Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 30, 1–21.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-1940

Internal aggregation models on comb lattices

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Abstract

The two-dimensional comb lattice $\mathcal{C}_2$ is a natural spanning tree of the Euclidean lattice $\mathbb{Z}^2$. We study three related cluster growth models on $\mathcal{C}_2$: *internal diffusion limited aggregation* (IDLA), in which random walkers move on the vertices of $\mathcal{C}_2$ until reaching an unoccupied site where they stop; *rotor-router aggregation* in which particles perform deterministic walks, and stop when reaching a site previously unoccupied; and the *divisible sandpile model* where at each vertex there is a pile of sand, for which, at each step, the mass exceeding 1 is distributed equally among the neighbours. We describe the shape of the divisible sandpile cluster on $\mathcal{C}_2$, which is then used to give inner bounds for IDLA and rotor-router aggregation.

Keywords: growth model; comb lattice; internal diffusion limited aggregation; rotor-router aggregation; divisible sandpile; asymptotic shape; random walk; rotor-router walk..
AMS MSC 2010: 60J10; 05C81.
Submitted to EJP on November 22, 2011, final version accepted on April 10, 2012.

1 Introduction

Let G be an infinite, locally finite and connected graph with a chosen origin $o \in G$. *Internal diffusion limited aggregation (IDLA)* is a random walk-based growth model, which was introduced by Diaconis and Fulton [7]. In IDLA n particles start at the origin of G, and each particle performs a simple random walk until it reaches a vertex which was not previously occupied. There the particle stops, and from now on occupies this vertex, and a new particle starts its journey at the origin. The resulting random set of n occupied sites in G is called the *IDLA cluster*, and will be denoted by A_n.

IDLA has received increased attention in the last years. In 1992, Lawler, Bramson and Griffeath [19] showed that for simple random walk on $\mathbb{Z}^d$, with $d \geq 2$, the limiting shape of IDLA, when properly rescaled, is almost surely an Euclidean ball of radius 1. In 1995, Lawler [18] refined this result by giving estimates on the fluctuations. Recently several improvements have been obtained. Asselah and Gaudillière [2, 3] proved an upper bound of order $\log(n)$ for the inner fluctuation δ_I and of order $\log^2(n)$ for the outer fluctuation δ_O in all dimensions $d \geq 2$. In [4] they improve the upper bound on the inner fluctuation to $\sqrt{\log(\text{radius})}$, for $d \geq 3$. Independently, and by different methods,

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Jerison, Levine and Sheffield [13, 12], proved also that both δ_I and δ_O are of order $\log(n)$ for IDLA on $\mathbb{Z}^2$ and of order $\sqrt{\log(\text{radius})}$ for $d \geq 3$.

Rotor-router aggregation is a deterministic version of IDLA, where particles perform *rotor-router walks*, which are deterministic analogues to random walks. They have been first introduced into the physics literature under the name *Eulerian walks* by Priezzhev, D.Dhar et al [23]. At each vertex of the graph G, we have an arrow (rotor) pointing to one of the neighbours of the vertex. A particle performing a *rotor-router walk* first changes the rotor at its current position to point to the next neighbour, in a fixed order chosen at the beginning, and then moves to the neighbour the rotor is now pointing at. In rotor-router aggregation each particle performs a rotor-router walk until it reaches an unoccupied site, where it stops. Then a new particle starts at the origin, without resetting the configuration of rotors. The resulting deterministic set R_n of n occupied sites is called the *rotor-router cluster*.

Rotor-router aggregation on the Euclidean lattice $\mathbb{Z}^d$ has been studied by Levine and Peres [20], who showed that the cluster R_n is a ball in the Euclidean distance. On the homogeneous tree Landau and Levine [16] proved that, provided the start configuration of rotors is acyclic, the rotor-router cluster forms a perfect ball with respect to the graph metric, whenever it has the right amount of particles. Kager and Levine [14] studied the shape of the rotor-router cluster on a modified two dimensional lattice, which they call the *layered square lattice*. In each of the known examples the limiting shape of rotor-router aggregation is the same as the one for IDLA, but with much smaller fluctuations compared to IDLA.

In order to prove inner bounds for the above models, we use a third growth model, the so-called *divisible sandpile*, which has been introduced by Levine and Peres [20] as a tool for studying internal growth models on $\mathbb{Z}^d$. In the divisible sandpile model each vertex can have an arbitrary amount of mass. If a vertex has mass at least 1, it is called *unstable* and it can *topple* by distributing the mass exceeding 1 equally among its neighbours. At each timestep a vertex is chosen and toppled if it is unstable. Provided every vertex is chosen infinitely often, the masses converge to a limiting distribution ≤ 1. The set of vertices with limit mass equal to 1 is called the *divisible sandpile cluster*. If we start with a mass of n concentrated at the origin, the corresponding sandpile cluster will be denoted by S_n.

All three growth models have very similar behaviour. This was first noticed by Levine and Peres [20, 21] for the case when the state space is an Euclidean lattice. Computer simulations suggest that the connection between the three growth models holds in wide generality, but only partial results are available for other state spaces. All these three models have the so-called *abelian property*, which makes them amenable to rigorous analysis. In the case of IDLA and the rotor-router model this means that, if we let several particles run at the same time, instead of one after another, it is irrelevant for the end result in which order the particles make their moves. In the case of the divisible sandpile model, it means that the limiting distribution is independent of the order in which vertices topple.

The aim of this paper is to study the three aggregation models described above on the *comb lattice* $\mathcal{C}_2$, which is the spanning tree of the two-dimensional Euclidean lattice $\mathbb{Z}^2$, obtained by removing all horizontal edges of $\mathbb{Z}^2$ except the ones on the x-axis. The graph $\mathcal{C}_2$ can also be constructed from a two-sided infinite path $\mathbb{Z}$ (the "*backbone*" of the comb), by attaching copies of $\mathbb{Z}$ (the "*teeth*") at every vertex of the backbone.

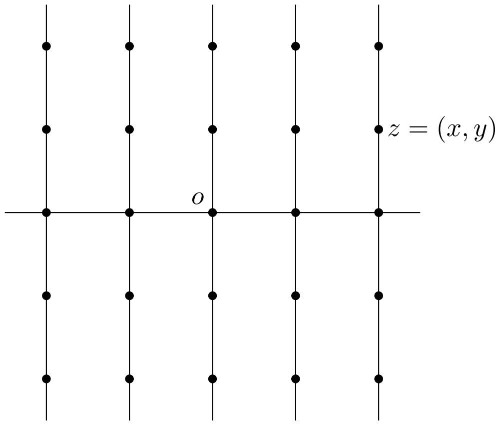

Figure 1: The comb $\mathcal{C}_2$

We use the standard embedding of the comb into $\mathbb{Z}^2$, and use Cartesian coordinates $z=(x,y)\in\mathbb{Z}^2$ to denote vertices of $\mathcal{C}_2$. The vertex $o=(0,0)$ will be the *root* or the *origin*; see Figure 1. For functions g on the vertex set of $\mathcal{C}_2$ we will often write $g(x,y)$ instead of $g(z)$, when $z=(x,y)$.

While $\mathcal{C}_2$ is a very simple graph, it has some remarkable properties. For example, the so-called *Einstein relation* between the spectral-, walk- and fractal-dimension is violated on the comb, see Bertacchi [5]. Peres and Krishnapur [15] showed that on $\mathcal{C}_2$ and other similar recurrent graphs two independent simple random walks meet only finitely often. Random walks on $\mathcal{C}_2$ have been studied by various authors, the first being Havlin and Weiss [24] and Gerl [9].

The paper is organized as follows. In Section 2 we introduce some notations and basic facts which will be used through the rest of the work. Section 3 is dedicated to the study of the divisible sandpile on the comb $\mathcal{C}_2$. We show in Theorem 3.5 that the sandpile cluster S_n on $\mathcal{C}_2$ has up to constant fluctuations the shape

$$\mathcal{B}_n=\left\{(x,y)\in\mathcal{C}_2:\ \frac{|x|}{k}+\left(\frac{|y|}{l}\right)^{1/2}\leq n^{1/3}\right\} \tag{1.1}$$

where

$$k=\left(\frac{3}{2}\right)^{2/3},\quad l=\frac{1}{2}\left(\frac{3}{2}\right)^{1/3}.$$

Section 4 deals with IDLA on $\mathcal{C}_2$. Using the results obtained for the sandpile model, we prove an inner bound for IDLA, which is of the type (1.1). Finally, in Section 5, we give an inner estimate for the rotor-router model on $\mathcal{C}_2$ which is weaker than the result obtained for IDLA. For a fixed initial configuration of rotors the exact shape of the rotor-router cluster on the comb has been obtained by the authors in [11] using a purely combinatorial approach.

2 Preliminaries

Let $\big(G,E(G)\big)$ be an infinite, undirected and connected graph, with vertex set G, equipped with a symmetric *adjacency relation* $\sim$, which defines the set of edges $E(G)$ (as a subset of $G\times G$). We write (x,y) for the edge between the pair of neighbours x,y. In order to simplify the notation, instead of writing $\big(G,E(G)\big)$ for a graph, we shall write only G, and it will be clear from the context whether we are considering edges or vertices. Let $o\in G$ be some fixed reference vertex called the *origin*. For $x,y\in G$, let

$\mathrm{d}(x,y)$ be the length of the shortest path from x to y. Also, write $d(x)$ for the *degree* of x, i.e, the number of neighbours of x. For a subset $A \subset G$ we denote by

$$\partial A = \big\{x \in G \setminus A : \; \exists y \in A \text{ with } x \sim y\big\} \quad \text{and} \quad \partial_I A = \big\{x \in A : \; \exists y \notin A \text{ with } x \sim y\big\}$$

the *(outer) boundary* respectively the *inner boundary* of A.

Let $P = \big(p(x,y)\big)_{x,y\in G}$ be the one-step transition probabilities of the simple random walk on G, i.e., $p(x,y) = 1/d(x)$ if $y \sim x$ and 0 otherwise. We write X_t for the position of the random walker at the discrete timestep t. Probabilities will be written as $\mathbb{P}$, in particular $\mathbb{P}_x$ denotes the probability of a random walk which starts at $x \in G$. Similarly $\mathbb{E}$ and $\mathbb{E}_x$ will denote expectations using the same convention. For $y, z \in G$ the *Green function* is defined as

$$G(y,z) = \mathbb{E}_y\Big[\sum_{t=0}^{\infty} \mathbf{1}_{\{X_t = z\}}\Big],$$

and represents the expected number of visits to z of the random walk X_t started at y. For a subset $A \subset G$, write G_A for the Green function of the random walk stopped upon leaving the set A. That is, if $\tau = \min\{t \geq 0 : X_t \notin A\}$, then

$$G_A(x,y) = \mathbb{E}_x\Big[\sum_{t=0}^{\tau-1} \mathbb{1}_{\{X_t = y\}}\Big].$$

For a function $f : G \to \mathbb{R}$, its *Laplace operator* Δf is defined as

$$\Delta f(x) = \frac{1}{d(x)} \sum_{y\sim x} \big(f(y) - f(x)\big).$$

A function $f : G \to \mathbb{R}$ is called *superharmonic* on a set $A \subset G$ if $\Delta f \leq 0$, and *harmonic* if $\Delta f = 0$, for all $x \in A$. For a function $g : G \to \mathbb{R}$, define its *least superharmonic majorant* as

$$s(x) = \inf\big\{f(x) : \; f \text{ superharmonic }, \; f \geq g\big\}.$$

Remark that the function s is itself superharmonic on G. The following is widely known.

Lemma 2.1 (Minimum principle). *If f is a superharmonic function on G and there exists $x \in G$ such that $f(x) = \min_G f$, then f is constant.*

3 Divisible Sandpile

Let $\mathcal{C}_2$ be the comb as in Figure 1, and let μ_0 be an *initial mass distribution* on $\mathcal{C}_2$, i.e., a function $\mu_0 : \mathcal{C}_2 \to \mathbb{R}_+$ with finite support. The *divisible sandpile* is a sequence $(\mu_k)_{k\geq 0}$ of mass distributions, which are created according to the following rule. At each time step k, choose a vertex $x \in \mathcal{C}_2$. If $\mu_k(x) \geq 1$, the pile of sand at x is unstable and topples, which means that x keeps mass 1 for itself and the remaining mass $\mu_k(x) - 1$ is distributed equally among the neighbours y of x, that is, according to the transition probabilities $p(x,y)$ of the simple random walk on $\mathcal{C}_2$. Given a mass distribution μ_k at time k and a vertex $x \in \mathcal{C}_2$, the *toppling operator* can be defined as

$$T_x \mu_k(y) = \mu_k(y) + \alpha_k(x) d(y) \Delta \delta'_x(y), \text{ for } y \in \mathcal{C}_2,$$

where $\delta'_x(y) = \frac{\delta_x(y)}{d(y)}$ and $\alpha_k(y) = \max\{\mu_k(y) - 1, 0\}$. Let $(x_k)_{k\geq 0}$ be a sequence of vertices in $\mathcal{C}_2$ called the *toppling sequence*, which contains each vertex of $\mathcal{C}_2$ infinitely often. Then the mass distribution of the sandpile after k steps is defined as

$$\mu_{k+1} = T_{x_k} \mu_k = T_{x_k} \cdots T_{x_0} \mu_0.$$

Hence, $\mu_{k+1}(y)$ is the amount of mass present at y after toppling the sites $x_0,\dots,x_k$ in succession. One of the tools that will be used throughout this work in various incarnations is the so-called *odometer function*, which was introduced by Levine and Peres [20].

Definition 3.1. *The* odometer function v_k *is defined as*

$$v_k(y)=\sum_{j\le k:\, x_j=y}\mu_j(y)-\mu_{j+1}(y)=\sum_{j\le k:\, x_j=y}\alpha_j(y),\ y\in\mathcal{C}_2,$$

and represents the total mass emitted from y during the first k topplings.

For simple random walks on $\mathcal{C}_2$ it is easier to work with the *normalized odometer function* $u_k(x)=\frac{v_k(x)}{d(x)}$. Lemma 3.1 of Levine and Peres [20] can be easily adapted to our case, in order to show that, as k goes to infinity, μ_k and u_k converge to limit functions μ and u respectively. Define

$$S=\{x\in\mathcal{C}_2:\ \mu(x)=1\}.$$

The set S is called the *sandpile cluster* with initial mass distribution μ_0. The limit functions μ and u satisfy

$$\mu(x)=\mu_0(x)+d(x)\Delta u(x),\ \text{for all } x\in S, \tag{3.1}$$

and

$$\mu(x)\le 1,\ \text{for all } x\in\mathcal{C}_2. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\Delta u(x)=\frac{1}{d(x)}\big(1-\mu_0(x)\big)\quad\text{for all } x\in S, \tag{3.3}$$

and $u(x)=0$, if $x\notin S$. The following result provides a method for solving this free boundary problem. For a proof, see once again Levine and Peres [20, Lemma 3.2].

Lemma 3.2. *Consider a function $\gamma:\mathcal{C}_2\to\mathbb{R}$ with*

$$\Delta\gamma(x)=\frac{1}{d(x)}\big(1-\mu_0(x)\big),\quad \textit{for all } x\in\mathcal{C}_2. \tag{3.4}$$

Then the normalized odometer function u of the sandpile satisfies $u=\gamma+s$, where s is the least superharmonic majorant of $-\gamma$.

Lemma 3.2 gives a representation of the odometer function which is independent of the toppling sequence.

Remark 3.3 (Abelian property)**.** *The limit u of the normalized odometer function and the sandpile cluster S are independent of the toppling sequence $(x_k)_{k\ge 0}$.*

3.1 Divisible Sandpile on the Comb

With the help of Lemma 3.2, we shall next describe the limit shape of the sandpile cluster on the two-dimensional comb $\mathcal{C}_2$. Consider an initial mass distribution μ_0 concentrated at the origin o, that is $\mu_0=n\cdot\delta_o$, and denote by

$$S_n=\{z\in\mathcal{C}_2:\mu(z)=1\}$$

the sandpile cluster, and by u_n the limit of the normalized odometer function for this choice of initial distribution. We use another simple fact about u_n; for a proof see [20, Lemma 3.4].

Lemma 3.4. *If $x \in S_n \setminus \{o\}$ and $y \sim x$ with $\mathrm{d}(o,y) < \mathrm{d}(o,x)$, then $u_n(y) \geq u_n(x) + 1$.*

By (3.3), the normalized odometer function satisfies

$$\Delta u_n(z) = \frac{1}{d(z)}\big(1 - n \cdot \delta_o(z)\big), \text{ for } z \in S_n. \tag{3.5}$$

The odometer function u_n can be reduced to odometer functions of suitable divisible sandpiles on $\mathbb{Z}$, which are easy to compute. Let $\tilde{u}_n$ be the normalized odometer function of the divisible sandpile on $\mathbb{Z}$, with initial mass distribution $\tilde{\mu}_0$ concentrated at 0, that is, $\tilde{\mu}_0 = n \cdot \delta_0$. By Remark 3.3 it is clear that the sandpile cluster $\tilde{S}_n$ on $\mathbb{Z}$ in this case is a symmetric interval around the origin 0. In order to compute $\tilde{u}_n$, by Lemma 3.2, we need to construct a function $\tilde{\gamma}_n : \mathbb{Z} \to \mathbb{R}$ with Laplacian given in (3.4). It is easy to check that $\tilde{\gamma}_n$ defined by

$$\tilde{\gamma}_n(y) = \frac{1}{2}\left(|y| - \frac{n}{2}\right)^2, \tag{3.6}$$

satisfies the required property. Since $\tilde{\gamma}_n$ is nonnegative, the constant function 0 is a superharmonic majorant of $-\tilde{\gamma}_n$. Hence, by Lemma 3.2, we have $\tilde{u}_n \leq \tilde{\gamma}_n$. Now, consider $\gamma_n : \mathcal{C}_2 \to \mathbb{R}$ with

$$\gamma_n(x,y) = \tilde{\gamma}_{n_x}(y), \text{ for } (x,y) \in \mathcal{C}_2, \tag{3.7}$$

where $n_x \in \mathbb{R}$ for all $x \in \mathbb{Z}$. The quantities n_x can be interpreted as the total amount of mass that ends up in the copy of $\mathbb{Z}$ that is attached to the vertex $(x,0)$. Then γ_n satisfies (3.4) if and only if

$$n_x = n \cdot \mathbb{1}_{\{x=0\}} + \tilde{\gamma}_{n_{x-1}}(0) - 2\tilde{\gamma}_{n_x}(0) + \tilde{\gamma}_{n_{x+1}}(0) \tag{3.8}$$

holds for all $x \in \mathbb{Z}$. From (3.6) and (3.8), and using the fact that $n_x = n_{-x}$ by symmetry, we get the following recursion for the numbers n_x

$$n_0 = n + \frac{1}{4}n_1^2 - \frac{1}{4}n_0^2, \tag{3.9}$$

$$n_x = \frac{1}{8}n_{x-1}^2 - \frac{1}{4}n_x^2 + \frac{1}{8}n_{x+1}^2, \text{ for } x > 0. \tag{3.10}$$

Equation (3.10) has strictly positive solutions as quadratic polynomials of the form

$$n_x = \frac{2}{3}x^2 - t \cdot x + \frac{9t^2 + 4}{24}, \text{ with } t \in \mathbb{R}. \tag{3.11}$$

By the initial condition (3.9), the parameter t satisfies the equation

$$n = \frac{3}{16}t^3 + \frac{5}{12}t,$$

which has one real root given by

$$t = T(n) - \frac{20}{27}T(n)^{-1}, \tag{3.12}$$

with $T(n) = \left(\frac{8\sqrt{3}}{243}\sqrt{2187n^2 + 125} + \frac{8}{3}n\right)^{\frac{1}{3}}$. By a series expansion around $n = \infty$, one obtains

$$t = 2\left(\frac{2}{3}\right)^{1/3} n^{1/3} + \mathcal{O}(1). \tag{3.13}$$

Therefore, the function $\gamma_n(x,y) = \tilde{\gamma}_{n_x}(y)$ with n_x defined by (3.11) and (3.12) satisfies the conditions of Lemma 3.2, with $\mu_0 = n \cdot \delta_o$. See Figure 2 for a graphical representation of γ_n. We are now ready to prove the limit shape for the divisible sandpile on $\mathcal{C}_2$.

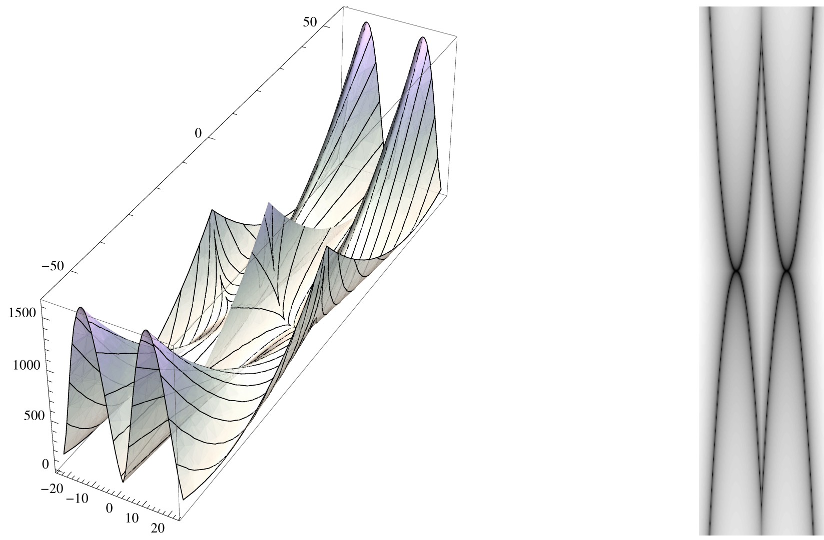

Figure 2: Two plots of γ_n for $n = 1000$. The graphic on the left is superimposed with contour lines representing the sets $\mathcal{B}_n$ for various values of n. In the density plot on the right, dark areas represent small values. By construction, the finite area which is surrounded by the local minima of γ_n coincides with the region S_n covered by the sandpile.

Theorem 3.5. *Let S_n be the divisible sandpile cluster on $\mathcal{C}_2$, with $\mu_0 = n \cdot \delta_o$. Then there exists a constant $c \geq 0$ such that, for $n \geq n_0$:*

$$\mathcal{B}_{n-c} \subset S_n \subset \mathcal{B}_{n+c},$$

where

$$\mathcal{B}_n = \left\{ (x,y) \in \mathcal{C}_2 : \ \frac{|x|}{k} + \left(\frac{|y|}{l} \right)^{1/2} \leq n^{1/3} \right\} \tag{3.14}$$

and

$$k = \left(\frac{3}{2}\right)^{2/3}, \qquad l = \frac{1}{2}\left(\frac{3}{2}\right)^{1/3}.$$

Proof. The upper bound $S_n \subset \mathcal{B}_{n+c}$: The mass distributions n_x are nonnegative for all x, therefore γ_n is nonnegative, and this implies that the constant function 0 is a superharmonic majorant of $-\gamma_n$. Thus, by Lemma 3.2, γ_n is an upper bound of the odometer function u_n. Moreover, Lemma 3.4 implies that u_n decreases by a fixed amount on the sandpile cluster S_n when we move away from the origin. Therefore, in order to get an upper bound for S_n, it suffices to calculate the minima of γ_n along each infinite ray starting at $o = (0,0)$. By the symmetry of $\mathcal{C}_2$, it is sufficient to consider only the first quadrant.

Consider the rays which lies entirely on the positive x-axis. We have $\gamma_n(x,0) = \frac{1}{8}n_x^2$. The minimum of this function is attained at $x_{\min} = \frac{3}{4}t$, with t given in (3.12). Using the series expansion (3.13) of t we get

$$x_{\min} = kn^{1/3} + \mathcal{O}(1), \quad \text{with } k = \left(\frac{3}{2}\right)^{2/3}, \tag{3.15}$$

which is also an upper bound of S_n on the x-axis by Lemma 3.4, since $\gamma_n(\lfloor x_{\min}\rfloor, 0)$ is bounded by a constant which is independent of n, and smaller than $1/10$.

To calculate the extent of the sandpile cluster on the "teeth", we need to compute the minima of γ_n in the y-direction. On each "tooth" of the comb, γ_n is a quadratic polynomial which attains its minimum at $y_{\min}(x) = \frac{n_x}{2}$. Moreover, $\gamma\big(x, \lfloor y_{\min}(x)\rfloor\big) \leq 1/2$. Using (3.11) and a series expansion around infinity we get

$$y_{\min}(x) = l\left(n^{1/3} - \frac{x}{k}\right)^2 + \frac{2}{3}x - \frac{1}{2l}n^{1/3} - \frac{7l}{9k}xn^{-1/3} + \mathcal{O}(1),$$

where $l = \frac{1}{2}\left(\frac{3}{2}\right)^{1/3}$. By the estimate in the x-direction we know that $(x,y) \in S_n$ only if $x \leq x_{\min}$. Thus, using the expansion (3.15) for $x_{\min}$ we obtain $(x,y) \in S_n$ if $|x| \leq kn^{1/3} + \mathcal{O}(1)$ and $|y| \leq l\left(n^{1/3} - \frac{x}{k}\right)^2 + \mathcal{O}(1)$, for $n \geq n_0$. This proves the upper bound $S_n \subset \mathcal{B}_{n+c}$.

The lower bound $\mathcal{B}_{n-c} \subset S_n$: On each infinite ray the minimum of $\gamma_n(z)$ is smaller than a constant $a > 0$, independent of n. Also, from the upper bound, we have $u_n(z) = 0$ for all $z \in \partial\mathcal{B}_{n+c}$. Hence $u_n(z) - \gamma_n(z) \geq -a$ for all $z \in \partial\mathcal{B}_{n+c}$. Since the function $u_n - \gamma_n$ is superharmonic, by the *Minimum Principle*, it attains its minimum on the boundary and the inequality $u_n(z) - \gamma_n(z) \geq -a$ holds for all $z \in \mathcal{B}_{n+c}$. Thus $\gamma_n - a$ is also a lower bound of the odometer function on $\mathcal{B}_{n+c}$, which gives the inner estimate $\mathcal{B}_{n-c} \subset S_n$, for some constant c. □

The next corollary follows directly from the proof of the previous theorem.

Corollary 3.6. *Let u_n be the normalized odometer function of the divisible sandpile on $\mathcal{C}_2$, with initial mass distribution $\mu_0 = n \cdot \delta_o$, and $\mathcal{B}_n \subset \mathcal{C}_2$ defined as in* (3.14). *There exists a constant $0 < a < 2$, such that, for all $n > n_0$ and all $z \in \mathcal{C}_2$*

$$\big(\gamma_n(z) - a\big)\mathbb{1}_{\mathcal{B}_n} \leq u_n(z) \leq \gamma_n(z).$$

4 Internal Diffusion Limited Aggregation

Let $\left(X_t^i\right)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed simple random walks on the comb $\mathcal{C}_2$, with common starting point $X_0^i = o$. Then X_t^i represents the position of the i-th particle at time t.

Definition 4.1. Internal diffusion limited aggregation (IDLA) *is a stochastic process of increasing subsets $\left(A_i\right)_{i\in\mathbb{N}}$ of $\mathcal{C}_2$, which are defined recursively as $A_1 = \{o\}$ and for $i \geq 2$*

$$\mathbb{P}\big[A_i = \mathbf{A} \cup \{x\} \,|\, A_{i-1} = \mathbf{A}\big] = \mathbb{P}\big[X^i_{\sigma^i} = x\big],$$

where $\sigma^i = \inf\{t \geq 0 :\ X_t^i \notin A_{i-1}\}$ is the first exit time of the random walk X_t^i from A_{i-1}.

The IDLA cluster is build up one site at a time. That is, suppose that we already have the cluster A_{i-1} after $i-1$ particles stopped, and we want to get A_i. For this, the i-th particle X_t^i starts at o, and evolves as long it stays inside the IDLA-cluster A_{i-1}. When X_t^i leaves A_{i-1} for the first time, it stops, and the point outside of the cluster that is visited by X_t^i is added to the new cluster A_i. The set A_i is called the *IDLA-cluster* of i particles. Figure 3 shows IDLA clusters on $\mathcal{C}_2$ with $100,\ 500$ and 1000 particles.

We will prove the following shape theorem for IDLA on $\mathcal{C}_2$.

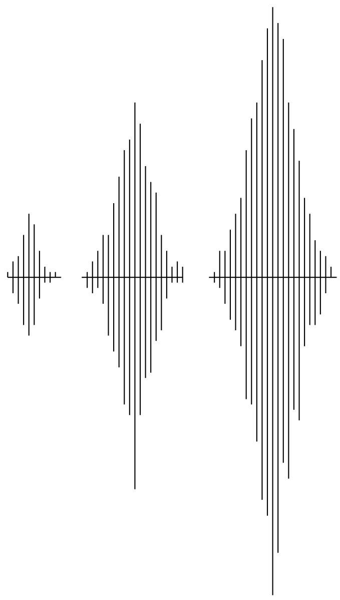

Figure 3: IDLA cluster

Theorem 4.2. *Let A_n be the IDLA cluster of n particles on $\mathcal{C}_2$. Then, for all $\varepsilon > 0$, we have with probability 1*

$$\mathcal{B}_{n(1-\varepsilon)} \subset A_n, \text{ for all sufficiently large } n, \tag{4.1}$$

where

$$\mathcal{B}_n = \left\{ (x,y) \in \mathcal{C}_2 : \frac{|x|}{k} + \left(\frac{|y|}{l}\right)^{1/2} \leq n^{1/3} \right\}$$

and

$$k = \left(\frac{3}{2}\right)^{2/3}, \quad l = \frac{1}{2}\left(\frac{3}{2}\right)^{1/3}.$$

The set $\mathcal{B}_n$ is the same as the limit shape of the divisible sandpile from Theorem 3.5. The proof of Theorem 4.2 uses ideas of Lawler, Bramson and Griffeath [19] and of Levine and Peres [21]. Following [19], we introduce the stopping times

$$\tau_n^i = \min\{t \geq 0 : X_t^i \notin \mathcal{B}_n\} \quad \text{and } \tau_z^i = \min\{t \geq 0 : X_t^i = z\},$$

for $z \in \mathcal{B}_n$. Consider the probability that a fixed vertex $z \in \mathcal{B}_n$ does not belong to the IDLA cluster A_n, which can be written in terms of the stopping times defined above as

$$\mathbb{P}[z \notin A_n] = \mathbb{P}\Big[\bigcap_{i \leq n} \sigma^i < \tau_z^i\Big].$$

Hence, by the Borel-Cantelli Lemma, convergence of the series

$$\sum_{n \geq n_0} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n], \tag{4.2}$$

is a sufficient condition for Theorem 4.2. Fix now n and $z \in \mathcal{B}_n$ and consider the random variables

$$N = \sum_{i=1}^n \mathbb{1}_{\{\tau_z^i < \sigma^i\}}, \quad M = \sum_{i=1}^n \mathbb{1}_{\{\tau_z^i < \tau_n^i\}} \quad \text{and} \quad L = \sum_{i=1}^n \mathbb{1}_{\{\sigma^i \leq \tau_z^i < \tau_n^i\}}$$

Then N represents the number of particles that visit z before leaving the cluster. The variable M counts the number of particles that visit z before leaving $\mathcal{B}_n$, and L is the number of particles that visit z after leaving the cluster A_i but before leaving $\mathcal{B}_n$. Remark that if $L < M$, then $z \in \mathcal{C}_2$ belongs to A_n. Moreover $N \geq M - L$. Therefore, in order to estimate $\mathbb{P}[z \notin A_n]$, we just need an upper bound for $\mathbb{P}[M = L]$. For any fixed number a

$$\begin{aligned} \mathbb{P}[z \notin A_n] = \mathbb{P}[N = 0] &\leq \mathbb{P}[M - L = 0] \leq \mathbb{P}[M \leq a \text{ or } L \geq a] \\ &\leq \mathbb{P}[M \leq a] + \mathbb{P}[L \geq a]. \end{aligned} \tag{4.3}$$

We shall show that for a suitable choice of a, the probabilities $\mathbb{P}[M \leq a]$ and $\mathbb{P}[L \geq a]$ are small enough, such that the series (4.2) converges. The derivation of a suitable value of a will be done differently, not like in the case of Euclidean lattices, studied by Lawler, Bramson and Griffeath in [19], who used classical asymptotics for the Green function stopped on a ball. Since in our case, the Green function $G_{\mathcal{B}_n}$ stopped on $\mathcal{B}_n$ is not directly available, we use the odometer function of the divisible sandpile as a replacement, as suggested by Levine and Peres in [21]. For simplicity of notation, we shall write $G_n(y,z)$ instead of $G_{\mathcal{B}_n}(y,z)$. The random variable M is a sum of i.i.d. indicator variables, with

$$\mathbb{E}[M] = n\mathbb{P}_o[\tau_z < \tau_n] = n\frac{G_n(o,z)}{G_n(z,z)}. \tag{4.4}$$

Even though L is a sum of dependent indicator variables, following [19], L can be bounded by a sum of independent indicators as follows. Only those particles with $X^i_{\sigma^i} \in \mathcal{B}_n$ contribute to L and for each $y \in \mathcal{B}_n$ there is at most one index i with $X^i_{\sigma^i} = y$. The corresponding post-τ_y random walks are independent. In order to avoid dependencies in L, enlarge the index set to all of $\mathcal{B}_n$ and define

$$\tilde{L} = \sum_{y \in \mathcal{B}_n} \mathbb{1}^y_{\{\tau_z < \tau_n\}},$$

where the indicators $\mathbb{1}^y$ correspond to independent random walks starting at y. Then $L \leq \tilde{L}$, and the expectation of $\tilde{L}$ is given by

$$\mathbb{E}[\tilde{L}] = \sum_{y \in \mathcal{B}_n} \mathbb{P}_y[\tau_z < \tau_n] = \frac{1}{G_n(z,z)} \sum_{y \in \mathcal{B}_n} G_n(y,z). \tag{4.5}$$

Now (4.3) can be rewritten as

$$\mathbb{P}[z \notin A_n] \leq \mathbb{P}[M \leq a] + \mathbb{P}[\tilde{L} \geq a]. \tag{4.6}$$

We shall relate the random variables $\tilde{L}$ and M with the odometer function of the divisible sandpile. For this, consider the function $f_n : \partial\mathcal{B}_n \cup \mathcal{B}_n \to \mathbb{R}$,

$$f_n(z) = \frac{G_n(z,z)}{d(z)} \mathbb{E}[M - \tilde{L}]. \tag{4.7}$$

Set $h_x(y) = \frac{G_n(x,y)}{d(y)}$, then for $x, y \in \mathcal{B}_n$ one has

$$\Delta h_x(y) = \begin{cases} -\dfrac{1}{d(x)}, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}, \tag{4.8}$$

By linearity of the Laplace operator, f_n solves the following Dirichlet problem

$$\begin{cases} \Delta f_n(z) = \dfrac{1}{d(z)}\big(1 - n \cdot \delta_o(z)\big), & \text{for } z \in \mathcal{B}_n \\ f_n(z) = 0, & \text{for } z \in \partial\mathcal{B}_n. \end{cases}$$

Recall that the odometer function u_n of the divisible sandpile with initial distribution $\mu_0 = n \cdot \delta_o$ solves the same Dirichlet problem (3.5) on the sandpile cluster S_n (whose shape is given by $\mathcal{B}_n$ by Theorem 3.5).

The uniqueness of the solution of a Dirichlet problem gives that $f_n = u_n$ on the set $\mathcal{B}_n$, and u_n is approximated (up to an additive constant) by the function γ_n defined in (3.7). Since $u_n > 0$, it follows that $f_n(z) > 0$, for all $z \in \mathcal{B}_n$, which is equivalent to $\mathbb{E}[M] > \mathbb{E}[\tilde{L}]$.

We will use the following large deviations estimate for sums of independent indicators. For a proof, see Alon and Spencer [1, Cor. A.1.14].

Lemma 4.3. *If N is a sum of finitely many independent indicator random variables, then for all $\lambda > 0$,*

$$\mathbb{P}\big[|N - \mathbb{E}N| > \lambda \mathbb{E}N\big] < 2e^{-c_\lambda \mathbb{E}N},$$

where c_λ is a constant depending only on λ.

In order to find an upper bound for the right hand-side of (4.6) we use the previous Lemma and choose $\lambda > 0$ and a such that

$$(1+\lambda)\mathbb{E}[\tilde{L}] \leq a \leq (1-\lambda)\mathbb{E}[M]. \tag{4.9}$$

Hence λ has to satisfy the relation

$$0 < \lambda \leq \frac{\mathbb{E}[M - \tilde{L}]}{\mathbb{E}[M + \tilde{L}]} = \frac{f_n(z)}{g_n(z)}, \tag{4.10}$$

and g_n defined as

$$g_n(z) = \frac{G_n(z,z)}{d(z)} \mathbb{E}[M + \tilde{L}]. \tag{4.11}$$

To obtain λ, we have to bound $f_n(z)/g_n(z)$ away from 0. For this, we first have to calculate the function g_n which is, like f_n, defined on $\partial\mathcal{B}_n \cup \mathcal{B}_n$.

4.1 The function g_n

By (4.4) and (4.5), the function g_n is the solution of the Dirichlet Problem

$$\begin{cases} \Delta g_n(z) = \dfrac{1}{d(z)}\big(-1 - n\cdot\delta_o(z)\big), & \text{for } z \in \mathcal{B}_n, \\ g_n(z) = 0, & \text{for } z \in \partial\mathcal{B}_n, \end{cases} \tag{4.12}$$

and can therefore be obtained by solving some linear recursions. For simplicity, we first shift the set $\mathcal{B}_n$ by $kn^{1/3}$ in the direction of the positive x-axis. This shifted set will be denoted by $\mathcal{B}_n^t$, which is the set of all $(x,y) \in \mathcal{C}_2$ with $0 \leq x \leq 2kn^{1/3}$ and

$$|y| \leq \frac{x^2}{3}, \quad \text{for } 0 \leq x \leq kn^{1/3},$$
$$|y| \leq \frac{(2kn^{1/3} - x)^2}{3}, \quad \text{for } kn^{1/3} < x \leq 2kn^{1/3}.$$

On the shifted set we define the function $g_n^t : \mathcal{B}_n^t \to \mathbb{R}$, by

$$g_n^t(x,y) = g_n(x + kn^{1/3}, y), \tag{4.13}$$

which solves the same Dirichlet problem (4.12) on $\mathcal{B}_n^t$ with the origin moved to $(kn^{1/3}, 0)$. By symmetry of g_n, it is enough to compute g_n^t for vertices (x,y) with $0 \leq x \leq kn^{1/3}$ and

$y \geq 0$. For $z = (x, y) \in \mathcal{B}_n^t$, with $y \neq 0$, the Laplace $\Delta g_n^t(z)$ is equal to $-1/2$, hence on each "tooth" of the comb, g_n^t satisfies the linear recursion

$$2g_n^t(x,y) = g_n^t(x, y+1) + g_n^t(x, y-1) + 1,$$

which has the general solution

$$g_n^t(x,y) = \frac{1}{2}(y - y^2) + c_1(x) + yc_2(x), \tag{4.14}$$

where $c_1(x)$ and $c_2(x)$ are functions of x, to be determined. For $(x,0), (x,1) \in \mathcal{C}_2$, we have

$$g_n^t(x,0) = c_1(x) \quad \text{and} \quad g_n^t(x,1) = c_1(x) + c_2(x). \tag{4.15}$$

From (4.12) we have the boundary conditions $g_n^t(0,0) = 0$ and $g_n^t(2kn^{1/3}, 0) = 0$ and for $0 \leq x \leq kn^{1/3}$, we have $g_n^t(x, x^2/3) = 0$. On the other hand, from equation (4.14), we get

$$g_n^t(x, x^2/3) = \frac{x^2}{6}\Big(1 - \frac{x^2}{3}\Big) + c_1(x) + \frac{x^2}{3}c_2(x) = 0,$$

which implies that the function $c_2(x)$ can be written as

$$c_2(x) = \frac{1}{2}\Big(\frac{x^2}{3} - 1\Big) - \frac{3}{x^2}c_1(x). \tag{4.16}$$

Moreover, on the x-axis the Laplace operator of g_n^t satisfies

$$\Delta g_n^t(x,0) = \begin{cases} -\frac{1}{4}, & \text{if } x \neq kn^{1/3} \\ -\frac{1}{4}(n+1), & \text{if } x = kn^{1/3}. \end{cases} \tag{4.17}$$

For $x \neq kn^{1/3}$, that is, when $(x,0)$ is not the center of $\mathcal{B}_n^t$, we have

$$g_n^t(x+1, 0) = 4g_n^t(x,0) - g_n^t(x-1,0) - 2g_n^t(x,1) - 1,$$

and using (4.15) we obtain

$$c_1(x+1) = 2c_1(x) - c_1(x-1) - 2c_2(x) - 1,$$

which together with (4.16) gives an equation for c_1, namely

$$c_1(x+1) = \Big(2 + \frac{6}{x^2}\Big)c_1(x) - c_1(x-1) - \frac{x^2}{3}.$$

This has an explicit solution as a polynomial of degree 4, given by

$$c_1(x) = -\frac{1}{18}x^4 + bx^3 - \frac{1}{36}x^2, \tag{4.18}$$

where b is a free parameter which can be computed using the other boundary conditions for g_n^t. Since $\Delta g_n^t(kn^{1/3}, 0) = -\frac{1}{4}(n+1)$, using equations (4.15), (4.16), and (4.18), we obtain

$$b = \frac{5K + 27n}{18(1 + 3K^2)},$$

where $K = kn^{1/3}$, and the constant $k = \big(\frac{3}{2}\big)^{2/3}$ is the same as in Theorem 4.2. Since we are interested in the form of g_n^t for n sufficiently large, we expand b around $n = \infty$, giving

$$b(n) = \frac{1}{6l}n^{1/3} + \mathcal{O}(n^{-1/3}).$$

Putting everything together we get $g_n(x,y) = g_n^t\big(kn^{1/3} - |x|, |y|\big)$, with

$$g_n^t(x,y) = \Big(\frac{1}{6l}n^{1/3} + \mathcal{O}(n^{-1/3})\Big)(x^3 - 3xy) + \frac{1}{36}(3y - 18y^2 - 2x^4 - x^2 + 12x^2y).$$

4.2 IDLA inner bound

We are now able to conclude the proof of Theorem 4.2.

Lemma 4.4. *For all $\varepsilon > 0$ there exists n_ε, such that for all $n \geq n_\varepsilon$ and all $z \in \mathcal{B}_{n(1-\varepsilon)}$*

$$\frac{\varepsilon}{4} \leq \frac{\mathbb{E}[M-\tilde{L}]}{\mathbb{E}[M+\tilde{L}]}.$$

Proof. By (4.10), one needs to study the function $\lambda_n(x,y) = \frac{f_n(x,y)}{g_n(x,y)}$. We have

$$\lambda_n(x,y) = \frac{\big(|y| - \frac{n_x}{2}\big)^2}{2c_1(x) + \big(2c_2(x)+1\big)y - y^2},$$

where $c_1(x)$, $c_2(x)$ and n_x are defined in (4.18), (4.16) and (3.11), respectively. It suffices to consider the first quadrant. For every fixed x, the function $\lambda_n(x,y)$ is decreasing in y for $0 \leq y \leq \frac{n_x}{2}$. From the proof of Theorem 3.5 we already know that

$$\frac{n_x}{2} = l\left(n^{1/3} - \frac{x}{k}\right)^2 + \mathcal{O}(1). \tag{4.19}$$

For $0 < \varepsilon < 1$ consider the set

$$\mathcal{B}_{n,\varepsilon} = \left\{(x,y) \in \mathcal{C}_2 :\ |x| \leq (1-\varepsilon)kn^{1/3} \text{ and } |y| \leq (1-\varepsilon)l\left(n^{1/3} - \frac{|x|}{k}\right)^2\right\}.$$

Obviously $\mathcal{B}_{n,\varepsilon} \subset \mathcal{B}_n$ for all ε, hence $\frac{f_n}{g_n}$ is well defined on this set. Furthermore, by (4.19), $\frac{f_n}{g_n}$ is also decreasing on $\mathcal{B}_{n,\varepsilon}$ as a function of y, for all $\varepsilon > 0$ and n big enough. This means that it is enough to study $\frac{f_n}{g_n}$ at the inner boundary of $\mathcal{B}_{n,\varepsilon}$. For this, let $z = (x,y) \in \mathcal{B}_{n,\varepsilon}$ with $|y| = (1-\varepsilon)l\left(n^{1/3} - \frac{|x|}{k}\right)^2$ be such a boundary point. Then

$$\lim_{n\to\infty} \frac{f_n(z)}{g_n(z)} = \frac{\varepsilon}{4-\varepsilon} > \frac{\varepsilon}{4}.$$

The statement follows from the fact that for each $\varepsilon > 0$ one can find an $\varepsilon' > 0$ such that $\mathcal{B}_{n(1-\varepsilon)} \subset \mathcal{B}_{n,\varepsilon'}$. □

Proof of Theorem 4.2. Recall that we need to show the convergence of the series (4.2). Fix $z \in \mathcal{B}_{n(1-\varepsilon)}$. We set $\lambda = \frac{\varepsilon}{4} > 0$ in Lemma 4.4, and choose

$$a = (1+\lambda)\mathbb{E}[\tilde{L}] = \Big(1 + \frac{\varepsilon}{4}\Big)\mathbb{E}[\tilde{L}]$$

in equation (4.9). Apply now Lemma 4.3 to M and $\tilde{L}$. Recall also that $\mathbb{E}[M] > \mathbb{E}[\tilde{L}]$. Then

$$\begin{aligned}\mathbb{P}[M \leq a] + \mathbb{P}[\tilde{L} \geq a] &= \mathbb{P}\Big[M \leq \Big(1+\frac{\varepsilon}{4}\Big)\mathbb{E}[\tilde{L}]\Big] + \mathbb{P}\Big[\tilde{L} \geq \Big(1+\frac{\varepsilon}{4}\Big)\mathbb{E}[\tilde{L}]\Big] \\ &\leq 4\exp\big\{-c_\lambda \mathbb{E}[\tilde{L}]\big\} \leq 4\exp\Big\{-c_\lambda \frac{g_n(z) - f_n(z)}{G_n(z,z)}\Big\},\end{aligned}$$

where c_λ is a constant depending only on λ. Hence, for all $n \geq n_\varepsilon$, we have

$$\sum_{n\geq n_\varepsilon} \sum_{z\in\mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n] \leq 4 \sum_{n\geq n_\varepsilon} \sum_{z\in\mathcal{B}_{n(1-\varepsilon)}} \exp\Big\{-c_\lambda \frac{g_n(z)-f_n(z)}{G_n(z,z)}\Big\}. \tag{4.20}$$

In order to estimate the stopped Green function $G_n(z,z)$ upon exiting $\mathcal{B}_n$, with $z=(x,y)$, note that

$$|y| \le b_n(x) := l\Big(n^{1/3} - \frac{|x|}{k}\Big)^2.$$

We have the trivial upper bound $G_n(z,z) \le 2G_A(y,y)$ where G_A is the Green function of the simple random walk on the integer line, stopped at the interval $A = \big[-b_n(x), b_n(x)\big]$. Using Proposition 1.6.3 and Theorem 1.6.4 from Lawler [17], this can be bounded by

$$G_A(y,y) = \frac{b_n(x)^2 - y^2}{b_n(x)} \le l\Big(n^{1/3} - \frac{|x|}{k}\Big)^2. \tag{4.21}$$

For every $\varepsilon > 0$, the function $g_n(z) - f_n(z)$ is again decreasing on every non-crossing path which starts at o and stays inside $\mathcal{B}_{n(1-\varepsilon)}$. Hence, it attains its minimum on the inner boundary $\partial_I \mathcal{B}_{n(1-\varepsilon)}$ of $\mathcal{B}_{n(1-\varepsilon)}$. Taking limits, we get for every sequence $z_n = (x, y_n)$ with x fixed and $z_n \in \partial_I \mathcal{B}_{n(1-\varepsilon)}$

$$\lim_{n\to\infty} \frac{g_n(z_n) - f_n(z_n)}{n^{4/3}} = \frac{k}{4}(2-\varepsilon)\varepsilon,$$

and for the sequence $z_n' = (x_n, 0)$ with $x_n = kn^{1/3}(1-\varepsilon)^{1/3}$

$$\lim_{n\to\infty} \frac{g_n(z_n') - f_n(z_n')}{n^{4/3}} = \frac{k}{4}\big(3 - 2\varepsilon - (\varepsilon-3)(1-\varepsilon)^{1/3}\big).$$

Hence for all $\varepsilon > 0$ and n big enough

$$\min_{z \in \mathcal{B}_{n(1-\varepsilon)}} \big(g_n(z) - f_n(z)\big) \ge C_\varepsilon \cdot n^{4/3},$$

for a constant C_ε which depends only on ε. Since, by (4.21) the stopped Green function $G_A(z,z)$ is of order $\mathcal{O}(n^{2/3})$, this implies

$$\min_{z \in \mathcal{B}_{n(1-\varepsilon)}} \frac{g_n(z) - f_n(z)}{G_n(z,z)} \ge C_\varepsilon' \cdot n^{2/3}.$$

Hence, (4.20) can be bounded by

$$\sum_{n \ge n_\varepsilon} \sum_{z \in \mathcal{B}_{n(1-\varepsilon)}} \mathbb{P}[z \notin A_n] \le 4 \sum_{n \ge n_\varepsilon} n \exp\{-c_\lambda C_\varepsilon' n^{2/3}\} < \infty,$$

which concludes the proof. □

4.3 The recurrent potential kernel

In the recurrent lattice case $\mathbb{Z}^2$, the limiting shape of IDLA is derived using estimates for the *recurrent potential kernel* which is defined as follows

$$A(x,o) = \lim_{n\to\infty} \sum_{t=0}^{n} \big(\mathbb{P}_o[X_t = o] - \mathbb{P}_x[X_t = o]\big), \quad \text{for } x \in \mathcal{C}_2.$$

See Lawler, Bramson and Griffeath [19] for more details. For some constant $N > 0$, the *level sets of the potential kernel* are sets of the form $\{x \in \mathcal{C}_2 : \ A(x,o) \ge N\}$, and the *level sets of the Green function* are of the form $\{x \in \mathcal{C}_2 : \ G(x,o) \ge N\}$.

In all previously known cases, the limiting shape of the IDLA-cluster is determined by the level sets of the potential kernel in the recurrent case and by level sets of the Green function in the transient case. Nevertheless, this is not the case for comb lattices

$\mathcal{C}_2$, even if the simple random walk is recurrent. In order to show this, let us consider the generating function of the potential kernel

$$\begin{aligned} A(x,o|z) &= \sum_{t=0}^{\infty} \big(\mathbb{P}_o[X_t = o] - \mathbb{P}_x[X_t = o]\big) z^t \\ &= G(o,o|z) - G(x,o|z), \end{aligned}$$

where $G(x,y|z) = \sum_{t=0}^{\infty} \mathbb{P}_x[X_t = y] z^t$ is the generating function of the Green function of simple random walk on $\mathcal{C}_2$. Using standard techniques for generating functions one gets for $x = (x_1, x_2) \in \mathcal{C}_2$

$$G(x,o|z) = F_1(z)^{|x_2|} F_2(z)^{|x_1|} G(o,o|z)$$

with

$$F_1(z) = \frac{1 - \sqrt{1-z^2}}{z}, \qquad F_2(z) = \frac{1 + \sqrt{1-z^2} - \sqrt{2}\sqrt{1 - z^2 + \sqrt{1-z^2}}}{z}$$

and

$$G(o,o|z) = \frac{\sqrt{2}}{\sqrt{1 - z^2 + \sqrt{1-z^2}}}.$$

See Bertacchi and Zucca [6] for details. Therefore

$$A(x,o|z) = G(o,o|z) \left(1 - F_1(z)^{|x_2|} F_2(z)^{|x_1|}\right)$$

and $\lim_{z \to 1^-} A(x,o|z) = 2|x_1|$. By Abel's theorem for power series, the potential kernel on the comb $\mathcal{C}_2$, if it exists, has to be equal to $A(x,o) = 2|x_1|$. Hence we have an example of an IDLA cluster whose behaviour is not given by the level sets of the potential kernel.

5 Rotor-Router Aggregation

A *rotor-router walk* on a graph G is defined as follows. For each vertex x fix a cyclic ordering $c(x)$ of its neighbours, i.e., $c(x) = (x_0, x_1, \dots, x_{d(x)-1})$, where $x \sim x_i$ for all $i = 0, 1, \dots, d(x) - 1$. The ordering $c(x)$ is called the *rotor sequence* of x. A *rotor configuration* is a function $\rho : G \to G$, with $\rho(x) \sim x$, for all $x \in G$. Hence ρ assigns to every vertex one of its neighbours. A *particle configuration* is a function $\sigma : G \to \mathbb{N}_0$, with finite support. If $\sigma(x) = m > 0$, we say that there are m particles at vertex x. A particle located at a vertex x with current rotor $\rho(x) = x_i$, performs a rotor-router walk like this: it first sets $\rho(x) = x_{i+1}$, where addition is modulo $d(x)$, and then it moves to the new vertex x_{i+1}.

Rotor-router aggregation is a deterministic process of increasing subsets $(R_i)_{i \in \mathbb{N}}$ of G defined recursively as $R_1 = \{o\}$, and

$$R_i = R_{i-1} \cup \{z_i\} \quad \text{for } i \geq 2,$$

where z_i is the first vertex outside of R_{i-1} that is visited by a particle performing a rotor-router walk, started at o. The particle stops at z_i, and a new particle starts its tour at the origin, without resetting the rotor configuration. The set R_n of occupied sites in G is called the *rotor-router cluster* of n particles. The *odometer function* $u_R(x)$ of rotor-router aggregation is defined as the total number of particles which are sent out by the vertex x during the creation of the rotor-router cluster R_n.

In this section we study rotor-router aggregation on $\mathcal{C}_2$, and we give an inner bound for the cluster R_n which holds for arbitrary initial configuration of rotors and is independent of the rotor sequence. The approach below relies on an idea of Holroyd and Propp [10], who use rotor weights in order to prove a variety of inequalities concerning rotor-walks and random walks.

5.1 Rotor Weights

Let G be a locally finite and connected graph, $\sigma_0 : G \to \mathbb{N}$ an initial particle configuration with finite support, and $\rho_0 : G \to G$ an initial rotor configuration with $\rho_0(x) = x_0$ for all $x \in G$, that is, all initial rotors point to the first element in the rotor sequence $c(x)$. Routing particles in the system, such that at each time step t exactly one particle makes one step of a rotor-router walk, gives rise to a sequence $(\rho_t, \sigma_t)_{t\in\mathbb{N}_0}$ of rotor- and particle-configurations. To each of the possible states (ρ_t, σ_t) of the system, we will assign a weight. In order to do so, let us fix a function $h : G \to \mathbb{R}$. Define the *particle weights* at time t to be

$$\mathbf{W_P}(t) = \sum_{x\in G} \sigma_t(x)h(x), \tag{5.1}$$

and the *rotor weights* of single vertices $x \in G$ as

$$w(x, k) = \begin{cases} 0, & \text{for } k = 0 \\ w(x, k-1) + h(x) - h\big(x_{k \bmod d(x)}\big), & \text{for } k > 0, \end{cases} \tag{5.2}$$

where x_i is the i-th neighbour of x in the rotor sequence $c(x)$. Notice that, for $k \geq d(x)$,

$$w(x, k) = w\big(x, k - d(x)\big) - d(x)\Delta h(x). \tag{5.3}$$

The total *rotor weights* at time t are given by

$$\mathbf{W_R}(t) = \sum_{x\in G} w(x, u_t(x)),$$

where $u_t(z)$ is the odometer function of this process, that is, the number of particles sent out by x in the first t steps. Note that ρ_0 is chosen in such a way that, if $i \equiv u_t(x) \bmod d(x)$, then $x_i = \rho_t(x)$ for all $t \geq 0$ and $x \in G$. It is easy to check that the sum of particle- and rotor-weights are invariant under routing of particles, i.e., for all times $t, t' \geq 0$

$$\mathbf{W_P}(t) + \mathbf{W_R}(t) = \mathbf{W_P}(t') + \mathbf{W_R}(t'). \tag{5.4}$$

For rotor-router aggregation on G we start with n particles at the origin, that is, $\sigma_0 = n \cdot \delta_o$, and we route a particle only if there is at least one other particle at the same position. The process terminates when no two particles are at the same position. Denote by $t^\star = t^\star(n)$ the number of steps it takes to finish the process, and by $(\sigma_{t^\star}, \rho_{t^\star})$ the final configuration. By the abelian property, the configuration $(\sigma_{t^\star}, \rho_{t^\star})$ does not depend on the order the particles made their steps, and by definition $\sigma_{t^\star}(x) = \mathbb{1}_{\{x\in R_n\}}$. We use the following weight function: for some $y \in G$, let $h_y : G \to \mathbb{R}$ given by

$$h(x) = h_y(x) = \frac{G_n(y, x)}{d(x)}, \tag{5.5}$$

where G_n is the Green function of the simple random walk on G, stopped upon exiting the sandpile cluster S_n with initial mass distribution $\mu_0 = n \cdot \delta_o$ on G. Take now some $y \in S_n$. Here S_n is the sandpile cluster of the divisible sandpile on some general graph G. Recall that the Laplace of $h_y(x)$ on G is given by (4.8). The particle weights at the beginning are

$$\mathbf{W_P}(0) = nh_y(o), \tag{5.6}$$

while the rotor weights are $\mathbf{W_R}(0) = 0$. At the end of the process, i.e., at time $t^\star$ when the rotor-router cluster R_n is formed, we have

$$\mathbf{W_P}(t^\star) = \sum_{x\in R_n} h_y(x) \leq \sum_{x\in S_n} h_y(x), \tag{5.7}$$

since h_y is equal to 0 outside of S_n. For the rotor weights we get from (5.3)

$$\mathbf{W_R}(t^\star) = \sum_{x\in R_n} \left\lfloor \frac{u_R(x)}{d(x)} \right\rfloor \big(- d(x)\Delta h_y(x)\big) + \sum_{x\in R_n} w(x,k_x), \tag{5.8}$$

where u_R is the rotor odometer function and $k_x = u_R(x) \bmod d(x)$. By (4.8) and (5.2)

$$\begin{aligned}\mathbf{W_R}(t^\star) &= \left\lfloor \frac{u_R(y)}{d(y)} \right\rfloor + \sum_{x\in R_n}\sum_{i=0}^{k_x} \big(h_y(x) - h_y(x_i)\big)\\ &\leq \frac{u_R(y)}{d(y)} + \sum_{x\in S_n}\sum_{z\sim x} |h_y(x) - h_y(z)|.\end{aligned} \tag{5.9}$$

Hence by the invariance of the total weights (5.4), we obtain

$$\sum_{x\in S_n} \big(n\delta_0(x) - 1\big) h_y(x) \leq \frac{u_R(y)}{d(y)} + \sum_{x\in S_n}\sum_{z\sim x} |h_y(x) - h_y(z)|. \tag{5.10}$$

Denote by $v(y)$ the lefthand side of (5.10). Then $v(y)$ solves the Dirichlet problem

$$\begin{cases}\Delta v(y) = \frac{1}{d(y)}\big(1 - n\delta_0(y)\big), \text{ for } y\in S_n,\\ v(y) = 0, \text{ for } y\notin S_n.\end{cases}$$

By (3.3), the normalized odometer function u_n of the divisible sandpile on G with initial mass distribution $\mu_0 = n\cdot\delta_o$ satisfies exactly the same Dirichlet problem, hence $v(y) = u_n(y)$. From the previous calculatio we get the next result, which compares the odometer function u_R of rotor-router aggregation with the odometer function u_n of the divisible sandpile. This result holds for any locally finite and connected graph G.

Proposition 5.1. *Let u_n be the normalized odometer function of the divisible sandpile with initial mass distribution $\mu_0 = n\cdot\delta_o$, and u_R the odometer function of rotor-router aggregation with n particles starting at the origin $o\in G$. Then, for all $y\in G$,*

$$u_n(y) \leq \frac{u_R(y)}{d(y)} + \widetilde{\mathbf{W}}_\mathbf{R}(y), \tag{5.11}$$

with

$$\widetilde{\mathbf{W}}_\mathbf{R}(y) = \sum_{x\in S_n}\sum_{z\sim x}\left| \frac{G_n(y,x)}{d(x)} - \frac{G_n(y,z)}{d(z)} \right|. \tag{5.12}$$

Levine and Peres [20] derived an inequality similar to (5.11) in the case of $\mathbb{Z}^d$ using a different method. For trees, $\widetilde{\mathbf{W}}_\mathbf{R}(y)$ can be expressed in terms of the expected distance from the starting point of a random walk to the point where it first exits S_n.

Proposition 5.2. *If G is a tree and $\mathrm{d}(\cdot,\cdot)$ is the graph distance on G, then*

$$\widetilde{\mathbf{W}}_\mathbf{R}(x) = 2\mathbb{E}_x\big[\mathrm{d}(x, X_T)\big] - 2,$$

where $T = \inf\big\{t\geq 0:\ X_t\notin S_n\big\}$, and (X_t) is the simple random walk on G.

Proof. For $y\sim z$ let N_{yz} be the number of transitions from y to z before the random walk exits S_n. Then

$$\mathbb{E}_x\big[N_{yz} - N_{zy}\big] = \frac{G_n(x,y)}{d(y)} - \frac{G_n(x,z)}{d(z)}.$$

See also [22, Proposition 2.2] for more details. Since G is a tree, the net number of crossings of each edge is smaller or equal to one, i.e.,

$$\big|\mathbb{E}_x\big[N_{yz} - N_{zy}\big]\big| \leq 1.$$

We consider G as a tree rooted at x, and denote by $\pi_{x,z}$ the shortest path from x to z. For $y \neq x$, write y^- for the parent of y, i.e., the unique neighbour of y that lies on the shortest path $\pi_{x,y}$. With this notation we get

$$\sum_{\substack{y,z\in S_n\\ y\sim z}} \left|\frac{G_n(x,y)}{d(y)} - \frac{G_n(x,z)}{d(z)}\right| = \sum_{\substack{y,z\in S_n\\ y\sim z}} \big|\mathbb{E}_x\big[N_{yz} - N_{zy}\big]\big| = 2\sum_{\substack{y\in S_n\\ y\neq x}} \mathbb{E}_x\big[N_{y^-y} - N_{yy^-}\big],$$

where the last equality is due to the antisymmetry of $N_{yz} - N_{zy}$. Let

$$C_y = \big\{z \in S_n :\ y \in \pi_{x,z}\big\}$$

be the cone of y. The random variable $N_{y^-y} - N_{yy^-}$ is either zero or one, the latter if the random walk exits S_n in the cone C_y, hence

$$\widetilde{\mathbf{W}}_{\mathbf{R}}(x) = 2\sum_{\substack{y\in S_n\\ y\neq x}} \mathbb{P}_x\big[X_T \in C_y\big] = 2\sum_{\substack{y\in S_n\\ y\neq x}}\sum_{z\in C_y} \mathbb{P}_x\big[X_T = z\big].$$

For all $z \in \partial S_n$ we have $\#\big\{y \in S_n \setminus \{x\} :\ z \in C_y\big\} = \mathrm{d}(x,z) - 1$, therefore

$$\widetilde{\mathbf{W}}_{\mathbf{R}}(x) = 2\sum_{z\in\partial S_n} \mathbb{P}_x\big[X_T = z\big]\big(\mathrm{d}(x,z) - 1\big) = 2\mathbb{E}_x\big[\mathrm{d}(x,X_T)\big] - 2,$$

which completes the proof. □

5.2 Rotor-router Aggregation on the Comb

Since $\mathcal{C}_2$ is a tree, by Proposition 5.1 and Proposition 5.2, one needs an upper bound for the expected distance from the starting point of a random walk (X_t) on $\mathcal{C}_2$ to the point where it first exits S_n, in order to derive an inner estimate of the rotor-router cluster. Recall that on $\mathcal{C}_2$, the sandpile cluster S_n has the shape $\mathcal{B}_n$ given in Theorem 3.5. Using the trivial upper estimate

$$\mathbb{E}_z\big[\mathrm{d}(z,X_T)\big] \leq \max\big\{\mathrm{d}(z,w) : w \in \partial S_n\big\} = |x| + |y| + ln^{2/3}, \tag{5.13}$$

with $z = (x,y)$ and $l = \frac{1}{2}\big(\frac{3}{2}\big)^{1/3}$ as in Theorem 3.5, we can show the following inner bound.

Theorem 5.3. *Let R_n be the rotor-router cluster of n particles on $\mathcal{C}_2$. Then, for $n \geq n_0$ and for any initial rotor configuration and choice of rotor sequence, we have*

$$\tilde{\mathcal{B}}_n \subset R_n,$$

where

$$\tilde{\mathcal{B}}_n = \left\{(x,y) \in \mathcal{C}_2 :\ |x| \leq kn^{1/3} - c_1 n^{1/6}, |y| \leq l\left(n^{1/3} - \frac{x}{k}\right)^2 + c_2 x - c_3 n^{1/3}\right\},$$

where

$$k = \left(\frac{3}{2}\right)^{2/3}, \quad l = \frac{1}{2}\left(\frac{3}{2}\right)^{1/3}$$

and c_1, c_2, c_3 are constants.

Proof. By the definition of rotor-router aggregation, $\big\{z \in \mathcal{C}_2 : u_R(z) > 0\big\} \subset R_n$, and by Proposition 5.1 together with Proposition 5.2, we have for vertices $z = (x, y)$

$$\frac{u_R(z)}{d(z)} \geq u_n(z) - 2\mathbb{E}_z\big[\mathrm{d}(z, X_T)\big] + 2 \geq u_n(z) - 2\big(|x| + |y| + ln^{2/3}\big) + 2,$$

The last inequality uses equation (5.13). By Corollary 3.6, we have a lower bound of the sandpile odometer u_n for $z \in S_n$

$$\gamma_n(z) - a \leq u_n(z),$$

where a is a positive constant smaller than 2, and γ_n is the function defined in (3.7). Thus, to derive an inner bound, it suffices to check for which $z = (x, y) \in S_n$ the inequality

$$\gamma_n(x, y) - 2\big(|x| + |y| + ln^{2/3}\big) > 0 \tag{5.14}$$

holds. By symmetry it is enough to consider $x, y \geq 0$. We first check inequality (5.14) on a “tooth” of the comb, that is, for a fixed x. The function γ_n is given as

$$\gamma_n(x, y) = \frac{1}{2}\left(y - \frac{n_x}{2}\right)^2,$$

where n_x is the amount of mass that ends up in the x-“tooth” of the sandpile. Since x is fixed, we can treat n_x as a constant. Hence the right hand side of (5.14) is a quadratic polynomial in y with smallest root

$$y_x = 2 + \frac{n_x}{2} - \sqrt{4 + \frac{k}{l}n^{2/3} + 2n_x + 4x}.$$

Substituting n_x as calculated in (3.11), an expansion around $n = \infty$ gives

$$y_x = ln^{2/3} - \frac{1}{2l}n^{1/3}x + \frac{x^2}{3} + \frac{2+\sqrt{6}}{3}x + c_1 n^{1/3} - c_2\frac{x^4}{n}.$$

Since $(x, y) \in S_n$, we have the bound $x \leq kn^{1/3}$, hence

$$y_x = l\left(n^{1/3} - \frac{x}{k}\right)^2 + \frac{2+\sqrt{6}}{3}x - cn^{1/3}, \tag{5.15}$$

for $n \geq n_0$, and a positive constant c. To get a bound on the x-axis, we calculate for which $x > 0$ the inequality $y_x > 0$ is satisfied. Since y_x is a polynomial of degree 2 in x this is easy to do, and again by series expansion around $n = \infty$ we obtain

$$x \leq k \cdot n^{1/3} - c_3 n^{1/6}, \tag{5.16}$$

for $n \geq n_0$. The inner bound for R_n now follows from (5.16) together with (5.15). □

Figure 4 shows the inner estimate of the rotor-router cluster from Theorem 5.3 in comparison to sandpile cluster S_n, for $n = 1000$. The white area is the area where the inequality (5.14) is valid, and corresponds to the set $\tilde{\mathcal{B}}_n$ of Theorem 5.3. The colouring is based on the value of the right-hand side of (5.14).

The inner bound could be improved if one has a substantially better upper bound for $\mathbb{E}_z[\mathrm{d}(z, X_T)]$. For regular graphs, one can also give an universal inner estimate for rotor-router aggregation, which relates the rotor-router cluster to a divisible sandpile cluster with a smaller mass. Using the methods of Levine and Peres in [20] one can deduce the following.

Proposition 5.4. *Let G be a regular graph with degree d and root o, and let R_n be the rotor-router cluster of n particles starting at o. Further, let S_n be the divisible sandpile cluster with mass distribution $\mu_0(x) = n \cdot \delta_o(x)$. Then $S_{n/(2d-1)} \subset R_n$.*

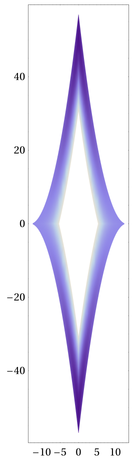

Figure 4: Inner bound

Remarks In the IDLA model, for an upper bound of the type $A_n \subset \mathcal{B}_{n(1+\varepsilon)}$, we still do not have sufficient tools. In all previous studied cases, the IDLA cluster grows uniformly, and this makes easy the study of random walks. This is of course violated in our case, since the set $\mathcal{B}_n$ defined in (1.1) grows with rate $n^{1/3}$ in the x-direction, and with rate $n^{2/3}$ in the y-direction. The *harmonic measure* for random walks stopped when exiting the set $\mathcal{B}_n$, was studied in [11], by making use of a special rotor-router type process. This is the first example where IDLA aggregate is not a set of uniform harmonic measure. In [11], we also give subsets of $\mathcal{C}_2$ with uniform harmonic measure.

Recently Duminil-Copin et al. [8] developed a method for proving an outer bound for IDLA without needing a harmonic measure estimate, provided an inner bound is known. Nevertheless, their method cannot be applied in our setting, since one of the required regularity conditions (weaker lower bound) is not satisfied on the comb.

Acknowledgments. The research of Wilfried Huss was partially supported by the FWF program P19115-N18 and the research of Ecaterina Sava was supported by the FWF program W 1230-N13. Part of the work was done during the stay of the first author at the University of Siegen, Germany.

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