

Spectral theory for symmetric one-dimensional Lévy processes killed upon hitting the origin^{*}

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Abstract

Spectral theory for transition operators of one-dimensional symmetric Lévy process killed upon hitting the origin is studied. Under very mild assumptions, an integral-type formula for eigenfunctions is obtained, and eigenfunction expansion of transition operators and the generator is proved. As an application, and the primary motivation, integral formulae for the transition density and the distribution of the hitting time of the origin are given in terms of the eigenfunctions.

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1 Introduction

In two recent articles [12, 13], spectral theory for some symmetric Lévy processes killed upon leaving the half-line was developed. One of the main motivations for this research came from fluctuation theory for Lévy processes: the distribution of the supremum functional and the first passage time can be expressed in terms of the eigenfunctions of the corresponding transition semigroup. The purpose of the present paper is to obtain similar results for processes killed upon hitting the origin, and apply them to the study of the hitting time of a single point. The following theorem is our main result.

Theorem 1.1. *Let X_t be a symmetric one-dimensional Lévy process, starting at 0, with Lévy-Khintchine exponent $\Psi(\xi)$, and suppose that $\Psi'(\xi) > 0$ and $2\xi\Psi''(\xi) \leq \Psi'(\xi)$ for $\xi > 0$, and that $1/(1 + \Psi(\xi))$ is integrable. Let τ_x be the first hitting time of $\{x\}$. Then*

$$\mathbf{P}(t < \tau_x < \infty) = \frac{1}{\pi} \int_0^\infty \frac{\cos \vartheta_\lambda e^{-t\Psi(\lambda)} \Psi'(\lambda) F_\lambda(x)}{\Psi(\lambda)} d\lambda \quad (1.1)$$

for $t > 0$ and almost all $x \in \mathbf{R}$. Here F_λ is a bounded, continuous function, defined by

$$F_\lambda(x) = \sin(|\lambda x| + \vartheta_\lambda) - G_\lambda(x) \quad (1.2)$$

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for $x \in \mathbf{R}$, where

$$\vartheta_\lambda = \arctan \left(-\frac{1}{\pi} \operatorname{pv} \int_0^\infty \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} d\xi \right) \quad (1.3)$$

(the notion of the principal value integral, denoted $\operatorname{pv} \int$, is discussed in Section 2), and G_λ is an $L^2(\mathbf{R}) \cap C_0(\mathbf{R})$ function with (integrable) Fourier transform

$$\mathcal{F}G_\lambda(\xi) = \cos \vartheta_\lambda \left(\frac{2\lambda}{\lambda^2 - \xi^2} - \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} \right) \quad (1.4)$$

for $\xi \in \mathbf{R} \setminus \{-\lambda, \lambda\}$.

Remark 1.2. It is easy to check that formula (1.1) can be differentiated in the time variable t under the integral sign. By combining it with estimates of F_λ , in many cases one can obtain asymptotic estimates of the density function of τ_x , as well as its derivatives, in a similar manner as in [13] for the first passage time. This topic will be addressed in a separate article.

Remark 1.3. When the Lévy measure of X_t has a completely monotone density on $(0, \infty)$, then $G_\lambda(x)$ is completely monotone on $[0, \infty)$ and in many cases can be given by a more straightforward formula; see Theorem 1.9.

Let us discuss shortly the assumptions of Theorem 1.1. Since the process X_t is assumed to be symmetric, $\Psi(\xi)$ is a real-valued function and $\Psi(\xi) \geq 0$. The assumption $\Psi'(\xi) > 0$ and $2\xi\Psi''(\xi) \leq \Psi'(\xi)$ for $\xi > 0$ is equivalent to the condition $\psi'(\xi) > 0$, $\psi''(\xi) \leq 0$ for $\xi > 0$ for the function $\psi(\xi) = \Psi(\sqrt{\xi})$. This is clearly satisfied by all subordinate Brownian motions (and hence for symmetric stable processes and mixtures of such), but also for many less regular processes, such as truncated symmetric stable processes. Examples are discussed in Section 5. Integrability of $1/(1 + \Psi(\xi))$ asserts that the process X_t hits single points with positive probability.

The class of Lévy processes studied in [12, 13] consisted of symmetric processes, whose Lévy measure admits a completely monotone density function on $(0, \infty)$. This regularity assumption was needed for an application of the Wiener-Hopf method for solving a certain integral equation in half-line. For the case of hitting a single point, considered below, a more direct approach is available, and therefore much more general processes can be dealt with.

From now on we consider the Lévy process X_t starting at a fixed point $x \in \mathbf{R}$, and denote the corresponding probability and expectation by \mathbf{P}_x and \mathbf{E}_x . The functions F_λ in Theorem 1.1 are eigenfunctions of transition operators of X_t killed upon hitting $\{0\}$. These operators are defined by the formula

$$P_t^{\mathbf{R} \setminus \{0\}} f(x) = \mathbf{E}_x(f(X_t) \mathbf{1}_{\{t < \tau_0\}})$$

for $t > 0$, $x \in \mathbf{R} \setminus \{0\}$, and they act on $L^p(\mathbf{R} \setminus \{0\})$ for arbitrary $p \in [1, \infty]$. By $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$ and $\mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^p)$ we denote the generator of the transition semigroup $P_t^{\mathbf{R} \setminus \{0\}}$ acting on $L^p(\mathbf{R} \setminus \{0\})$, and its domain; a more detailed discussion of these notions is given in Preliminaries. Our main results about F_λ are contained in the three theorems stated below.

Theorem 1.4. Let X_t be a symmetric one-dimensional Lévy process with Lévy-Khintchine exponent $\Psi(\xi)$, and suppose that $\Psi'(\xi) > 0$ for $\xi > 0$, and that $1/(1 + \Psi(\xi))$ is integrable. Fix $\lambda > 0$, and let F_λ be defined as in Theorem 1.1. Then $F_\lambda(0) = 0$, $F_\lambda \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^\infty)$, and

$$\mathcal{A}_{\mathbf{R} \setminus \{0\}} F_\lambda = -\Psi(\lambda) F_\lambda, \quad P_t^{\mathbf{R} \setminus \{0\}} F_\lambda = e^{-t\Psi(\lambda)} F_\lambda \quad (1.5)$$

for $t > 0$. In addition, for $\xi > 0$,

$$\int_{-\infty}^{\infty} F_{\lambda}(x) e^{-\xi|x|} dx = 2 \frac{\xi \sin \vartheta_{\lambda}}{\xi^2 + \lambda^2} + \frac{2 \cos \vartheta_{\lambda}}{\pi} \operatorname{pv} \int_0^{\infty} \frac{\xi}{\xi^2 + \zeta^2} \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\zeta)} d\zeta. \quad (1.6)$$

In the sense of Schwartz distributions,

$$\mathcal{F}F_{\lambda}(\xi) = \cos \vartheta_{\lambda} \operatorname{pv} \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} + \pi \sin \vartheta_{\lambda} (\delta_{\lambda}(\xi) + \delta_{-\lambda}(\xi)), \quad (1.7)$$

and

$$\mathcal{A}F_{\lambda}(x) = -\Psi(\lambda)F_{\lambda}(x) + \Psi'(\lambda) \cos \vartheta_{\lambda} \delta_0(x), \quad (1.8)$$

where δ_{ξ} is the Dirac delta distribution at ξ .

Remark 1.5. The assertions of the theorem can be interpreted correctly also when $1/(1 + \Psi(\xi))$ is not integrable. In this case $\vartheta_{\lambda} = \arctan(\infty) = \pi/2$, and therefore $\cos \vartheta_{\lambda} = 0$, $G_{\lambda} = 0$, and finally $F_{\lambda}(x) = \cos(\lambda x)$, as for the free process. This reflects the fact that X_t does not hit 0 from almost all starting points, and therefore $P_t^{\mathbf{R} \setminus \{0\}} f(x) = P_t f(x)$ for almost all $x \in \mathbf{R}$ (P_t is the transition semigroup of the free process X_t).

Remark 1.6. For a given $\lambda > 0$, the statement of the theorem remains true if the condition $\Psi'(\xi) > 0$ for all $\xi > 0$ is replaced by the condition $\Psi'(\lambda) > 0$ and $\Psi(\xi) \neq \Psi(\lambda)$ for all $\xi \neq \pm\lambda$. It is an interesting open problem to study the remaining cases, namely: when Ψ is increasing on $(0, \infty)$ and $\Psi'(\lambda) = 0$, and when $\Psi(\xi) = \Psi(\lambda)$ has more than one solution on $(0, \infty)$.

Remark 1.7. The primitive function of $2\lambda/(\lambda^2 - \xi^2)$ is $\log((\lambda + \xi)/(\lambda - \xi))$. This implies that for $\lambda > 0$,

$$\frac{1}{\pi} \operatorname{pv} \int_0^{\infty} \frac{2\lambda}{\lambda^2 - \xi^2} d\xi = 0.$$

By a similar direct calculation (we omit the details), for $\lambda, \xi > 0$,

$$\frac{1}{\pi} \operatorname{pv} \int_0^{\infty} \frac{\xi}{\xi^2 + \zeta^2} \frac{2\lambda}{\lambda^2 - \zeta^2} d\zeta = \frac{\lambda}{\lambda^2 + \xi^2}.$$

Hence, formula (1.3) can be rewritten without the principal value integral as

$$\vartheta_{\lambda} = \arctan \left(\frac{1}{\pi} \int_0^{\infty} \left(\frac{2\lambda}{\lambda^2 - \xi^2} - \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} \right) d\xi \right) \quad (1.9)$$

In a similar manner, (1.6) is equivalent to

$$\int_{-\infty}^{\infty} F_{\lambda}(x) e^{-\xi|x|} dx = 2 \frac{\xi \sin \vartheta_{\lambda} + \lambda \cos \vartheta_{\lambda}}{\xi^2 + \lambda^2} - \frac{2 \cos \vartheta_{\lambda}}{\pi} \int_0^{\infty} \frac{\xi}{\xi^2 + \zeta^2} \left(\frac{2\lambda}{\lambda^2 + \xi^2} - \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\zeta)} \right) d\zeta. \quad (1.10)$$

For estimates and more detailed properties of the eigenfunctions, further regularity of the Lévy-Khintchine exponent $\Psi(\xi)$ is needed. The required assumption is the same as in [12, 13], and it can be put in the following three equivalent forms. The notion of a complete Bernstein function is discussed in Preliminaries.

Proposition 1.8 (Proposition 2.14 in [12]). *Let X_t be a one-dimensional Lévy process. The following conditions are equivalent:*

- (a) $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function $\psi(\xi)$;
- (b) X_t is a subordinate Brownian motion (that is, $X_t = B_{Z_t}$, where B_s is the Brownian motion, Z_t is a non-decreasing Lévy process, and B_s and Z_t are independent processes), and the Lévy measure of the underlying subordinator Z_t has a completely monotone density function;
- (c) X_t is symmetric and the Lévy measure of X_t has a completely monotone density function on $(0, \infty)$.

Theorem 1.9. *Let $\lambda > 0$. Suppose that the assumptions of Theorem 1.4 hold true.*

- (a) *If $2\xi\Psi''(\xi) \leq \Psi'(\xi)$ for all $\xi > 0$, then we have $\vartheta_\lambda \in [0, \pi/2)$ and $|G_\lambda(x)| \leq G_\lambda(0) = \sin \vartheta_\lambda$.*
- (b) *Suppose that any of the equivalent conditions of Proposition 1.8 holds true. Then $\vartheta_\lambda \in [0, \pi/2)$, G_λ is completely monotone on $[0, \infty)$, and $G_\lambda(x) = \mathcal{L}\gamma_\lambda(|x|)$ (where \mathcal{L} is the Laplace transform) for a finite measure γ_λ . In particular, $G_\lambda(x) \geq 0$, and G_λ is integrable,*

$$\int_{-\infty}^{\infty} G_\lambda(x) dx = \mathcal{F}G_\lambda(0) = \frac{\cos \vartheta_\lambda}{\lambda} \left(1 - \frac{\lambda \Psi'(\lambda)}{2\Psi(\lambda)} \right). \quad (1.11)$$

If in addition Ψ extends to a function $\Psi^+(\xi)$ holomorphic in the right complex half-plane $\{\xi \in \mathbf{C} : \operatorname{Re} \xi > 0\}$ and continuous in $\{\xi \in \mathbf{C} : \operatorname{Re} \xi \geq 0\}$, and $\Psi^+(i\xi) \neq \Psi(\lambda)$ for all $\xi > 0$, then for all $x \in \mathbf{R}$,

$$G_\lambda(x) = \frac{\Psi'(\lambda) \cos \vartheta_\lambda}{\pi} \int_0^\infty \operatorname{Im} \frac{1}{\Psi(\lambda) - \Psi^+(i\xi)} e^{-\xi|x|} d\xi. \quad (1.12)$$

Note that the assumptions for part (a) of the theorem are exactly the same as in Theorem 1.1.

Before the statement of the third main result about the eigenfunctions F_λ , let us make the following remark. The ‘sine-cosine’ Fourier transform $\tilde{\mathcal{F}}$, defined by the formula

$$\tilde{\mathcal{F}}f(\lambda) = \left(\int_{-\infty}^{\infty} f(x) \cos(\lambda x) dx, \int_{-\infty}^{\infty} f(x) \sin(\lambda x) dx \right)$$

for $f \in C_c(\mathbf{R})$ and $\lambda > 0$, extends to a unitary (up to a constant factor $\sqrt{\pi}$) mapping of $L^2(\mathbf{R})$ onto $L^2((0, \infty)) \oplus L^2((0, \infty))$, in which the free transition operators P_t take a diagonal form. Namely, P_t transforms to a multiplication operator $e^{-t\Psi(\lambda)}$, that is,

$$\tilde{\mathcal{F}}P_t f(\lambda) = e^{-t\Psi(\lambda)} \tilde{\mathcal{F}}f(\lambda);$$

see Remark 2.7. The above formula is an explicit spectral decomposition, or *generalised eigenfunction expansion* of P_t . The next theorem provides a similar result for transition operators $P_t^{\mathbf{R} \setminus \{0\}}$ of the killed process. The corresponding transform is given by a similar formula as $\tilde{\mathcal{F}}$, with $\cos(\lambda x)$ replaced by $F_\lambda(x)$.

Theorem 1.10. Suppose that the assumptions of Theorem 1.4 are satisfied, and let F_λ be the eigenfunctions from Theorem 1.4. Define

$$\begin{aligned}\Pi_{\text{even}}f(\lambda) &= \int_{-\infty}^{\infty} f(x)F_\lambda(x)dx, \\ \Pi_{\text{odd}}f(\lambda) &= \int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx, \\ \Pi f(\lambda) &= (\Pi_{\text{even}}f(\lambda), \Pi_{\text{odd}}f(\lambda))\end{aligned}$$

for $f \in C_c(\mathbf{R} \setminus \{0\})$ and $\lambda > 0$. Furthermore, define

$$\Pi^*(f_1, f_2)(x) = \int_0^\infty f_1(\lambda)F_\lambda(x)d\lambda + \int_0^\infty f_2(\lambda)\sin(\lambda x)d\lambda$$

for $f_1, f_2 \in C_c((0, \infty))$ and $x > 0$. Then $\pi^{-1/2}\Pi$ and $\pi^{-1/2}\Pi^*$ extend to unitary operators, which map $L^2(\mathbf{R} \setminus \{0\})$ onto $L^2((0, \infty)) \oplus L^2((0, \infty))$, and $L^2((0, \infty)) \oplus L^2((0, \infty))$ onto $L^2(\mathbf{R} \setminus \{0\})$, respectively. Furthermore, $\Pi\Pi^* = \pi$, $\Pi^*\Pi = \pi$, and

$$\Pi P_t^{\mathbf{R} \setminus \{0\}} f(\lambda) = e^{-t\Psi(\lambda)} \Pi f(\lambda), \quad (1.13)$$

for all $f \in L^2(\mathbf{R} \setminus \{0\})$. In addition, $f \in L^2(\mathbf{R} \setminus \{0\})$ belongs to $\mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^2)$ if and only if $\Psi(\lambda)\Pi f(\lambda)$ is square integrable on $(0, \infty)$, and in this case for $\lambda > 0$,

$$\Pi \mathcal{A}_{\mathbf{R} \setminus \{0\}} f(\lambda) = -\Psi(\lambda)\Pi f(\lambda). \quad (1.14)$$

In a rather standard way, explicit spectral representation for $P_t^{\mathbf{R} \setminus \{0\}}$ yields a formula for the kernel of the operator, that is, for the transition density.

Corollary 1.11. Suppose that the assumptions of Theorem 1.4 are satisfied, and $2\xi\Psi''(\xi) \leq \Psi'(\xi)$ for all $\xi > 0$. Then the transition density of the process killed upon hitting the origin (that is, the kernel function of $P_t^{\mathbf{R} \setminus \{0\}}$) is given by

$$\begin{aligned}p_t^{\mathbf{R} \setminus \{0\}}(x, y) &= \frac{1}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} (F_\lambda(x)F_\lambda(y) + \sin(\lambda x)\sin(\lambda y))d\lambda \\ &= \frac{p_t(x-y) - p_t(x+y)}{2} + \frac{1}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x)F_\lambda(y)d\lambda.\end{aligned} \quad (1.15)$$

Formula (1.15), however, is of rather limited applications due to many cancellations in the integral with two oscillatory terms. Nevertheless, it is one of the very few explicit descriptions of the transition density of a killed Lévy process.

By symmetry, we have $\mathbf{P}_x(\tau_0 > t) = \mathbf{P}_0(\tau_x > t)$. Hence, Theorem 1.1 follows trivially from the following result. In the theorem, an extra condition on Ψ is required in order to use Theorem 1.9(a).

Theorem 1.12. Suppose that the assumptions of Theorem 1.4 are satisfied, and $2\xi\Psi''(\xi) \leq \Psi'(\xi)$ for all $\xi > 0$. Then

$$\mathbf{P}_x(t < \tau_0 < \infty) = \frac{1}{\pi} \int_0^\infty \frac{\cos \vartheta_\lambda e^{-t\Psi(\lambda)} \Psi'(\lambda) F_\lambda(x)}{\Psi(\lambda)} d\lambda \quad (1.16)$$

for $t > 0$ and almost all $x \in \mathbf{R} \setminus \{0\}$.

Since $\mathbf{P}_x(\tau_0 > t) = \int_{\mathbf{R} \setminus \{0\}} p_t^{\mathbf{R} \setminus \{0\}}(x, y)dy$, one could naively try to derive formula (1.16) by integrating (1.15) over $y \in \mathbf{R} \setminus \{0\}$. Heuristically, changing the order of integration and substituting $\mathcal{F}F_\lambda(0) = \cos \vartheta_\lambda \Psi'(\lambda)/\Psi(\lambda)$ for the integral $\int_{-\infty}^\infty F_\lambda(x)dx$ (which is not

absolutely convergent, or even convergent in the usual way) would yield (1.16) with $\mathbf{P}_x(t < \tau_0)$ instead of $\mathbf{P}_x(t < \tau_0 < \infty)$. Hence, during this illegitimate use of Fubini, $\mathbf{P}_0(\tau_0 = \infty)$ is lost.

Relatively little is known about the distribution of the hitting time of single points, τ_x . In the general case, the double integral transform (Fourier in space, Laplace in time) is well-known,

$$\int_{\mathbf{R}} e^{i\xi x} \mathbf{E}_x e^{-z\tau_0} dx = \frac{c(z)}{z + \Psi(\xi)}$$

for $z > 0$ and $\xi \in \mathbf{R}$, where $c(z)$ is the normalisation constant, chosen so that the integral of the right-hand side is 1 (see Preliminaries). Hence,

$$\mathbf{E}_x e^{-z\tau_0} = \frac{u_z(x)}{u_z(0)}$$

for $z > 0$ and $x \in \mathbf{R}$, where $u_z(x)$ is the resolvent (or z -potential) kernel for X_t . However, $u_z(x)$ is typically given by an oscillatory integral (namely, the inverse Fourier transform of $1/(z + \Psi(\xi))$), and therefore it is not easy to invert the Laplace transform in time, even for symmetric α -stable processes. Some asymptotic analysis of $\mathbf{P}_x(\tau_0 > t)$ (and many other related objects) can be found, for example, in [4, 9, 15, 21]. Some more detailed results in this area have been obtained for spectrally negative Lévy processes, see, for example, [6] and Section 46 in [17]. For totally asymmetric α -stable processes, a series representation for $\mathbf{P}_x(\tau_0 > t)$ was obtained in [16].

The hitting time of a single point is closely related to questions about local time and excursions of a Lévy process away from the origin, see [5, 20]. Similar questions arise when killing a Lévy process is replaced by penalizing it whenever it hits 0, cf. [22]. In fact, using the same method, one can find generalized eigenfunctions for the transition operators of the penalized semigroup. Finally, spectral theory for the half-line has been successfully applied in the study of processes killed upon leaving the interval [10, 11, 14] (see also [1, 2]) and higher-dimensional domains [8]. One can expect similar applications of the spectral theory for $\mathbf{R} \setminus \{0\}$, developed in the present article.

The remaining part of the article is divided into four sections. In Preliminaries, we describe the background on Schwartz distributions, complete Bernstein and Stieltjes functions, and Lévy processes and their generators. Section 3 contains the derivation of the formula for the eigenfunctions F_λ . The generalised eigenfunction expansion and formulae for the transition density and the distribution of the hitting time are proved in Section 4. Finally, examples are studied in Section 5.

2 Preliminaries

2.1 Schwartz distributions

Some background on the theory of Schwartz distributions is required for the proof of Theorem 1.4. A general reference here is [19]; a closely related setting is described with more details in [12].

The class of Schwartz functions in \mathbf{R} is denoted by \mathcal{S} , and \mathcal{S}' is the space of tempered distributions in \mathbf{R} . If $\varphi \in \mathcal{S}$ and $F \in \mathcal{S}'$, we write $\langle F, \varphi \rangle$ for the pairing of F and φ . The Fourier transform of $\varphi \in \mathcal{S}$ is defined by $\mathcal{F}\varphi(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} \varphi(x) dx$. For $F \in \mathcal{S}'$, $\mathcal{F}F$ is the tempered distribution satisfying $\langle \mathcal{F}F, \varphi \rangle = \langle F, \mathcal{F}\varphi \rangle$ for all $\varphi \in \mathcal{S}$. This definition extends the usual definition of the Fourier transform of $L^1(\mathbf{R})$ or $L^2(\mathbf{R})$ functions and finite signed measures on \mathbf{R} .

The convolution of $\varphi_1, \varphi_2 \in \mathcal{S}$ is defined as usual by $\varphi_1 * \varphi_2(x) = \int_{-\infty}^{\infty} \varphi_1(y) \varphi_2(x-y) dy$.

When $\varphi \in \mathcal{S}$ and $F \in \mathcal{S}'$, then $F * \varphi$ is the infinitely smooth function defined by

$$F * \varphi(x) = \langle F, \varphi_x \rangle, \quad \text{where} \quad \varphi_x(y) = \varphi(x - y).$$

The convolution does not extend continuously to entire space \mathcal{S}' . We say that $F_1, F_2 \in \mathcal{S}'$ are *\mathcal{S}' -convolvable* if for all $\varphi_1, \varphi_2 \in \mathcal{S}$, the functions $F_1 * \varphi_1$ and $F_2 * \varphi_2$ are convolvable in the usual sense, i.e. the integral $\int_{-\infty}^{\infty} (F_1 * \varphi_1)(y)(F_2 * \varphi_2)(x - y)dy$ exists for all x . When this is the case, the *\mathcal{S}' -convolution* $F_1 * F_2$ is the unique distribution F satisfying $F * \varphi_1 * \varphi_2 = (F_1 * \varphi_1) * (F_2 * \varphi_2)$ for $\varphi_1, \varphi_2 \in \mathcal{S}$. There are other non-equivalent definitions of the convolution of distributions; see [19].

The *support* of a distribution F is the smallest closed set $\text{supp } F$ such that $\langle F, \varphi \rangle = 0$ for all $\varphi \in \mathcal{S}$ such that $\varphi(x) = 0$ for $x \in \text{supp } F$. If any of the tempered distributions F_1, F_2 has compact support, or if $F_1 * \varphi$ is bounded and $F_2 * \varphi_2$ is integrable for all $\varphi \in \mathcal{S}$, then F_1 and F_2 are automatically \mathcal{S}' -convolvable. Note that \mathcal{S}' -convolution is *not* associative in general.

If distributions F_1, F_2 are \mathcal{S}' -convolvable, then $\mathcal{F}(F_1 * F_2) = \mathcal{F}F_1 \cdot \mathcal{F}F_2$ (the *exchange formula*), where the multiplication of distributions extends standard multiplication of functions in an appropriate manner. Below only a few special cases are discussed, and for the general notion of multiplication of distributions, we refer the reader to [19].

When F_1 is a measure and F_2 is a function, then their multiplication $F_1 \cdot F_2$ is the measure $F_2(x)F_1(dx)$, as expected. Next example requires the following definition. By writing $F_1 = \text{pv}(1/(x - x_0))$ we mean that

$$\langle F_1, \varphi \rangle = \text{pv} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x - x_0} dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R} \setminus [x_0 - \varepsilon, x_0 + \varepsilon]} \frac{\varphi(x)}{x} dx;$$

the *principal value integral* $\text{pv} \int$ is defined by the right-hand side. When $F_1 = \text{pv}(1/(x - x_0))$ and F_2 is a function vanishing at x_0 and differentiable at x_0 , then $F_1 \cdot F_2$ is the ordinary function $F_2(x)/(x - x_0)$. The third example is the extension of the previous one. Let f be a function vanishing at a finite number of points x_1, x_2, \dots, x_n , continuously differentiable in a neighborhood of each x_j with $f'(x_j) \neq 0$, and such that $1/f(x)$ is bounded outside any neighborhood of $\{x_1, \dots, x_n\}$. In this case, $F_1 = \text{pv}(1/f(x))$ is formally defined by

$$\langle F_1, \varphi \rangle = \text{pv} \int_{-\infty}^{\infty} \frac{\varphi(x)}{f(x)} dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R} \setminus \bigcup_{j=1}^n [x_j - \varepsilon, x_j + \varepsilon]} \frac{\varphi(x)}{f(x)} dx.$$

Clearly, $F_1 = \sum_{j=1}^n f'(x_j) \text{pv}(1/(x - x_j)) + F$, where F is a genuine function. Hence, by the previous example, we obtain the following result: when $F_1 = \text{pv}(1/f(x))$ is as above and F_2 is a function which vanishes at each x_j and is differentiable at each x_j , then $F_1 \cdot F_2$ is the function $F_2(x)/f(x)$.

The Laplace transform of a function F on $[0, \infty)$ is defined by $\mathcal{L}F(\xi) = \int_0^{\infty} F(x)e^{-\xi x} dx$. For a (possibly signed) Radon measure m on $[0, \infty)$, we have $\mathcal{L}m(\xi) = \int_0^{\infty} e^{-\xi x} m(dx)$.

2.2 Stieltjes functions and complete Bernstein functions

A general reference here is [18]. A function $f(x)$ is said to be a *complete Bernstein function* (CBF in short) if

$$f(x) = c_1 + c_2 x + \frac{1}{\pi} \int_0^{\infty} \frac{x}{x + s} \frac{m(ds)}{s}, \quad x > 0, \quad (2.1)$$

where $c_1 \geq 0$, $c_2 \geq 0$, and m is a Radon measure on $(0, \infty)$ such that $\int \min(s^{-1}, s^{-2})m(ds) < \infty$. The right-hand side clearly extends to a holomorphic function of $x \in \mathbf{C} \setminus (-\infty, 0]$,

and we often identify f with its holomorphic extension. A function $f(x)$ is a *Stieltjes function* if

$$f(x) = \frac{\tilde{c}_1}{x} + \tilde{c}_2 + \frac{1}{\pi} \int_0^\infty \frac{1}{x+s} \tilde{m}(ds), \quad x > 0, \quad (2.2)$$

where $\tilde{c}_1 \geq 0$, $\tilde{c}_2 \geq 0$, and \tilde{m} is a Radon measure on $(0, \infty)$ such that $\int \min(1, s^{-1}) \tilde{m}(ds) < \infty$. Below we collect the properties of Stieltjes and complete Bernstein functions used in the article.

Proposition 2.1 (see [18]). (a) A function f is a CBF if and only if $f(z) \geq 0$ for $z > 0$ and either f is constant, or f extends to a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$, which leaves the upper and the lower complex half-planes invariant.

(b) A function f is a Stieltjes function if and only if $f(z) \geq 0$ for $z > 0$ and either f is constant, or f extends to a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$, which swaps the upper and the lower complex half-planes.

(c) If f is positive, then the following four conditions are mutually equivalent: $f(x)$ is a CBF, $x/f(x)$ is a CBF, $1/f(x)$ is a Stieltjes function, $f(x)/x$ is a Stieltjes function.

(d) A function f is a Stieltjes function with representation (2.2) if and only if f is the Laplace transform of the measure $\tilde{c}_2 \delta_0(ds) + (\tilde{c}_1 + \mathcal{L}\tilde{m}(s))ds$. \square

Proposition 2.2 (Proposition 7.14(a) in [18]). If ψ is a non-constant CBF, then

$$\psi_\lambda(\xi) = \frac{1 - \xi/\lambda^2}{1 - \psi(\xi)/\psi(\lambda^2)} \quad (2.3)$$

(extended continuously at λ^2 , so that $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2\psi'(\lambda^2))$) is again a CBF. \square

Proposition 2.3 (Corollary 6.3 in [18], Proposition 2.18 in [12]).

(a) Let f be a CBF with representation (2.1). Then

$$c_1 = \lim_{z \rightarrow 0^+} f(z), \quad c_2 = \lim_{z \rightarrow \infty} \frac{f(z)}{z}, \quad (2.4)$$

and

$$m(ds) = \lim_{\varepsilon \rightarrow 0^+} (\operatorname{Im} f(-s + i\varepsilon) ds), \quad (2.5)$$

with the limit understood in the sense of weak convergence of measures.

(b) Let f be a Stieltjes function with representation (2.2). Then

$$\tilde{c}_1 = \lim_{z \rightarrow 0^+} (zf(z)), \quad \tilde{c}_2 = \lim_{z \rightarrow \infty} f(z), \quad (2.6)$$

and

$$\tilde{m}(ds) + \pi \tilde{c}_2 \delta_0(ds) = \lim_{\varepsilon \rightarrow 0^+} (-\operatorname{Im} f(-s + i\varepsilon) ds), \quad (2.7)$$

with the limit understood in the sense of weak convergence of measures. \square

Proposition 2.4 (Proposition 2.21 in [12]). (a) If f is a complete Bernstein function, then $|f(z)| \leq C(\varepsilon)f(|z|)$ and $|f(z)| \leq C(f, \varepsilon)(1 + |z|)$ for $z \in \mathbb{C}$, $|\operatorname{Arg} z| \leq \pi - \varepsilon$, $\varepsilon \in (0, \pi)$;

(b) If f is a Stieltjes function, then $|f(z)| \leq C(\varepsilon)f(|z|)$ and $|f(z)| \leq C(f, \varepsilon)(1 + |z|^{-1})$ for $z \in \mathbb{C}$, $|\operatorname{Arg} z| \leq \pi - \varepsilon$, $\varepsilon \in (0, \pi)$. \square

2.3 Lévy processes

Below we recall some standard notions related to Lévy processes and describe the setting for the present article. General references for Lévy processes are [3, 17]; for the properties of semigroups of killed Lévy processes, see [12] and the references therein.

Throughout this article, X_t is a one-dimensional Lévy process with Lévy-Khintchine exponent $\Psi(\xi)$; that is,

$$\mathbf{E}_0 e^{i\xi X_t} = e^{-t\Psi(\xi)}, \quad t > 0, \xi \in \mathbf{R}. \quad (2.8)$$

The probability and expectation corresponding to the process starting at $x \in \mathbf{R}$ are denoted by \mathbf{P}_x and \mathbf{E}_x , respectively. The following assumptions are in force throughout the article:

- (A) X_t is a symmetric process;
- (B) X_t hits every single point in finite time with positive probability.

The first assumption is equivalent to the condition $\Psi(\xi) = \Psi(-\xi)$ for all $\xi \in \mathbf{R}$, and to the condition $\Psi(\xi) \geq 0$ for $\xi \in \mathbf{R}$. For symmetric processes, Assumption (B) is equivalent to integrability of $1/(z + \Psi(\xi))$ over $\xi \in \mathbf{R}$ for some (and hence for all) $z > 0$, see Theorem 2 in [9], Theorem II.19 in [3] or Theorem 43.3 in [17]. This equivalent form is more convenient in applications.

Both conditions are natural for the spectral theory. When Assumption A fails, i.e. when X_t is not symmetric, the operators $P_t^{\mathbf{R} \setminus \{0\}}$ are not self-adjoint (in fact, they even fail to be normal operators). Assumption B asserts that the operators $P_t^{\mathbf{R} \setminus \{0\}}$ are not equal to P_t ; the spectral theory for the latter is trivial, see Remark 2.7.

The transition operators P_t of the free process X_t are defined by

$$P_t f(x) = \mathbf{E}_x f(X_t) = \int_{\mathbf{R}} f(y) \mathbf{P}_x(X_t \in dy), \quad t > 0, x \in \mathbf{R},$$

whenever the integral is absolutely convergent. The operators P_t (acting on $L^2(\mathbf{R})$) are Fourier multipliers with Fourier symbol $e^{-t\Psi(\xi)}$, and hence they are convolution operators. By assumption (B) and Fourier inversion formula, the convolution kernel is an integrable function $p_t(x)$, called the transition density.

Let $D \subseteq \mathbf{R}$ be an open set, and let $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ be the first time the process X_t exits D . The transition operators of the process X_t killed upon leaving D are given by

$$P_t^D f(x) = \mathbf{E}_x(f(X_t) \mathbf{1}_{t < \tau_D}) = \int_D f(y) \mathbf{P}_x(X_t \in dy; t < \tau_D), \quad t > 0, x \in D.$$

The kernel function $p_t^D(x, y)$ of P_t^D (the transition density of the killed process) exists, and $0 \leq p_t^D(x, y) \leq p_t(x - y)$ ($t > 0, x, y \in D$).

By $L^p(D)$, $p \in [1, \infty)$, we denote the space of real-valued functions f with finite p -norm $\|f\|_{L^p(D)} = (\int_D |f(x)|^p dx)^{1/p}$. The space of essentially bounded functions on D is denoted by $L^\infty(D)$. The space of bounded continuous functions is denoted by $C_b(\mathbf{R})$, and $C_0(\mathbf{R})$ is the set of continuous functions vanishing at $\pm\infty$. By $C_0(D)$ we denote the space of $C_0(\mathbf{R})$ functions vanishing on the complement of D .

In this article, we only consider $D = \mathbf{R} \setminus \{0\}$, so that $\tau_D = \tau_0$ is the hitting time of the origin. In this case clearly $L^p(\mathbf{R} \setminus \{0\})$ can be identified with $L^p(\mathbf{R})$; nevertheless, we use the notation $L^p(\mathbf{R} \setminus \{0\})$ to emphasise that the corresponding operators are related to the process killed upon hitting the origin.

The following properties of transition semigroups are described in detail in [12]. The operators P_t and $P_t^{\mathbf{R} \setminus \{0\}}$ form a contraction semigroup on $L^p(\mathbf{R})$ and $L^p(\mathbf{R} \setminus \{0\})$,

respectively, where $p \in [1, \infty]$. This semigroup is strongly continuous when $p < \infty$. Also, P_t is a contraction semigroup on $C_b(\mathbf{R})$ and $C_0(\mathbf{R})$, strongly continuous on the latter space. By Theorem II.19 in [3] or Theorem 43.3 in [17], 0 is regular for itself, and so $P_t^{\mathbf{R} \setminus \{0\}}$ is a strongly continuous semigroup of operators on $C_0(\mathbf{R} \setminus \{0\})$ (see also [12]).

We denote the generators of the semigroups P_t and $P_t^{\mathbf{R} \setminus \{0\}}$ (on appropriate function spaces) by \mathcal{A} and $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$, respectively. The corresponding domains are denoted by $\mathcal{D}(\mathcal{A}; \mathcal{X})$ and $\mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; \mathcal{X})$, respectively, where \mathcal{X} indicates the underlying function space. For example, $\mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^\infty)$ is the domain of the generator of the semigroup $P_t^{\mathbf{R} \setminus \{0\}}$ of contractions on $L^\infty(\mathbf{R} \setminus \{0\})$, that is, the set of those $f \in L^\infty(\mathbf{R} \setminus \{0\})$ for which the limit

$$\mathcal{A}_{\mathbf{R} \setminus \{0\}} f = \lim_{t \rightarrow 0^+} \frac{P_t^{\mathbf{R} \setminus \{0\}} f - f}{t} \quad (2.9)$$

exists in $L^\infty(\mathbf{R} \setminus \{0\})$. Note that the limit (2.9), if it exists, does not depend on the choice of the underlying function space. Therefore, using a single symbol \mathcal{A} for operators acting on different domains $\mathcal{D}(\mathcal{A}; \mathcal{X})$ causes no confusion (again, see [12]).

The resolvent kernel, or the z -potential kernel, is defined by $u_z(x) = \int_0^\infty p_t(x) e^{-zt} dt$. For each $z > 0$, it is a bounded, continuous function of $x \in \mathbf{R}$, and $\mathcal{F}u_z(\xi) = 1/(z + \Psi(\xi))$. Furthermore, $u_z(x)$ is the convolution kernel of the inverse operator of $z - \mathcal{A}$ (acting on $L^2(\mathbf{R})$). On the other hand, by Theorem II.19 in [3] or Theorem 43.3 in [17], $u_z(x)/u_z(0)$ is the Laplace transform of the distribution of τ_0 ,

$$\mathbf{E}_x e^{-z\tau_0} = \frac{u_z(x)}{u_z(0)}, \quad z > 0, x \in \mathbf{R}.$$

This identity will be used in the proof of Theorem 1.12.

Following [12], we introduce the *distributional generator* of X_t , denoted by an italic letter A : we let A be the tempered distribution satisfying $\mathcal{A}\varphi = A * \varphi$ for all $\varphi \in \mathcal{S}$. By symmetry of X_t , equivalently we have $\langle A, \varphi \rangle = \mathcal{A}\varphi(0)$ for $\varphi \in \mathcal{S}$. The distributional generator can be thought of as a pointwise extension of the generator \mathcal{A} . The following result from [12] describes the connection between A and $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$, the generator of the killed semigroup.

Lemma 2.5 (see Lemma 2.10 in [12]). *Let $F \in C_0(\mathbf{R} \setminus \{0\})$. If the distribution $A * F$ is equal to a $C_0(\mathbf{R} \setminus \{0\})$ function plus a distribution supported at $\{0\}$, then $F \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; C_0)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}} F(x) = A * F(x)$ for $x \in \mathbf{R} \setminus \{0\}$. \square*

The following technical result has been proved in [12] for general domains. We record its version for $\mathbf{R} \setminus \{0\}$.

Lemma 2.6 (see Lemma 2.11 in [12]). *Let $F \in C_b^\infty(\mathbf{R})$, that is, F and all derivatives of F are bounded functions on \mathbf{R} . Suppose that for some $r > 0$, $F(x) = 0$ for $x \in [-r, r]$, and $AF(0) = 0$. Then $F \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; C_0)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}} F(x) = AF(x)$ for all $x \in \mathbf{R} \setminus \{0\}$. \square*

Remark 2.7. *The spectral theory for the free process is very simple, thanks to the Lévy Khintchine formula (2.8). Indeed, the function $e^{i\lambda x}$ ($\lambda \in \mathbf{R}$) is the eigenfunction of P_t and \mathcal{A} , with eigenvalue $e^{-t\Psi(\lambda)}$ and $-\Psi(\lambda)$, respectively. Clearly, these eigenfunctions yield the generalised eigenfunction expansion of P_t and \mathcal{A} by means of Fourier transform.*

Since X_t is symmetric, $\Psi(\lambda) = \Psi(-\lambda)$ is real. It follows that $\sin(\lambda x)$ and $\cos(\lambda x)$ ($\lambda > 0$) are also eigenfunctions of P_t and \mathcal{A} . Consequently, $\sin(\lambda x)$ yields generalised eigenfunction expansion for odd $L^2(\mathbf{R})$ functions (via Fourier sine transform), while $\cos(\lambda x)$ plays a similar role for even $L^2(\mathbf{R})$ functions (via Fourier cosine transform). \square

3 Eigenfunctions in $\mathbf{R} \setminus \{0\}$

In this section the formula for F_λ is derived, and some properties of eigenfunctions are studied. More precisely, we prove Theorems 1.4 and 1.9. As in Theorem 1.4, $\Psi(\xi)$ is the Lévy-Khintchine exponent of a symmetric Lévy process X_t , and we assume that $1/(1 + \Psi(\xi))$ is integrable, and $\Psi'(\xi) > 0$ for $\xi > 0$. Our argument follows closely the approach of [12], where the case of the half-line $(0, \infty)$ is studied. Noteworthy, in our case there is no need to employ the Wiener-Hopf method, which makes the proof significantly shorter and simpler.

Mimicking the definition of distributional eigenfunctions in half-line (Definition 4.1 in [12]), we introduce the notion of distributional eigenfunctions of $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$. Note that the condition $F(0) = 0$ has no meaning for general Schwartz distributions F , so in contrast to the definition of [12], at this stage we do not assume that F vanishes in the complement of the domain (that is, at the origin).

Definition 3.1. A tempered distribution F is said to be a distributional eigenfunction of $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$, corresponding to the eigenvalue $-\Psi(\lambda)$, if F is S' -convolvable with A , and $A * F + \Psi(\lambda)F$ is supported in $\{0\}$.

Note that, in particular, $\cos(\lambda x)$ is a distributional eigenfunction of $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$. However, it is not the one we are looking for, as it does not vanish at 0. By copying the proof of Lemma 4.2 in [12] nearly verbatim, one obtains the following result. We only sketch the prove and omit the technical details, referring the interested reader to [12].

Lemma 3.2. Let $\lambda > 0$, and suppose that F is a distributional eigenfunction of $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$, corresponding to the eigenvalue $-\Psi(\lambda)$. If F is bounded, continuous on \mathbf{R} , $F(0) = 0$ and

$$\lim_{x \rightarrow \pm\infty} (F(x) - C \sin(|\lambda x| + \vartheta)) = 0$$

for some $C, \vartheta \in \mathbf{R}$, then $F \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^\infty)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}}F = -\Psi(\lambda)F$.

Sketch of the proof. It is possible to find an infinitely smooth function f_1 such that $f_1(x) = 0$ when $|x| \leq 1$, $f_1(x) = C \sin(|\lambda x| + \vartheta)$ when $|x|$ is large enough, and $\mathcal{A}f_1(0) = 0$. We let $f_2(x) = F(x) - f_1(x)$. By Lemma 2.6, $f_1 \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; C_b)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}}f_1(x) = \mathcal{A}f_1(x) = A * f_1(x)$. With a little effort, one shows that Lemma 2.5 applies to f_2 . It follows that $f_2 \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; C_0)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}}f_2(x) = A * f_2(x)$. Hence, $F = f_1 + f_2 \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; C_b)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}}F = A * F = -\Psi(\lambda)F$, as desired \square

It is relatively easy to find a formula for distributional eigenfunctions of $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$ satisfying the assumptions of Lemma 3.2. First, we give a brief, heuristic derivation of the formula. Suppose that F_λ is a bounded, continuous, even function on \mathbf{R} such that $F_\lambda(0) = 0$ and $A * F + \Psi(\lambda)F$ is supported in $\{0\}$. A tempered distribution is supported in $\{0\}$ if and only if its Fourier transform is a polynomial. Hence, $(-\Psi(\xi) + \Psi(\lambda))\mathcal{F}F_\lambda(\xi)$ is a polynomial $Q(\xi)$. It follows that the distribution $\mathcal{F}F_\lambda$ is expected to have form

$$\text{pv} \frac{Q(\xi)}{\Psi(\lambda) - \Psi(\xi)}$$

(the principal value corresponds to singularities at $\pm\lambda$), plus some distribution supported in $\{-\lambda, \lambda\}$ (the zeros of $\Psi(\lambda) - \Psi(\xi)$). The function F_λ should be as regular as possible, so we assume that Q is constant (say, $Q(\xi) = a_\lambda \Psi'(\lambda)$), and that the distribution supported in $\{-\lambda, \lambda\}$ is a combination of Dirac measures (say, $\pi b_\lambda(\delta_\lambda + \delta_{-\lambda})$). This suggests the following definition:

$$\mathcal{F}F_\lambda(\xi) = a_\lambda \text{pv} \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} + \pi b_\lambda(\delta_\lambda(\xi) + \delta_{-\lambda}(\xi))$$

for some $a_\lambda, b_\lambda \in \mathbf{R}$. We can normalize this definition by assuming that $a_\lambda^2 + b_\lambda^2 = 1$ and $a_\lambda \geq 0$, so that $a_\lambda = \cos \vartheta_\lambda$ and $b_\lambda = \sin \vartheta_\lambda$ for some $\vartheta_\lambda \in [-\pi/2, \pi/2)$. Furthermore, the condition $F_\lambda(0) = 0$ can be formally rewritten as $\int_{-\infty}^{\infty} \mathcal{F}F_\lambda(\xi) d\xi = 0$, which gives a linear equation in a_λ and b_λ . After solving this equation, we obtain formulae given in Theorem 1.4.

Proof of Theorem 1.4. Since $\Psi(\xi)$ is smooth near $\xi = \lambda$, the integrand in (1.9) is a bounded function, and the integral is finite by the assumption. By a similar argument, the right-hand side of (1.4) is a bounded integrable function. Hence, (1.4) indeed defines an $L^2(\mathbf{R}) \cap C_0(\mathbf{R})$ function $G_\lambda(x)$, and so $F_\lambda(x)$ is well defined and belongs to $C_b(\mathbf{R})$.

Denote

$$K_\lambda = -\frac{1}{\pi} \operatorname{pv} \int_0^\infty \frac{\Psi'(\lambda) d\xi}{\Psi(\lambda) - \Psi(\xi)}, \quad \lambda > 0, \quad (3.1)$$

so that $\vartheta_\lambda = \arctan(K_\lambda)$ (by (1.3)). Let

$$a_\lambda = \cos \vartheta_\lambda = \frac{1}{\sqrt{1 + K_\lambda^2}}, \quad b_\lambda = \sin \vartheta_\lambda = \frac{K_\lambda}{\sqrt{1 + K_\lambda^2}}. \quad (3.2)$$

Note that $\mathcal{F}G_\lambda(\xi) = a_\lambda(2\lambda/(\lambda^2 - \xi^2) - \Psi'(\lambda)/(\Psi(\lambda) - \Psi(\xi)))$. We define

$$F_\lambda^*(x) = \sin(|\lambda x| + \vartheta_\lambda) = a_\lambda \sin |\lambda x| + b_\lambda \cos(\lambda x),$$

so that $F_\lambda(x) = F_\lambda^*(x) - G_\lambda(x)$.

By the Fourier inversion formula,

$$\begin{aligned} G_\lambda(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}G_\lambda(\xi) d\xi \\ &= \frac{a_\lambda}{\pi} \left(\operatorname{pv} \int_0^\infty \frac{2\lambda}{\lambda^2 - \xi^2} d\xi - \operatorname{pv} \int_0^\infty \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} d\xi \right). \end{aligned}$$

The first principal value integral vanishes (cf. Remark 1.7). It follows that

$$G_\lambda(0) = a_\lambda K_\lambda = \cos(\vartheta_\lambda) \tan(\vartheta_\lambda) = \sin(\vartheta_\lambda) = F_\lambda^*(0),$$

and therefore $F_\lambda(0) = 0$, as desired.

Note that $2\pi\delta_\lambda(\xi)$ is the Fourier transform of $e^{-i\lambda x}$, while $\operatorname{pv} 2i/(\lambda - \xi)$ is the Fourier transform of $-e^{-i\lambda x} \operatorname{sign} x$. Hence, $\pi(\delta_{-\lambda}(\xi) + \delta_\lambda(\xi))$ is the Fourier transform of $\cos(\lambda x)$, and $\operatorname{pv}(2\lambda/(\lambda^2 - \xi^2))$ is the Fourier transform of $\sin |\lambda x|$. It follows that,

$$\begin{aligned} \mathcal{F}F_\lambda(\xi) &= \mathcal{F}F_\lambda^*(\xi) - \mathcal{F}G_\lambda(\xi) \\ &= \left(\pi b_\lambda(\delta_{-\lambda}(\xi) + \delta_\lambda(\xi)) + a_\lambda \operatorname{pv} \frac{2\lambda}{\lambda^2 - \xi^2} \right) - a_\lambda \left(\frac{2\lambda}{\lambda^2 - \xi^2} - \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} \right) \\ &= a_\lambda \operatorname{pv} \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\xi)} + \pi b_\lambda(\delta_\lambda(\xi) + \delta_{-\lambda}(\xi)), \end{aligned}$$

and (1.7) is proved. As any bounded function, F_λ is \mathcal{S}' -convolvable with A , and by the exchange formula we have (see Preliminaries)

$$\mathcal{F}(A * F_\lambda + \Psi(\lambda)F_\lambda)(\xi) = (-\Psi(\xi) + \Psi(\lambda))\mathcal{F}F_\lambda(\xi) = a_\lambda\Psi'(\lambda).$$

In particular, $A * F_\lambda + \Psi(\lambda)F_\lambda = a_\lambda\Psi'(\lambda)\delta_0$, which proves (1.8). Formula (1.6) follows from (1.7) by the exchange formula, since the Fourier transform of $e^{-a|x|}$ is $2a/(a^2 + \xi^2)$. It remains to prove that $F_\lambda \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^\infty)$ and that (1.5) holds true.

By (1.8), F_λ is a distributional eigenfunction of $\mathcal{A}_{\mathbf{R} \setminus \{0\}}$. Furthermore, $F_\lambda \in C_b(\mathbf{R})$, $F_\lambda(0) = 0$ and $F_\lambda(x) - F_\lambda^*(x) = G_\lambda(x)$ converges to 0 as $x \rightarrow \pm\infty$. By Lemma 3.2, $F_\lambda \in \mathcal{D}(\mathcal{A}_{\mathbf{R} \setminus \{0\}}; L^\infty)$ and $\mathcal{A}_{\mathbf{R} \setminus \{0\}} F_\lambda = -\Psi(\lambda)F_\lambda$. Finally, for $x \in \mathbf{R} \setminus \{0\}$ and $t > 0$ we have (see, for example, [7])

$$\begin{aligned} P_t^{\mathbf{R} \setminus \{0\}} F_\lambda(x) &= F_\lambda(x) + \int_0^t P_s^{\mathbf{R} \setminus \{0\}} \mathcal{A}_{\mathbf{R} \setminus \{0\}} F_\lambda(x) ds \\ &= F_\lambda(x) - \Psi(\lambda) \int_0^t P_s^{\mathbf{R} \setminus \{0\}} F_\lambda(x) ds. \end{aligned}$$

For a fixed $x \in \mathbf{R} \setminus \{0\}$, this integral equation is easily solved, and we obtain $P_t^{\mathbf{R} \setminus \{0\}} F_\lambda(x) = e^{-t\Psi(\lambda)} F_\lambda(x)$, as desired. \square

Suppose that $\Psi(\xi) = \psi(\xi^2)$ for some function $\psi(\xi)$ ($\xi > 0$). Note that with the notation of (2.3), we have

$$\begin{aligned} \mathcal{F}G_\lambda(\xi) &= a_\lambda \left(\frac{2\lambda}{\lambda^2 - \xi^2} - \frac{2\lambda\psi'(\lambda^2)}{\psi(\lambda^2) - \psi(\xi^2)} \right) \\ &= a_\lambda \frac{2\lambda}{\lambda^2 - \xi^2} \left(1 - \frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)} \right) = \frac{2a_\lambda}{\lambda} \frac{1}{(\psi_\lambda)_\lambda(\xi^2)}. \end{aligned} \quad (3.3)$$

Proof of Theorem 1.9. Part (a). Let $\psi(\xi) = \Psi(\sqrt{\xi})$. Then we have $\psi'(\xi) > 0$ and $\psi''(\xi) \leq 0$ for $\xi > 0$; hence, ψ is increasing and concave. First, we prove that $\vartheta_\lambda \geq 0$. Let K_λ be defined by (3.1). Observe that since $\psi_\lambda(\xi^2)$ is infinitely smooth near $\xi = \lambda$, we have

$$\begin{aligned} K_\lambda &= -\frac{1}{\pi} \frac{\lambda^2 \Psi'(\lambda)}{\Psi(\lambda)} \text{pv} \int_0^\infty \frac{\psi_\lambda(\xi^2)}{\lambda^2 - \xi^2} d\xi \\ &= -\frac{1}{\pi} \lim_{q \rightarrow 1^-} \frac{\lambda^2 \Psi'(\lambda)}{\Psi(\lambda)} \left(\int_0^{q\lambda} + \int_{\lambda/q}^\infty \right) \frac{\psi_\lambda(\xi^2)}{\lambda^2 - \xi^2} d\xi. \end{aligned}$$

By substituting $\xi = \lambda s$ for $\xi < \lambda$ and $\xi = \lambda/s$ for $\xi > \lambda$, and then combining the two integrals, we obtain that

$$K_\lambda = \frac{1}{\pi} \frac{\lambda \Psi'(\lambda)}{\Psi(\lambda)} \int_0^1 \frac{(\psi_\lambda(\lambda^2/s^2) - \psi_\lambda(\lambda^2 s^2))}{1 - s^2} ds.$$

Since ψ is concave, ψ_λ is a non-decreasing function (e.g. by a geometric argument: $\lambda^{-2}\psi(\lambda^2)/\psi_\lambda(\xi) = (\psi(\xi^2) - \psi(\lambda^2))/(\xi^2 - \lambda^2)$ is the difference quotient of ψ , and hence a non-increasing function of ξ). It follows that $K_\lambda \geq 0$, and therefore $\vartheta_\lambda = \arctan(K_\lambda) \geq 0$. Monotonicity of ψ_λ and formula (3.3) imply also that $\mathcal{F}G_\lambda(\xi) \geq 0$ for all ξ , which, by Fourier inversion formula, proves that $|G_\lambda(\xi)| \leq G_\lambda(0)$.

Part (b). Suppose now that ψ defined as above is a complete Bernstein function. Clearly, ψ is concave, so, by part (a), $\vartheta_\lambda \in [0, \pi/2)$. The proof of (1.12) is very similar to the proof of formula (1.4) in [12].

Let $\tilde{G}_\lambda(x) = \mathbf{1}_{(0, \infty)}(x) G_\lambda(x)$. We will show that $\mathcal{L}\tilde{G}_\lambda$ is a Stieltjes function. For $a > 0$, the Fourier transform of $e^{-a|x|}$ is $2a/(a^2 + \xi^2)$. Hence, by Plancherel's theorem, for $\xi > 0$ we have

$$\mathcal{L}\tilde{G}_\lambda(\xi) = \frac{1}{2} \int_{-\infty}^\infty e^{-\xi|x|} G_\lambda(x) dx = \frac{1}{\pi} \int_0^\infty \frac{\xi \mathcal{F}G_\lambda(\zeta)}{\xi^2 + \zeta^2} d\zeta.$$

Substituting $\zeta = s\sqrt{\xi}$, we obtain

$$\mathcal{L}\tilde{G}_\lambda(\xi) = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{\xi} \mathcal{F}G_\lambda(s\sqrt{\xi})}{\xi + s^2} ds. \quad (3.4)$$

By (3.3),

$$\mathcal{F}G_\lambda(\xi) = \frac{2 \cos \vartheta_\lambda}{\lambda} \frac{1}{(\psi_\lambda)_\lambda(\xi^2)}.$$

By Proposition 2.2, ψ_λ and $(\psi_\lambda)_\lambda$ are complete Bernstein functions. Hence, by Proposition 2.1(c) $1/(\psi_\lambda)_\lambda$ is a Stieltjes function, bounded on $(0, \infty)$, and so $\mathcal{F}G_\lambda$ extends to a holomorphic function in the right complex half-plane $\operatorname{Re} \xi > 0$. Abusing the notation, we denote this extension with the same symbol $\mathcal{F}G_\lambda$. By Proposition 2.4(b), $\mathcal{F}G_\lambda(\xi)$ is bounded on every region $-\pi/2 + \varepsilon \leq \operatorname{Arg} \xi \leq \pi/2 - \varepsilon$. It follows that the right-hand side of (3.4) defines a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$, which is the holomorphic extension of $\mathcal{L}\tilde{G}_\lambda$. We denote this extension by $g(\xi)$.

Assume that $\operatorname{Im} \xi > 0$ and $\operatorname{Re} \xi < 0$. By substituting $s = -\zeta\sqrt{\xi}$ (ζ is a complex variable here) in the right-hand side of (3.4), we obtain

$$g(\xi) = -\frac{1}{\pi} \int_0^{-\xi^{-1/2}\infty} \frac{\mathcal{F}G_\lambda(-\xi\zeta)}{1+\zeta^2} d\zeta.$$

The integrand has a simple pole at $\zeta = i$, with the corresponding residue $(-i/2)\mathcal{F}G_\lambda(-i\xi)$. By an appropriate contour integration, the residue theorem and passage to a limit (for the details, see Lemma 3.8 in [12]),

$$g(\xi) = -\frac{1}{\pi} \int_0^\infty \frac{\mathcal{F}G_\lambda(-\xi\zeta)}{1+\zeta^2} d\zeta + \mathcal{F}G_\lambda(-i\xi). \quad (3.5)$$

When $\operatorname{Im} \xi > 0$, $\operatorname{Re} \xi > 0$, we have in a similar manner

$$g(\xi) = \frac{1}{\pi} \int_0^\infty \frac{\mathcal{F}G_\lambda(\xi\zeta)}{1+\zeta^2} d\zeta. \quad (3.6)$$

Recall that $\mathcal{F}G_\lambda(\zeta) = (2\lambda^{-1} \cos \vartheta_\lambda)/(\psi_\lambda)_\lambda(\zeta^2)$, and $1/(\psi_\lambda)_\lambda$ is a Stieltjes function. Hence, $\operatorname{Im} \mathcal{F}G_\lambda(\zeta) > 0$ when $\operatorname{Re} \zeta > 0$ and $\operatorname{Im} \zeta > 0$, and $\operatorname{Im} \mathcal{F}G_\lambda(\zeta) < 0$ when $\operatorname{Re} \zeta > 0$ and $\operatorname{Im} \zeta < 0$. By Proposition 2.1(b), formulae (3.5) and (3.6) imply that $g = \mathcal{L}\tilde{G}_\lambda$ is indeed a Stieltjes function. Hence, by Proposition 2.1(d), \tilde{G}_λ is the Laplace transform of a Radon measure γ_λ on $[0, \infty)$. Since $\tilde{G}_\lambda \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$, γ_λ does not charge $\{0\}$, and it is a finite measure.

The assumption for the last part of theorem can be rephrased as follows: the holomorphic extension of ψ to the upper complex half-plane has continuous boundary limit on $(-\infty, 0)$, which will be denoted by ψ^+ , and $\psi^+(-\xi^2) \neq \psi(\lambda^2)$ for all $\xi > 0$. In this case, by (3.5) and (3.3), the continuous boundary limit g^+ of g on $(-\infty, 0)$ approached from the upper half-plane exists, and it satisfies

$$g^+(-\xi) = -\frac{1}{\pi} \int_0^\infty \frac{\mathcal{F}G_\lambda(\xi\zeta)}{1+\zeta^2} d\zeta + a_\lambda \left(\frac{2\lambda}{\lambda^2 + \xi^2} - \frac{2\lambda\psi'(\lambda^2)}{\psi(\lambda^2) - \psi^+(-\xi^2)} \right).$$

By Proposition 2.3(b), it follows that

$$\gamma_\lambda(d\xi) = \frac{1}{\pi} \operatorname{Im} g^+(-\xi) d\xi = \frac{2\lambda\psi'(\lambda^2)a_\lambda}{\pi} \operatorname{Im} \frac{1}{\psi(\lambda^2) - \psi^+(-\xi^2)} d\xi.$$

Formula (1.12) is proved. \square

4 Eigenfunction expansion in $\mathbf{R} \setminus \{0\}$

In this section we prove Theorem 1.10, which states that the system of generalised eigenfunctions $F_\lambda(x)$ and $\sin(\lambda x)$, $\lambda > 0$, is complete in $L^2(\mathbf{R} \setminus \{0\})$. Throughout this

section assumptions of Theorem 1.4 are in force, that is, $\Psi(\xi)$ is the Lévy-Khintchine exponent of a symmetric Lévy process X_t , $1/(1 + \Psi(\xi))$ is integrable, and $\Psi'(\xi) > 0$ for $\xi > 0$. Our argument is modelled after the proof of Theorem 1.9 in [13], providing a similar result for the half-line. Noteworthy, in contrast to the previous section, here the case of $\mathbf{R} \setminus \{0\}$ appears to require essentially more work than the half-line $(0, \infty)$.

We begin with some auxiliary definitions and four technical lemmas. For $z \in \mathbf{C} \setminus (-\infty, 0]$, we define

$$\begin{aligned}\varphi(z) &= \frac{1}{\pi} \int_0^\infty \frac{1}{\Psi(\zeta) + z} d\zeta, \\ \varphi(\xi, z) &= \frac{1}{\pi} \int_0^\infty \frac{\xi^2}{\xi^2 + \zeta^2} \frac{1}{\Psi(\zeta) + z} d\zeta, \\ \varphi(\xi_1, \xi_2, z) &= \frac{1}{\pi} \int_0^\infty \frac{\xi_1^2}{\xi_1^2 + \zeta^2} \frac{\xi_2^2}{\xi_2^2 + \zeta^2} \frac{1}{\Psi(\zeta) + z} d\zeta.\end{aligned}$$

The integrals are convergent by integrability of $1/(1 + \Psi(\xi))$. By a substitution $\Psi(\zeta) = s$, one easily sees that $\varphi(z)$, $\varphi(\xi, z)$ and $\varphi(\xi_1, \xi_2, z)$ are Stieltjes functions of z . Representing measures (as in Proposition 2.3) for these and related Stieltjes functions play an important role in the sequel.

We remark that the above three functions are related to the resolvent (or z -potential) kernel $u_z(x)$ of the operator \mathcal{A} , for example, $\varphi(z) = u_z(0)$ and $\varphi(\xi, z)/\xi = \int_{-\infty}^\infty u_z(x) e^{-\xi|x|} dx$ (see Preliminaries). These connections are only used in the proof of Theorem 1.12.

Lemma 4.1. *For any $\xi > 0$, the function $\varphi(\xi, z)/\varphi(z)$ is a Stieltjes function of z .*

Proof. Clearly, $\varphi(\xi, z)/\varphi(z) > 0$ for $z > 0$. By Proposition 2.1(b), it suffices to show that $\text{Im}(\varphi(\xi, z)/\varphi(z)) \leq 0$ when $\text{Im } z > 0$; then automatically $\text{Im}(\varphi(\xi, z)/\varphi(z)) = -\text{Im}(\varphi(\xi, \bar{z})/\varphi(\bar{z})) \geq 0$ when $\text{Im } z < 0$. Let $z = x + iy$ for real x, y . We have

$$\text{Re} \frac{1}{\Psi(\zeta) + z} = \frac{\Psi(\zeta) + x}{(\Psi(\zeta) + x)^2 + y^2}, \quad \text{Im} \frac{1}{\Psi(\zeta) + z} = -\frac{y}{(\Psi(\zeta) + x)^2 + y^2}.$$

Hence, by expanding the integrals and a short calculation, we obtain

$$\begin{aligned}\text{Im} \frac{\varphi(\xi, z)}{\varphi(z)} &= \frac{\text{Im} \varphi(\xi, z) \text{Re} \varphi(z) - \text{Re} \varphi(\xi, z) \text{Im} \varphi(z)}{|\varphi(z)|^2} \\ &= \frac{1}{\pi^2 |\varphi(z)|^2} \int_0^\infty \int_0^\infty \frac{\xi^2}{\xi^2 + \zeta_1^2} \times \frac{y(\Psi(\zeta_1) - \Psi(\zeta_2))}{((\Psi(\zeta_1) + x)^2 + y^2)((\Psi(\zeta_2) + x)^2 + y^2)} d\zeta_1 d\zeta_2.\end{aligned}$$

Another version of the above formula is obtained by changing the roles of ζ_1 and ζ_2 . By adding the sides of the two versions of the formula, we obtain a symmetrised version,

$$\begin{aligned}2 \text{Im} \frac{\varphi(\xi, z)}{\varphi(z)} &= \frac{1}{\pi^2 |\varphi(z)|^2} \int_0^\infty \int_0^\infty \left(\frac{\xi^2}{\xi^2 + \zeta_1^2} - \frac{\xi^2}{\xi^2 + \zeta_2^2} \right) \times \\ &\quad \times \frac{y(\Psi(\zeta_1) - \Psi(\zeta_2))}{((\Psi(\zeta_1) + x)^2 + y^2)((\Psi(\zeta_2) + x)^2 + y^2)} d\zeta_1 d\zeta_2.\end{aligned}$$

Note that $\Psi(\zeta)$ is increasing on $(0, \infty)$, while $\xi^2/(\xi^2 + \zeta^2)$ decreases with $\zeta > 0$. It follows that when $y > 0$, the integrand on the right-hand side is non-positive, and so $\text{Im}(\varphi(\xi, z)/\varphi(z)) \leq 0$. \square

Lemma 4.2. *For any $\xi_1, \xi_2 > 0$, the function*

$$\tilde{\varphi}(\xi_1, \xi_2, z) = \varphi(\xi_1, \xi_2, z) - \frac{\varphi(\xi_1, z)\varphi(\xi_2, z)}{\varphi(z)}$$

is a Stieltjes function of z .

Proof. The argument is very similar to the proof of Lemma 4.1. For simplicity, we denote $\varphi(\xi_1, z) = a_1 + ib_1$, $\varphi(\xi_2, z) = a_2 + ib_2$, $\varphi(z) = a + ib$ for real a, a_1, a_2, b, b_1, b_2 . We have

$$\begin{aligned} \operatorname{Im} \frac{\varphi(\xi_1, z)\varphi(\xi_2, z)}{\varphi(z)} &= \frac{(a_1b_2 + b_1a_2)a - (a_1a_2 - b_1b_2)b}{a^2 + b^2} \\ &= \frac{(a^2 + b^2)b_1b_2 - (ab_1 - a_1b)(ab_2 - a_2b)}{b(a^2 + b^2)} \\ &= \frac{\operatorname{Im} \varphi(\xi_1, z) \operatorname{Im} \varphi(\xi_2, z)}{\operatorname{Im} \varphi(z)} + \frac{\operatorname{Im}(\varphi(\xi_1, z)/\varphi(z)) \operatorname{Im}(\varphi(\xi_2, z)/\varphi(z))}{\operatorname{Im}(1/\varphi(z))}. \end{aligned} \quad (4.1)$$

By Lemma 4.1, $\varphi(\xi_1, z)/\varphi(z)$ and $\varphi(\xi_2, z)/\varphi(z)$ are Stieltjes functions, and by Proposition 2.1(c), $1/\varphi(z)$ is a complete Bernstein function. Hence, the second summand on the right-hand side of (4.1) is non-negative when $\operatorname{Im} z > 0$ (Proposition 2.1(a,b)). The first one is non-positive, but below we prove that it is bounded below by $\operatorname{Im} \varphi(\xi_1, \xi_2, z)$.

For $z = x + iy$ with real x, y , we have as in the proof of Lemma 4.1,

$$\begin{aligned} &\operatorname{Im} \varphi(\xi_1, z) \operatorname{Im} \varphi(\xi_2, z) - \operatorname{Im} \varphi(z) \operatorname{Im} \varphi(\xi_1, \xi_2, z) \\ &= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{\xi_1^2}{\xi_1^2 + \zeta_1^2} \left(\frac{\xi_2^2}{\xi_2^2 + \zeta_2^2} - \frac{\xi_2^2}{\xi_2^2 + \zeta_1^2} \right) \times \\ &\quad \times \frac{y^2}{((\Psi(\zeta_1) + x)^2 + y^2)((\Psi(\zeta_2) + x)^2 + y^2)} d\zeta_1 d\zeta_2 \end{aligned}$$

By a similar symmetrisation procedure as in the proof of Lemma 4.1, we obtain that

$$\begin{aligned} &2 \operatorname{Im} \varphi(\xi_1, z) \operatorname{Im} \varphi(\xi_2, z) - 2 \operatorname{Im} \varphi(z) \operatorname{Im} \varphi(\xi_1, \xi_2, z) \\ &= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \left(\frac{\xi_1^2}{\xi_1^2 + \zeta_1^2} - \frac{\xi_1^2}{\xi_1^2 + \zeta_2^2} \right) \left(\frac{\xi_2^2}{\xi_2^2 + \zeta_2^2} - \frac{\xi_2^2}{\xi_2^2 + \zeta_1^2} \right) \times \\ &\quad \times \frac{y^2}{((\Psi(\zeta_1) + x)^2 + y^2)((\Psi(\zeta_2) + x)^2 + y^2)} d\zeta_1 d\zeta_2, \end{aligned}$$

and again the integrand on the right-hand side is non-positive. Hence,

$$\operatorname{Im} \varphi(\xi_1, z) \operatorname{Im} \varphi(\xi_2, z) - \operatorname{Im} \varphi(z) \operatorname{Im} \varphi(\xi_1, \xi_2, z) \leq 0.$$

When $y > 0$, we have $\operatorname{Im} \varphi(z) < 0$, and so

$$\frac{\operatorname{Im} \varphi(\xi_1, z) \operatorname{Im} \varphi(\xi_2, z)}{\operatorname{Im} \varphi(z)} \geq \operatorname{Im} \varphi(\xi_1, \xi_2, z).$$

By (4.1), it follows that $\operatorname{Im} \tilde{\varphi}(\xi_1, \xi_2, z) \leq 0$ whenever $\operatorname{Im} z > 0$. Clearly, $\operatorname{Im} \tilde{\varphi}(\xi_1, \xi_2, z) = -\operatorname{Im} \tilde{\varphi}(\xi_1, \xi_2, \bar{z}) \geq 0$ when $\operatorname{Im} z < 0$. By Proposition 2.1(b), it remains to prove that $\tilde{\varphi}(\xi_1, \xi_2, z) \geq 0$ when $z > 0$.

Since $\tilde{\varphi}(\xi_1, \xi_2, z)$ is real when $z > 0$ and $\operatorname{Im} \tilde{\varphi}(\xi_1, \xi_2, z) \leq 0$ when $\operatorname{Im} z > 0$, we have $(\partial/\partial z)\tilde{\varphi}(\xi_1, \xi_2, z) \leq 0$ for $z > 0$. It follows that $\tilde{\varphi}(\xi_1, \xi_2, z)$ is a non-increasing function of $z > 0$, and hence it suffices to show that $\lim_{z \rightarrow \infty} \tilde{\varphi}(\xi_1, \xi_2, z) \geq 0$. When $z \rightarrow \infty$, by the definition of $\varphi(\xi, z)$ and $\varphi(\xi_1, \xi_2, z)$ and dominated convergence, the functions $\varphi(\xi_1, z)$ and $\varphi(\xi_1, \xi_2, z)$ converge to 0. Also, $\varphi(\xi_2, z)/\varphi(z) \leq 1$ for all $z > 0$. We conclude that $\tilde{\varphi}(\xi_1, \xi_2, z)$ converges to 0 as $z \rightarrow \infty$, and the proof is complete. \square

For $\lambda, \xi > 0$, we define

$$K_\lambda(\xi) = -\frac{1}{\pi} \operatorname{pv} \int_0^\infty \frac{\xi^2}{\xi^2 + \zeta^2} \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Psi(\zeta)} d\zeta, \quad L_\lambda(\xi) = \frac{\xi^2}{\xi^2 + \lambda^2},$$

and, as in (3.1),

$$K_\lambda = -\frac{1}{\pi} \operatorname{pv} \int_0^\infty \frac{\Psi'(\lambda) d\xi}{\Psi(\lambda) - \Psi(\xi)}.$$

We denote the boundary limits of $\varphi(z)$, $\varphi(\xi, z)$ and $\varphi(\xi_1, \xi_2, z)$ along $(-\infty, 0)$ approached from the upper half-plane by $\varphi^+(-z)$, $\varphi^+(\xi, -z)$ and $\varphi^+(\xi_1, \xi_2, -z)$ ($z > 0$), respectively. The existence of these limits is a part of the next result.

Lemma 4.3. *For $\xi, \lambda > 0$, we have*

$$\varphi^+(-\Psi(\lambda)) = \frac{K_\lambda - i}{\Psi'(\lambda)}, \quad \varphi^+(\xi, -\Psi(\lambda)) = \frac{K_\lambda(\xi) - iL_\lambda(\xi)}{\Psi'(\lambda)}.$$

Furthermore (cf. Lemma 4.1),

$$\operatorname{Im} \frac{\varphi^+(\xi, -\Psi(\lambda))}{\varphi^+(-\Psi(\lambda))} = -\frac{K_\lambda L_\lambda(\xi) - K_\lambda(\xi)}{1 + K_\lambda^2},$$

and for $\xi_1, \xi_2, \lambda > 0$ (cf. Lemma 4.2),

$$\begin{aligned} \operatorname{Im} \tilde{\varphi}^+(\xi_1, \xi_2, -\Psi(\lambda)) \\ = -\frac{1}{\Psi'(\lambda)} \frac{K_\lambda L_\lambda(\xi_1) - K_\lambda(\xi_1)}{\sqrt{1 + K_\lambda^2}} \frac{K_\lambda L_\lambda(\xi_2) - K_\lambda(\xi_2)}{\sqrt{1 + K_\lambda^2}}. \end{aligned}$$

Proof. For $\lambda > 0$ and any bounded, continuously differentiable function f on $(0, \infty)$, we have (see Section 1.8 in [19])

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \frac{1}{\Psi(\zeta) - \Psi(\lambda) + i\varepsilon} f(\zeta) d\zeta \\ = \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \frac{\Psi(\zeta) - \Psi(\lambda)}{(\Psi(\zeta) - \Psi(\lambda))^2 + \varepsilon^2} f(\zeta) d\zeta - \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \frac{i\varepsilon}{(\Psi(\zeta) - \Psi(\lambda))^2 + \varepsilon^2} f(\zeta) d\zeta \\ = -\operatorname{pv} \int_0^\infty \frac{1}{\Psi(\lambda) - \Psi(\zeta)} f(\zeta) d\zeta - \frac{i}{\Psi'(\lambda)} f(\lambda). \end{aligned}$$

Hence, the boundary limits $\varphi^+(-\Psi(\lambda))$, $\varphi^+(\xi, -\Psi(\lambda))$ and $\varphi^+(\xi_1, \xi_2, -\Psi(\lambda))$ (approached from the upper half-plane; here and below $\xi, \xi_1, \xi_2 > 0$) are continuous functions of $\lambda > 0$, given by

$$\begin{aligned} \varphi^+(-\Psi(\lambda)) &= -\frac{1}{\pi} \operatorname{pv} \int_0^\infty \frac{1}{\Psi(\lambda) - \Psi(\zeta)} d\zeta - \frac{i}{\Psi'(\lambda)} = \frac{K_\lambda - i}{\Psi'(\lambda)}, \\ \varphi^+(\xi, -\Psi(\lambda)) &= -\frac{1}{\pi} \operatorname{pv} \int_0^\infty \frac{\xi^2}{\xi^2 + \zeta^2} \frac{1}{\Psi(\lambda) - \Psi(\zeta)} d\zeta - \frac{i}{\Psi'(\lambda)} \frac{\xi^2}{\xi^2 + \lambda^2} = \frac{K_\lambda(\xi) - iL_\lambda(\xi)}{\Psi'(\lambda)}, \end{aligned}$$

and (we only need the imaginary part)

$$\operatorname{Im} \varphi^+(\xi_1, \xi_2, -\Psi(\lambda)) = -\frac{1}{\Psi'(\lambda)} \frac{\xi_1^2}{\xi_1^2 + \lambda^2} \frac{\xi_2^2}{\xi_2^2 + \lambda^2} = -\frac{L_\lambda(\xi_1) L_\lambda(\xi_2)}{\Psi'(\lambda)}.$$

We obtain that

$$\operatorname{Im} \frac{\varphi^+(\xi, -\Psi(\lambda))}{\varphi^+(-\Psi(\lambda))} = -\frac{K_\lambda L_\lambda(\xi) - K_\lambda(\xi)}{1 + K_\lambda^2},$$

and therefore, by (4.1),

$$\begin{aligned} & \operatorname{Im} \frac{\varphi^+(\xi_1, -\Psi(\lambda))\varphi^+(\xi_2, -\Psi(\lambda))}{\varphi^+(-\Psi(\lambda))} \\ &= -\frac{L_\lambda(\xi_1)L_\lambda(\xi_2)}{\Psi'(\lambda)} + \frac{K_\lambda L_\lambda(\xi_1) - K_\lambda(\xi_1)}{1 + K_\lambda^2} \frac{K_\lambda L_\lambda(\xi_2) - K_\lambda(\xi_2)}{1 + K_\lambda^2} \frac{1 + K_\lambda^2}{\Psi'(\lambda)} \\ &= \frac{1}{\Psi'(\lambda)} \left(\frac{K_\lambda L_\lambda(\xi_1) - K_\lambda(\xi_1)}{\sqrt{1 + K_\lambda^2}} \frac{K_\lambda L_\lambda(\xi_2) - K_\lambda(\xi_2)}{\sqrt{1 + K_\lambda^2}} - L_\lambda(\xi_1)L_\lambda(\xi_2) \right). \end{aligned}$$

The proof is complete. \square

Lemma 4.4. For all $\xi_1, \xi_2 > 0$ and $t \geq 0$, let

$$\Phi(\xi_1, \xi_2, t) = \int_0^\infty \frac{K_\lambda L_\lambda(\xi_1) - K_\lambda(\xi_1)}{\sqrt{1 + K_\lambda^2}} \frac{K_\lambda L_\lambda(\xi_2) - K_\lambda(\xi_2)}{\sqrt{1 + K_\lambda^2}} e^{-t\Psi(\lambda)} d\lambda. \quad (4.2)$$

Then

$$\Phi(\xi_1, \xi_2, 0) = \frac{\pi}{2} \frac{\xi_1 \xi_2}{\xi_1 + \xi_2}, \quad (4.3)$$

and for all $z > 0$,

$$\frac{1}{\pi} \int_0^\infty \Phi(\xi_1, \xi_2, t) e^{-zt} dt = \tilde{\varphi}(\xi_1, \xi_2, z). \quad (4.4)$$

Proof. Note that by Lemma 4.3, we have

$$\begin{aligned} \Phi(\xi_1, \xi_2, t) &= - \int_0^\infty \operatorname{Im} \tilde{\varphi}^+(\xi_1, \xi_2, -\Psi(\lambda)) e^{-t\Psi(\lambda)} \Psi'(\lambda) d\lambda \\ &= - \int_0^\infty \operatorname{Im} \tilde{\varphi}^+(\xi_1, \xi_2, -s) e^{-ts} ds. \end{aligned}$$

By Lemma 4.2, $\operatorname{Im} \tilde{\varphi}^+(\xi_1, \xi_2, -s) \leq 0$. Hence, by Fubini,

$$\int_0^\infty \Phi(\xi_1, \xi_2, t) e^{-zt} dt = - \int_0^\infty \frac{1}{s+z} \operatorname{Im} \tilde{\varphi}^+(\xi_1, \xi_2, -s) ds.$$

By the definition, the coefficients c_1, c_2 in the representation (2.2) of the Stieltjes function $\varphi(\xi_1, \xi_2, z)$ vanish. Hence, by Proposition 2.3(b) and the inequality $\tilde{\varphi}(\xi_1, \xi_2, z) \leq \varphi(\xi_1, \xi_2, z)$ ($z > 0$), the coefficients c_1, c_2 in the representation (2.2) for $\tilde{\varphi}(\xi_1, \xi_2, z)$ vanish too. By (2.2) and another application of Proposition 2.3(b),

$$- \int_0^\infty \frac{1}{s+z} \operatorname{Im} \tilde{\varphi}^+(\xi_1, \xi_2, -s) ds = \pi \tilde{\varphi}(\xi_1, \xi_2, z).$$

This proves (4.4).

By monotone convergence, $\Phi(\xi_1, \xi_2, t)$ is right-continuous at $t = 0$. This and (4.4) imply that

$$\Phi(\xi_1, \xi_2, 0) = \lim_{z \rightarrow \infty} z \int_0^\infty \Phi(\xi_1, \xi_2, t) e^{-zt} dt = \pi \lim_{z \rightarrow \infty} (z \tilde{\varphi}(\xi_1, \xi_2, z)).$$

As $z \rightarrow \infty$, by monotone convergence, the function $z\varphi(\xi, z)$ converges to ξ , but $z\varphi(z)$ diverges to infinity. Hence, $z\varphi(\xi_1, z)\varphi(\xi_2, z)/\varphi(z)$ converges to 0. It follows that (see the definition of $\tilde{\varphi}(\xi_1, \xi_2, z)$ in Lemma 4.2)

$$\Phi(\xi_1, \xi_2, 0) = \pi \lim_{z \rightarrow \infty} (z\varphi(\xi_1, \xi_2, z)) = \int_0^\infty \frac{\xi_1^2}{\xi_1^2 + \zeta^2} \frac{\xi_2^2}{\xi_2^2 + \zeta^2} d\zeta;$$

monotone convergence was used again for the second equality. The integral can be easily evaluated (we omit the details), and (4.3) follows. \square

With the above technical background, we can prove generalized eigenfunction expansion of the operators $P_t^{\mathbf{R} \setminus \{0\}}$. Let Π_{even} , Π_{odd} and Π be defined as in Theorem 1.10. First, we prove that the operator Π_{even} is isometric, up to a constant factor.

Lemma 4.5. *For any even functions $f, g \in L^2(\mathbf{R} \setminus \{0\})$,*

$$\langle \Pi_{\text{even}} f, \Pi_{\text{even}} g \rangle_{L^2((0, \infty))} = \pi \langle f, g \rangle_{L^2(\mathbf{R} \setminus \{0\})}.$$

Proof. Let $e_\xi(x) = e^{-\xi|x|}$, $x \in \mathbf{R} \setminus \{0\}$, $\xi > 0$. By Theorem 1.4, for $\xi, \lambda > 0$ we have

$$\begin{aligned} \Pi_{\text{even}} e_\xi(\lambda) &= \int_{-\infty}^{\infty} F_\lambda(x) e^{-\xi|x|} dx \\ &= \frac{-2K_\lambda(\xi) \cos \vartheta_\lambda + 2L_\lambda(\xi) \sin \vartheta_\lambda}{\xi} \\ &= \frac{2}{\xi} \frac{K_\lambda L_\lambda(\xi) - K_\lambda(\xi)}{\sqrt{1 + K_\lambda^2}}. \end{aligned} \quad (4.5)$$

By (4.2) and (4.3), it follows that for $\xi_1, \xi_2 > 0$,

$$\int_0^\infty \Pi_{\text{even}} e_{\xi_1}(\lambda) \Pi_{\text{even}} e_{\xi_2}(\lambda) d\lambda = \frac{4\Phi(\xi_1, \xi_2, 0)}{\xi_1 \xi_2} = \frac{2\pi}{\xi_1 + \xi_2} = \pi \langle e_{\xi_1}, e_{\xi_2} \rangle_{L^2(\mathbf{R} \setminus \{0\})}.$$

The family of functions e_ξ is linearly dense in the space of even $L^2(\mathbf{R} \setminus \{0\})$ functions. The result follows by approximation. \square

The following side-result is interesting. It is obvious when $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ . In the general case, however, a direct approach seems problematic.

Proposition 4.6. *We have*

$$\int_{-\infty}^{\infty} F_\lambda(x) e^{-\xi|x|} dx \geq 0, \quad \lambda, \xi > 0.$$

Proof. By Lemma 4.3 and (4.5), we see that for $e_\xi(x) = e^{-\xi|x|}$,

$$\begin{aligned} \int_{-\infty}^{\infty} F_\lambda(x) e^{-\xi|x|} dx &= \Pi_{\text{even}} e_\xi(\lambda) = \frac{2}{\xi} \frac{K_\lambda L_\lambda(\xi) - K_\lambda(\xi)}{\sqrt{1 + K_\lambda^2}} \\ &= -\frac{2\sqrt{1 + K_\lambda^2}}{\xi} \operatorname{Im} \frac{\varphi^+(\xi, -\Psi(\lambda))}{\varphi^+(-\Psi(\lambda))}, \end{aligned}$$

which is nonnegative by Lemma 4.1. \square

The space $L^2(\mathbf{R} \setminus \{0\})$ is the direct sum of the spaces $L_{\text{even}}^2(\mathbf{R} \setminus \{0\})$ and $L_{\text{odd}}^2(\mathbf{R} \setminus \{0\})$, consisting of even and odd $L^2(\mathbf{R} \setminus \{0\})$ functions, respectively. Recall that Π_{odd} is the ‘sine part’ of the Fourier transform. Hence, $\pi^{-1/2}\Pi_{\text{odd}}$ is a unitary mapping of $L_{\text{odd}}^2(\mathbf{R} \setminus \{0\})$ onto $L^2((0, \infty))$, and Π_{odd} is a zero operator on $L_{\text{even}}^2(\mathbf{R} \setminus \{0\})$. In a similar manner, $\pi^{-1/2}\Pi_{\text{even}}$ is an isometric mapping of $L_{\text{even}}^2(\mathbf{R} \setminus \{0\})$ into $L^2((0, \infty))$, and Π_{even} vanishes on $L_{\text{odd}}^2(\mathbf{R} \setminus \{0\})$. It remains to prove that $\pi^{-1/2}\Pi_{\text{even}}$ is a unitary operator on $L_{\text{even}}^2(\mathbf{R} \setminus \{0\})$ onto $L^2((0, \infty))$. An equivalent condition is that the kernel of the adjoint operator $\Pi_{\text{even}}^* : L^2((0, \infty)) \rightarrow L^2(\mathbf{R} \setminus \{0\})$ is trivial. This is proved in the following result.

Lemma 4.7. For all $f, g \in L^2((0, \infty))$,

$$\langle \Pi_{\text{even}}^* f, \Pi_{\text{even}}^* g \rangle_{L^2(\mathbf{R} \setminus \{0\})} = \pi \langle f, g \rangle_{L^2((0, \infty))}, \quad (4.6)$$

and

$$P_t^{\mathbf{R} \setminus \{0\}} \Pi_{\text{even}}^* f = \Pi_{\text{even}}^* (e^{-t\Psi} f), \quad t > 0. \quad (4.7)$$

Proof. As in the proof of Lemma 5.2 in [12], formula (4.7) for $f \in C_c((0, \infty))$ follows directly from Theorem 1.4 and Fubini. Indeed, let $\text{supp } f \subseteq [\lambda_1, \lambda_2]$, $0 < \lambda_1 < \lambda_2$. Using (1.2), (1.4) and Fourier inversion formula, it is easy to prove that $F_\lambda(y)$ is bounded uniformly in $\lambda \in [\lambda_1, \lambda_2]$ and $y \in \mathbf{R} \setminus \{0\}$ (we omit the details). Furthermore, $p_t^{\mathbf{R} \setminus \{0\}}(x, y)$ is integrable in $y \in \mathbf{R} \setminus \{0\}$. Hence,

$$\begin{aligned} P_t^{\mathbf{R} \setminus \{0\}} \Pi_{\text{even}}^* f &= \int_{\mathbf{R} \setminus \{0\}} \int_{\lambda_1}^{\lambda_2} p_t^{\mathbf{R} \setminus \{0\}}(x, y) F_\lambda(y) f(\lambda) d\lambda dy \\ &= \int_{\lambda_1}^{\lambda_2} e^{-t\Psi(\lambda)} F_\lambda(x) f(\lambda) d\lambda = \Pi_{\text{even}}^* (e^{-t\Psi} f). \end{aligned}$$

Since Π_{even}^* is a bounded operator, (4.7) extends to general $f \in L^2((0, \infty))$ by an approximation argument.

The adjoint of an isometric operator is isometric if and only if the kernel of the adjoint is trivial. Hence, by Lemma 4.5, it suffices to show that the kernel of Π_{even}^* is trivial.

Suppose that $f \in L^2((0, \infty))$ and $\Pi_{\text{even}}^* f = 0$. We claim that this implies that $f = 0$. By (4.7), $\Pi_{\text{even}}^* (e^{-t\Psi} f) = 0$ for any $t > 0$. Let $e_\xi(x) = e^{-\xi|x|}$, $x \in \mathbf{R} \setminus \{0\}$. We obtain that for all $t > 0$ and $\xi > 0$,

$$\begin{aligned} 0 &= \langle \Pi_{\text{even}}^* (e^{-t\Psi} f), e_\xi \rangle_{L^2(\mathbf{R} \setminus \{0\})} = \langle e^{-t\Psi} f, \Pi_{\text{even}} e_\xi \rangle_{L^2((0, \infty))} \\ &= \int_0^\infty e^{-t\Psi(\lambda)} f(\lambda) \Pi_{\text{even}} e_\xi(\lambda) d\lambda. \end{aligned}$$

By substituting $s = \Psi(\lambda)$, we obtain that

$$\int_0^\infty e^{-ts} \frac{f(\Psi^{-1}(s)) \Pi_{\text{even}} e_\xi(\Psi^{-1}(s))}{\Psi'(\Psi^{-1}(s))} ds = 0, \quad t, \xi > 0.$$

It follows that for each $\xi > 0$, $f(\Psi^{-1}(s)) \Pi_{\text{even}} e_\xi(\Psi^{-1}(s)) = 0$ for almost all $s > 0$.

In other words, for each $\xi > 0$, $f(\lambda) \Pi_{\text{even}} e_\xi(\lambda) = 0$ for almost all $\lambda > 0$. By Fubini, for almost every $\lambda > 0$, $f(\lambda) \Pi_{\text{even}} e_\xi(\lambda) = 0$ for almost all $\xi > 0$. But for each fixed $\lambda > 0$, $\Pi_{\text{even}} e_\xi(\lambda)$ (as a function of $\xi > 0$) is the Laplace transform of a nonzero function (namely, $2F_\lambda(x) \mathbf{1}_{(0, \infty)}(x)$), and so it cannot vanish for almost all $\xi > 0$. This implies that $f(\lambda) = 0$ for almost every $\lambda > 0$, proving our claim. \square

Proof of Theorem 1.10. The operator $\Pi f = (\Pi_{\text{even}} f, \Pi_{\text{odd}} f)$ is unitary by Lemmas 4.5 and 4.7, and the discussion preceding the statement of Lemma 4.7.

Let $f \in L^2(\mathbf{R} \setminus \{0\})$, and let $f = f_{\text{even}} + f_{\text{odd}}$ be the decomposition into the even and odd part. Clearly, $P_t^{\mathbf{R} \setminus \{0\}} f_{\text{even}}$ is an even function, and $P_t^{\mathbf{R} \setminus \{0\}} f_{\text{odd}}$ is an odd function. Hence, $\Pi_{\text{even}} f_{\text{odd}} = \Pi_{\text{even}} P_t^{\mathbf{R} \setminus \{0\}} f_{\text{odd}} = 0$ and $\Pi_{\text{odd}} f_{\text{even}} = \Pi_{\text{odd}} P_t^{\mathbf{R} \setminus \{0\}} f_{\text{even}} = 0$. By Lemma 4.7,

$$\begin{aligned} \Pi_{\text{even}} P_t^{\mathbf{R} \setminus \{0\}} f_{\text{even}} &= \pi^{-1} \Pi_{\text{even}} (P_t^{\mathbf{R} \setminus \{0\}} \Pi_{\text{even}}^* \Pi_{\text{even}} f_{\text{even}}) \\ &= \pi^{-1} \Pi_{\text{even}} (\Pi_{\text{even}}^* (e^{-t\Psi} \Pi_{\text{even}} f_{\text{even}})) = e^{-t\Psi} \Pi_{\text{even}} f_{\text{even}}. \end{aligned} \quad (4.8)$$

Furthermore, by the strong Markov property, $p_t(x - y) - p_t^{\mathbf{R} \setminus \{0\}}(x, y) = \mathbf{E}_x(p_t(y); \tau_0 \leq t)$ (the *Hunt's formula*), and the right-hand side is an even function of y (for any $x \in \mathbf{R} \setminus \{0\}$). Therefore $P_t^{\mathbf{R} \setminus \{0\}} f_{\text{odd}} = P_t f_{\text{odd}}$. By the remark preceding the statement of Theorem 1.10 (see also Remark 2.7), it follows that

$$\Pi_{\text{odd}} P_t^{\mathbf{R} \setminus \{0\}} f_{\text{odd}} = \Pi_{\text{odd}} P_t f_{\text{odd}} = e^{-t\Psi} \Pi_{\text{odd}} f_{\text{odd}}. \quad (4.9)$$

The above properties combined together prove (1.13). \square

Proof of Corollary 1.11. Let $f \in C_c(\mathbf{R} \setminus \{0\})$. Then $\Pi f \in L^2((0, \infty)) \oplus L^2((0, \infty))$, and hence $e^{-t\Psi} \Pi f$ is a pair of integrable functions. Hence,

$$\begin{aligned} P_t^{\mathbf{R} \setminus \{0\}} f(x) &= \pi^{-1} \Pi(e^{-t\Psi} \Pi^* f)(x) \\ &= \frac{1}{\pi} \left(\int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x) \left(\int_{-\infty}^\infty f(y) F_\lambda(y) dy \right) d\lambda \right. \\ &\quad \left. + \frac{1}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} \sin(\lambda x) \left(\int_{-\infty}^\infty f(y) \sin(\lambda y) dy \right) d\lambda \right) \end{aligned}$$

By Theorem 1.9(a), $F_\lambda(y)$ is bounded uniformly in $\lambda > 0$ and $y \in \mathbf{R} \setminus \{0\}$. By Fubini, it follows that

$$P_t^{\mathbf{R} \setminus \{0\}} f(x) = \frac{1}{\pi} \int_{-\infty}^\infty f(y) \int_0^\infty e^{-t\Psi(\lambda)} (F_\lambda(x) F_\lambda(y) + \sin(\lambda x) \sin(\lambda y)) d\lambda dy.$$

This proves the first equality in (1.15). For the second one, note that

$$\begin{aligned} &\frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} \sin(\lambda x) \sin(\lambda y) d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-t\Psi(\lambda)} (\cos(\lambda(x - y)) - \cos(\lambda(x + y))) d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-t\Psi(\lambda)} (e^{-i\lambda(x - y)} - e^{-i\lambda(x + y)}) d\lambda \\ &= p_t(x - y) - p_t(x + y). \end{aligned} \quad \square$$

Proof of Theorem 1.12. Recall that we need to prove that

$$\mathbf{P}_x(t < \tau_0 < \infty) = \frac{1}{\pi} \int_0^\infty \frac{\cos \vartheta_\lambda e^{-t\Psi(\lambda)} \Psi'(\lambda) F_\lambda(x)}{\Psi(\lambda)} d\lambda. \quad (4.10)$$

Note that $G_\lambda(0) = \sin \vartheta_\lambda - F_\lambda(0) = \sin \vartheta_\lambda$. For $\lambda, t > 0$ and $x \in \mathbf{R} \setminus \{0\}$ we decompose the integrand in (4.10),

$$\begin{aligned} \frac{\cos \vartheta_\lambda e^{-t\Psi(\lambda)} \Psi'(\lambda) F_\lambda(x)}{\Psi(\lambda)} &= \frac{e^{-t\Psi(\lambda)} \Psi'(\lambda)}{\Psi(\lambda)} (\sin(|\lambda x| + \vartheta_\lambda) - \sin \vartheta_\lambda) \\ &\quad + \frac{e^{-t\Psi(\lambda)} \Psi'(\lambda)}{\Psi(\lambda)} (G_\lambda(0) - G_\lambda(x)). \end{aligned} \quad (4.11)$$

By the assumption, $\Psi(\xi) = \psi(\xi^2)$, where ψ is non-negative, increasing and concave. Hence, $\Psi'(\lambda) = 2\lambda\psi'(\lambda^2) \leq 2\lambda(\psi(\lambda^2)/\lambda^2) = 2\Psi(\lambda)/\lambda$. It follows that

$$\left| \frac{e^{-t\Psi(\lambda)} \Psi'(\lambda)}{\Psi(\lambda)} (\sin(|\lambda x| + \vartheta_\lambda) - \sin \vartheta_\lambda) \right| \leq 2|x|e^{-t\Psi(\lambda)}.$$

Furthermore, the second summand on the right-hand side of (4.11) is non-negative by Theorem 1.9. It follows that the integral in (4.10) is well-defined (possibly infinite). Denote the right-hand side of (4.10) by $f(t, x)$ ($t > 0$, $x \in \mathbf{R} \setminus \{0\}$). If $e_\xi(x) = e^{-\xi|x|}$, then for $\xi, z > 0$ we have, by Fubini,

$$\int_0^\infty \int_{-\infty}^\infty f(t, x) e^{-\xi|x|} e^{-zt} dx dt = \frac{1}{\pi} \int_0^\infty \frac{\cos \vartheta_\lambda \Psi'(\lambda) \Pi_{\text{even}} e_\xi(\lambda)}{(z + \Psi(\lambda)) \Psi(\lambda)} d\lambda;$$

the use of Fubini here is justified by the decomposition of the integrand into an absolutely integrable part and a non-negative part, as in (4.11). Using $\cos \vartheta_\lambda = (1 + K_\lambda^2)^{-1/2}$ and formula (4.5), we obtain that

$$\int_0^\infty \int_{-\infty}^\infty f(t, x) e^{-\xi|x|} e^{-zt} dx dt = \frac{2}{\pi \xi} \int_0^\infty \frac{\Psi'(\lambda)}{(z + \Psi(\lambda)) \Psi(\lambda)} \frac{K_\lambda L_\lambda(\xi) - K_\lambda(\xi)}{1 + K_\lambda^2} d\lambda.$$

By Lemma 4.3, substitution $\Psi(\lambda) = s$ and Proposition 2.3(b),

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty f(t, x) e^{-\xi|x|} e^{-zt} dx dt &= -\frac{2}{\pi \xi z} \int_0^\infty \left(\frac{1}{\Psi(\lambda)} - \frac{1}{z + \Psi(\lambda)} \right) \operatorname{Im} \frac{\varphi^+(\xi, -\Psi(\lambda))}{\varphi^+(-\Psi(\lambda))} \Psi'(\lambda) d\lambda \\ &= -\frac{2}{\pi \xi z} \int_0^\infty \left(\frac{1}{s} - \frac{1}{z + s} \right) \operatorname{Im} \frac{\varphi^+(\xi, -s)}{\varphi^+(-s)} ds \\ &= \frac{2}{\xi z} \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\varphi(\xi, \varepsilon)}{\varphi(\varepsilon)} - \frac{\varphi(\xi, z)}{\varphi(z)} \right). \end{aligned}$$

Note that $\varphi(\xi, z) \leq \varphi(z)$, so that the above expression is finite. In particular, $f(t, x)$ is finite for almost all $t > 0$, $x \in \mathbf{R} \setminus \{0\}$.

On the other hand, for $z > 0$ and $x \in \mathbf{R}$, denote $g_z(x) = \mathbf{E}_x \exp(-z\tau_0)$. Then $g_z(x) = u_z(x)/u_z(0)$, where $u_z(x)$ is the resolvent (or z -potential) kernel for the operator \mathcal{A} (see Preliminaries). Hence,

$$\mathcal{F}g_z(\xi) = \frac{1}{u_z(0)} \frac{1}{\Psi(\xi) + z}, \quad z > 0, \xi \in \mathbf{R},$$

where, by the Fourier inversion formula,

$$u_z(0) = \frac{1}{\pi} \int_0^\infty \frac{1}{\Psi(\zeta) + z} d\zeta = \varphi(z).$$

Furthermore, for $z > 0$ and $x \in \mathbf{R} \setminus \{0\}$,

$$\int_0^\infty \mathbf{P}_x(\tau_0 \leq t) e^{-zt} dt = \frac{\mathbf{E}_x \exp(-z\tau_0)}{z} = \frac{g_z(x)}{z}.$$

By Fubini and Plancherel's theorem,

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \mathbf{P}_x(\tau_0 \leq t) e^{-\xi|x|} e^{-zt} dx dt &= \frac{1}{z} \int_{-\infty}^\infty g_z(x) e^{-\xi|x|} dx \\ &= \frac{2}{\pi z} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \mathcal{F}g_z(\zeta) d\zeta = \frac{2}{\pi \xi z} \frac{1}{\varphi(z)} \int_0^\infty \frac{\xi^2}{\xi^2 + \zeta^2} \frac{1}{\Psi(\zeta) + z} d\zeta = \frac{2}{\xi z} \frac{\varphi(\xi, z)}{\varphi(z)}. \end{aligned}$$

In particular, we have

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \mathbf{P}_x(\tau_0 < \infty) e^{-\xi|x|} e^{-zt} dx dt &= \frac{1}{z} \int_{-\infty}^\infty \mathbf{P}_x(\tau_0 < \infty) e^{-\xi|x|} dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\varepsilon}{z} \int_{-\infty}^\infty \int_0^\infty \mathbf{P}_x(\tau_0 < t) e^{-\xi|x|} e^{-\varepsilon t} dx dt \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{2}{\xi z} \frac{\varphi(\xi, \varepsilon)}{\varphi(\varepsilon)} \right), \end{aligned}$$

and so finally

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \mathbf{P}_x(t < \tau_0 < \infty) e^{-\xi|x|} e^{-zt} dx dt &= \frac{2}{\xi z} \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\varphi(\xi, \varepsilon)}{\varphi(\varepsilon)} - \frac{\varphi(\xi, z)}{\varphi(z)} \right) \\ &= \int_0^\infty \int_{-\infty}^\infty f(t, x) e^{-\xi|x|} e^{-zt} dx dt. \end{aligned}$$

By the uniqueness of the Laplace transform, formula (4.10) follows for almost every pair $t > 0$, $x \in \mathbf{R} \setminus \{0\}$. By dominated convergence, both sides of (4.10) are continuous in $t > 0$, and the theorem is proved. \square

Remark 4.8. One could try the following argument for the proof of Theorem 1.12: By monotone convergence,

$$\mathbf{P}_x(\tau_0 > t) = \int_{-\infty}^\infty p_t^{\mathbf{R} \setminus \{0\}}(x, y) dy = \lim_{\xi \rightarrow 0^+} \int_{-\infty}^\infty p_t^{\mathbf{R} \setminus \{0\}}(x, y) e^{-\xi|y|} dy = \lim_{\xi \rightarrow 0^+} P_t^{\mathbf{R} \setminus \{0\}} e_\xi(x),$$

where $e_\xi(x) = e^{-\xi|x|}$. For each $\xi > 0$, by Theorem 1.10 we have

$$\begin{aligned} P_t^{\mathbf{R} \setminus \{0\}} e_\xi(x) &= \frac{1}{\pi} \Pi_{\text{even}}^*(e^{-t\Psi} \Pi_{\text{even}} e_\xi)(x) \\ &= \frac{1}{\pi} \int_0^\infty F_\lambda(x) e^{-t\Psi(\lambda)} \Pi_{\text{even}} e_\xi(\lambda) d\lambda. \end{aligned} \quad (4.12)$$

Recall that as $\xi \rightarrow 0^+$, $\Pi_{\text{even}} e_\xi(\lambda)$ converges to $\cos \vartheta_\lambda \Psi'(\lambda) / \Psi(\lambda)$. Hence, if the above integrals were uniformly integrable as $\xi \rightarrow 0^+$, we would obtain

$$\mathbf{P}_x(\tau_0 > t) = \frac{1}{\pi} \int_0^\infty F_\lambda(x) e^{-t\Psi(\lambda)} \frac{\cos \vartheta_\lambda \Psi'(\lambda)}{\Psi(\lambda)} d\lambda.$$

However, if $\mathbf{P}_x(\tau_0 < \infty) < 1$ (that is, when X_t is transient), we know by Theorem 1.12 that this formula is false! This proves that the integrand in (4.12) in some cases fails to be uniformly integrable, and its limit (in the sense of distributions) may contain a point mass at 0. For this reason, a more direct approach to the proof of Theorem 1.12, similar to the argument sketched above, seems to be problematic and require stronger regularity conditions on Ψ . \square

5 Examples

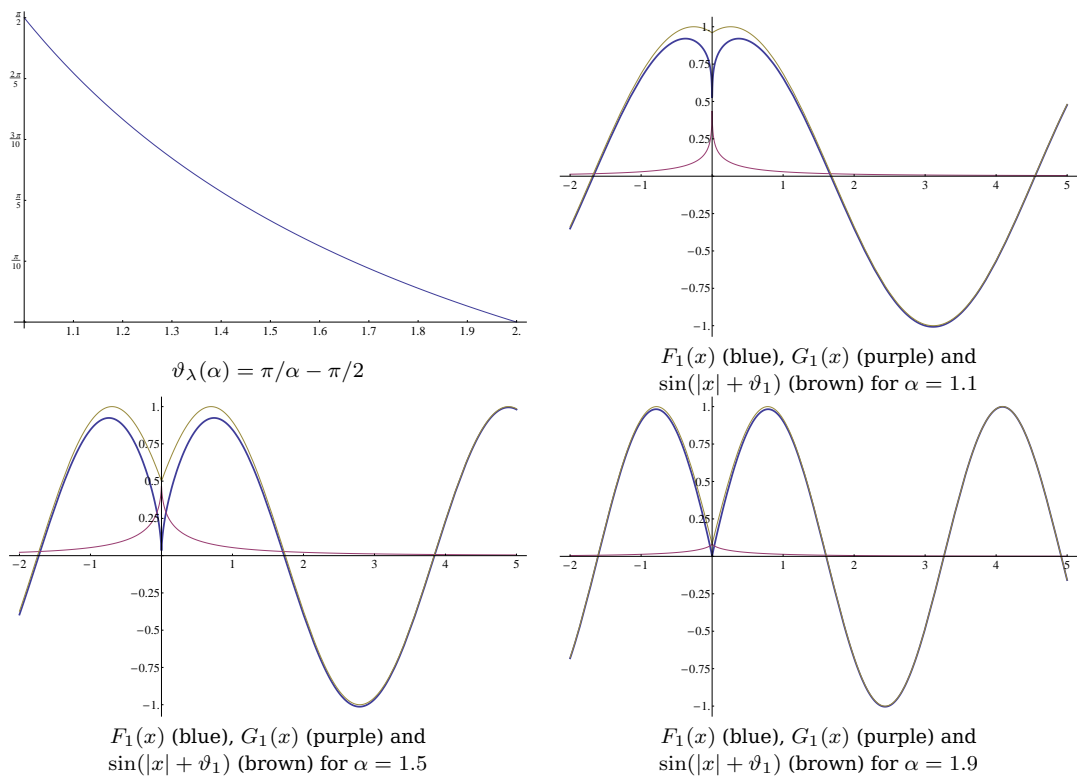
In this section we consider a few examples: processes frequently found in the literature and an irregular case. We focus on spectral theory, that is, on Theorems 1.4, 1.9 and 1.10, and only record formulae given in Corollary 1.11 and 1.12. As mentioned in the introduction, detailed properties of first hitting times to points will be studied in a separate article.

Plots have been prepared using *Mathematica 8.1* at the Wrocław University of Technology. Numerical integration was used for the computation of ϑ_λ and fast Fourier transform for the calculation of $G_\lambda(x)$.

Example 5.1. Let $\Psi(\xi) = |\xi|^\alpha$, $\alpha \in (1, 2]$, be the Lévy-Khintchine exponent of the one-dimensional symmetric α -stable Lévy process. The distribution of the hitting times to points for these processes have been previously studied in [21].

By substituting $\xi = \lambda s$ in the integral over $(0, \lambda)$ and $\xi = \lambda/s$ in the integral over (λ, ∞) , we obtain (see (3.1))

$$K_\lambda = -\frac{1}{\pi} \text{pv} \int_0^\infty \frac{\alpha \lambda^{\alpha-1}}{\lambda^\alpha - \xi^\alpha} d\xi = -\frac{\alpha}{\pi} \int_0^1 \left(\frac{1}{1-s^\alpha} - \frac{1}{s^{2-\alpha}(1-s^\alpha)} \right) ds.$$


 Figure 1: Plots for $\Psi(\xi) = |\xi|^\alpha$.

By series expansion and Fubini,

$$K_\lambda = \frac{\alpha}{\pi} \sum_{n=0}^{\infty} \left(\int_0^1 (s^{\alpha-2} - 1) s^{n\alpha} ds \right) = \frac{\alpha}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{(n+1)\alpha - 1} - \frac{1}{n\alpha + 1} \right).$$

Recall that for $z \in (0, \pi)$,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

Hence, after some rearrangement of the sum,

$$\begin{aligned} K_\lambda &= -\frac{1}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{1/\alpha + n} + \frac{1}{1/\alpha - (n+1)} \right) \\ &= -\frac{1}{\pi} \left(\frac{1}{1/\alpha} + \sum_{k=1}^{\infty} \left(\frac{1}{1/\alpha + k} + \frac{1}{1/\alpha - k} \right) \right) = -\cot \frac{\pi}{\alpha}. \end{aligned}$$

Hence,

$$\vartheta_\lambda = \arctan(K_\lambda) = \frac{\pi}{\alpha} - \frac{\pi}{2}, \quad (5.1)$$

and $\cos \vartheta_\lambda = \sin(\pi/\alpha)$, $\sin(\vartheta_\lambda) = -\cos(\pi/\alpha)$. Furthermore, by (1.11),

$$\begin{aligned} G_\lambda(x) &= \frac{\alpha \lambda^{\alpha-1} \sin(\pi\alpha/2) \sin(\pi/\alpha)}{\pi} \int_0^\infty \frac{\xi^\alpha}{\lambda^{2\alpha} - 2\lambda^\alpha \xi^\alpha \cos(\pi\alpha/2) + \xi^{2\alpha}} e^{-\xi|x|} d\xi \\ &= \frac{\alpha \sin(\pi\alpha) \sin(\pi/\alpha)}{\pi} \int_0^\infty \frac{s^\alpha}{1 - 2s^\alpha \cos(\pi\alpha) + s^{2\alpha}} e^{-\lambda s|x|} ds. \end{aligned}$$

Hence,

$$F_\lambda(x) = \sin\left(|\lambda x| + \frac{\pi}{\alpha} - \frac{\pi}{2}\right) - \frac{\alpha \sin(\pi\alpha/2) \sin(\pi/\alpha)}{\pi} \times \\ \times \int_0^\infty \frac{s^\alpha}{1 - 2s^\alpha \cos(\pi\alpha/2) + s^{2\alpha}} e^{-\lambda s|x|} ds. \quad (5.2)$$

Note that $F_\lambda(x) = F_1(\lambda x)$, which could be deduced a priori using scaling properties of X_t . As expected, in the limiting case $\alpha \rightarrow 1^+$ we have $\vartheta_\lambda \rightarrow \pi/2$ and $G_\lambda(x) \rightarrow 0$ (the Cauchy process never hits 0, hence $F_\lambda(x) = \cos(\lambda x) = \sin(|\lambda x| + \pi/2)$ is the even eigenfunction in $\mathbf{R} \setminus \{0\}$). Also, for $\alpha = 2$, we obtain $\vartheta_\lambda = 0$ and $G_\lambda(x) = 0$ ($F_\lambda(x) = \sin|\lambda x|$ is the even eigenfunction for the Brownian motion in $\mathbf{R} \setminus \{0\}$). See Figure 1 for plots of F_λ .

Since X_t is recurrent, we have $\mathbf{P}_x(\tau_0 = \infty) = 0$. Hence, by Theorem 1.12 and scaling,

$$\mathbf{P}_x(\tau_0 > t) = \frac{\alpha \sin(\pi/\alpha)}{\pi} \int_0^\infty \frac{e^{-t\lambda^\alpha} F_1(\lambda x)}{\lambda} d\lambda.$$

By Corollary 1.11,

$$p_t^{\mathbf{R} \setminus \{0\}}(x, y) = \frac{1}{\pi} \int_0^\infty e^{-t\lambda^\alpha} (F_1(\lambda x) F_1(\lambda y) + \sin(\lambda x) \sin(\lambda y)) d\lambda.$$

These formulae are applicable for numerical computation.

Example 5.2. Let $\alpha \in (0, 2)$, $\beta > 0$, and $\Psi(\xi) = \xi^2 + \beta|\xi|^\alpha$. The corresponding Lévy process X_t is a mixture of Brownian motion and the symmetric α -stable process. Note that X_t is recurrent if and only if $\alpha > 1$. We have

$$\vartheta_\lambda = \arctan\left(-\frac{1}{\pi} \text{pv} \int_0^\infty \frac{2\lambda + \alpha\beta\lambda^{\alpha-1}}{\lambda^2 + \beta\lambda^\alpha - \xi^2 - \beta\xi^\alpha} d\xi\right).$$

This integral does not have a closed form, but it can be proved that $\lim_{\lambda \rightarrow \infty} \vartheta_\lambda = 0$ and $\lim_{\lambda \rightarrow 0^+} \vartheta_\lambda = \min(\pi/\alpha - \pi/2, \pi/2)$. By Theorem 1.9,

$$F_\lambda(x) = \sin(|\lambda x| + \vartheta_\lambda) - \frac{(2\lambda + \alpha\beta\lambda^{\alpha-1}) \cos \vartheta_\lambda}{\pi} \times \\ \times \int_0^\infty \frac{\sin(\alpha\pi/2)\xi^\alpha}{(\lambda^2 + \beta\lambda^\alpha - \xi^2)^2 + \xi^{2\alpha} - 2\cos(\alpha\pi/2)(\lambda^2 + \beta\lambda^\alpha - \xi^2)\xi^\alpha} e^{-\xi|x|} d\xi.$$

Example 5.3. Suppose that $\Psi(\xi) = (\xi^2 + 1)^{\alpha/2} - 1$ is the Lévy-Khintchine exponent of the relativistic α -stable Lévy process, $\alpha \in (1, 2)$, corresponding to unit mass. In this case again

$$\vartheta_\lambda = \arctan\left(-\frac{1}{\pi} \text{pv} \int_0^\infty \frac{\alpha\lambda(\lambda^2 + 1)^{\alpha/2-1}}{(\lambda^2 + 1)^{\alpha/2} - (\xi^2 + 1)^{\alpha/2}} d\xi\right)$$

cannot be evaluated symbolically. However, with some effort, one can show that $\lim_{\lambda \rightarrow 0^+} \vartheta_\lambda = 0$ and $\lim_{\lambda \rightarrow \infty} \vartheta_\lambda = \pi/\alpha - \pi/2$. We also have

$$F_\lambda(x) = \sin(|\lambda x| + \vartheta_\lambda) - \frac{\alpha\lambda(\lambda^2 + 1)^{\alpha/2-1} \cos \vartheta_\lambda}{\pi} \times \\ \times \int_1^\infty \frac{\sin(\alpha\pi/2)(1 - \xi^2)^{\alpha/2}}{(\lambda^2 + 1)^\alpha + (1 - \xi^2)^\alpha - 2\cos(\alpha\pi/2)(\lambda^2 + 1)^{\alpha/2}(1 - \xi^2)^{\alpha/2}} e^{-\xi|x|} d\xi.$$

In particular, $F_\lambda(x) - \sin(|\lambda x| + \vartheta_\lambda)$ decays exponentially fast as $|x| \rightarrow \infty$.

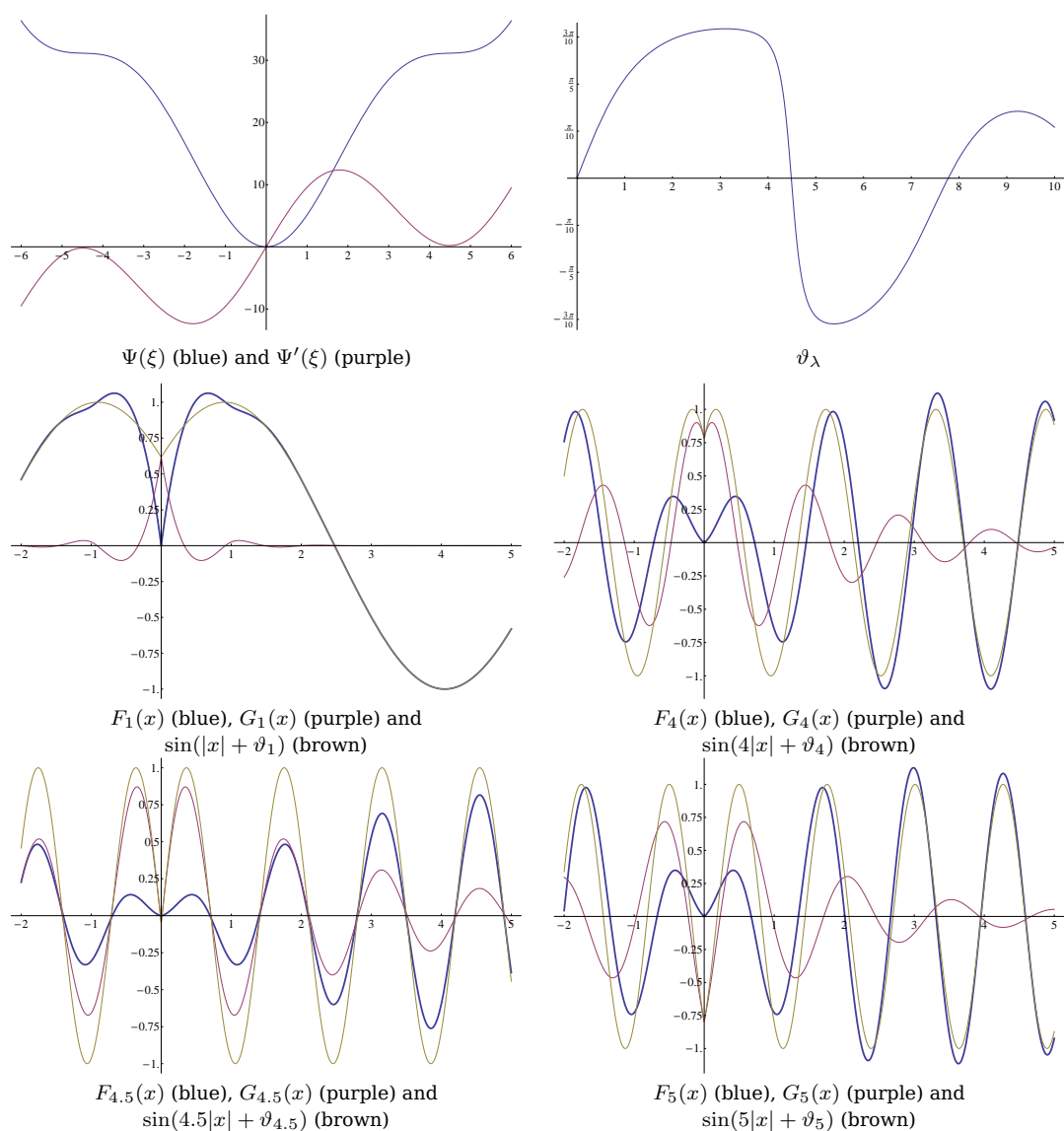


Figure 2: Plots for $\Psi(\xi) = \xi^2 + 9(1 - \cos \xi)$. Note the differences between $\lambda = 1$ ($\vartheta_\lambda > 0$, G_λ attains global maximum at 0), $\lambda = 4$ ($\vartheta_\lambda > 0$, G_λ has a positive local minimum at 0), $\lambda = 4.5$ (ϑ_λ and $G_\lambda(0)$ close to 0, slow convergence of $F_\lambda(x)$ to $\sin(|\lambda x| + \vartheta_\lambda)$ as $|x| \rightarrow \infty$) and $\lambda = 5$ ($\vartheta_\lambda < 0$, G_λ has a negative global minimum at 0).

Example 5.4. Let $\Psi(\xi) = \xi^2 + 9(1 - \cos \xi)$. The corresponding process X_t is the sum of the Brownian motion (with variance $2t$) and an independent compound Poisson process with jumps ± 1 occurring at rate 9. It can be easily checked that $\Psi'(\xi) > 0$ for $\xi > 0$, and therefore Theorem 1.4 applies. However, ϑ_λ is negative for some $\lambda > 0$, and $G_\lambda(x)$ is a signed function, see Figure 2.

Since $\psi(\xi) = \Psi(\sqrt{\xi}) = \xi + 9(1 - \cos \sqrt{\xi})$ is not concave, Theorem 1.12 (or Theorem 1.1) cannot be applied to X_t .

Example 5.5. Consider the truncated symmetric α -stable Lévy process X_t , that is, a pure-jump Lévy process X_t with Lévy measure $c|x|^{-1-\alpha}\mathbf{1}_{(-1,1)}(x)$ for some $c > 0$. Then,

$$\psi(\xi) = \Psi(\sqrt{\xi}) = c \int_{-1}^1 \frac{1 - \cos(\sqrt{\xi}x)}{|x|^{1+\alpha}} dx = 2c\xi^{\alpha/2} \int_0^{\sqrt{\xi}} \frac{1 - \cos s}{s^{1+\alpha}} ds.$$

Clearly, $\psi'(\xi) > 0$ for all $\xi > 0$. We claim that $\psi''(\xi) \leq 0$ for all $\xi > 0$. We have

$$\frac{2\psi''(\xi)}{c} = \frac{\sin \sqrt{\xi}}{\xi^{3/2}} - (2 - \alpha) \frac{1 - \cos \sqrt{\xi}}{\xi^2} - \frac{\alpha(2 - \alpha)}{\xi^{2-\alpha/2}} \int_0^{\sqrt{\xi}} \frac{1 - \cos s}{s^{1+\alpha}} ds. \quad (5.3)$$

Since $\sin s \leq s - s^3/6 + s^5/120$ and $1 - \cos s \geq s^2/2 - s^4/24$ for $s > 0$, we have

$$\frac{2\psi''(\xi)}{c} \leq \frac{1}{\xi} - \frac{1}{6} + \frac{\xi}{120} - \frac{2 - \alpha}{2\xi} + \frac{2 - \alpha}{24} - \frac{\alpha}{2\xi} + \frac{\alpha(2 - \alpha)}{24(4 - \alpha)} = -\frac{1}{3(4 - \alpha)} + \frac{\xi}{120},$$

and hence $\psi''(\xi) \leq 0$ for $\xi \leq 40/(4 - \alpha)$. When $\xi > 40/(4 - \alpha)$, then in particular $\xi > \sqrt{12}$, and therefore, by (5.3),

$$\begin{aligned} \frac{2\psi''(\xi)}{c} &\leq \frac{1}{\xi^{3/2}} - \frac{\alpha(2 - \alpha)}{\xi^{2-\alpha/2}} \int_0^{\sqrt{12}} \frac{s^2/2 - s^4/24}{s^{1+\alpha}} ds \\ &\leq \frac{1}{\xi^{3/2}} \left(1 - \frac{12^{1-\alpha/2}\alpha}{(4 - \alpha)} \xi^{(\alpha-1)/2} \right) \leq \frac{1}{\xi^{3/2}} (1 - \xi^{(\alpha-1)/2}) \leq 0. \end{aligned}$$

Our claim is proved.

In particular, we may apply Theorems 1.4, 1.10 and 1.12 to truncated stable processes. Due to a rather involved description of Ψ , the computations are significantly harder in this case. See Figure 3 for plots of ϑ_λ and F_λ .

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Symmetric Lévy processes killed upon hitting 0

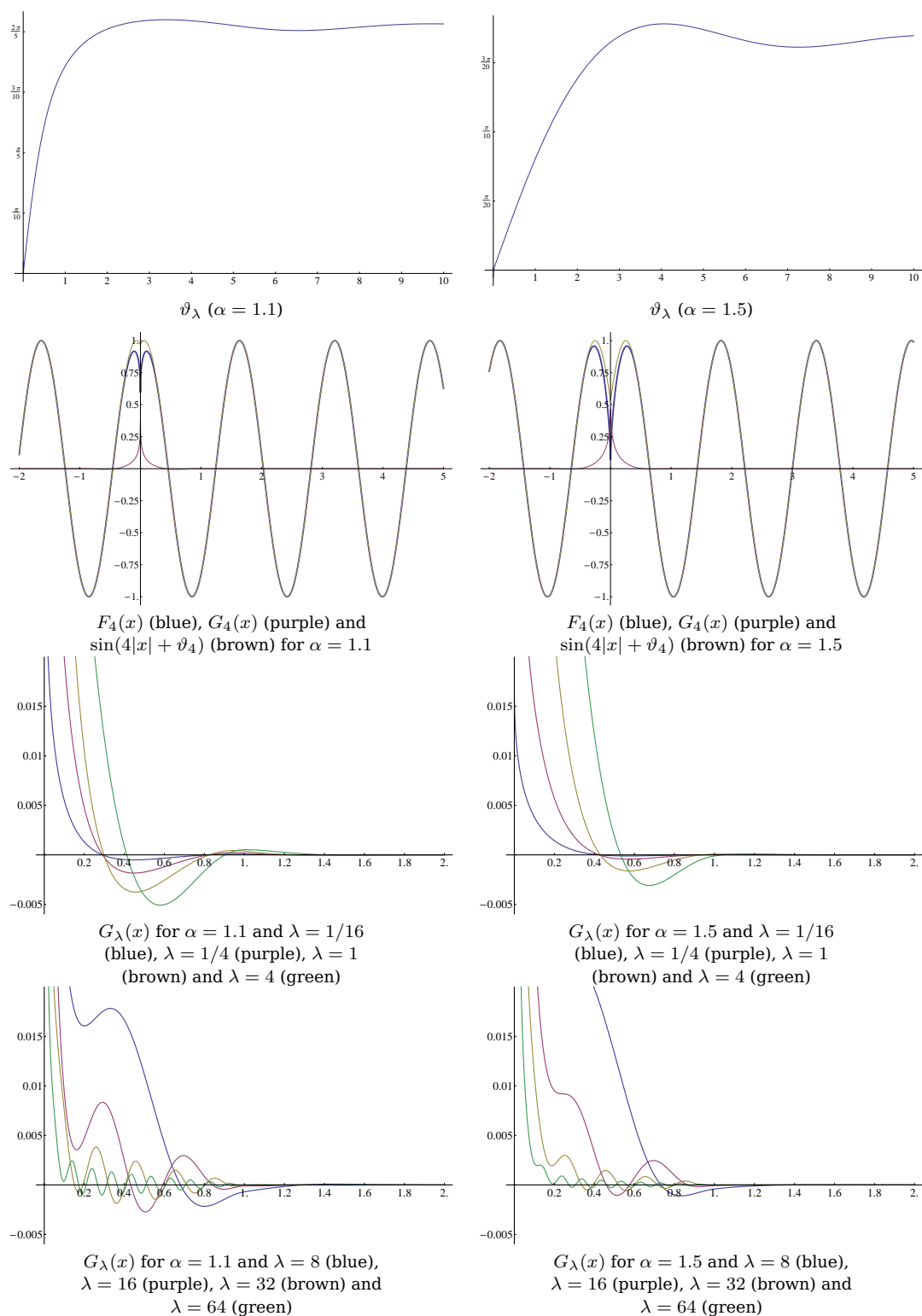


Figure 3: Plots for the truncated α -stable process for $\alpha = 1.1$ and $\alpha = 1.5$. Note that $|G_\lambda(x)| \leq G_\lambda(0)$, for all $\lambda > 0$, $x \in \mathbf{R}$, but $G_\lambda(x)$ fails to be everywhere nonnegative.

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