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Stochastic representation of entropy solutions of semilinear elliptic obstacle problems with measure data*

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Abstract

We consider semilinear obstacle problem with measure data associated with uniformly elliptic divergence form operator. We prove existence and uniqueness of entropy solution of the problem and provide stochastic representation of the solution in terms of some generalized reflected backward stochastic differential equations with random terminal time.

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1 Introduction

Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with regular boundary. In this paper we investigate obstacle problem with measure data associated with semilinear operator of the form

$$\mathcal{A}u = Au + f_u,$$

where

$$Au = \frac{1}{2}\sum_{i,j=1}^{d} D_i(a_{ij}D_j u), \quad f_u = f(\cdot, u) \tag{1.1}$$

and $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function such that

$$(1/\Lambda)|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d \quad \text{a.e. on } D \tag{1.2}$$

for some $\Lambda \geq 1$, $f : D \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying some assumptions to be specified later on.

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Let $\mathcal{M}_b(D)$ denote the set of all bounded signed measures on D and let $\mathcal{M}_b^2(D)$ denote the subset of $\mathcal{M}_b(D)$ consisting of all smooth measures, i.e. measures that charge no set of zero capacity (see Section 2 for details). Suppose we are given a quasi-continuous obstacle $\psi : D \to \mathbb{R}$ and $\mu \in \mathcal{M}_b^2(D)$. Roughly speaking, we consider the problem of finding the smallest quasi-continuous function $u : D \to \mathbb{R}$ such that

$$\mathcal{A}u \geq -\mu, \quad u|_{\partial D} = 0, \quad u \geq \psi \text{ q.e.}. \tag{1.3}$$

If $\mu \in \mathcal{M}_b^2(D) \cap H^{-1}(D)$ and the set

$$K_\psi = \{v \in H_0^1(D) : v \text{ is quasi-continuous, } v \geq \psi \text{ q.e. in } D\}$$

is nonempty, then the problem reduces to the following elliptic variational inequality problem (denoted by EVI(f, μ, ψ)): find $u \in K_\psi(D)$ such that

$$-\langle \mathcal{A}u, v - u\rangle \geq \langle \mu, v - u\rangle \quad \forall v \in K_\psi(D). \tag{1.4}$$

From (1.4) it follows that

$$-\langle \mathcal{A}u + \mu, v\rangle \geq 0 \quad \forall v \in H_0^1(D),\ v \geq 0.$$

Hence, by the Riesz-Schwartz theorem, there exists a positive Radon measure γ on D such that $\mathcal{A}u = -\mu - \gamma$, i.e.

$$\frac{1}{2}(a\nabla u, \nabla v)_2 - (f_u, v)_2 = \int_D v\, d\mu + \int_D v\, d\gamma, \quad v \in L^\infty(D) \cap H_0^1(D). \tag{1.5}$$

The measure γ is uniquely determined by (1.5), and is called the obstacle reaction associated with u. In the general case where $\mu \in \mathcal{M}_b^2(D)$ we consider entropy solutions of (1.3) in the sense defined in [13], i.e. we call $u : D \to \mathbb{R}$ a solution of (1.3) if there exists a positive measure $\gamma \in \mathcal{M}_b^2(D)$ such that u is a quasi-continuous entropy solution of the problem

$$\mathcal{A}u = -\mu - \gamma, \quad u|_{\partial D} = 0, \quad u \geq \psi \text{ q.e.} \tag{1.6}$$

and u is minimal in the sense that for any positive measure $\bar{\gamma} \in \mathcal{M}_b^2(D)$, if v is a quasi-continuous entropy solution of (1.6) with γ replaced by $\bar{\gamma}$, then $v \geq u$ q.e..

We will make the following assumptions:

(H1) $f : D \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function ($f(x, y)$ is continuous in y for a.e. $x \in D$ and measurable in x for every $y \in \mathbb{R}$) such that

- (a) $f(x, 0) = 0$, $f(x, \cdot)$ is nonincreasing for a.e. $x \in D$ (it follows in particular that $f(x, y)y \leq 0$ for $y \in \mathbb{R}$ and a.e. $x \in D$),
- (b) $F_c \in L^1(D)$ for every $c > 0$, where $F_c(x) = \sup_{|y|\leq c} |f(x, y)|$, $x \in D$,

(H2) $\mu \in \mathcal{M}_b^2(D)$,

(H3) $\psi : D \to \mathbb{R}$ is quasi-continuous and there is $\mu^* \in H^{-1}(D) \cap \mathcal{M}_b^+(D)$ such that $\psi \leq \psi^*$ q.e. on D, where $\psi^* \in H_0^1(D)$ is a variational solution of $\mathcal{A}\psi^* = -\mu^*$.

In the analytical part of the paper we show that under (H1)–(H3) the obstacle problem (1.3) has a unique entropy solution. In the proof of that part we combine ideas from [13], where the obstacle problem with $f = 0$ but more general than A nonlinear elliptic operator of monotone type $\bar{A}$ mapping $W^{1,p}(D)$, $p > 1$, into its dual is considered, and from [1, 2], where problems $\bar{A}u + f_u = -\mu$, $u|_{\partial D} = 0$ (i.e. $\psi = -\infty$) with f satisfying (H1) are considered.

It is known that if $\mu \in L^p(D)$ with $p > d$ and f satisfies the Lipschitz and the linear growth condition in u then the Dirichlet problem (1.6) with $\psi = -\infty$ has a unique continuous weak solution which can be represented by solutions of some backward stochastic differential equation (BSDE) with random terminal time (see [16, 17]). It is also known that viscosity solutions of some problems of the form (1.6) with nondivergence form operator in place of A may be represented by solutions of some reflected BSDEs (RBSDEs) with random terminal time (see [15]). Therefore it is natural to try to relate solutions of (1.6) to some reflected BSDE with forward driving process associated with A. In the paper we show that this is indeed possible and leads to investigation of interesting generalized RBSDEs involving additive functionals associated with measures μ and γ.

Let $\mathbb{X} = \{(X, P_x); x \in \mathbb{R}^d\}$ be a Markov process associated with the operator A (see Section 2) and let $\mathbb{X}_D$ be the part of $\mathbb{X}$ on D, i.e. $\mathbb{X}_D = \{(X^D, P_x); x \in D \cup \{\partial\}\}$, where ∂ is an extra point adjoint to D,

$$X_t^D = \begin{cases} X_t & \text{on } \{t < \tau\}, \\ \partial & \text{on } \{t \geq \tau\} \end{cases} \tag{1.7}$$

and

$$\tau = \inf\{t \geq 0 : X_t \notin D\}.$$

It is known that to every $\mu \in \mathcal{M}_b^2(D)$ corresponds a unique continuous additive functional (CAF) R of $\mathbb{X}_D$ whose Revuz measure is μ. The main result of the paper says that if (H1)–(H3) are satisfied then there exists a unique solution to (1.6) which has a quasi-continuous and quasi-everywhere (q.e. for short) finite version u. Secondly, for q.e. $x \in D$ the triple (Y, Z, K), where K is a positive CAF of $\mathbb{X}_D$ in Revuz correspondence with γ and

$$Y_t = u(X_t^D), \quad Z_t = \sigma \nabla u(X_t^D), \quad t \geq 0, \tag{1.8}$$

where σ is a symmetric square root of a, is a unique solution to the generalized reflected BSDE of the form

$$\begin{cases} Y_t = \int_{t\wedge\tau}^{\tau} f(X_s, Y_s)\,ds + R_\tau - R_{t\wedge\tau} + K_\tau - K_{t\wedge\tau} - \int_{t\wedge\tau}^{\tau} Z_s\,dB_s,\ t \geq 0,\ P_x\text{-a.s.}, \\ Y_t \geq \psi(X_t^D), \quad P_x\text{-a.s. for } t \geq 0, \\ K \text{ is a continuous increasing, } K_0 = 0,\ \int_0^\tau (Y_s - \psi(X_s))\,dK_s = 0,\ P_x\text{-a.s.}, \end{cases}$$

where B is a Wiener process. It follows immediately that for q.e. $x \in D$,

$$u(x) = Y_0, \quad P_x\text{-a.s.}. \tag{1.9}$$

Thus, the above RBSDE provides stochastic representation of quasi-continuous solutions of (1.6). With this representation in hand the minimality of u in the sense defined in [13] is a consequence of comparison results for solutions of generalized BSDEs proved in Section 3. From (1.8) and the fact that K increases only when $Y = \psi(X^D)$ we also deduce that

$$\int_D (u - \psi)\,d\gamma = 0,$$

i.e. the obstacle reaction γ associated with u is minimal in the sense that it acts only when $u = \psi$. Finally, let us mention that the representation (1.9) makes it posssible to give simple probabilistic definition of a solution of the problem (1.6).

Notation. As usual, for $p \in [1, \infty]$ we denote by $L^p(D)$ and $W^{1,p}(D)$ the standard Lebesgue and Sobolev spaces, $W_0^{1,p}(D)$ is the closure of $C_0^\infty(D)$ in $W^{1,p}(D)$. If $p = 2$ we write $H_0^1(D)$ instead of $W_0^{1,2}(D)$. $H^{-1}(D)$ is the dual space to $H_0^1(D)$. By $\|\cdot\|_2$ and $(\cdot\,,\cdot)_2$ we denote the usual norm and scalar product in $L^2(D)$.

2 Additive functionals of symmetric diffusions and smooth measures

In this section we are concerned with additive functionals of killed symmetric diffusions associated with A which are in the Revuz correspondence with smooth measures on D.

2.1 Symmetric diffusions

Let $\Omega = C([0,\infty),\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $[0,\infty)$ equipped with the topology of uniform convergence on compact intervals and let X be the canonical process on Ω. It is known that given operator A defined by (1.1) with a satisfying (1.2) one can construct a weak fundamental solution p for A and then a time-homogeneous Markov process $\mathbb{X} = \{(X,P_x); x\in\mathbb{R}^d\}$ for which p is the transition density function, i.e.

$$P_x(X_0=x)=1,\quad P_x(X_t\in B)=\int_B p(t,x,y)\,dy,\quad t>0$$

for any Borel $B\subset\mathbb{R}^d$ (see, e.g., [20]). Alternatively, one can define $\mathbb{X}$ as the Markov process associated with the Dirichlet form

$$\mathcal{E}(u,v)=\int_{\mathbb{R}^d}(a\nabla u,\nabla v)\,dx,\quad u,v\in D[\mathcal{E}]=H^1(\mathbb{R}^d)$$

(see [8, Example 4.5.2]).

Set $\mathcal{F}^0_\infty=\sigma(X_s,s<\infty)$, $\mathcal{F}^0_t=\sigma(X_s,s\le t)$ and for fixed $T>0$ set $\mathcal{F}^{T,0}_t=\sigma(\bar X^T_s,s\in[0,t])$, where $\bar X^T_t=X_{T-t}$, $t\in[0,t]$. Let $\mathcal{P}$ denote the set of all probability measures on $\mathbb{R}^d$ and let $P_\mu(\Gamma)=\int_{\mathbb{R}^d}P_x(\Gamma)\,\mu(dx)$, $\mu\in\mathcal{P}$, $\Gamma\in\mathcal{F}^0_\infty$. Let $\mathcal{F}^\mu_\infty$ denote the completion of $\mathcal{F}^0_\infty$ with respect to P_μ and let $\mathcal{F}^\mu_t$ (resp. $\bar{\mathcal{F}}^{T,\mu}_t$) denote the completion of $\mathcal{F}^0_t$ (resp. $\bar{\mathcal{F}}^{T,0}_t$) in $\mathcal{F}^\mu_\infty$ with respect to P_μ. Finally, let $\mathcal{F}_\infty=\bigcap_{\mu\in\mathcal{P}}\mathcal{F}^\mu_\infty$, $\mathcal{F}_t=\bigcap_{\mu\in\mathcal{P}}\mathcal{F}^\mu_t$, $\bar{\mathcal{F}}^T_t=\bigcap_{\mu\in\mathcal{P}}\mathcal{F}^{T,\mu}_t$.

Let X^D denote the process X killed outside D, i.e. X^D is defined by (1.7), where ∂ is an isolated point regarded as the point at infinity of D, and let $\mathbb{X}_D$ denote the part of $\mathbb{X}$ on D, i.e. $\mathbb{X}_D=\{(X,P_x),x\in D\cup\{\partial\}\}$, where $P_\partial(X_t=\partial)=1$, $t\ge0$. By [8, Theorem 4.4.2], the Dirichlet form of $\mathbb{X}_D$ is

$$\mathcal{E}(u,v)=(a\nabla u,\nabla v)_2,\quad u,v\in D[\mathcal{E}]=H^1_0(D).$$

Let $\mathcal{X}=\mathbb{R}^d$ or $\mathcal{X}=D$. Recall that a set $N\subset\mathcal{X}$ is called exceptional if there is a Borel set $B\supset N$ such that $P_m(\sigma_B<\infty)=0$, where $\sigma_B=\inf\{t>0:X_t\in B\}$ and m is the Lebesgue measure on $\mathcal{X}$.

We call an $\{\mathcal{F}_t\}$-adapted process $A=\{A_t,t\ge0\}$ a continuous additive functional (CAF) of $\mathbb{X}$ (resp. $\mathbb{X}_D$) if there is a set $\Lambda\in\mathcal{F}_\infty$ (called defining set for A) and an exceptional set $N\subset\mathbb{R}^d$ (resp. $N\subset D$) such that $P_x(\Lambda)=1$ for $x\in N^c$, $\theta_s\Lambda\subset\Lambda$ for $s\ge0$, where $\theta_s:\Omega\to\Omega$, $(\theta_s\omega)_t=\omega_{s+t}$, and for $\omega\in\Lambda$, $A_0(\omega)=0$, $A_\cdot(\omega)$ is continuous and $A_{s+t}(\omega)=A_s(\omega)+A_t(\theta_s\omega)$, $s,t\ge0$. If $N=\emptyset$, A is called AF in the strict sense. An $[0,\infty)$-valued CAF is called positive CAF (PCAF). Two AF's A^1,A^2 of $\mathbb{X}$ ($\mathbb{X}_D$) are said to be equivalent if there is an exceptional set $N\subset\mathbb{R}^d$ ($N\subset D$) such that for every $t>0$, $P_x(A^1_t=A^2_t)=1$ for $x\in N^c$.

Given a CAF A of $\mathbb{X}$ we define its energy by

$$e(A)=\lim_{t\downarrow0}\frac{1}{2t}\int_{\mathbb{R}^d}E_x|A_t|^2\,dx$$

whenever the limit exists. A CAF A of $\mathbb{X}$ such that $e(A)=0$ and for $t>0$, $E_x|A_t|<\infty$ for $x\in N^c$ is called a CAF of zero energy with exceptional set $N\subset\mathbb{R}^d$. We call M a martingale AF (MAF) of $\mathbb{X}$ with exceptional set $N \subset \mathbb{R}^d$ if for every $t > 0$, $E_x|M_t|^2 < \infty$, $E_xM_t = 0$ for $x \in N^c$. Recall that if M is a MAF of $\mathbb{X}$ with exceptional set N then M is an $(\{\mathcal{F}_t\}, P_x)$-square-integrable martingale for each $x \in N^c$. If $N = \emptyset$, M is a MAF in the strict sense.

We say that a CAF A of $\mathbb{X}$ in the strict sense is of zero quadratic variation if for each $x \in \mathbb{R}^d$,

$$Q^m_T(A) \equiv \sum_{k=0}^{m} |A_{t^m_{k+1}} - A_{t^m_k}|^2 \to 0 \text{ in probability } P_x \text{ as } m \to \infty$$

for any $T > 0$ and any sequence $\{\Pi^m = \{0 = t^m_0 < t^m_1 < \cdots < t^m_m = T\}\}$ of partitions of $[0,T]$ such that $\|\Pi^m\| = \max_{0\le k\le m-1} |t^m_{k+1} - t^m_k| \to 0$ as $m \to \infty$.

Let us recall that from [16, Theorem 3.4] it follows that there exist a continuous MAF M of $\mathbb{X}$ in the strict sense and a CAF A of $\mathbb{X}$ in the strict sense of zero quadratic variation such that

$$X_t - X_0 = M_t + A_t, \quad t \ge 0, \quad P_x\text{-a.s.}. \tag{2.1}$$

Thus, for each $x \in \mathbb{R}^d$ the canonical process X is an $(\{\mathcal{F}_t\}, P_x)$-Dirichlet process in the sense of Föllmer. Note also that the decomposition (2.1) coincides with the Fukushima strict decomposition of X into a MAF of $\mathbb{X}$ of locally zero energy and a CAF of $\mathbb{X}$ of zero energy (see [8, Theorem 5.5.1]).

From [16, Theorem 3.4] one can conclude that for every $T > 0$ there exists a unique continuous $\{\bar{\mathcal{F}}^T_t\}$-adapted process N^T such that N^T is a square-integrable martingale on $[0,T]$ under P_x for each $x \in \mathbb{R}^d$ and

$$A_t = \frac{1}{2}(-M_t + N^T_{T-t} - N^T_T - V_t), \quad t \in [0,T], \quad P_x\text{-a.s.}, \tag{2.2}$$

where

$$V^i_t = \sum_{j=1}^{d} \int_0^t a_{ij}(X_s) \frac{D_j p}{p}(s, X_0, X_s)\, ds, \quad t \ge 0 \tag{2.3}$$

(here D_jp stands for the generalized derivative of $y_j \mapsto p(t,x,y)$). Moreover, the co-variation processes of $M = (M^1, \dots, M^d)$ and $N^T = (N^{T,1}, \dots, N^{T,d})$ are given by

$$\langle M^i, M^j\rangle_t = \int_0^t a^{ij}(X_s)\, ds, \quad \langle N^{T,i}, N^{T,j}\rangle_t = \int_0^t a^{ij}(\bar{X}^T_s)\, ds, \quad t \in [0,T]. \tag{2.4}$$

Decomposition (2.1)–(2.3) may be called the strict Lyons-Zheng decomposition of $\mathbb{X}$.

2.2 Capacity, smooth measures

Let F be a compact subset of D. Recall that the capacity of F with respect to D is defined as

$$\text{cap}(F) = \inf\{\int_D |\nabla u(x)|^2\, dx : u \in C_0^\infty(D), u \ge \mathbf{1}_F\}$$

(we use the convention that $\inf \emptyset = \infty$). The capacity of an open subset $U \subset D$ is defined as

$$\text{cap}(U) = \sup\{\text{cap}(F) : F \text{ is compact}, F \subset U\}.$$

Finally, the capacity of any $B \subset D$ is defined as

$$\text{cap}(B) = \inf\{\text{cap}(U) : U \text{ is open}, B \subset U\}.$$

By [8, Theorem 4.2.1(ii)], $N \subset D$ is exceptional iff $\text{cap}(N) = 0$. Hence, in particular, for any Borel $B \subset D$,

$$\text{cap}(B) = 0 \text{ iff } P_m(\exists t > 0,\, X^D_t \in B) = 0. \tag{2.5}$$

Let $B \subset D$. In what follows a statement depending on $x \in B$ is said to hold quasi-everywhere on B (q.e. for short) if there is a set $N \subset B$ of zero capacity such that the statement holds for every $B \setminus N$.

A function $u : D \to \bar{\mathbb{R}}$ is quasi-continuous if for every $\varepsilon > 0$ there is an open set E such that $\mathrm{cap}(E) < \varepsilon$ and $u|_{D \setminus E}$ is continuous in $D \setminus E$.

It is known that every $u \in H^1_0(D)$ has a quasi-continuous representative that will always be identified with u.

Let $\mathcal{M}(D)$ denote the set of all signed Radon measures on D and let $\mathcal{M}_b(D)$ denote the subset of $\mathcal{M}(D)$ consisting of all measures whose total variation $|\mu|$ on D is finite. As usual, we identify $\mathcal{M}_b(D)$ with the dual of the Banach space $C^0(D)$ of continuous functions on D which vanish on the boundary of D, so that the duality is given by $\langle \mu, u \rangle = \int_D u\, d\mu$, $u \in C^0(D)$, and $\|\mu\|_{\mathcal{M}_b(D)} = |\mu|(D)$. By $\mathcal{M}^2_b(D)$ we denote the space of all measures μ in $\mathcal{M}_b(D)$ such that $\mu(B) = 0$ for every set $B \subset D$ such that $\mathrm{cap}(B) = 0$. By $\mathcal{M}^{2,+}_b(D)$ we denote the subset of $\mathcal{M}^2_b(D)$ consisting of all positive measures.

It is known that if $\mu \in H^{-1}(D) \cap \mathcal{M}_b(D)$ then $\mu \in \mathcal{M}^2_b(D)$, every $u \in H^1_0(D) \cap L^\infty(D)$ is summable with respect to μ and

$$\langle \mu, u \rangle = \int_D u\, d\mu,$$

where now $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(D)$ and $H^1_0(D)$ and u on the right hand-side is a quasi-continuous representative of u on the left hand-side.

Finally, let us recall that in [3] the following important result is proved: if $\mu \in \mathcal{M}_b(D)$ then $\mu \in \mathcal{M}^2_b(D)$ iff μ admits decomposition of the form

$$\mu = g + G \tag{2.6}$$

with $g \in L^1(D)$, $G \in H^{-1}(D)$.

2.3 Additive functionals of killed diffusions and smooth measures

In this subsection we use decomposition (2.6) to investigate structure of additive functionals of $\mathbb{X}_D$ corresponding to measures of the class $\mathcal{M}^2_b(D)$.

Let $S^+_0(D)$ denote the family of positive Radon measures on D of finite energy integrals, i.e. such that

$$\int_D |v(x)|\, d\mu(x) \le C \|v\|_{H^1_0(D)}, \quad v \in H^1_0(D) \cap C_0(D)$$

for some $C \ge 0$ ($C_0(D)$ is the space of all continuous functions on D having compact support). It is known (see [8, Section 2.2]) that $\mu \in S^+_0(D)$ iff for each $\alpha > 0$ there exists a unique function $U_\alpha \mu \in H^1_0(D)$, called α-potential of μ, such that

$$\frac{1}{2}(a \nabla U_\alpha \mu, \nabla v)_2 + \alpha (U_\alpha \mu, v)_2 = \int_D v(x)\, \mu(dx), \quad v \in H^1_0(D) \cap C_0(D).$$

Notice that by [8, Lemma 2.2.3], if $\mu \in S^+_0(D)$ and μ is bounded then $\mu \in \mathcal{M}^{2,+}_b(D)$. Let

$$S^+_{00}(D) = \{\mu \in S^+_0(D) : \mu(D) < \infty, \|U_1 \mu\|_\infty < \infty\}, \quad S_{00}(D) = S^+_{00}(D) - S^+_{00}(D).$$

By [8, Theorem 2.2.3], for any Borel set $B \subset D$,

$$\mathrm{cap}(B) = 0 \text{ iff } \mu(B) = 0 \text{ for every } \mu \in S^+_{00}(D). \tag{2.7}$$

By [8, Theorem A.2.10] the part process $\mathbb{X}_D$ is a Markov process on D (with respect to the filtration $\{\mathcal{F}_t\}$) with the transition function

$$p^D(t, x, B) = P_x(X_t \in B, t < \tau), \quad t > 0,\ x \in D,\ B \in \mathcal{B}(D).$$

Therefore the semigroup $\{P_t^D\}$ of operators associated with $\mathbb{X}_D$ is given by

$$P_t^D f(x) = E_x \mathbf{1}_{t<\tau} f(X_t), \quad t > 0,\ x \in D,\ f \in \mathcal{B}^+(D),$$

where E_x denotes the expectation with respect to P_x and $\mathcal{B}^+(D)$ is the space of positive measurable functions on D. By [5, Theorem 2.4], $p^D(t,x,\cdot)$ admits the transition density $p^D(t,x,y)$, which is symmetric and continuous on $D \times D$.

From now on we will use the following useful convention: any numerical function f on D will automatically be extended to $\bar{D} \cup \{\partial\}$ by setting $f(x) = 0$, $x \in \partial D$, $f(\partial) = 0$. With this convention,

$$f(X_t^D) = f(X_{t\wedge\tau}), \quad t \geq 0.$$

Let $\{R_\alpha, \alpha > 0\}$ denote the resolvent of $\mathbb{X}_D$, i.e.

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t^D f(x)\,dt = E_x \int_0^\tau e^{-\alpha t} f(X_t)\,dt, \quad f \in \mathcal{B}^+(D),$$

and let

$$U_A^\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t^D)\,dA_t, \quad f \in \mathcal{B}^+(D).$$

Definition. We say that a PCAF A of $\mathbb{X}_D$ and $\mu \in \mathcal{M}_b^{2,+}(D)$ are in the Revuz correspondence if for any $\alpha > 0$ and $f, g \in \mathcal{B}^+(D)$,

$$(g, U_A^\alpha f)_2 = \int_D f(x) R_\alpha g(x)\,d\mu(x). \tag{2.8}$$

In that case we call μ the Revuz measure of A and we write $\mu \sim A$ or $A \sim \mu$.

It is known (see [8, Theorem 5.1.3]) that (2.8) is equivalent to the following condition: for any $t > 0$, $g, f \in \mathcal{B}^+(D)$,

$$E_{g\cdot m} \int_0^t f(X_s^D)\,dA_s = \int_0^t \langle P_s^D g, f\cdot\mu\rangle\,ds. \tag{2.9}$$

By Lemma 5.1.8 and Theorem 5.1.3 in [8], any $\mu \in \mathcal{M}_b^{2,+}(D)$ admits a PCAF A of $\mathbb{X}_D$ whose Revuz measure is μ and that the PCAF A related to given $\mu \in \mathcal{M}_b^{2,+}(D)$ is unique up to the equivalence. In particular, if $\mu(dx) = f(x)\,dx$ for some positive $f \in L^1(D)$ then the unique PCAF A of $\mathbb{X}_D$ associated with μ is given by

$$A_t = \int_0^t f(X_s^D)\,ds = \int_0^{t\wedge\tau} f(X_s)\,ds, \quad t \geq 0. \tag{2.10}$$

Notice that $\mu \in \mathcal{M}_b^2(D)$ iff $\mu = \mu_1 - \mu_2$ for some $\mu_1, \mu_2 \in \mathcal{M}_b^{2,+}(D)$ (for instance one can apply the Jordan decomposition). Similarly, $\mu \in S_0(D)$ iff $\mu = \mu_1 - \mu_2$ for some $\mu_1, \mu_2 \in S_0^+(D)$. Given a signed measure $\mu \in S_0$ we decompose it as $\mu = \mu_1 - \mu_2$ in the above way and set

$$A = A^1 - A^2,$$

where $A^1 \sim \mu_1$, $A^2 \sim \mu_2$. Clearly A is a finite CAF of bounded variation and does not depend on the choice of μ_1, μ_2.

Lemma 2.1. *Let A be a CAF of $\mathbb{X}_D$ of finite variation associated with $\mu \in \mathcal{M}_b^{2,+}(D)$. Then for any $\nu \in S_{00}^+(D)$,*

$$E_\nu |A|_t = E_\nu(A_t^1 + A_t^2) \leq (1+t)\|U_1\nu\|_\infty \cdot \|\mu\|_{\mathcal{M}_b(D)}, \quad t > 0. \tag{2.11}$$

Proof. Follows from [8, Lemma 5.1.9]. □

Lemma 2.2. *If A is a CAF of $\mathbb{X}_D$ of finite variation associated with some measure $\mu \in \mathcal{M}_b^{2,+}(D)$ then $E_\nu A_\tau < \infty$ for every $\nu \in S_{00}(D)$.*

Proof. By (2.11), for any $\nu \in S_{00}(D)$ and $N > 0$ we have

$$P_\nu(A_\tau = \infty) = P_\nu(A_\tau = \infty, \tau \le N) + P_\nu(A_\tau = \infty, \tau > N) \le P_\nu(\tau > N) \le N^{-1} E_\nu \tau.$$

But $E_x \tau = E_x \int_0^\tau 1\, dt \equiv u(x)$, so u is a solution to the problem $Au = -1$, $u|_{\partial D} = 0$. Since $1 \in L^2(D)$, $u \in H_0^1(D)$. Hence $E_\nu \tau < \infty$ since $\nu \in S_0(D)$. □

Lemma 2.3. *Let $\mu, \mu_n \in \mathcal{M}_b^2(D)$, $A \sim \mu$, $A^n \sim \mu_n$. If $\mu_n \to \mu$ in $\mathcal{M}_b(D)$ then for any $\nu \in S_{00}^+(D)$ and $T \in [0, \infty]$, $|A^n - A|_{T \wedge \tau} \to 0$ in measure P_ν.*

Proof. We only consider the case $T = \infty$. The proof of the lemma in case $T \in [0, \infty)$ is similar and therefore we omit it. For any $\varepsilon > 0$ i $N > 0$ we have

$$\begin{aligned} P_\nu(|A^n - A|_\tau > \varepsilon) &\le P_\nu(|A^n - A|_\tau > \varepsilon) + P_\nu(\tau > N) \\ &\le \varepsilon^{-1} E_\nu |A^n - A|_{N \wedge \tau} + N^{-1} E_\nu \tau \\ &\le \varepsilon^{-1}(1 + N) \|U_1 \nu\|_\infty \cdot \|\mu_n - \mu\|_{\mathcal{M}_b(D)} + N^{-1} E_\nu \tau, \end{aligned}$$

from which the result follows. □

Following [18] given $T > 0$ and $h = (h^1, \dots, h^d) \in L^p(D)^d$ with $p > d$ we set

$$H_t^T(h) = -\sum_{i=1}^d \int_0^{t\wedge\tau} h^i(X_s)\, d(M_s^i + V_s^i) - \sum_{i=1}^d \int_{T - t\wedge\tau}^T h^i(\bar{X}_s^T)\, dN_s^{T,i}, \quad t \in [0, T],$$

where M, V, N^T are processes of the decomposition (2.2). One can show (see [18]) that $H_t^T(h) = H_t^{T+1}(h)$, $t \in [0, T]$, P_x-a.s. for every $x \in D$. Therefore we may define $H(h)$ on $[0, \infty)$ by putting $H_t(h) = H_t^T(h)$, $t \in [0, T]$. In the sequel we will use the notation

$$H_t(h) = \int_0^{t\wedge\tau} h(X_s) * dX_s, \quad t \ge 0.$$

It is known (see [18, Lemma 1]) that if $h \in W^{1,p}(D)^d$ with $p > d$ then for every $x \in D$,

$$\int_0^{t\wedge\tau} \mathrm{div} h(X_s)\, ds = \int_0^{t\wedge\tau} (a^{-1}h)(X_s) * dX_s = H_t(a^{-1}h), \quad t \ge 0, \quad P_x\text{-a.s.},$$

where a^{-1} denotes the inverse of a. From the above and (2.10) it follows that if $h \in W^{1,p}(D)^d$ for some $p > d$ then $H(a^{-1}h)$ is a CAF of $\mathbb{X}_D$ in the strict sense. Applying approximation arguments one can show that in fact it is a CAF of $\mathbb{X}_D$ in the strict sense for any $h \in L^p(D)^d$ with $p > d$.

The following proposition is a variant of [10, Proposition 3.5].

Proposition 2.4. *Let $h \in L^2(D)^d$ and let $\{h_n\} \subset L^\infty(D)^d$ be a sequence such that $h_n \to g$ in $L^2(D)^d$. Then*

(i) *There is a subsequence (still denoted by n) and a CAF A of $\mathbb{X}_D$ such that for every $T > 0$,*

$$E_x \sup_{t \le T} \Big| \int_0^{t\wedge\tau} (a^{-1}h_n)(X_s) * dX_s - A_t \Big| \to 0$$

for q.e. $x \in D$. In fact,

$$A_t = \int_0^{t\wedge\tau} (a^{-1}h)(X_s) * dX_s, \quad t \ge 0, \quad P_x\text{-a.s.}$$

for a.e. $x \in D$.

(ii) *If, in addition,* $\mathrm{div} h \equiv \mu \in \mathcal{M}_b(D)$ *then* $A \sim \mu$.

Proof. (i) Let

$$A^n_t = \int_0^{t\wedge\tau} (a^{-1}h_n)(X_s) * dX_s, \quad t \geq 0. \tag{2.12}$$

First we are going to show that for every $T > 0$,

$$\lim_{n,k\to\infty} \int_D E_x Y^{n,k}_T \, dx = 0, \tag{2.13}$$

where

$$Y^{n,k}_t = \sup_{s\leq t\wedge T} |A^n_s - A^k_s|.$$

By the definition of A^n,

$$\begin{aligned} E_x Y^{n,k}_T &= E_x \sup_{t\leq T} |\int_0^{t\wedge\tau} \mathrm{div}(h_n - h_k)(X_s)\, ds| \\ &= E_x \sup_{t\leq T} |\int_0^{t\wedge\tau} a^{-1}(h_n - h_k)(X_s)\, d(M_s + V_s) + \int_{T-t\wedge\tau}^{T} a^{-1}(h_n - h_k)(\bar{X}^T_s)\, dN^T_s|. \end{aligned}$$

By Doob's L^2-inequality and symmetry of the transition density $p^D(t,\cdot,\cdot)$,

$$\begin{aligned} E_m \sup_{t\leq T} | & \int_0^{t\wedge\tau} a^{-1}(h_n - h_k)(X_s)\, dM_s + \int_{T-t\wedge\tau}^{T} a^{-1}(h_n - h_k)(\bar{X}^T_s)\, dN^T_s| \\ & \leq C\|h_n - h_k\|_2^2. \end{aligned} \tag{2.14}$$

Furthermore,

$$\begin{aligned} & E_m \sup_{t\leq T} |\int_0^{t\wedge\tau} a^{-1}(h_n - h_k)(X_s)\, dV_s| \\ & \leq C \left(\int_D (E_x \int_0^t s^{-1/2} |(h_n - h_k)(X_s)|^2 \, ds)\, dx \right)^{1/2} \\ & \quad \times \left(\int_D (E_x \int_0^t s^{1/2} \frac{|\nabla p|^2}{p^2}(s, x, X_s)\, ds)\, dx \right)^{1/2} \\ & \leq C \left(\int_0^t s^{-1/2} (\int_D \int_D |(h_n - h_k)(y)|^2 p(s,x,y)\, dx\, dy) ds \right)^{1/2} \\ & \quad \times \left(\int_D (\int_0^t s^{1/2} \int_D \frac{|\nabla p|^2}{p}(s,x,y)\, ds\, dy)\, dx \right)^{1/2} \\ & \leq C\|h_n - h_k\|_2 (m(D))^{1/2}, \end{aligned} \tag{2.15}$$

the last inequality being a consequence of symmetry of $p(t,\cdot,\cdot)$ and [17, Lemma 5.2]. From (2.14), (2.15) we get (2.13). Now, set $B = \{x \in D : E_x \sup_{t\leq T} |A^n_t - A^k_t| \nrightarrow 0\}$. Let F be a compact subset in B and let $\sigma = \inf\{t \geq 0 : X_t \in F\}$. By the definition of σ and the strong Markov property,

$$\begin{aligned} P_x(\sigma \leq T) &\leq P_x(E_{X^D_\sigma} Y^{n,k}_t \nrightarrow 0) = P_x(E_x(\theta_\sigma Y^{n,k}_t | \mathcal{G}_\sigma) \nrightarrow 0) = P_x(E_x(Y^{n,k}_{t+\sigma}|\mathcal{G}_\sigma) \nrightarrow 0) \\ &= P_x(E_x(\sup_{s\leq(t+\sigma)\wedge T} |A^n_s - A^k_s| |\mathcal{G}_\sigma \nrightarrow 0) \leq P_x(E_x(Y^{n,k}_T|\mathcal{G}_\sigma) \nrightarrow 0). \end{aligned}$$

From (2.13) it follows that there is a subsequence such that if $n,k \to \infty$ along this subsequence then $E_x(Y^{n,k}_T|\mathcal{G}_\sigma) \to 0$, P_x-a.s. for a.e. $x \in D$. Hence $P_x(\sigma < \infty) = 0$ for a.e. $x \in D$, and so $P_m(\sigma < \infty) = 0$. From (2.5) we conclude now that cap$(F) = 0$, hence that cap$(B) = 0$. Thus,

$$\lim_{n,k\to\infty} E_x \sup_{t\le T} |A^n_t - A^k_t| = 0$$

for q.e. $x \in D$. Hence for q.e. $x \in D$ there exists a continuous process A^x such that

$$\lim_{n\to\infty} E_x \sup_{t\le T} |A^n_t - A^x_t| = 0. \tag{2.16}$$

To complete the proof of (i) we use arguments from the proof of [8, Lemma A.3.2]. Set $n_0(x) = 0$,

$$n_k(x) = \inf\{m > n_{k-1}(x) : \sup_{p,q\ge m} P_x(\sup_{t\le T} |A^p_t - A^q_t| > 2^{-k}) \le 2^{-k}\}, \quad k \ge 1,$$

and $Z^{x,k} = A^{n_k(x)}$, $Z^k = Z^{X_0,k}$, $\Lambda = \{\omega \in \Omega : \{Z^k(\omega)\}$ converges uniformly on $[0, T]\}$. Since

$$P_x(\sup_{t\le T} |Z^{k+1}_t - Z^k_t| > 2^{-k}) \le 2^{-k}, \quad k \ge 1$$

for q.e. $x \in D$, applying the Borel-Cantelli lemma shows that $P_x(\Lambda) = 1$ for q.e. $x \in D$. Set now $A_t(\omega) = \liminf_{k\to\infty} Z^k_t(\omega)$ for $\omega \in \Lambda$ and $A_t(\omega) = 0$ for $\omega \notin \Lambda$. Then A is a CAF of $\mathbb{X}_D$ with defining set Λ and $P_x(A_t = A^x_t, t \in [0, T]\} = 1$ for q.e. $x \in D$. From this and (2.16),

$$\lim_{n\to\infty} E_x \sup_{t\le T} |A^n_t - A_t| = 0$$

for q.e. $x \in D$, which proves (i).

(ii) Without loss of generality we may and will assume that $\mu \ge 0$. Let j_n be a mollifier and let $\mu_n = \mu * j_n$. Then $\mu_n = \mathrm{div} g_n$, where $g_n = g * j_n$. Since $\|\mu_n\|_1 \le \|\mu\|_{\mathcal{M}_b(D)}$, $\{\mu_n\}$ is relatively compact in the weak* topology in $\mathcal{M}_b(D)$ by Alaoglu's theorem. Therefore choosing a subsequence if necessary we may assume that $\mu_n \rightharpoonup \mu$ weakly* in $\mathcal{M}_b(D)$. Let A^n be the AF defined by (2.12). Since $A^n \sim \mu_n$, for every $f, g \in \mathcal{B}(D)$ and $\alpha > 0$,

$$(g, U^\alpha_{A^n} f)_2 = \langle f \cdot \mu_n, R_\alpha g\rangle.$$

Suppose now that $f \in C_b(D)$ and $g \in \mathcal{B}_b(D)$. Then $R_\alpha g$ is a continuous solution of the problem $(-\alpha + A)u = -g$, $u_{|\partial D} = 0$. In particular, $R_\alpha g \in C^0(D)$, and hence

$$\langle f \cdot \mu_n, R_\alpha g\rangle \to \langle f \cdot \mu, R_\alpha g\rangle.$$

On the other hand, by part (i),

$$(g, U^\alpha_{A^n} f)_2 \to (g, U^\alpha_A f)_2\,.$$

Thus,

$$(g, U^\alpha_A f)_2 = \langle f \cdot \mu, R_\alpha g\rangle \tag{2.17}$$

for $\alpha > 0$, $f \in C_b(D)$. By [8, Lemma 5.1.7] there exists a smooth measure $\mu_A \in \mathcal{M}^{2,+}(D)$ such that $\mu_A \sim A$. Since (2.17) is satisfied with μ replaced by μ_A, repeating arguments from the proof of [8, Theorem 5.1.3] shows that $\langle f \cdot \mu_A, g\rangle = \langle f \cdot \mu, g\rangle$ for any 0-excessive function h. Since for every $x \in \partial D$, $P^D_t 1(x) = P_x(\tau > t) \uparrow 1$ as $t \downarrow 0$, $g \equiv 1$ is 0-excessive. Hence $\langle f \cdot \mu_A, 1\rangle = \langle f \cdot \mu, 1\rangle$ for $f \in C_b(D)$ which implies that $\mu = \mu_A$. Thus, $\mu \sim A$, and the proof is complete. □

Lemma 2.5. *If $G_n \to G$ w $H^{-1}(D)$ then there exist $g^0, g^0_n \in L^2(D)$ and $g, g_n \in L^2(D)^d$ such that $G = g^0 - \mathrm{div} g$, $G_n = g^0_n - \mathrm{div} g_n$ and $\|g^0_n - g^0\|_2 + \|g_n - g\|_2 \to 0$.*

Proof. By Riesz's theorem there exist $u, u_n \in H_0^1(D)$ such that

$$\langle G, v\rangle = (u, v)_{H_0^1(D)}, \quad \langle G_n, v\rangle = (u_n, v)_{H_0^1(D)}$$

for $v \in H_0^1(D)$. Set $g^0 = u, g_n^0 = u_n$ and $g^i = \frac{\partial u}{\partial x_i}$, $g_n^i = \frac{\partial u_n}{\partial x_i}$, $i = 1, \dots, d$. Then $G = g^0 - \mathrm{div} g$, $G_n = g_n^0 - \mathrm{div} g_n$ and

$$\|G_n - G\|_{H^{-1}(D)} \le (\int_D \sum_{i=0}^d |g_n^i - g^i|^2\,dx)^{1/2} \equiv c_n.$$

On the other hand, putting $v_n = (u_n - u)/\|u_n - u\|_{H_0^1(D)}$ we see that

$$\langle G_n - G, v_n\rangle = \|u_n - u\|_{H_0^1(D)} = c_n\,,$$

which shows that $\|G_n - G\|_{H^{-1}(D)} = c_n$. Thus $c_n \to 0$, which proves the lemma. ☐

Lemma 2.6. *Let $\mu, \mu_n \in H^{-1}(D) \cap \mathcal{M}_b(D)$ and let $A \sim \mu$, $A^n \sim \mu_n$. If $\mu_n \to \mu$ in $H^{-1}(D)$ then there is a subsequence (still denoted by n) such that for any $\nu \in S_{00}^+(D)$ and $T > 0$,*

$$\sup_{t\le T} |A^n_{t\wedge\tau} - A_{t\wedge\tau}| \to 0 \quad \text{in measure } P_\nu.$$

Proof. In view of Lemmas 2.3 and 2.5 it suffices to prove the proposition in the case $\mu = \mathrm{div} g$, $\mu_n = \mathrm{div} g_n$ for some $g, g_n \in L^2(D)^d$ such that $g_n \to g$ in $L^2(D)^d$. But then, by Proposition 2.4,

$$A_t = \int_0^{t\wedge\tau} (a^{-1}g)(X_s) * dX_s, \quad A^n_t = \int_0^{t\wedge\tau} (a^{-1}g_n)(X_s) * dX_s, \quad t \ge 0, \quad P_x\text{-a.s.}$$

for a.e. $x \in D$, and therefore in much the same way as in the proof of (2.13) one can show that for any $T > 0$, $E_m \sup_{t\le T} |A^n_t - A_t| \to 0$. To prove the lemma it suffices now to repeat arguments from the proof of (2.16) to show that up to a subsequence, $\sup_{t\le T} |A^n_{t\wedge\tau} - A_{t\wedge\tau}| \to 0$ in measure P_x for q.e. $x \in D$, and hence, by (2.7), in measure P_ν for any $\nu \in S_{00}^+(D)$. ☐

3 RBSDEs and the obstacle problem - uniqueness of solutions

Let σ denote the symmetric square-root of a. Set

$$B_t = \int_0^t \sigma^{-1}(X_s)\,dM_s, \quad t \ge 0 \tag{3.1}$$

and observe that from (2.4) it follows that B is an $(\{\mathcal{F}_t\}, P_x)$-standard Brownian motion for each $x \in \mathbb{R}^d$.

Definition We say that a triple (Y^x, Z^x, K^x) of $\{\mathcal{F}_t\}$-adapted processes is a solution of $\mathrm{RBSDE}_x(f, \mu, \psi)$ if

(i) $Y^x_t = \int_{t\wedge\tau}^\tau f(X_s, Y^x_s)\,ds + R_\tau - R_{t\wedge\tau} + K^x_\tau - K^x_{t\wedge\tau} - \int_{t\wedge\tau}^\tau Z^x_s\,dB_s$, $t \ge 0$, P_x-a.s., where $R \sim \mu$,

(ii) Y^x is P_x-a.s. continuous, $\{Y^x_t, t \le T\} \in \mathcal{D}(P_x)$ for $T > 0$, i.e. for every $T > 0$ the family of random variables $\{Y^x_\sigma$, σ is an $\{\mathcal{F}_t\}$-stopping time, $\sigma \le T\}$ is uniformly integrable under P_x, $\lim_{T\to\infty} E_x|Y^x_{T\wedge\tau}| = 0$,

(iii) $\int_0^\tau |Z^x_t|^2\,dt < \infty$, P_x-a.s.,

(iv) $Y^x_t \ge \psi(X^D_t)$, $t \ge 0$, P_x-a.s.,

(v) K^x is a continuous increasing process such that $K_0^x = 0$, $\int_0^\tau (Y_s^x - \psi(X_s))\, dK_s^x = 0$, P_x-a.s.

A pair (Y^x, Z^x) of $\{\mathcal{F}_t\}$-adapted processes is a solution of $\mathrm{BSDE}_x(f,\mu)$ if Y^x, Z^x satisfy (ii), (iii) and condition (i) is satisfied with $K^x \equiv 0$.

For a given constant $k > 0$ we define the truncature operator $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(y) = \begin{cases} y & \text{if } |y| \le k, \\ k\,\mathrm{sign}(y) & \text{if } |y| > k, \end{cases}$$

and for a function $u : \mathbb{R} \to \mathbb{R}$ we define the truncated function $T_k u$ pointwise, i.e. $(T_k u)(x) = T_k(u(x))$.

Definition Let $\mu \in \mathcal{M}_b^2(D)$. We say that a measurable and almost everywhere finite function $u : D \to \mathbb{R}$ is an entropy solution of the problem

$$\mathcal{A}u = -\mu, \quad u|_{\partial D} = 0 \tag{3.2}$$

if

$$\forall k > 0,\ T_k u \in H_0^1(D) \tag{3.3}$$

and

$$\frac{1}{2}(a\nabla u, \nabla T_k(u-v))_2 - (f_u, T_k(u-v))_2 \le 2\int_D T_k(u-v)\, d\mu \tag{3.4}$$

for every $v \in H_0^1(D) \cap L^\infty(D)$ and $k > 0$.

Following [13] we adopt the following definition.

Definition We say that $u : D \to \mathbb{R}$ is an entropy solution of $\mathrm{OP}(f,\mu,\psi)$ if

(i) there exists $\gamma \in \mathcal{M}_b^{2,+}(D)$ such that u is an entropy solution of the problem

$$\mathcal{A}u = -\mu - \gamma, \quad u|_{\partial D} = 0 \tag{3.5}$$

such that $u \ge \psi$ q.e. in D,

(ii) for any $\bar\gamma \in \mathcal{M}_b^{2,+}(D)$, if v is an entropy solution of the problem

$$\mathcal{A}v = -\mu - \bar\gamma, \quad v|_{\partial D} = 0 \tag{3.6}$$

such that $v \ge \psi$ q.e. in D, then $v \ge u$ q.e. in D.

Let us remark that by the definition, if there exists a solution to $\mathrm{OP}(f,\mu,\psi)$ then it is unique, and if u denotes the solution then the measure γ satisfying (i), (ii) is uniquely determined. We call γ the obstacle reaction associated with u.

Since entropy solution u to $\mathrm{OP}(f,\mu,\psi)$ satisfies (3.3), it has a quasi-continuous representative. Therefore we will always assume that u denotes the quasi-continuous representative of a given entropy solution. If, in addition, $\|\nabla T_k u\|_2 \le C(1+k)$ for some $C > 0$ then the quasi-continuous representative is q.e. finite, i.e. $\mathrm{cap}\{|u| = \infty\} = 0$ (see [7, Remark 2.11]).

We know that solution of $\mathrm{OP}(f,\mu,\psi)$ if exists is unique by the definition. Uniqueness of solutions of associated RBSDEs with data f,μ,ψ under monotonicity condition on f follows from the following comparison result.

Theorem 3.1. *Let $f, f' : D\times\mathbb{R} \to \mathbb{R}$, $\psi : D \to \mathbb{R}$ be measurable functions, $\mu, \mu' \in \mathcal{M}_b^2(D)$ and $x \in D$. Suppose that (Y,Z,K) is a solution of $\mathrm{RBSDE}_x(f,\mu,\psi)$ and (Y',Z') is a solution of $\mathrm{BSDE}_x(f',\mu')$ such that $Y'_t \ge \psi(X_t)$, $t \ge 0$, P_x-a.s. If $f(z,\cdot)$ is nonincreasing and $f(z,\cdot) \le f'(z,\cdot)$ for a.e. $z \in D$, $\mu \le \mu'$ and $\psi(X)$ is continuous under P_x then $Y'_t \ge Y_t$, $t \ge 0$, P_x-a.s.*

Proof. Fix $T>0$. Let $\tau_n=\inf\{t\ge 0:\int_0^t|Z_s-Z'_s|^2\,ds>n\}\wedge T$ and let $R\sim\mu$, $R'\sim\mu'$. By the Itô-Tanaka formula, for $t\le T$ we have

$$
\begin{aligned}
&(Y_{t\wedge\tau_n}-Y'_{t\wedge\tau_n})^+ +\frac{1}{2}(L^0_{\tau\wedge\tau_n}(Y-Y')-L^0_{t\wedge\tau\wedge\tau_n}(Y-y'))\\
&\quad=(Y_{\tau\wedge\tau_n}-Y'_{\tau\wedge\tau_n})^+ +\int_{t\wedge\tau_n\wedge\tau}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}(f(X_s,Y_s)-f'(X_s,Y'_s))\,ds\\
&\qquad+\int_{t\wedge\tau_n\wedge\tau}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}\,d(R_s-R'_s+K_s)-\int_{t\wedge\tau_n\wedge\tau}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}(Z_s-Z'_s)\,dB_s,
\end{aligned}
$$

where $L^0(Y-Y')$ denote the local time at 0 of the semimartingale $Y-Y'$. By the assumptions on f,f',

$$
\begin{aligned}
&\int_{t\wedge\tau\wedge\tau_n}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}(f(X_s,Y_s)-f'(X_s,Y'_s))\,ds\\
&\qquad\le\int_{t\wedge\tau\wedge\tau_n}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}(f(X_s,Y_s)-f(X_s,Y'_s))\,ds\le 0.
\end{aligned}
$$

Moreover, since $\{Y_t>Y'_t\}\subset\{Y_t>\psi(X^D_t)\}$ and $R'-R\sim\mu'-\mu\ge 0$ is an increasing process,

$$
\int_{t\wedge\tau\wedge\tau_n}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}\,dK_s=0,\quad\int_{t\wedge\tau\wedge\tau_n}^{\tau\wedge\tau_n}\mathbf{1}_{\{Y_s>Y'_s\}}\,d(R_s-R'_s)\le 0.
$$

Hence

$$
E_x(Y_{t\wedge\tau_n}-Y'_{t\wedge\tau_n})^+\le E_x(Y_{\tau\wedge\tau_n}-Y'_{\tau\wedge\tau_n})^+.
$$

Since $\{Y_t,t\le T\}$, $\{Y'_t,t\le T\}\in\mathcal{D}(P_x)$ and $\tau_n\uparrow T$, P_x-a.s., letting $n\to\infty$ in the above inequality gives

$$
E_x(Y_t-Y'_t)^+\le E_x(Y_{T\wedge\tau}-Y'_{T\wedge\tau})^+=E_x(Y_T-Y'_T)^+.
$$

Since $\lim_{T\to\infty}E_x(Y_T-Y'_T)^+=0$, it follows that $E_x(Y_t-Y'_t)^+=0$ for $t\le T$, which proves the theorem. □

Corollary 3.2. *Assume that $f:D\times\mathbb{R}\to\mathbb{R}$, $\psi:D\to\mathbb{R}$ are measurable functions and $\mu\in\mathcal{M}^2_b(D)$. If $f(z,\cdot)$ is nonincreasing for a.e. $z\in D$ and the process $\psi(X)$ is continuous under P_x for some $x\in D$ then the solution of RBSDE$_x(f,\mu,\psi)$ is unique.*

Remark 3.3. *Assume that f,f',μ,μ' satisfy the assumptions of Theorem 3.1. From its proof (with $K\equiv 0$) it follows that if (Y,Z) is a solution of BSDE$_x(f,\mu)$ and (Y',Z') is a solution of BSDE$_x(f',\mu')$ for some $x\in D$ then $Y'_t\ge Y_t$, $t\ge 0$, P_x-a.s.*

4 Existence and stochastic representation of solutions of the obstacle problem

Our main goal is to prove existence and stochastic representation of solutions of the obstacle problem with data f,μ,ψ satisfying (H1)–(H3). Since the proof of this result is rather lengthy, we first assume additionally that $\mu\in H^{-1}(D)$ and f is bounded, and then we consider the general case.

4.1 The case $\mu\in H^{-1}(D)$

Assume that $\mu\in H^1_0(D)\cap\mathcal{M}_b(D)$ and that f is bounded and satisfies (H1a). Since the set K_ψ is nonempty by (H3), convex and closed (see, e.g., [8, Theorem 2.1.4]), and the operator $-\mathcal{A}:K_\psi\to H^{-1}(D)$ is strongly monotone, coercive and weakly continuous, there exists a unique solution of the elliptic variational inequality (1.4) (see, e.g., [9, Corollary III.1.8]).

Proposition 4.1. *Assume that f is bounded and satisfies (H1a), $\mu \in H^{-1}(D) \cap \mathcal{M}_b^2(D)$ and ψ satisfies (H3). Let $u \in H_0^1(D)$ be a solution of EVI(f,μ,ψ) and let γ be the obstacle reaction associated with u. Then for q.e. $x \in D$ the triple (Y,Z,K) defined by*

$$Y_t = u(X_t^D), \quad Z_t = \sigma\nabla u(X_t^D), \quad t \geq 0, \quad K \sim \gamma \tag{4.1}$$

is a solution of RBSDE$_x(f,\mu,\psi)$.

Proof. Step 1. We first assume additionally that $\mu \in L^\infty(D)$. Let $u_n \in H_0^1(D)$ be a unique weak solution of the problem

$$Au_n = -\mu - f_{u_n} - n(u_n-\psi)^-, \quad u_n|_{\partial D} = 0.$$

Then

$$\begin{aligned}\frac{1}{2}(a\nabla(u_n-\psi^*),\nabla(u_n-\psi^*))_2 &= (f_{u_n}-f_{\psi^*},u_n-\psi^*)_2 + (\mu-\mu^*,u_n-\psi^*)_2 \\ &\quad + n((u_n-\psi)^-,u_n-\psi^*)_2 \\ &\leq (\mu-\mu^*,u_n-\psi^*)_2,\end{aligned}$$

the last inequality being a consequence of (H1a) and the fact that $\psi^* \geq \psi$ a.e.. From the above and Poincaré's inequality it follows that $\{u_n\}$ is bounded in $H_0^1(D)$. Therefore, taking a subsequence if necessary we may and will assume that there is $w \in H_0^1(D)$ such that $u_n \to w$ weakly in $H_0^1(D)$. It is known (see [17]) that for every $x \in D$ the pair $(Y^n,Z^n) = (u_n(X^D),\sigma\nabla u_n(X^D))$ is a unique solution of the BSDE$(f,\mu+n(u_n-\psi)^-)$. In particular, for $T>0$,

$$\begin{aligned}u_n(X_{t\wedge\tau}) - u_n(X_{T\wedge\tau}) &= \int_{t\wedge\tau}^{T\wedge\tau}(f_{u_n}+\mu)(X_s)\,ds + n\int_{t\wedge\tau}^{T\wedge\tau}(Y_s^n-\psi(X_s))^-\,ds \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau}\sigma\nabla u_n(X_s)\,dB_s, \quad t\in[0,T], \quad P_x\text{-a.s.}.\end{aligned} \tag{4.2}$$

By Remark 3.3, $Y^n \leq Y^{n+1}$, P_x-a.s. for every $x \in D$. It follows in particular that $u_n \leq u_{n+1}$, $n \in \mathbb{N}$, and hence that $u_n \leq w$ a.e.. As a consequence,

$$\begin{aligned}\frac{1}{2}(a\nabla(u_n-w),\nabla(u_n-w))_2 &= (f_{u_n}-f_w,u_n-w)_2 + (f_w,u_n-w)_2 \\ &\quad + (\mu,u_n-w)_2 + n((u_n-\psi)^-,u_n-w)_2 \\ &\leq (f_w,u_n-w)_2 + (\mu,u_n-w)_2,\end{aligned}$$

from which we conclude that $u_n \to w$ in $H_0^1(D)$. Since for any $v \in K_\psi$,

$$\begin{aligned}\frac{1}{2}(a\nabla u_n,\nabla(v-u_n))_2 - (f_{u_n},v-u_n)_2 &= (\mu,v-u_n)_2 + n((u_n-\psi)^-,v-u_n)_2 \\ &\geq (\mu,v-u_n)_2,\end{aligned}$$

letting $n \to \infty$ and using the fact that $f_{u_n} \to f_w$ in $L^2(D)$ by Nemitskii's theorem (see [11, Theorem 2.1]), we see that

$$\frac{1}{2}(a\nabla w,\nabla(v-w))_2 - (f_w,v-w)_2 \geq (\mu,v-w)_2$$

for $v \in K_\psi$, which shows that w is a solution of EVI(f,μ,ψ), i.e. $w=u$. Now, let $R \sim \mu$, $K \sim \gamma$ and let

$$K_t^n = n\int_0^t (Y_s^n-\psi(X_s))^-\,ds, \quad t \geq 0,$$

i.e. $K^n \sim \gamma_n$, where $\gamma_n = n(u_n - \psi)^- \, dx$. Since $\gamma = f_u + \mu - Au$, $\gamma_n = f_{u_n} + \mu - Au_n$ and

$$\|A(u_n - u)\|_{H^{-1}(D)} \le \frac{\Lambda}{2} \|u_n - u\|_{H_0^1(D)},$$

we see that $\gamma_n \to \gamma$ in $H^{-1}(D)$. Hence, by Lemma 2.6 and (2.7), there is a subsequence (still denoted by n) such that for any $\nu \in \mathcal{S}_{00}^+(D)$,

$$\sup_{t \le T} |K^n_{t\wedge\tau} - K_{t\wedge\tau}| \to 0 \quad \text{in measure } P_\nu. \tag{4.3}$$

Moreover, by Doob's inequality and Lemma 2.1, for any $\nu \in \mathcal{S}_{00}^+(D)$,

$$\begin{aligned} E_\nu \sup_{t \le T} \Big| \int_0^{t\wedge\tau} \sigma \nabla(u_n - u)(X_s) \, dB_s \Big|^2 &\le C E_\nu \int_0^{T\wedge\tau} |\nabla(u_n - u)|^2(X_s) \, ds \\ &\le C(1+T) \|U_1 \nu\|_\infty \|\nabla(u_n - u)\|_2 \to 0 \end{aligned}$$

and, since $f_{u_n} \to f_u$ in $L^1(D)$,

$$E_\nu \int_0^{T\wedge\tau} |(f_{u_n} - f_u)(X_s)| \, ds \le C(1+T) \|U_1 \nu\|_\infty \|f_{u_n} - f_u\|_1 \to 0.$$

Finally, from [8, Lemma 5.1.5] and (2.7) it follows that there is a subsequence (still denoted by n) such that P_ν-a.s. the sequence $\{u_n(X)\}$ converges to $u(X)$ uniformly in $[0,T]$ in probability P_ν. Therefore letting $n \to \infty$ in (4.2) gives

$$\begin{aligned} u(X_{t\wedge\tau}) - u(X_{T\wedge\tau}) &= \int_{t\wedge\tau}^{T\wedge\tau} f_u(X_s) \, ds + R_{T\wedge\tau} - R_{t\wedge\tau} + K_{T\wedge\tau} - K_{t\wedge\tau} \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau} \sigma \nabla u(X_s) \, dB_s, \quad t \in [0,T], \quad P_\nu\text{-a.s.}. \end{aligned} \tag{4.4}$$

Since

$$Y^n_t \ge \psi(X^D_t), \quad t \in [0,T], \quad \int_0^{T\wedge\tau} (u_n(X_t) - \psi(X_t)) \, dK^n_t = 0, \quad P_\nu\text{-a.s.},$$

it follows that

$$Y_t \ge \psi(X^D_t), \quad t \in [0,T], \quad \int_0^{T\wedge\tau} (u(X_t) - \psi(X_t)) \, dK_t = 0, \quad P_\nu\text{-a.s.}. \tag{4.5}$$

Letting $T \to \infty$ in (4.4), (4.5) shows that under P_ν the triple defined by (4.1) is a solution of RBSDE with data f, μ, ψ and hence, by (2.7), is a solution of RBSDE$_x(f, \mu, \psi)$ for q.e. $x \in D$.

Step 2. We now show how to dispense with the assumption that $\mu \in L^\infty(D)$. If $\mu \in H^{-1}(D) \cap \mathcal{M}_b(D)$ then there exist $g^0 \in L^2(D)$, $g \in L^2(D)^d$ such that $\mu = g^0 - \text{div} g$. Let j_n be a mollifier and let $\mu_n = T_n g^0 - \text{div}((T_n g) * j_n)$. Let $u_n \in H_0^1(D)$ be a weak solution of EVI(f, μ_n, ψ) and let γ_n be the obstacle reaction associated with u_n so that

$$Au_n + f_{u_n} = -\mu_n - \gamma_n.$$

By *Step 1*, for q.e. $x \in D$ the triple (Y^n, Z^n, K^n) defined by

$$Y^n_t = u_n(X^D_t,), \quad Z^n_t = \sigma \nabla u_n(X^D_t), \quad t \ge 0, \quad K^n \sim \gamma_n \tag{4.6}$$

is a solution of RBSDE$_x(f,\mu_n,\psi)$. Hence, for any $T>0$,

$$u_n(X_{t\wedge\tau})-u_n(X_{T\wedge\tau})=\int_{t\wedge\tau}^{T\wedge\tau} f_{u_n}(X_s)\,ds+R^n_{T\wedge\tau}-R^n_{t\wedge\tau}+K^n_{T\wedge\tau}-K^n_{t\wedge\tau}$$
$$-\int_{t\wedge\tau}^{T\wedge\tau}\sigma\nabla u_n(X_s)\,dB_s,\quad t\in[0,T],\quad P_x\text{-a.s.} \tag{4.7}$$

for q.e. $x\in D$, where $R^n\sim\mu_n$, $K^n\sim\gamma_n$. Let $u\in H^1_0(D)$ be a weak solution of EVI(f,μ,ψ) and let γ be the obstacle reaction associated with u. Taking $v=u_n$ as a test function in (1.4) we get

$$-\langle\mathcal{A}u,u_n-u\rangle\geq\langle\mu,u_n-u\rangle.$$

Since $-\langle\mathcal{A}u_n,v-u_n\rangle\geq\langle\mu_n,v-u_n\rangle$ for $v\in K_\psi$, we also have

$$-\langle\mathcal{A}u_n,u-u_n\rangle\geq\langle\mu_n,u-u_n\rangle.$$

Hence

$$\begin{aligned}\frac12(a\nabla(u_n-u),\nabla(u_n-u))_2&\leq(f_{u_n}-f_u,u_n-u)_2+\langle\mu_n-\mu,u_n-u\rangle\\&\leq\langle\mu_n-\mu,u_n-u\rangle\\&\leq\|\mu_n-\mu\|_{H^1(D)}\|u_n-u\|_{H^1_0(D)},\end{aligned}$$

from which it follows that $u_n\to u$ in $H^1_0(D)$. As a consequence, $f_{u_n}\to f_u$ in $L^2(D)$ by Nemitskii's theorem, and hence $\gamma_n\to\gamma$ in $H^{-1}(D)$. From Lemma 2.6 and (2.7) it follows now that there is a subsequence (still denoted by n) such that for any $\nu\in\mathcal{S}^+_{00}(D)$,

$$\sup_{t\leq T}(|R^n_{t\wedge\tau}-R_{t\wedge\tau}|+|K^n_{t\wedge\tau}-K_{t\wedge\tau}|)\to0\quad\text{in measure } P_\nu,$$

where $R\sim\mu$, $K\sim\gamma$. To complete the proof it suffices now to let $n\to\infty$ in (4.7) and repeat step by step arguments following (4.3) in *Step 1*. □

Proposition 4.2. *Let f,μ satisfy the assumptions of Proposition 4.1 and let $u\in H^1_0(D)$ be a weak solution of the problem*

$$\mathcal{A}u=-\mu.$$

Then for q.e. $x\in D$ the pair

$$Y_t=u(X^D_t),\quad Z_t=\sigma\nabla u(X^D_t),\quad t\geq0$$

is a solution of BSDE$_x(f,\mu)$.

Proof. Follows from the proof of Proposition 4.1 with $\psi=-\infty$ and $\gamma=0$, $K=0$. □

Proposition 4.3. *Under the assumptions of Proposition 4.1,*

$$\|\gamma\|_{\mathcal{M}_b(D)}\leq\|(\mu-\mu^*)^-\|_{\mathcal{M}_b(D)}.$$

Proof. Let us define the operator $\mathcal{B}:H^1_0(D)\to H^{-1}(D)$ by

$$\mathcal{B}w=Aw+(f_{w+\psi^*}-f_{\psi^*}).$$

Let $u_n\in H^1_0(D)$ be a solution of EVI(f,μ,ψ_n) with $\psi_n=\psi-n^{-1}$ and let $w_n\in H^1_0(D)$ be a solution of the elliptic variational inequality with the operator $\mathcal{B}$, measure $\mu-\mu^*$ and obstacle $\psi_n-\psi^*$, i.e.

$$\begin{cases}\langle-\mathcal{B}w_n,\xi-w_n\rangle\geq\langle\mu-\mu^*,\xi-w_n\rangle\quad\forall\xi\in K_{\psi_n-\psi^*}(D),\\ w_n\in K_{\psi_n-\psi^*}(D).\end{cases}$$

Since $\mathcal{A}\psi^* = -\mu^*$, for every $\eta \in K_{\psi_n}$ we have

$$\begin{aligned}\langle -\mathcal{A}(w_n+\psi^*), \eta - w_n - \psi^*\rangle &= \langle -Aw_n - (f_{w_n+\psi^*} - f_{\psi^*}) - A\psi^* - f_{\psi_*}, \eta - \psi^* - w_n\rangle \\ &= \langle -\mathcal{B}w_n, \eta - \psi^* - w_n\rangle + \langle \mu^*, \eta - \psi^* - w_n\rangle \\ &\geq \langle \mu, \eta - \psi^* - w_n\rangle.\end{aligned}$$

From the above it follows that $w_n+\psi^*$ is a solution of EVI(f,μ,ψ_n), i.e. $u_n = w_n+\psi^*$. As a consequence, the obstacle reaction β_n associated with w_n coincides with the obstacle reaction γ_n associated with u_n because

$$Au_n + f_{u_n} = Aw_n + f_{w_n+\psi^*} + A\psi^* = \mathcal{B}w_n + A\psi^* + f_{\psi^*} = -(\mu-\mu^*) - \beta_n - \mu^* = -\mu_n - \beta_n.$$

Let $v \in H_0^1(D)$ be a solution to the equation

$$\mathcal{B}v = -(\mu-\mu^*)^+$$

and let $\xi = w_n \wedge\ v$. Since $g(x,y) = f(x, y+\psi^*(x)) - f(x,\psi^*(x))$ is bounded and satisfies (H1a), it follows from Proposition 4.2 and Remark 3.3 that $v \geq 0$. Hence $\xi \in K_{\psi_n-\psi^*}$ and

$$\frac{1}{2}(a\nabla w_n, \nabla(\xi - w_n))_2 - (f_{w_n+\psi^*} - f_{\psi^*}, \xi - w_n)_2 - \langle \mu-\mu^*, \xi - w_n\rangle \geq 0.$$

Since $\xi - w_n \leq 0$,

$$\begin{aligned}&\frac{1}{2}(a\nabla v, \nabla(\xi - w_n))_2 - (f_{v+\psi^*} - f_{\psi^*}, \xi - w_n)_2 - \langle \mu-\mu^*, \xi - w_n\rangle \\ &\quad = \langle (\mu-\mu^*)^-, \xi - w_n\rangle \leq 0.\end{aligned}$$

By the above inequalities,

$$\begin{aligned}0 &\geq (a\nabla(v-w_n), \nabla(\xi - w_n))_2 - 2(f_{v+\psi^*} - f_{w_n+\psi^*}, \xi - w_n)_2 \\ &= \int_D \mathbf{1}_{\{v<w_n\}}(a\nabla(v-w_n), \nabla(\xi-w_n))\,dx \\ &\quad - 2\int_D \mathbf{1}_{\{v<w_n\}}(f_{v+\psi^*} - f_{w_n+\psi^*})(v-w_n)\,dx \\ &\geq \int_D \mathbf{1}_{\{v<w_n\}}(a\nabla(\xi-w_n), \nabla(\xi-w_n))\,dx \\ &\geq (a\nabla(\xi-w_n), \nabla(\xi-w_n))_2.\end{aligned}$$

Hence $\xi = w_n$, and so $w_n \leq v$. Since

$$\mathcal{B}v - \mathcal{B}w_n = -(\mu-\mu^*)^+ + \mu - \mu^* + \gamma_n = -(\mu-\mu^*)^- + \gamma_n,$$

we have

$$\begin{aligned}\langle (\mu-\mu^*)^- - \gamma_n, T_\varepsilon(v-w_n)\rangle &= \frac{1}{2}(a\nabla(v-w_n), \nabla T_\varepsilon(v-w_n))_2 \\ &\quad - (f_{v+\psi^*} - f_{w_n+\psi^*}, T_\varepsilon(v-w_n))_2 \geq 0,\end{aligned}$$

the last inequality being a consequence of monotonicity of $f(x,\cdot)$. Consequently,

$$\int_D T_\varepsilon(v-w_n)\,d\gamma_n \leq \int_D T_\varepsilon(v-w_n)\,d(\mu-\mu^*)^-,$$

from which we deduce that $\gamma_n(\{v-w_n>\varepsilon\}) \leq (\mu-\mu^*)^-(D)$ for $\varepsilon > 0$, and hence that $\gamma_n(\{v-w_n>0\}) \leq (\mu-\mu^*)^-(D)$. From this it may be concluded that

$$\gamma_n(D) \leq \|\mu-\mu^*\|_{\mathcal{M}_b(D)}, \quad n \in \mathbb{N}. \tag{4.8}$$

Indeed, since $v \geq w_n$, (4.8) will be proved once we prove that $\gamma_n(\{v = w_n\}) = 0$. Since $v \geq 0$ and $\psi_n - \psi^* < 0$,

$$\gamma_n(\{v = w_n\}) \leq \gamma_n(\{w_n \geq 0\}) \leq \gamma_n(\{w_n > \psi_n - \psi^*\}) = \gamma_n(\{u_n > \psi_n\}).$$

On the other hand, by (2.9),

$$0 = E_m \int_0^t (u_n - \psi_n)(X_s^D)\, dK_s^n = \langle \int_0^t P_s^D 1\, ds, (u_n - \psi_n) \cdot \gamma_n \rangle,$$

where $K^n \sim \gamma_n$. Since X has continuous trajectories, $\int_0^t P_s^D 1(x)\, ds = \int_0^t P_x(\tau > s)\, ds > 0$ for $x \in D$. From this we conclude that

$$\int_D (u_n - \psi_n)\, d\gamma_n = 0, \tag{4.9}$$

and hence that $\gamma_n(\{u_n > \psi_n\}) = 0$. Thus, $\gamma_n(\{v = w_n\}) = 0$, and (4.8) is proved. To complete the proof of the proposition it suffices now to prove that $\gamma_n \to \gamma$ in $\mathcal{M}_b(D)$, where γ is the obstacle reaction associated with the solution u of EVI(f, μ, ψ). To see this, we first show that $\{u_n\}$ is bounded in $H_0^1(D)$. Since $u_n \leq u_{n+1}$ by Proposition 4.1 and Theorem 3.1, it follows that there is $w \in H_0^1(D)$ such that $u_n \to w$ weakly in $H_0^1(D)$. In fact, as in *Step 1* of the proof of Proposition 4.1 one can show that $w = u$ and $u_n \to u$ strongly in $H_0^1(D)$. By [13, Proposition 3.8], u_n is a solution of OP$(0, \bar{\mu}_n, \psi_n)$ whereas u is a solution of OP$(0, \bar{\mu}, \psi)$, where $\bar{\mu}_n = f_{u_n} + \mu$, $\bar{\mu} = f_u + \mu$. Since $\bar{\mu}_n \to \bar{\mu}$ in $\mathcal{M}_b(D)$, it follows from [13, Theorem 2.7] that $\gamma_n \to \gamma$ in $\mathcal{M}_b(D)$, which is the desired conclusion. □

Proposition 4.4. *Under the assumptions of Proposition 4.1, if $u \in H_0^1(D)$ is a solution of EVI(f, μ, ψ) then it is a solution of OP(f, μ, ψ).*

Proof. Our method of proof will be adaptation of the proof of [13, Proposition 3.8]. Let γ be the obstacle reaction associated with u. By Proposition 4.3, $\gamma \in \mathcal{M}_b^+(D)$ and hence $\gamma \in \mathcal{M}_b^{2,+}(D)$ since $\gamma \in H^{-1}(D)$. Therefore u is an entropy solution of the problem (3.5). Let $\bar{\gamma} \in \mathcal{M}_b^{2,+}(D)$ and let v be an entropy solution of the problem $\mathcal{A}v = -\mu - \bar{\gamma}$, $v|_{\partial D} = 0$ such that $v \geq \psi$ q.e. in D. What is left is to show that $v \geq u$ q.e. in D. By [13, Remark 4.5] there is a sequence $\{\bar{\gamma}_n\} \subset H^{-1}(D) \cap \mathcal{M}_b^{2,+}(D)$ such that $\bar{\gamma}_n \uparrow \bar{\gamma}$ strongly in $\mathcal{M}_b(D)$. Let $v_n \in H_0^1(D)$ be a weak solution of the problem $\mathcal{A}v_n = -\mu - \bar{\gamma}_n$, and let $u_n \in H_0^1(D)$ be a solution of EVI(f, μ, ψ_n) with $\psi_n = \psi \wedge v_n$. By Propositions 4.1, 4.2 and Theorem 3.1, $v_n \geq u_n$, so the proof will be completed by showing that $v_n \uparrow v$, $u_n \uparrow u$ q.e. in D. To see this, let us first observe that by Proposition 4.2 and Remark 3.3, $v_n \leq v_{n+1}$ q.e. Hence $\psi_n \leq \psi_{n+1}$, so using once again Propositions 4.1, 4.2 and Theorem 3.1 we see that $u_n \leq u_{n+1}$. By the above there are v^*, u^* such that $v_n \uparrow v^*$, $u_n \uparrow u^*$ q.e. Let $w \in H_0^1(D)$ be a weak solution of the problem $Aw = -f_{v^*} - \mu - \bar{\gamma}$, $w|_{\partial D} = 0$. Since $f_{v_n} \to f_{v^*}$ in $L^1(D)$, it follows from the stability results for entropy solutions (see Theorem 2.3 and Corollary 3.2 in [13]) that $T_k v_n \to T_k w$ in $H_0^1(D)$ for every $k > 0$. It follows that $w = v^*$, hence that v^* is a weak solution of the problem $Av^* = -f_{v^*} - \mu - \bar{\gamma}$, $v^*|_{\partial D} = 0$. Since the last problem has a unique solution, $v = v^*$, and consequently, $v_n \uparrow v$ q.e. in D. On the other hand, by the definition of a weak solution of EVI,

$$-\langle \mathcal{A}u_n, v - u_n \rangle \geq \langle \mu, v - u_n \rangle, \quad v \in K_{\psi_n}. \tag{4.10}$$

From (4.10) with $v = \psi^*$ and the fact that $\mathcal{A}\psi^* = -\mu^*$ it follows that

$$\begin{aligned} \frac{1}{2}(a\nabla(\psi^* - u_n), \nabla(\psi^* - u_n))_2 &\leq (f_{\psi^*} - f_{u_n}, \psi^* - u_n)_2 + \langle \mu^* - \mu, \psi^* - u_n \rangle \\ &\leq \langle \mu^* - \mu, \psi^* - u_n \rangle, \end{aligned}$$

hence that $\{u_n\}$ is bounded in $H^1_0(D)$. Therefore we may assume that $u_n \to u^*$ weakly in $H^1_0(D)$. In fact, since we already know that $u^* \geq \psi \wedge v = \psi$ q.e.,

$$\begin{aligned}\frac{1}{2}(a\nabla(u^*-u_n),\nabla(u^*-u_n))_2 &\leq (f_{u^*}-f_{u_n},u^*-u_n)_2+\langle Au^*-\mu,u^*-u_n\rangle\\ &\leq \langle Au^*-\mu,u^*-u_n\rangle,\end{aligned}$$

from which it follows that $u_n \to u^*$ strongly in $H^1_0(D)$. Therefore letting $n\to\infty$ in (4.10) shows that u^* is a solution of EVI(f,μ,ψ). Accordingly $u=u^*$, and consequently, $u_n \uparrow u$ q.e. in D. □

4.2 General measure data

Let $M^q(D)$, $q\geq 1$, denote the Marcinkiewicz space of order q (see, e.g., [12, Section 2.18]). Recall that $M^q(D)$ can be defined as the set of measurable functions $u: D\to\mathbb{R}$ such that the corresponding distribution function

$$\lambda(t)=m(\{x\in D: |u(x)|>t\}),\quad t>0$$

satisfies an estimate of the form

$$\lambda(t)\leq Ct^{-q}$$

for some $C\geq 0$. One can check that $L^q(D)\subset M^q(D)\subset L^p(D)$ for $1\leq p<q$.

In the proof of the existence of a solution to the problem OP(f,μ,ψ) under (H1)–(H3) we will need the following stability result for entropy solutions of (3.2).

Theorem 4.5. *Assume that f satisfies (H2) and $\mu,\mu_n\in\mathcal{M}^2_b(D)$. Let u be an entropy solution of the problem (3.2) and u_n be an entropy solution of the problem*

$$\mathcal{A}u_n=-\mu_n,\quad u_n|_{\partial D}=0.$$

If $\mu_n\to\mu$ in $\mathcal{M}_b(D)$ then $u_n\to u$ in $W^{1,q}_0(D)$ for $q\in[1,d/(d-1))$ and $T_ku_n\to T_ku$ in $H^1_0(D)$ for every $k>0$.

Proof. The proof follows closely the proof of [1, Theorem 6.1] (see also [2]). Nevertheless we provide its main ingredients because we will use them in the proof of our main result.

We first assume that $d\geq 3$. By (3.4) with $v=0$ and the fact that $u_nf_{u_n}\leq 0$,

$$\begin{aligned}\Lambda^{-1}\|\nabla T_ku_n\|_2^2=\Lambda^{-1}\int_{\{|u_n|<k\}}|\nabla u_n|^2\,dx &\leq \int_{\{|u_n|<k\}}(a\nabla u_n,\nabla u_n)\,dx\\ &=2\int_{\{|u_n|<k\}}u_nf_{u_n}\,dx+2\int_{\{|u_n|<k\}}u_n\,d\mu_n\leq 2k\|\mu_n\|_{\mathcal{M}_b(D)}.\end{aligned}\tag{4.11}$$

It follows that $\{\nabla T_ku_n\}_n$ is bounded in $L^2(D)$ and hence, by Poincaré's inequality, in $H^1_0(D)$. Let $q\in[1,2d/(d-2))$. Since the imbedding $H^1_0(D)\hookrightarrow L^q(D)$ is compact, we may and will assume that $\{T_ku_n\}_n$ is a Cauchy sequence in $L^q(D)$. From this and estimates of meas$\{|u_n-u_m|>t\}$ on pages 256–256 in [1] it follows that $\{u_n\}$ is a Cauchy sequence in measure. Hence there is u such that $u_n\to u$ in measure in D. Extracting a subsequence if necessary we may and will assume that $u_n\to u$ a.e.. Since $f(x,\cdot)$ is continuous, $f_{u_n}\to f_u$ a.e.. Let $\{\xi_i\}$ be a sequence of real smooth increasing functions such that $\xi_i(y)\to\xi(y)$, where $\xi(y)=0$ if $|y|\leq k$ and $\xi(y)=\text{sign}(y)$ if $|y|>k$. Since

$$0\leq(a\nabla u_n,\nabla\xi_i(u_n))_2=2\int_D f_{u_n}\xi_i(u_n)\,dx+2\int_D\xi_i(u_n)\,d\mu_n,$$

letting $i \to \infty$ gives

$$0 \le -\int_{\{u_n<-k\}} f_{u_n}\,dx + \int_{\{u_n>k\}} f_{u_n}\,dx + \int_{\{|u_n|>k\}} d\mu_n,$$

that is

$$\int_{\{|u_n|>k\}} |f_{u_n}|\,dx \le \int_{\{|u_n|>k\}} d\mu_n. \tag{4.12}$$

Letting $k \downarrow 0$ in (4.12) we see that $\|f_{u_n}\|_1 \le \|\mu_n\|_{\mathcal{M}_b(D)}$. Hence, by Fatou's lemma, $\|f_u\|_1 \le \liminf_{n\to\infty} \|f_{u_n}\|_1 = \|\mu\|_{\mathcal{M}_b(D)}$. By (4.11) and [1, Lemma 4.2], for every $\varepsilon > 0$ there exists $A > 0$ such that meas$\{|\nabla u_n| > A\} \le \varepsilon$ for all n. Moreover,

$$\begin{aligned}
&\int_{\{|u_n-u_m|\le k\}} (a\nabla(u_n-u_m), \nabla(u_n-u_m))\,dx \\
&\quad\le 2\int_{\{|u_n-u_m|\le k\}} |f_{u_n}-f_{u_m}|\cdot|u_n-u_m|\,dx + 2\int_{\{|u_n-u_m|\le k\}} |u_n-u_m|\,d\mu_n \\
&\quad\le 2k\|\mu_m\|_{\mathcal{M}_b(D)} + 6k\|\mu_n\|_{\mathcal{M}_b(D)} \le 8kC.
\end{aligned}$$

Using the above estimate one can show as in [1] (see pages 257–258) that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Hence $\{\nabla u_n\}$ converges in measure to some function v. Since we know that for each $k > 0$, $\{\nabla T_k u_n\}_n$ is bounded in $L^2(D)$, it converges weakly in $L^2(D)$ to $\nabla T_k u$ for $k > 0$ and $v = \nabla u$. Thus,

$$u_n \to u, \quad \nabla u_n \to \nabla u \text{ in measure.} \tag{4.13}$$

By (4.11) and [1, Lemma 4.1], the sequence $\{u_n\}$ is bounded in the Marcinkiewicz space $M^{d/(d-2)}(D)$. Moreover, again by [1, Lemma 4.1], $\{\nabla u_n\}$ is bounded in $M^{d/(d-1)}(D)$. Since we already know that $\nabla u_n \to \nabla u$ in measure, it follows from this that $\nabla u_n \to \nabla u$ in $L^q(D)$ for $q < d/(d-1)$, and hence, by Poincaré's inequality, that

$$u_n \to u \text{ in } W_0^{1,q}(D), \quad q \in [1, d/(d-1)). \tag{4.14}$$

From (4.12) it follows also that the sequence $\{f_{u_n}\}$ is equiintegrable. Hence $\{f_{u_n} - f_u\}$ is equiintegrable, and consequently,

$$f_{u_n} \to f_u \text{ in } L^1(D) \tag{4.15}$$

since we know that $f_{u_n} \to f_u$ a.e.. Finally, to show that u is an entropy solution to (3.2) let us consider an entropy solution w to the problem

$$Aw = -f_u - \mu, \quad w|_{\partial D} = 0.$$

By [13, Corollary 3.2] one can find $g, g_n \in L^1(D)$, $G, G_n \in H^{-1}(D)$ such that $\mu = g + G$, $\mu_n = g_n + G_n$ and $g_n \to g$ in $L^1(D)$, $G_n \to G$ in $H^{-1}(D)$. From this, (4.15) and known stability results for entropy solutions (see [13, Theorem 2.3] or [14, Theorem 1.2]) it follows that $T_k u_n \to T_k w$ in $H_0^1(D)$ for $k > 0$. Thus, $w = u$. As a consequence, $T_k u_n \to T_k u$ in $H_0^1(D)$ for $k > 0$ and u is an entropy solution of (3.2), which completes the proof in case $d \ge 3$.

Now assume that $d = 2$. Then the imbedding $H_0^1(D) \hookrightarrow L^2(D)$ is compact, so the same proof as in case $d \ge 3$ shows that (4.13) holds true. By (4.11), for any $p \in [1, 2)$,

$$\int_{\{|u_n|<k\}} |\nabla u|^p\,dx \le Ck^{p/2},$$

from which in much the same way as in the proof of [1, Lemma 4.1] it follows that $\{u_n\}$ is bounded in the space $M^{p/(2-p)}(D)$. Thus, $\{u_n\}$ is bounded in $M^q(D)$ for any $q \geq 1$. Moreover, from (4.11) and the proof of [1, Lemma 4.2] it follows that $\{\nabla u_n\}$ is bounded in $M^q(D)$ for $q \in [1,2)$. In particular, it follows that $\{u_n\}$ is bounded in $W_0^{1,q}(D)$ for $q \in [1, d/(d-1))$. From this and (4.13) we get (4.14). The rest of the proof runs as before. □

In part (ii) of the following main theorem we use some ideas from [4], where L^1 solutions of non-reflected BSDEs with deterministic terminal time and coefficients satisfying the monotonicity condition are considered. L^1 solutions of similar reflected BSDEs are considered in [19].

Theorem 4.6. *Assume that (H1)–(H3) are satisfied and $d \geq 2$.*

(i) *There exists a quasi-continuous q.e. finite entropy solution u of OP(f, μ, ψ). Moreover, if $d \geq 3$ then*

$$u \in M^{2d/(d-2)}(D), \quad |\nabla u| \in M^{d/(d-1)}(D),$$

and if $d = 2$ then

$$u \in M^p(D), \quad |\nabla u| \in M^q(D)$$

for any $p \geq 1$, $q \in [1,2)$. In particular, in both cases, $u \in W_0^{1,q}(D)$ for any $q \in [1, d/(d-1))$.

(ii) *Let γ be the obstacle reaction associated with u. Then for q.e. $x \in D$ the triple (Y, Z, K), where*

$$Y_t = u(X_t^D), \quad Z_t = \sigma \nabla u(X_t^D), \quad t \geq 0, \quad K \sim \gamma \tag{4.16}$$

is a unique solution of RBSDE$_x(f, \mu, \psi)$. Moreover, for every $T > 0$ and $\beta \in (0,1)$,

$$E_x \sup_{t \leq T} |Y_t|^\beta < \infty, \quad E_x\Big(\int_0^{T\wedge\tau} |Z_s|^2\, ds\Big)^{\beta/2} < \infty \tag{4.17}$$

for q.e. $x \in D$.

Proof. By (2.6), $\mu = g + G$ for some $g \in L^1(D)$, $G \in H^{-1}(D) \cap \mathcal{M}_b^2(D)$. Let $g_n = T_n g$ and let $\mu_n = g_n + G$. Since $g_n, G \in H^{-1}(D) \cap \mathcal{M}_b(D)$, $\mu_n \in H^{-1}(D) \cap \mathcal{M}_b(D)$. Let $u_n \in H_0^1(D)$ be a solution to EVI$(T_n f, \mu_n, \psi)$ and let γ_n be the obstacle reaction associated with u_n, i.e.

$$Au_n + (T_n f)_{u_n} = -\mu_n - \gamma_n. \tag{4.18}$$

By Proposition 4.4, u_n is a solution of OP$(T_n f, \mu_n, \psi)$. Furthermore, by (4.18), the fact that $(T_n f)_{u_n} u_n \leq 0$ and Proposition 4.3 we have

$$\begin{aligned}
\Lambda^{-1} \int_{\{|u_n|<k\}} |\nabla u_n|^2\, dx &\leq \int_{\{|u_n|<k\}} (a\nabla u_n, \nabla u_n)\, dx \\
&= 2\int_{\{|u_n|<k\}} (T_n f)_{u_n} u_n\, dx + 2\int_{\{|u_n|<k\}} u_n\, d(\mu_n + \gamma_n) \\
&\leq 2k(\|\mu_n\|_{\mathcal{M}_b(D)} + \|\gamma_n\|_{\mathcal{M}_b(D)}) \\
&\leq 2k(\|\mu_n\|_{\mathcal{M}_b(D)} + \|(\mu_n - \mu^*)^-\|_{\mathcal{M}_b(D)} \\
&\leq 2k(2\|g\|_1 + \|\mu^*\|_{\mathcal{M}_b(D)}) \equiv kC.
\end{aligned} \tag{4.19}$$

Let us define ξ_i as in the proof of Theorem 4.5. Then

$$0 \leq (a\nabla u_n, \nabla \xi_i(u_n))_2 = 2\int_D (T_n f)_{u_n} \xi_i(u_n)\, dx + 2\int_D \xi_i(u_n)\, d(\mu_n + \gamma_n),$$

from which as in the proof of (4.12) it follows that

$$\int_{\{|u_n|>k\}} |(T_nf)_{u_n}|\,dx \le \int_{\{|u_n|>k\}} d(\mu_n+\gamma_n). \tag{4.20}$$

Letting $k \downarrow 0$ in (4.20) we get

$$\|(T_nf)_{u_n}\|_1 \le \|\mu_n\|_{\mathcal{M}_b(D)} + \|\gamma_n\|_{\mathcal{M}_b(D)} \le C.$$

Moreover,

$$\begin{aligned}
&\int_{\{|u_n-u_m|\le k\}} (a\nabla(u_n-u_m),\nabla(u_n-u_m))\,dx \\
&\quad\le 2\int_{\{|u_n-u_m|\le k\}} |(T_nf)_{u_n}-(T_mf)_{u_m}|\cdot|u_n-u_m|\,dx \\
&\qquad+2\int_{\{|u_n-u_m|\le k\}} |u_n-u_m|\,d(\mu_n+\gamma_n) \\
&\quad\le 2k(\|\mu_m\|_{\mathcal{M}_b(D)}+\|\gamma_m\|_{\mathcal{M}_b(D)})+6k(\|\mu_n\|_{\mathcal{M}_b(D)}+\|\gamma_n\|_{\mathcal{M}_b(D)})\le 8kC.
\end{aligned}\tag{4.21}$$

Using (4.19)–(4.21) in much in the same way as in the proof of Theorem 4.5 we show that there is $u \in W_0^{1,q}(D)$, $q\in[1,d/(d-1))$, such that (4.14), (4.15) are satisfied. Set $\bar\mu_n=(T_nf)_{u_n}$, $\bar\mu=f_u+\mu$ and denote by v the solution of the problem OP$(0,\bar\mu,\psi)$. Since $\|\bar\mu_n-\bar\mu\|_{\mathcal{M}_b(D)} \le |(T_nf)_{u_n}-f_u\|_1+\|g_n-g\|_1$, $\bar\mu_n\to\bar\mu$ in $\mathcal{M}_b(D)$, and hence, by [13, Theorem 2.7], $T_k(u_n)\to T_k(v)$ in $H_0^1(D)$ for every $k>0$ and $\gamma_n\to\gamma$ in $\mathcal{M}_b(D)$, where $\gamma\in\mathcal{M}_b^{2,+}(D)$ is the obstacle reaction associated with v. From this and (4.14) it follows in particular that $v=u$. Therefore u is an entropy solution of the problem (3.5) and $T_ku_n\to T_k(u)$ in $H_0^1(D)$ for $k>0$. By the last statement and (4.19), $\|\nabla T_ku\|_2\le Ck$, $k>0$, so according to the remark following the definition of a solution of the obstacle problem, u is q.e. finite. Before we prove that u satisfies the minimality condition (ii) in the definition of the obstacle problem we will show that for q.e. $x\in D$ the triple (Y,Z,K) defined by (4.16) is a solution of RBSDE$_x(f,\mu,\psi)$ satisfying (4.17). By Proposition 4.1, for each $x\in D$ the triple (Y^n,Z^n,K^n) defined by (4.6) is a solution of RBSDE$_x(T_nf,\mu_n,\psi)$, i.e.

$$Y_t^n=\int_{t\wedge\tau}^{\tau}(T_nf)_{u_n}(X_s)\,ds+R_\tau^n-R_{t\wedge\tau}^n+K_\tau^n-K_{t\wedge\tau}^n-\int_{t\wedge\tau}^{\tau}Z_s^n\,dB_s,\quad t\ge0,\quad P_x\text{-a.s.},$$

where $R^n\sim\mu_n$. Write $c_n=(T_nf)_{u_n}+g_n$ and let $C^n\sim c_n(x)\,dx$, $A\sim G$. Then the above equation takes the form

$$Y_t^n=A_\tau-A_{t\wedge\tau}+C_\tau^n-C_{t\wedge\tau}^n+K_\tau^n-K_{t\wedge\tau}^n-\int_{t\wedge\tau}^{\tau}Z_s^n\,dB_s,\quad t\ge0.$$

To simplify notation set

$$\delta\Phi=\Phi^n-\Phi^{n+k},\quad \Phi:=C,K,Y,Z.$$

Using arguments from the proof of [4, Proposition 6.4] one can show that for any $0\le t\le T$ and $x\in D$,

$$|\delta Y_t|\le E_x\left(|\delta Y_{T\wedge\tau}|+|\delta C|_{T\wedge\tau}+|\delta K|_{T\wedge\tau}|\mathcal{F}_{t\wedge\tau}\right)\equiv M_t. \tag{4.22}$$

Hence $E_x|\delta Y_t|\le E_x|M_t|$ for $x\in D$, and consequently, for any $\nu\in S_{00}^+(D)$,

$$E_\nu|\delta Y_t|\le E_\nu|M_t|. \tag{4.23}$$

By Lemma 2.1,

$$\lim_{n\to\infty}\sup_{k\geq 1}E_\nu(|\delta C|_{T\wedge\tau}+|\delta K|_{T\wedge\tau})=0. \tag{4.24}$$

On the other hand, since $u_n\to u$ in $W^{1,q}_0(D)$ for $q<d/(d-1)$ and $\lim_{T\to\infty}u(X^D_T)=0$, $\lim_{T\to\infty}u_n(X^D_T)=0$, we have

$$\lim_{T\to\infty}\sup_{n,k\geq 1}E_\nu|\delta Y_{T\wedge\tau}|\leq 2\lim_{T\to\infty}\sup_{n\geq 1}E_\nu|u_n(X^D_T)|=0. \tag{4.25}$$

Since (4.24) holds for every $T>0$, it follows from (4.23) and (4.25) that

$$\lim_{n\to\infty}\sup_{k\geq 1}E_\nu|\delta Y_t|=0,\quad t\geq 0. \tag{4.26}$$

Since M defined by (4.22) is a martingale under P_x for every $x\in D$, applying [4, Lemma 6.1] yields

$$E_x(\sup_{t\leq T}|M_t|^\beta)\leq(1-\beta)^{-1}(E_x|M_T|)^\beta.$$

Hence, for every $x\in D$,

$$E_x\sup_{t\leq T}|\delta Y_t|^\beta\leq(1-\beta)^{-1}(E_x(|\delta Y_{T\wedge\tau}|+|\delta C|_{T\wedge\tau}+|\delta K|_{T\wedge\tau}))^\beta.$$

Integrating the above inequality with respect to ν and using Hölder's inequality we get

$$\begin{aligned}E_\nu\sup_{t\leq T}|\delta Y_t|^\beta&\leq\frac{1}{1-\beta}(\nu(D))^{1-\beta}(E_\nu(|\delta Y_{T\wedge\tau}|+|\delta C|_{T\wedge\tau}+|\delta K|_{T\wedge\tau}))^\beta\\&\leq\frac{1}{1-\beta}(\nu(D))^{1-\beta}\{(E_\nu|\delta Y_{T\wedge\tau}|)^\beta+(E_\nu(|\delta C|_{T\wedge\tau}+|\delta K|_{T\wedge\tau}))^\beta\}.\end{aligned}$$

From this as in the proof of (4.26) we get

$$\lim_{n\to\infty}\sup_{k\geq 1}E_\nu\sup_{t\leq T}|\delta Y_t|^\beta=0,\quad T\geq 0. \tag{4.27}$$

As in the proof of [4, Lemma 3.1] one can show that for every $\beta\in(0,1)$ there exists $C_\beta\geq 0$ such that for $x\in D$,

$$E_x(\int_0^{T\wedge\tau}|\delta Z_s|^2\,ds)^{\beta/2}\leq C_\beta E_x\{\sup_{t\leq T}|\delta Y_{T\wedge\tau}|^\beta+(|\delta C|_{T\wedge\tau})^\beta+(|\delta K|_{T\wedge\tau}))^\beta\}.$$

Hence

$$\begin{aligned}&E_\nu(\int_0^{T\wedge\tau}|\delta Z_s|^2\,ds)^{\beta/2}\\&\quad\leq C_\beta E_\nu\{\sup_{t\leq T}|\delta Y_{T\wedge\tau}|^\beta+(\nu(D))^{1-\beta}(E_\nu|\delta C|_{T\wedge\tau})^\beta+(E_\nu|\delta K|_{T\wedge\tau})^\beta\}.\end{aligned}$$

Using (4.27) and arguing as before we conclude from the above that

$$\lim_{n\to\infty}\sup_{k\geq 1}E_\nu(\int_0^{T\wedge\tau}|\delta Z_t|^2\,dt)^{\beta/2}=0,\quad T\geq 0. \tag{4.28}$$

Let $\mathcal{S}^\beta(\mathbb{R}^d)$ (resp. $M^\beta(\mathbb{R}^d)$) denote the space of progressively measurable $\mathbb{R}^d$-valued processes on $[0,\infty)$ equipped with the metric

$$\varrho(X,X')=\sum_{N=1}^\infty 2^{-N}(E_\nu\sup_{t\leq N}|X_t-X'_t|^\beta\wedge 1)$$

$$\Big(\text{resp.}\quad \varrho(Z,Z')=\sum_{N=1}^{\infty}2^{-N}((E_\nu(\int_0^N|Z_t-Z'_t|^2\,dt)^{\beta/2})\wedge 1)\Big).$$

Obviously $\mathcal{S}^\beta(\mathbb{R}^d)$, $M^\beta(\mathbb{R}^d)$ are complete spaces. By (4.27) and (4.28), $\{(Y^n,Z^n)\}$ is a Cauchy sequence in $\mathcal{S}^\beta(\mathbb{R}^d)\times M^\beta(\mathbb{R}^d\times\mathbb{R}^d)$. Let (Y^ν,Z^ν) denote its limit. Clearly Y^ν, Z^ν do not depend on β and are adapted. Moreover, Y^ν is P_ν-a.s. continuous, because the processes Y^n are continuous and

$$\sup_{t\le T}|Y^n_t-Y^\nu_t|\to 0 \text{ in measure } P_\nu \tag{4.29}$$

for every $T>0$. By Doob's inequality for continuous local martingales,

$$E_\nu\sup_{t\le T}|\int_0^{t\wedge\tau}(Z^n_s-Z^\nu_s)\,dB_s|^\beta\le\frac{4-\beta}{2-\beta}E_\nu(\int_0^{t\wedge\tau}|Z^n_s-Z^\nu_s|^2\,ds)^{\beta/2}\to 0.$$

Since $c_n\to c$ in $L^1(D)$ and $\gamma_n\to\gamma$ in $\mathcal{M}_b(D)$, it follows from the above and Lemma 2.3 that

$$Y^\nu_t=\int_{t\wedge\tau}^{\tau}c(X_s)\,ds+K_\tau-K_{t\wedge\tau}-\int_{t\wedge\tau}^{\tau}Z^\nu_s\,dB_s,\quad t\ge 0,\quad P_\nu\text{-a.s.},\tag{4.30}$$

where $K\sim\gamma$. Since $Y^n_t\ge\psi(X^D_t)$, $t\ge 0$ for $n\in\mathbb{N}$ and $\psi(X^D)$ has continuous trajectories under P_ν, from (4.29) it also follows that

$$Y^\nu_t\ge\psi(X^D_t),\quad t\ge 0,\quad P_\nu\text{-a.s.}\tag{4.31}$$

and

$$\int_0^\tau(Y^\nu_t-\psi(X^D_t))\,dK_t=0,\quad P_\nu\text{-a.s.},\tag{4.32}$$

the last equality being a consequence of that fact that

$$\sup_{t\le T}|\int_0^{t\wedge\tau}(Y^n_s-\psi(X^D_s))\,dK^n_s-\int_0^{t\wedge\tau}(Y^\nu_s-\psi(X^D_s))\,dK_s|\to 0$$

in measure P_ν for $T>0$. Since (4.30)–(4.32) hold for every $\nu\in S^+_{00}(D)$, to complete the proof it suffices to show that

$$u(X^D)=Y^\nu,\quad P_\nu\text{-a.s.}\tag{4.33}$$

and

$$\int_0^{T\wedge\tau}|Z^\nu_t-\sigma\nabla u(X_t)|^2\,dt=0,\quad P_\nu\text{-a.s.}\tag{4.34}$$

for $T>0$. We know that $Y^n_t=u_n(X^D_t)$, $t\ge 0$. Since $T_ku_n\to T_ku$ in $H^1_0(D)$ for $k>0$, it follows from [8, Lemma 5.1.2] that there is a subsequence, still denoted by n, such that for every $k\in\mathbb{N}$ and $T>0$,

$$\sup_{t\le T}|T_k(u_n(X^D_t))-T_k(u(X^D_t))|\to 0 \text{ in measure } P_\nu\,.$$

On the other hand, by (4.29), for $k\in\mathbb{N}$, $T>0$,

$$\sup_{t\le T}|T_k(Y^n_t)-T_k(Y^\nu_t)|\to 0 \text{ in measure } P_\nu\,.$$

Hence $T_ku(X^D)=T_k(Y^\nu)$ under P_ν for every $k\in\mathbb{N}$, which yields (4.33). To show (4.34) let us first observe that from the fact that $\int_0^{T\wedge\tau}|Z^n_t-Z^\nu_t|^2\,dt\to 0$ in measure P_ν it follows that

$$\int_0^{T\wedge\tau}|T_k(Z^n_t)-T_k(Z^\nu_t)|^2\,dt\to 0 \text{ in measure } P_\nu \tag{4.35}$$

for every $k \in \mathbb{N}$. On the other hand, since $\sigma\nabla u_n \to \sigma\nabla u$ in $L^q(D)$ with $q < d/(d-1)$,

$$\int_0^{T\wedge\tau} |T_k(\sigma\nabla u_n(X_t)) - T_k(\sigma\nabla u(X_t))|^2\,dt \to 0 \text{ in measure } P_\nu\,. \tag{4.36}$$

Combining (4.35), (4.36) with the fact that $Z^n = \sigma\nabla u_n(X^D)$, $dt\otimes P_\nu$-a.s. we conclude that $\int_0^{T\wedge\tau}|T_k(\sigma\nabla u(X_t)) - T_k(Z^\nu_t)|^2\,dt = 0$, P_ν-a.s. for $k>0$, $T>0$, from which we get (4.34). By what has already been proved and (2.7), for q.e. $x\in D$ the triple (4.16) is a solution of $\mathrm{RBSDE}_x(f,\mu,\psi)$ and satisfies (4.17), so the proof is completed by showing that u satisfies the minimality condition. Let $\bar\gamma \in \mathcal{M}^{2,+}_b(D)$ and let $\bar u$ be an entropy solution of the problem

$$A\bar u = -f_{\bar u} - \mu - \bar\gamma, \quad \bar u|_{\partial D} = 0$$

such that $\bar u \ge \psi$ q.e. in D. We have to show that $\bar u \ge u$. Since $\bar\gamma\in\mathcal{M}^{2,+}_b(D)$, $\bar\gamma = h + H$ for some $h\in L^1(D)$, $H\in H^{-1}(D)$. Let $h_n = T_n h$, $\bar\gamma_n = h_n + H$ and let $\bar u_n \in H^1_0(D)$ be a weak solution of the problem

$$A\bar u_n = -(T_n f)_{\bar u_n} - \mu_n - \bar\gamma_n.$$

Since $\mu_n + \bar\gamma_n \to \mu + \bar\gamma$ in $\mathcal{M}_b(D)$, it follows from Theorem 4.5 that $\bar u_n \to \bar u$ in $W^{1,q}_0(D)$ and $T_k\bar u_n \to T_k\bar u$ in $H^1_0(D)$ for $k>0$. Moreover, as in the proof of (4.15) one can show that $(T_n f)_{\bar u_n} \to f_{\bar u}$ in $L^1(D)$. On the other hand, by Proposition 4.2, for q.e. $x\in D$ the pair $(\bar Y^n, \bar Z^n)$ defined by

$$\bar Y^n_t = \bar u_n(X^D_t), \quad \bar Z^n_t = \sigma\nabla\bar u_n(X^D_t), \quad t\ge 0$$

is a solution of $\mathrm{BSDE}_x(T_n f, \mu_n + \bar\gamma_n)$, i.e.

$$\bar Y^n_t = \int_{t\wedge\tau}^{\tau} (T_n f)_{\bar u_n}(X_s)\,ds + \bar R^n_\tau - \bar R^n_{t\wedge\tau} - \int_{t\wedge\tau}^{\tau} \bar Z^n_s\,dB_s, \quad t\ge 0, \quad P_x\text{-a.s.},$$

where $\bar R^n \sim \mu_n + \bar\gamma_n$. Since $(T_n f)_{\bar u_n} \to f_{\bar u}$ in $L^1(D)$ and $\mu_n + \bar\gamma_n \to \mu+\bar\gamma$ in $\mathcal{M}_b(D)$, in much the same way as in the proof of (4.30) we show that for any $\nu\in S_{00}(D)$ there exist a continuous proces $\bar Y^\nu \in \mathcal{S}^\beta(\mathbb{R}^d)$ and $\bar Z^\nu \in M^\beta(\mathbb{R}^d\times\mathbb{R}^d)$ such that

$$\bar Y^\nu_t = \int_{t\wedge\tau}^{\tau} c(X_s)\,ds + \bar R_\tau - \bar R_{t\wedge\tau} - \int_{t\wedge\tau}^{\tau} \bar Z^\nu_s\,dB_s, \quad t\ge 0, \quad P_\nu\text{-a.s.},$$

where $\bar R \sim \mu + \bar\gamma$. Then, using the fact that $T_k\bar u_n \to T_k\bar u$ in $H^1_0(D)$ for $k>0$ we show as in the proof of (4.33), (4.34) that

$$\bar u(X^D) = \bar Y^\nu, \quad \int_0^{T\wedge\tau} |\bar Z^\nu_t - \sigma\bar u(X_t)|^2\,dt = 0, \quad P_\nu\text{-a.s.},$$

from which it follows that for q.e. $x\in D$ the pair

$$\bar Y_t = \bar u(X^D_t), \quad \bar Z_t = \sigma\nabla\bar u(X^D_t), \quad t\ge 0$$

is a solution of $\mathrm{BSDE}_x(f, \mu+\bar\gamma)$. That $\bar u \ge u$ now follows from Theorem 3.1. □

Remark 4.7. *(i) Under the assumptions and notation of Theorem 4.6,*

$$\int_D (u-\psi)\,d\gamma = 0.$$

This follows from the fact that $\gamma\sim K$ *and* $\int_0^t (u-\psi)(X^D_s)\,dK_s = 0$ *for q.e.* $x\in D$ *(see the proof of (4.9)).*

(ii) If $\gamma = h+H$ for some $h \in L^1(D)$, $H \in H^{-1}(D)$ and $H = h^0 - \mathrm{div}\bar{h}$ for some $h^0 \in L^2(D)$, $\bar{h} \in L^2(D)^d$ then for a.e. $x \in D$,

$$K_t = \int_0^{t\wedge\tau} (h+h^0)(X_s)\,ds + \int_0^{t\wedge\tau} (a^{-1}\bar{h})(X_s) * dX_s, \quad t \geq 0, \quad P_x\text{-a.s.}$$

(see (2.10) and Proposition 2.4).

Remark 4.8. *The entropy solution u of Theorem 4.6 is the renormalized solution of OP(f,μ,ψ), that is if v is a renormalized solution of (3.6) such that $v \geq \psi$ q.e. then $v \geq u$ on D and u satisfies (3.5) in the sense of [7, Definition 2.13]. The last statement means that*

$$\frac{1}{2}(a\nabla u, \nabla w)_2 + (f_u, w)_2 = \int_D w\,d(\mu+\gamma) \tag{4.37}$$

for every $w \in H^1_0(D) \cap L^\infty(D)$ with the property that there exist $k > 0$ and $w_+, w_- \in W^{1,p}_0(D) \cap L^\infty(D)$ with $p > d$ such that $w = w_+$ a.e. on the set $\{u > k\}$ and $w = w_-$ a.e. on the set $\{u > k\}$. For equivalent definitions of renormalized solutions see [7]. The first statement, i.e. that $v \geq u$ q.e. follows immediately from the fact that the renormalized solution of (2.16) is the entropy solution (see [7, Remark 2.17]). To show (4.37) let us define u_n, μ_n, γ_n as in the proof of Theorem 4.6. Since u_n is a weak solution of (4.18), it is a renormalized solution of (4.18). Hence

$$\frac{1}{2}(a\nabla u_n, \nabla w)_2 + (f_{u_n}, w)_2 = \int_D w\,d(\mu_n+\gamma_n). \tag{4.38}$$

We know that $T_k u_n \to T_k u$ in $H^1_0(D)$, $u_n \to u$ in $W^{1,q}_0(D)$ for $q \in [1, d/(d-1))$ and $\mu_n + \gamma_n \to \mu + \gamma$ in $\mathcal{M}_b(D)$. Therefore letting $n \to \infty$ in (4.38) and using [13, Corollary 3.2] and Lemma 2.5 we get (4.37).

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