expectation of the norm is of the order of the Orlicz norm $\|(d_1,\dots,d_n)\|_M$ where the Orlicz function is given by $M(s)=\sqrt{\frac{2}{\pi}}\int_0^s e^{-\frac{1}{2t^2}}\,dt$ [4]. This Orlicz norm is up to a logarithm of n equal to the norm $\max_{1\le i\le n}|d_i|$.

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These two cases are of very different structure and seem to present essentially what might occur concerning the structure of matrices. This leads us to conjecture that the expectation for arbitrary matrices is up to a logarithmic factor equal to

$$\max_{i=1,\dots,n} \left\|(a_{i,j})_{j=1}^n\right\|_2 + \max_{j=1,\dots,n} \|(a_{i,j})_{i=1}^n\|_2 \,. \tag{1.1}$$

Latała [5] showed for arbitrary matrices

$$\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to 2}\right) \le C \max_{i=1,\dots,n} \left\|(a_{i,j})_{j=1}^n\right\|_2 + \max_{j=1,\dots,n} \|(a_{i,j})_{i=1}^n\|_2 + \left\|(a_{i,j})_{i,j=1}^n\right\|_4 .$$

Seginer [12] showed for any $n \times m$ random matrix $(X_{i,j})_{i,j=1}^{n,m}$ of independent identically distributed random variables

$$\mathbb{E}\left\|(X_{i,j})_{i,j=1}^{n,m}\right\|_{2\to 2} \le c\left(\mathbb{E}\max_{1\le i\le n}\left\|(X_{i,j})_{j=1}^m\right\|_2 + \mathbb{E}\max_{1\le j\le n}\|(X_{i,j})_{i=1}^n\|_2\right).$$

The behavior of the smallest singular value has been determined in [3, 7, 9, 10, 14].

Theorem 1.1. *There is a constant $c > 0$ such that for all $a_{i,j} \in \mathbb{R}$, $i, j = 1, ..., n$ and all independent standard Gaussian random variables $g_{i,j}$, $i, j = 1, ..., n$,*

$$\begin{aligned}
&\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to 2}\right) \\
&\le c\left(\ln\left(e\frac{\left\|(a_{i,j})_{i,j=1}^n\right\|_1}{\left\|(a_{i,j})_{i,j=1}^n\right\|_\infty}\right)\right)\left(\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right) + \mathbb{E}\left(\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right)\right).
\end{aligned}$$

The norm ratio appearing in the logarithm is at most n^2. In the same way we prove Theorem 1.1 we can show the similar formula

$$\begin{aligned}
&\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to 2}\right) \\
&\le c\left(\ln\left(e\frac{\left\|(a_{i,j})_{i,j=1}^n\right\|_1}{\left\|(a_{i,j})_{i,j=1}^n\right\|_\infty}\right)\right)\left(\max_{i=1,\dots,n}\left\|(a_{i,j})_{j=1}^n\right\|_2 + \max_{j=1,\dots,n}\|(a_{i,j})_{i=1}^n\|_2\right).
\end{aligned}$$

The proof of this formula is essentially the same as the proof of Theorem 1.1. Throughout the proof we use the expression $\max_{i=1,\dots,n}\left\|(a_{i,j})_{j=1}^n\right\|_2$ instead of the expression $\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right)$.

The inequality of Theorem 1.1 is generalized to arbitrary random variables as in [5].

Theorem 1.2. *Let $X_{i,j}$, $i, j = 1, ..., n$, be independent, mean zero random variables. Then*

$$E\left(\left\|(X_{i,j})_{i,j=1}^n\right\|_{2\to 2}\right) \le c\,(\ln(en))^2\left(\mathbb{E}\max_{i=1,\dots,n}\left\|(X_{i,j})_{j=1}^n\right\|_2 + \mathbb{E}\max_{j=1,\dots,n}\|(X_{i,j})_{i=1}^n\|_2\right).$$

On the other hand,

$$\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right) + \mathbb{E}\left(\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right) \tag{1.2}$$

is obviously smaller than $\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to 2}\right)$. We show that the expression (1.2) is equivalent to the Musielak-Orlicz norm of the vector $(1, \dots, 1)$, where the Orlicz functions are given through the coefficients $a_{i,j}$, $i, j = 1, \dots, n$. Our formula (Theorem 3.1) enables us to estimate from below the expectation of the operator norm in many cases efficiently.

Moreover, we do not know of any matrix where the expectation of the norm is not of the same order as (1.2).

A convex function $M:[0,\infty)\to[0,\infty)$ with $M(0)=0$ that is not identically 0 is called an *Orlicz function*. Let M be an Orlicz function and $x\in\mathbb{R}^n$ then the *Orlicz norm* of x, $\|x\|_M$, is defined by

$$\|x\|_M=\inf\left\{t>0\left|\sum_{i=1}^n M\left(\frac{|x_i|}{t}\right)\le 1\right.\right\}.$$

We say that two Orlicz functions M and N are equivalent ($M\sim N$) if there are strictly positive constants c_1 and c_2 such that for all $s\ge 0$

$$M(c_1 s)\le N(s)\le M(c_2 s).$$

If two Orlicz functions are equivalent, so are their norms: For all $x\in\mathbb{R}^n$

$$c_1\|x\|_M\le\|x\|_N\le c_2\|x\|_M.$$

In addition, let M_i, $i=1,...,n$, be Orlicz functions and let $x\in\mathbb{R}^n$ then the *Musielak-Orlicz norm* of x, $\|x\|_{(M_i)_i}$, is defined by

$$\|x\|_{(M_i)_i}=\inf\left\{t>0\left|\sum_{i=1}^n M_i\left(\frac{|x_i|}{t}\right)\le 1\right.\right\}.$$

2 The upper estimate

In this section we are going to prove the upper estimate. We require the following known lemma. In a more general form see e.g. ([13], Lemma 10).

Lemma 2.1. *Let* $x^{(l)}=\frac{1}{\sqrt{l}}(\overbrace{1,...,1}^{l},\overbrace{0,...,0}^{n-l})$, $l=1,...,n$, *and let* B_T *be the convex hull of* $\left(\varepsilon_1 x^{(l)}_{\pi(1)},...,\varepsilon_n x^{(l)}_{\pi(n)}\right)$, *where* $\varepsilon_i=\pm1$, $i=1,...,n$, *and* π *denote permutations of* $\{1,...,n\}$. *Let* $\|\ \|_T$ *be the norm on* $\mathbb{R}^n$ *whose unit ball is* B_T. *Then, for all* $x\in\mathbb{R}^n$

$$\|x\|_2\le\|x\|_T\le\sqrt{\ln(en)}\|x\|_2.$$

Proof. The left hand inequality is obvious. Let $x\in\mathbb{R}^n$. Then $x_1^*,\dots,x_n^*$ denotes the decreasing rearrangement of the numbers $|x_1|,\dots,|x_n|$. Let $a_k=\sqrt{k}-\sqrt{k-1}$ for $k=1,\dots,n$. Then, for all $x\in\mathbb{R}^n$

$$\|x\|_T=\sum_{k=1}^n x_k^*(\sqrt{k}-\sqrt{k-1}).$$

Since $\sqrt{k}-\sqrt{k-1}\le\frac{1}{\sqrt{k}}$

$$\|x\|_T\le\left(\sum_{k=1}^n|\sqrt{k}-\sqrt{k-1}|^2\right)^{\frac12}\|x\|_2\le\left(\sum_{k=1}^n\frac1k\right)^{\frac12}\|x\|_2\le\sqrt{\ln(en)}\|x\|_2.$$

□

We denote

$$S_T^{n-1}=\left\{x=(x_1,...,x_n)\in S^{n-1}\left|\exists i=1,...,n\left|\left\{j=1,...,n|x_j=\pm\frac{1}{\sqrt{i}}\right\}\right|=i\right.\right\},$$

the set of extremal points of B_T. Then by our previous lemma we have

$$\|A\|_{2\to 2} = \sup_{x\in S^{n-1}} \|Ax\|_2 \le \sqrt{\ln(en)} \sup_{x\in S_T^{n-1}} \|Ax\|_2 . \tag{2.1}$$

We use now the concentration of sums of independent gaussian random variables $X = \sum_{i=1}^n g_i z_i$ in a Banach space ([6], formula (2.35)): For all $t > 0$

$$\mathbb{P}\{\|X\| \ge \mathbb{E}\|X\| + t\} \le \exp\left(-\frac{t^2}{2\sigma(X)^2}\right), \tag{2.2}$$

where

$$\sigma(X) = \sup_{\|\xi\|_*=1} \left(\sum_{i=1}^n |\xi(z_i)|^2\right)^{\frac{1}{2}} . \tag{2.3}$$

Applying (2.2) to

$$\mathbb{E}\|Gx\|_2 \le \left(\sum_{i,j=1}^n a_{ij}^2 x_j^2\right)^{\frac{1}{2}} \quad \text{and} \quad \sigma(Gx) = \max_{i=1\dots,n} \left(\sum_{j=1}^n a_{ij}^2 x_j^2\right)^{\frac{1}{2}},$$

we immediately get the following lemma.

Lemma 2.2. *For all $i,j = 1,...,n$ let $a_{i,j} \in \mathbb{R}$, let $g_{i,j}$ be independent standard gaussians, $G = (a_{i,j}g_{i,j})_{i,j=1}^n$ and let $x \in B_2^n$. For all $\beta \ge 1$ and all x with $\max_{i=1,\dots,n}\left(\sum_{j=1}^n a_{i,j}^2 x_j^2\right) > 0$ we have*

$$\mathbb{P}\left\{\|Gx\|_2 > \beta\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{ij}g_{ij})_{i=1}^n\|_2\right]\right)\right\}$$
$$\le \exp\left\{-\frac{1}{2}\frac{\left(\beta\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right) - \left(\sum_{i,j=1}^n a_{i,j}^2 x_j^2\right)^{\frac{1}{2}}\right)^2}{\max_{i=1,\dots,n}\left(\sum_{j=1}^n a_{i,j}^2 x_j^2\right)}\right\}.$$

Proposition 2.1. *For all $i,j = 1,...,n$ let $a_{i,j} \in \mathbb{R}$, let $g_{i,j}$ be independent standard gaussian random variables and let $G = (a_{i,j}g_{i,j})_{i,j=1}^n$. For all β with $\beta \ge \sqrt{\frac{\pi}{2}}$*

$$\mathbb{P}\left\{\|G\|_{2\to 2} > \beta\sqrt{\ln(en)}\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right)\right\}$$
$$\le \sum_{l=1}^n \exp\left(l\ln(2n) - l\frac{\beta^2}{\pi}\right).$$

Furthermore, we get for $\beta = \sqrt{3\pi\ln(2n)}$

$$\mathbb{P}\left\{\|G\|_{2\to 2} > \sqrt{3\pi}\ln(en)\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right)\right\}$$
$$\le \frac{1}{n^2}.$$

Proof. We shall apply Lemma 2.2. We may assume that $\max_{i=1,\dots,n}\left(\sum_{j=1}^{n} a_{i,j}^2 x_j^2\right) > 0$ for all $x \in B_2^n \setminus \{0\}$. By (2.1)

$$\|G\|_{2\to 2} \le \sqrt{\ln(en)} \sup_{x\in S_T^{n-1}} \|Gx\|_2 .$$

Therefore, for $\beta \in \mathbb{R}_{>0}$, we have

$$\begin{aligned}&\mathbb{P}\left(\|G\|_{2\to2} > \beta\sqrt{\ln(en)}\left(\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right) + \mathbb{E}\left(\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right)\right)\right)\\ &\le \mathbb{P}\left(\sup_{x\in S_T^{n-1}}\|Gx\|_2 > \beta\left(\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{ij}g_{ij})_{j=1}^n\right\|_2\right) + \mathbb{E}\left(\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right)\right)\right).\end{aligned}$$

For all $l = 1, \dots, n$ let M_l be the set of $x^{(l)} \in S_T^{n-1}$, such that $x_j^{(l)} \in \{0, \pm\frac{1}{\sqrt{l}}\}$ for all $j = 1, \dots, n$. Now we apply Lemma 2.2 and get

$$\begin{aligned}&\mathbb{P}\left\{\|G\|_{2\to2} > \beta\sqrt{\ln(en)}\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right)\right\}\\
&\le \mathbb{P}\left\{\sup_{x\in S_T^{n-1}}\|Gx\|_2 > \beta\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{ij}g_{ij})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right)\right\}\\
&\le \sum_{l=1}^{n}\sum_{x^{(l)}\in M_l}\mathbb{P}\left\{\left\|Gx^{(l)}\right\|_2 > \beta\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{ij}g_{ij})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right)\right\}\\
&\le \sum_{l=1}^{n}\sum_{x^{(l)}\in M_l}\exp\left\{-\frac{1}{2}\cdot l\left(\beta\frac{\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{ij}g_{ij})_{j=1}^n\right\|_2\right] + \mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right)}{\max_{i=1,\dots,n}\left(\sum_{j\in\{k|x_k^{(l)}\neq0\}}^{n} a_{ij}^2\right)^{\frac12}}\right.\right.\\
&\qquad\left.\left.-\frac{\frac{1}{\sqrt{l}}\left(\sum_{i=1}^{n}\sum_{j\in\{k|x_k^{(l)}\neq0\}} a_{i,j}^2\right)^{\frac12}}{\max_{i=1,\dots,n}\left(\sum_{j\in\{k|x_k^{(l)}\neq0\}}^{n} a_{i,j}^2\right)^{\frac12}}\right)^2\right\}.\end{aligned}$$

We have

$$\frac{1}{\sqrt{l}}\left(\sum_{i=1}^{n}\sum_{j\in\{k|x_k^{(l)}\neq0\}} a_{ij}^2\right)^{\frac12} \le \max_{j\in\{k|x_k^{(l)}\neq0\}}\|(a_{ij})_{i=1}^n\|_2 \le \max_{j=1,\dots,n}\|(a_{ij})_{i=1}^n\|_2$$

and by triangle inequality

$$\begin{aligned}\mathbb{E}\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2 &\ge \max_{j=1,\dots,n}\mathbb{E}\left(\sum_{i=1}^{n}|a_{i,j}g_{i,j}|^2\right)^{\frac12} \\ &\ge \max_{j=1,\dots,n}\left(\sum_{i=1}^{n}|a_{i,j}|^2\left(\mathbb{E}|g_{i,j}|\right)^2\right)^{\frac12} = \sqrt{\frac{2}{\pi}}\max_{j=1,\dots,n}\|(a_{i,j})_{i=1}^n\|_2 .\end{aligned} \tag{2.4}$$

Therefore, we have for all β with $\beta \geq \sqrt{\frac{\pi}{2}}$

$$\beta \mathbb{E} \max_{j=1,\ldots,n} \|(a_{i,j} g_{i,j})_{i=1}^n\|_2 - \frac{1}{\sqrt{l}} \left(\sum_{i=1}^n \sum_{j \in \{k | x_k^{(l)} \neq 0\}} a_{i,j}^2 \right)^{\frac{1}{2}} \geq 0.$$

Thus

$$\begin{aligned}
&\mathbb{P}\left\{ \|G\|_{2\to 2} > \beta \sqrt{\ln(en)} \left(\mathbb{E}\left[\max_{i=1,\ldots,n} \left\|(a_{i,j} g_{i,j})_{j=1}^n\right\|_2 \right] + \mathbb{E}\left[\max_{j=1,\ldots,n} \|(a_{i,j} g_{i,j})_{i=1}^n\|_2 \right] \right) \right\} \\
&\leq \sum_{l=1}^n \sum_{x^{(l)} \in M_l} \exp\left\{ -\frac{1}{2} \cdot l \left(\beta \frac{\mathbb{E}\left[\max_{i=1,\ldots,n} \left\|(a_{i,j} g_{i,j})_{j=1}^n\right\|_2 \right]}{\max_{i=1,\ldots,n} \left(\sum_{j \in \{k | x_k^{(l)} \neq 0\}}^n a_{i,j}^2 \right)^{\frac{1}{2}}} \right)^2 \right\}.
\end{aligned}$$

Again, by (2.4) we have for all β with $\beta \geq \sqrt{\frac{\pi}{2}}$

$$\begin{aligned}
&\mathbb{P}\left\{ \|G\|_{2\to 2} > \beta \sqrt{\ln(en)} \left(\mathbb{E}\left[\max_{i=1,\ldots,n} \left\|(a_{i,j} g_{i,j})_{j=1}^n\right\|_2 \right] + \mathbb{E}\left[\max_{j=1,\ldots,n} \|(a_{i,j} g_{i,j})_{i=1}^n\|_2 \right] \right) \right\} \\
&\leq \sum_{l=1}^n \sum_{x^{(l)} \in M_l} \exp\left(-l \frac{\beta^2}{\pi} \right) \leq \sum_{l=1}^n 2^l n^l \exp\left(-l \frac{\beta^2}{\pi} \right) \\
&= \sum_{l=1}^n \exp\left(l \ln(2n) - l \frac{\beta^2}{\pi} \right).
\end{aligned}$$

We choose $\beta = \sqrt{3\pi \ln(2n)}$, thus such that $3 \ln(2n) = \frac{\beta^2}{\pi}$. Then

$$\begin{aligned}
&\mathbb{P}\left\{ \|G\|_{2\to 2} > \sqrt{3\pi \ln(2n) \ln(en)} \left(\mathbb{E}\left[\max_{i=1,\ldots,n} \left\|(a_{ij} g_{ij})_{j=1}^n\right\|_2 \right] + \mathbb{E}\left[\max_{j=1,\ldots,n} \|(a_{ij} g_{ij})_{i=1}^n\|_2 \right] \right) \right\} \\
&\leq \sum_{l=1}^n \exp\left(-2l \ln(2n) \right) = \sum_{l=1}^n \left(\frac{1}{4n^2} \right)^l \leq \frac{1}{n^2} \sum_{l=1}^\infty \left(\frac{1}{4} \right)^l \leq \frac{1}{n^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{P}\left\{ \|G\|_{2\to 2} > \sqrt{3\pi} \ln(en) \left(\mathbb{E}\left[\max_{i=1,\ldots,n} \left\|(a_{ij} g_{ij})_{j=1}^n\right\|_2 \right] + \mathbb{E}\left[\max_{j=1,\ldots,n} \|(a_{ij} g_{ij})_{i=1}^n\|_2 \right] \right) \right\} \\
&\leq \frac{1}{n^2}.
\end{aligned}$$

□

Proposition 2.2. *Let $a_{i,j} \in \mathbb{R}$, $i,j = 1,\ldots,n$ and $g_{i,j}$, $i,j = 1,\ldots,n$, be independent standard Gaussian random variables, then*

$$\begin{aligned}
&\mathbb{E}\left(\left\|(a_{i,j} g_{i,j})_{i,j=1}^n\right\|_{2\to 2} \right) \\
&\leq \left(\sqrt{\frac{\pi}{2}} + \sqrt{3\pi} \ln(en) \right) \left(\mathbb{E}\left[\max_{i=1,\ldots,n} \left\|(a_{i,j} g_{i,j})_{j=1}^n\right\|_2 \right] + \mathbb{E}\left[\max_{j=1,\ldots,n} \|(a_{i,j} g_{i,j})_{i=1}^n\|_2 \right] \right).
\end{aligned}$$

Proof. We divide the estimate of $\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\right)$ into two parts. Let M be set of all points with

$$\begin{aligned}&\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\\&\le\sqrt{3\pi}\ln(en)\left(\mathbb{E}\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2+\mathbb{E}\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right).\end{aligned}$$

Clearly,

$$\begin{aligned}&\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\chi_M\right)\\&\le\sqrt{3\pi}\ln(en)\left(\mathbb{E}\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2+\mathbb{E}\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right).\end{aligned}$$

Furthermore, by Cauchy-Schwarz inequality and Proposition 2.1 we get

$$\begin{aligned}&\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\chi_{M^c}\right)\le\sqrt{\mathbb{P}\left(M^c\right)}\left(\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}^2\right)\right)^{\frac12}\\&\le\frac1n\left(\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{HS}^2\right)\right)^{\frac12}=\frac1n\left(\sum_{i,j=1}^n|a_{i,j}|^2\right)^{\frac12}\le\max_{1\le i\le n}\left(\sum_{j=1}^n|a_{i,j}|^2\right)^{\frac12}.\end{aligned}$$

As in (2.4)

$$\begin{aligned}&\max_{i=1,\dots,n}\left\|(a_{ij})_{j=1}^n\right\|_2+\max_{j=1,\dots,n}\|(a_{ij})_{i=1}^n\|_2\\&\le\sqrt{\frac{\pi}{2}}\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{ij}g_{ij})_{j=1}^n\right\|_2+\max_{j=1,\dots,n}\|(a_{ij}g_{ij})_{i=1}^n\|_2\right).\end{aligned}$$

Altogether, this yields

$$\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\chi_{M^c}\right)\le\sqrt{\frac{\pi}{2}}\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2+\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right).$$

Summing up, we get

$$\begin{aligned}&\mathbb{E}\left(\left\|(a_{i,j}g_{i,j})_{i,j=1}^n\right\|_{2\to2}\right)\\&\le\left(\sqrt{\frac{\pi}{2}}+\sqrt{3\pi}\ln(en)\right)\left(\mathbb{E}\left[\max_{i=1,\dots,n}\left\|(a_{i,j}g_{i,j})_{j=1}^n\right\|_2\right]+\mathbb{E}\left[\max_{j=1,\dots,n}\|(a_{i,j}g_{i,j})_{i=1}^n\|_2\right]\right).\end{aligned}$$

□

Proof. (Theorem 1.1) W.l.o.g. we assume $0\le a_{i,j}\le1$, $i,j=1,\dots,n$ and that there is a coordinate that equals 1. For all $i,j=1,\dots,n$ and $k\in\mathbb{N}$ we define

$$a_{i,j}^k=\begin{cases}\frac{1}{2^k} & \text{,if } \frac{1}{2^k}<a_{i,j}\le\frac{1}{2^{k-1}}\\ 0 & \text{,else.}\end{cases}$$

Let $G=(a_{i,j}g_{i,j})_{i,j=1}^n$ and $G^k=(a_{i,j}^kg_{i,j})_{i,j=1}^n$. We denote by $\phi(k)$ the number of nonzero entries in $(a_{i,j}^k)_{i,j=1}^n$ and we choose γ such that $\left\|(a_{i,j})_{i,j=1}^n\right\|_1=2^\gamma\left\|(a_{i,j})_{i,j=1}^n\right\|_\infty$. Thus, we get $\phi(k)\frac{1}{2^k}=\sum_{i,j=1}^n a_{i,j}^k\le2^\gamma$ and therefore $\phi(k)\le2^{k+\gamma}$. Therefore, the non-zero entries of G^k are contained in a submatrix of size $2^{k+\gamma}\times2^{k+\gamma}$. Taking this into account and applying Proposition 2.2 to G^k

$$\begin{aligned}&\mathbb{E}\left\|G^k\right\|_{2\to2}\\&\le\left(\sqrt{\frac{\pi}{2}}+\sqrt{3\pi}\ln(e2^{k+\gamma})\right)\mathbb{E}\left(\max_{i=1,\dots,n}\left\|(a_{i,j}^kg_{i,j})_{j=1}^n\right\|_2+\max_{j=1,\dots,n}\left\|(a_{i,j}^kg_{i,j})_{i=1}^n\right\|_2\right).\end{aligned}$$

Since

$$\sqrt{\frac{\pi}{2}} + \sqrt{3\pi}\ln(e2^{k+\gamma}) \leq \frac{13}{10} + \frac{31}{10}(1+\ln(2^{k+\gamma})) \leq 8(k+\gamma)$$

we get

$$\mathbb{E}\left\|G^k\right\|_{2\to 2} \leq 8(k+\gamma)\left(\mathbb{E}\sum_{i,j=1}^{n}|a_{i,j}^k g_{i,j}|^2\right)^{\frac{1}{2}} \leq 8(k+\gamma)2^{\frac{\gamma}{2}-\frac{k}{2}}.$$

Therefore,

$$\sum_{k\geq 2\gamma}\mathbb{E}\left\|G^k\right\|_{2\to 2} \leq 8\sum_{k\geq 2\gamma}\frac{k+\gamma}{2^{\frac{k}{4}}} \leq 16\sum_{k=1}^{\infty}\frac{k}{2^{\frac{k}{4}}}.$$

Since one of the coordinates of the matrix is 1

$$\mathbb{E}\left\|G^1\right\|_{2\to 2} \geq \int_{-\infty}^{\infty}|g|dt = \sqrt{\frac{2}{\pi}}.$$

Therefore, there is a constant c such that

$$\mathbb{E}\left\|G\right\|_{2\to 2} \leq 2\mathbb{E}\left\|\sum_{k\leq 2\gamma}G^k\right\|_{2\to 2} + 2\sum_{k>2\gamma}\mathbb{E}\|G^k\|_{2\to 2} \leq c\mathbb{E}\left\|\sum_{k\leq 2\gamma}G^k\right\|_{2\to 2}.$$

The matrix $\sum_{k\leq 2\gamma}G^k$ has at most

$$\sum_{k\leq 2\gamma}\phi(k) \leq \sum_{k\leq 2\gamma}2^{\gamma+k} \leq 2^{3\gamma+1} \leq 2\left(\frac{\left\|(a_{i,j})_{i,j=1}^n\right\|_1}{\left\|(a_{i,j})_{i,j=1}^n\right\|_\infty}\right)^3 \tag{2.5}$$

entries that are different from 0. Therefore, all nonzero entries of $\sum_{k\leq 2\gamma}G^k$ are contained in a square submatrix having less than (2.5) rows and columns. We may apply Proposition 2.2 and get with a proper constant c

$$\begin{aligned}\mathbb{E}\left(\|G\|_{2\to 2}\right) \leq & c\left(\sqrt{\frac{\pi}{2}} + \sqrt{3\pi}\ln\left[e\cdot 2\left(\frac{\left\|(a_{i,j})_{i,j=1}^n\right\|_1}{\left\|(a_{i,j})_{i,j=1}^n\right\|_\infty}\right)^3\right]\right)\times \\ & \mathbb{E}\left(\max_{i=1,\dots,n}\left\|\left(\sum_{k\leq 2\gamma}a_{i,j}^k g_{i,j}\right)_{j=1}^n\right\|_2 + \max_{j=1,\dots,n}\left\|\left(\sum_{k\leq 2\gamma}a_{i,j}^k g_{i,j}\right)_{i=1}^n\right\|_2\right).\end{aligned}$$

□

3 The lower estimate

Theorem 3.1. *For all $i,j=1,...,n$ let $a_{i,j}\in\mathbb{R}_{\geq 0}$ and $g_{i,j}$ be independent standard gaussians. For all $i=1,...,n$ and all s with $0\leq s<\left\|(a_{i,j})_{j=1}^n\right\|_2^{-1}$*

$$N_i(s) = s\max_{j=1,\dots,n}a_{i,j}\exp\left(-\frac{1}{s^2\max_{j=1,\dots,n}a_{i,j}^2}\right)$$

and for all $s\geq\left\|(a_{i,j})_{j=1}^n\right\|_2^{-1}$

$$N_i(s) = \frac{\max_{j=1,\dots,n}a_{i,j}}{\left\|(a_{i,j})_{j=1}^n\right\|_2}\exp\left(-\frac{\left\|(a_{i,j})_{j=1}^n\right\|_2^2}{\max_{j=1,\dots,n}a_{i,j}^2}\right) + \frac{3}{e}\left\|(a_{i,j})_{j=1}^n\right\|_2\left(s-\frac{1}{\left\|(a_{i,j})_{j=1}^n\right\|_2}\right).$$

Moreover, $\widetilde{N}_j$*,* $j = 1, ..., n$*, are defined in the same way for the transposed matrix* $(a_{i,j})_{j,i=1}^n$*. Then* N_i*,* $i = 1, \dots, n$ *and* $\widetilde{N}_j$*,* $j = 1, \dots, n$*, are Orlicz functions and*

$$\begin{aligned}
&c_1 \left(\|(1)_{i=1}^n\|_{(N_i)_i} + \left\|(1)_{j=1}^n\right\|_{(\widetilde{N}_j)_j} \right) \\
&\leq \mathbb{E} \left(\max_{i=1,\dots,n} \left\|(a_{i,j} g_{i,j})_{j=1}^n\right\|_2 \right) + \mathbb{E} \left(\max_{j=1,\dots,n} \|(a_{i,j} g_{i,j})_{i=1}^n\|_2 \right) \\
&\leq c_2 \left(\|(1)_{i=1}^n\|_{(N_i)_i} + \left\|(1)_{j=1}^n\right\|_{(\widetilde{N}_j)_j} \right),
\end{aligned}$$

where c_1 *and* c_2 *are absolute constants.*

The following example is an immediate consequence of Theorem 3.1. It covers Toeplitz matrices.

Example 3.2. *Let* A *be a* $n \times n$*-matrix such that for all* $i, = 1 \dots, n$ *and* $k = 1, \dots, n$

$$\left(\sum_{j=1}^n |a_{i,j}|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^n |a_{j,k}|^2 \right)^{\frac{1}{2}}$$

and

$$\max_{1 \leq j \leq n} |a_{i,j}| = \max_{1 \leq j \leq n} |a_{j,k}|.$$

Then

$$\begin{aligned}
&\mathbb{E} \left(\max_{i=1,\dots\dots,n} \left\|(a_{i,j} g_{i,j})_{j=1}^n\right\|_2 \right) + \mathbb{E} \left(\max_{j=1,\dots\dots,n} \|(a_{i,j} g_{i,j})_{i=1}^n\|_2 \right) \\
&\sim \max \left\{ \left(\sum_{j=1}^n |a_{1,j}|^2 \right)^{\frac{1}{2}}, \sqrt{\ln n} \max_{1 \leq j \leq n} |a_{1,j}| \right\}.
\end{aligned}$$

We associate to a random variable X a Orlicz function M by

$$M(s) = \int_0^s \int_{\frac{1}{t} \leq |X|} |X| d\mathbb{P} dt. \tag{3.1}$$

We have

$$\begin{aligned}
M(s) &= \int_0^s \int_{\frac{1}{t} \leq |X|} |X| d\mathbb{P} dt \\
&= \int_0^s \left(\frac{1}{t} \mathbb{P} \left(|X| \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^\infty \mathbb{P} \left(|X| \geq u \right) du \right) dt.
\end{aligned} \tag{3.2}$$

Lemma 3.3. *There are strictly positive constants* c_1 *and* c_2 *such that for all* $n \in \mathbb{N}$*, all independent random variables* $X_1, ..., X_n$ *with finite first moments and for all* $x \in \mathbb{R}^n$

$$c_1 \|x\|_{(M_i)_i} \leq \mathbb{E} \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_{(M_i)_i},$$

where $M_1, ..., M_n$ *are Orlicz functions that are associated to the random variables (3.1).*

Lemma 3.3 is a generalization of the same result for identically distributed random variables [4]. It can be generalized from the ℓ_∞-norm to Orlicz norms.

We use the fact [11] that for all $x > 0$

$$\frac{\sqrt{2\pi}}{(\pi - 1)x + \sqrt{x^2 + 2\pi}} e^{-\frac{1}{2}x^2} \leq \sqrt{\tfrac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}s^2} ds \leq \sqrt{\tfrac{2}{\pi}} \tfrac{1}{x} e^{-\frac{1}{2}x^2}. \tag{3.3}$$

Proof. (Theorem 3.1) We apply Lemma 3.3 to the random variables

$$X_i = \left(\sum_{j=1}^{n} |a_{i,j} g_{i,j}|^2\right)^{\frac{1}{2}} \qquad\qquad i = 1, \dots, n.$$

Now, it is enough to show that $M_i \sim N_i$ for all $i = 1, \dots, n$. We have two cases.

We consider first $s < \frac{1}{2}\left(\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\right)^{-1}$. There are constants $c_1, c_2 > 0$ such that for all u with $u > 2\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}$

$$\exp\left(-c_1^2 \frac{u^2}{\max\limits_{j=1,\dots,n} a_{i,j}^2}\right) \le \mathbb{P}\left(\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge u\right) \le \exp\left(-c_2^2 \frac{u^2}{\max\limits_{j=1,\dots,n} a_{i,j}^2}\right). \tag{3.4}$$

The right-hand side inequality follows from (2.2). The left-hand side inequality follows from

$$\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \ge a_{i,1}^2 g_{i,1}^2.$$

Since $\frac{1}{t} > 2\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}$ for $0 < t < s$, we can apply (3.4). Therefore,

$$\begin{aligned}
M_i(s) &= \int_0^s \left\{\frac{1}{t}\mathbb{P}\left(\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge \frac{1}{t}\right) + \int_{\frac{1}{t}}^{\infty} \mathbb{P}\left(\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge u\right) du\right\} dt \\
&\le \int_0^s \left\{\frac{1}{t}\exp\left(-\frac{c_2^2}{t^2 \max\limits_{j=1,\dots,n} a_{i,j}^2}\right) + \int_{\frac{1}{t}}^{\infty} \exp\left(-c_2^2 \frac{u^2}{\max\limits_{j=1,\dots,n} a_{i,j}^2}\right) du\right\} dt.
\end{aligned}$$

By (3.3)

$$M_i(s) \le \int_0^s \left\{\frac{1}{t}\exp\left(-\frac{c_2^2}{t^2 \max\limits_{j=1,\dots,n} a_{i,j}^2}\right) + \frac{t}{2c_2^2} \max_{j=1,\dots,n} a_{i,j}^2 \exp\left(-\frac{c_2^2}{t^2 \max\limits_{j=1,\dots,n} a_{i,j}^2}\right)\right\} dt.$$

Since $\frac{1}{t} > 2\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge \sqrt{\frac{2}{\pi}}\left\|(a_{i,j})_{j=1}^n\right\|_2$, we get

$$\frac{1}{t} + \frac{t}{2c_2^2} \max_{j=1,\dots,n} a_{i,j}^2 \le \frac{1}{t} + \frac{1}{2c_2^2}\sqrt{\frac{\pi}{2}} \frac{1}{\left\|(a_{i,j})_{j=1}^n\right\|_2} \left\|(a_{i,j})_{j=1}^n\right\|_2^2 \le \frac{c}{t},$$

where c is an absolute constant. Altogether, we get

$$M_i(s) \le \int_0^s \frac{c}{t}\exp\left(-\frac{c_2^2}{t^2 \max\limits_{j=1,\dots,n} a_{i,j}^2}\right) dt = \int_{\frac{1}{s}}^{\infty} \frac{c}{u}\exp\left(-\frac{c_2^2 u^2}{\max\limits_{j=1,\dots,n} a_{i,j}^2}\right) du.$$

Passing to a new constant c_2 and using (3.3) we get for all s with $0 \le s < \frac{1}{2}\left(\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\right)^{-1}$

$$M_i(s) \le c\int_{\frac{1}{s}}^{\infty} \frac{1}{u}\exp\left(-\frac{c_2^2 u^2}{\max\limits_{j=1,\dots,n} a_{i,j}^2}\right) du \le \frac{s}{c_2}(\max_{j=1,\dots,n} a_{i,j}) \exp\left(-\frac{c_2^2}{s^2 \max\limits_{j=1,\dots,n} a_{i,j}^2}\right). \tag{3.5}$$

From this and the definition of N_i we get that there is a constant c such that for all s with $0 \le s < \frac{1}{2}\left(\mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\right)^{-1}$

$$M_i(s) \le N_i(cs).$$

Indeed, the inequality follows immediately from (3.5) provided that $\frac{s}{c_2} \le \frac{1}{2}\left(\sum_{j=1}^n |a_{i,j}|^2\right)^{-\frac{1}{2}}$. If $\frac{c_2}{2}\left(\sum_{j=1}^n |a_{i,j}|^2\right)^{-\frac{1}{2}} \le s \le \frac{1}{2}\left(\mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\right)^{-1}$ then, by (3.5) and $\sqrt{\frac{2}{\pi}}\max_{1\le j\le n} a_{i,j} \le \mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}$,

$$M_i(s) \le \frac{2\max\limits_{j=1,\dots,n} a_{i,j}}{c_2\mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}} \exp\left(-\frac{c_2^2(\mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}})^2}{4\max\limits_{j=1,\dots,n} a_{i,j}^2}\right) \le \frac{\sqrt{2\pi}}{c_2}.$$

Moreover,

$$\frac{\sqrt{2\pi}}{c_2} \le N_i\left(\left(\frac{\sqrt{2\pi}}{c_2}+1\right)\|(a_{i,j})_{j=1}^n\|^{-1}\right).$$

Therefore, with a universal constant c the inequality $M_i(s) \le N_i(cs)$ also holds for those values of s. The inverse inequality is treated in the same way.

Now we consider s with $s \ge \frac{1}{2}\left(\mathbb{E}\left(\sum\limits_{j=1}^n a_{i,j}^2 g_{ij}^2\right)^{\frac{1}{2}}\right)^{-1}$ and denote $\alpha = \mathbb{E}\left(\sum\limits_{j=1}^n a_{ij}^2 g_{ij}^2\right)^{\frac{1}{2}}$. The following holds

$$\begin{aligned} M_i(s) &= \int_0^s \left\{\frac{1}{t}\mathbb{P}\left(\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge \frac{1}{t}\right) + \int_{\frac{1}{t}}^{\infty} \mathbb{P}\left(\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge u\right)du\right\}dt \\ &= \int_0^{\frac{1}{2\alpha}} \left\{\frac{1}{t}\mathbb{P}\left(\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge \frac{1}{t}\right) + \int_{\frac{1}{t}}^{\infty} \mathbb{P}\left(\left(\sum_{j=1}^n a_{ij}^2 g_{ij}^2\right)^{\frac{1}{2}} \ge u\right)du\right\}dt \\ &\quad + \int_{\frac{1}{2\alpha}}^s \left\{\frac{1}{t}\mathbb{P}\left(\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge \frac{1}{t}\right) + \int_{\frac{1}{t}}^{\infty} \mathbb{P}\left(\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} \ge u\right)du\right\}dt \end{aligned}$$

The first summand equals $M_i(\frac{1}{2\alpha})$. Therefore, by (3.5) the first summand is of the order

$$\frac{\max\limits_{j=1,\dots,n} a_{i,j}}{\mathbb{E}\left(\sum\limits_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}} \exp\left(-c_2^2\frac{\left(\mathbb{E}\left(\sum\limits_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\right)^2}{\max\limits_{j=1,\dots,n} a_{i,j}^2}\right).$$

We estimate the second summand. The second summand is less than or equal to

$$\int_{\frac{1}{2\alpha}}^s \left\{\frac{1}{t} + \mathbb{E}\left(\sum_{j=1}^n a_{ij}^2 g_{ij}^2\right)^{\frac{1}{2}}\right\}dt \le \int_{\frac{1}{2\alpha}}^s 3\mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} dt \le 3\mathbb{E}\left(\sum_{j=1}^n a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} s.$$

Therefore, with a universal constant c we have for all s with $s \geq \frac{1}{2}\left(\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{ij}^2\right)^{\frac{1}{2}}\right)^{-1}$

$$M_i(s) \leq (c-1)s\left(\sum_{j=1}^{n}|a_{i,j}|^2\right)^{\frac{1}{2}} \leq cs\left(\sum_{j=1}^{n}|a_{i,j}|^2\right)^{\frac{1}{2}} - 1 \leq N_i(cs).$$

Now, we sketch a proof of a lower estimate. By (3.1), for all s with $s \geq \frac{2}{\alpha}$

$$M_i(s) \geq \int_{\frac{2}{\alpha}}^{s} \int_{\frac{1}{t} \leq \left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}} \left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} d\mathbb{P}dt \geq \int_{\frac{2}{\alpha}}^{s} \int_{\frac{\alpha}{2} \leq \left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}} \left(\sum_{j=1}^{n} a_{i,j}^2 g_{ij}^2\right)^{\frac{1}{2}} d\mathbb{P}dt.$$

By the definition of α

$$\begin{aligned} M_i(s) &\geq \frac{1}{2}\int_{\frac{2}{\alpha}}^{s} \mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} dt \\ &= \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\left(s - 2\left(\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}}\right)^{-1}\right) \\ &= \frac{1}{2}\mathbb{E}\left(\sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2\right)^{\frac{1}{2}} s - 1. \end{aligned}$$

The rest is done as in the case of the upper estimate. □

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