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Global heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ in half-space-like domains*

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Abstract

Suppose that $d \geq 1$ and $\alpha \in (0,2)$. In this paper, we establish by using probabilistic methods sharp two-sided pointwise estimates for the Dirichlet heat kernels of $\{\Delta + a^\alpha \Delta^{\alpha/2};\ a \in (0,1]\}$ on half-space-like $C^{1,1}$ domains for all time $t > 0$. The large time estimates for half-space-like domains are very different from those for bounded domains. Our estimates are uniform in $a \in (0,1]$ in the sense that the constants in the estimates are independent of $a \in (0,1]$. Thus they yield the Dirichlet heat kernel estimates for Brownian motion in half-space-like domains by taking $a \to 0$. Integrating the heat kernel estimates with respect to the time variable t, we obtain uniform sharp two-sided estimates for the Green functions of $\{\Delta + a^\alpha \Delta^{\alpha/2};\ a \in (0,1]\}$ in half-space-like $C^{1,1}$ domains in $\mathbb{R}^d$.

Keywords: symmetric α-stable process; heat kernel; transition density; Green function; exit time; Lévy system; harmonic function; fractional Laplacian; Laplacian; Brownian motion.
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1 Introduction and Setup

Throughout this paper, we assume that $d \geq 1$ is an integer and $\alpha \in (0,2)$. Let $\Delta = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ be the Laplacian on $\mathbb{R}^d$ and $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ the fractional Laplacian on $\mathbb{R}^d$. On $C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact support, the operator $\Delta^{\alpha/2}$ coincides with the operator $\widehat{\Delta}^{\alpha/2}$ defined by

$$\widehat{\Delta}^{\alpha/2} u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d:\, |y-x|>\varepsilon\}} (u(y)-u(x)) \frac{\mathcal{A}(d,\alpha)}{|x-y|^{d+\alpha}}\, dy, \tag{1.1}$$

where $\mathcal{A}(d,\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Here Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$. For $a > 0$, define $\mathcal{L}^a = \Delta + a^\alpha \Delta^{\alpha/2}$ on $\mathbb{R}^d$. The

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non-local operator $\mathcal{L}^a$ is the infinitesimal generator of the Lévy process $X^a := X^0 + aY$ on $\mathbb{R}^d$, where $X^0 = (X^0_t, t \geq 0)$ is a Brownian motion in $\mathbb{R}^d$ with generator Δ and $Y = (Y_t, t \geq 0)$ is an independent rotationally symmetric α-stable process in $\mathbb{R}^d$ whose generator is $\Delta^{\alpha/2}$. We will call the process X^a the independent sum of the Brownian motion X^0 and the symmetric α-stable process Y with weight $a > 0$. The process X^a is a prototype of Lévy processes that have both diffusive and jumping components.

Due to their importance in theory and applications, fine potential theoretical properties of these Lévy processes have been under intense study recently. For any open set $D \subset \mathbb{R}^d$, let $p^a_D(t, x, y)$ be the Dirichlet heat kernel of $\mathcal{L}^a$ in D. The function $p^a_D(t, x, y)$ is also the transition density with respect to the Lebesgue measure on D of the subprocess $X^{a,D}$ of X^a killed upon leaving D. In a recent paper [6], we established sharp two-sided estimates of $p^a_D(t, x, y)$ on any $C^{1,1}$ open set D for $t \in (0, T]$ in a uniform form in $a \in (0, 1]$ for every fixed $T > 0$. If in addition D is bounded, sharp two-sided estimates on $p^a_D(t, x, y)$ for $t > T$ are also obtained in [6]. However, when D is unbounded, the large time behavior of $p^a_D(t, x, y)$ should be very different from that for bounded open sets, as one can see from the symmetric stable processes case treated in [12].

The main purpose of this paper is to derive a sharp two-sided estimate of $p^a_D(t, x, y)$ for all time on a large class of unbounded domains, namely, half-space-like $C^{1,1}$ domains. See below for the definition of half-space-like $C^{1,1}$ domains. Obtaining sharp two-sided Dirichlet heat kernel estimates for any Markov process is typically a non-trivial and demanding task. This is especially so for X^a due to the different scalings in Brownian motion and symmetric stable processes and the complications from the fact that X^a has both a continuous component and a pure jump component. The analysis of precise boundary behavior of $p^a_D(t, x, y)$ for large times turns out to be quite challenging and delicate. In [6], the correct boundary decay rate for $p^a_D(t, x, y)$ for small t was established by using some exit distribution estimates obtained in [8]. Unfortunately the estimates obtained in [8] are not suitable to use in the present case. Thus in this paper we need first to derive new exit distribution estimates that are suitable for large time heat kernel estimates. The first step is, similar to that in [3, 8, 13], to compute $\mathcal{L}^1\phi$ for certain test functions. But unlike in [8], to obtain the desired estimates, we do not use a combinations of test functions to construct suitable subharmonic and superharmonic functions. Instead, we use a generalization of Dynkin's formula to derive directly the needed exit distribution estimates presented in Lemma 2.4 below. We believe that our approach to obtain the correct boundary decay rate in this paper is quite general and may be used for other types of jump processes.

In the remainder of this section, we will state the main result (Theorem 1.4) of this paper, followed by some remarks, a conjecture (see Remark 1.5(i)) and an application to Green function estimates (Theorem 1.7). To do so, we need first to recall some known facts about X^a. Let $p^a(t, x, y)$ be the transition density of X^a with respect to the Lebesgue measure on $\mathbb{R}^d$. The function $p^a(t, x, y)$ is smooth on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. For any $\lambda > 0$, $(\lambda X^a_{\lambda^{-2}t}, t \geq 0)$ has the same distribution as $(X^{a\lambda^{(\alpha-2)/\alpha}}_t, t \geq 0)$ (see the second paragraph of [6, Section 2]), so we have

$$p^{a\lambda^{(\alpha-2)/\alpha}}(t, x, y) = \lambda^{-d} p^a(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{1.2}$$

For $a > 0$ and $C > 0$, define

$$h^a_C(t, x, y) := \left(t^{-d/2} \wedge (a^\alpha t)^{-d/\alpha}\right) \wedge \left(t^{-d/2} e^{-C|x-y|^2/t} + \left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right)\right). \tag{1.3}$$

Here and in the sequel, we use ":=" as a way of definition and, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The following sharp two-sided estimates on $p^a(t, x, y)$

follow from (1.2) and the main results in [11, 23] that give the sharp estimates on $p^1(t,x,y)$.

Theorem 1.1. *There are constants $c, C_1 \ge 1$ such that, for all $a \in [0,\infty)$ and $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$c^{-1}\, h^a_{C_1}(t,x,y) \le p^a(t,x,y) \le c\, h^a_{1/C_1}(t,x,y).$$

We record a simple but useful observation. Its proof will be given at the end of this section.

Proposition 1.2. *For every $c>0$ and $c_1>0$, there is a constant $c_2 \ge 1$ such that for any $a>0$,*

$$c_2^{-1}\left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right) \le h^a_c(t,x,y) \le c_2\left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right)$$

holds when either $t \ge c_1 a^{-2\alpha/(2-\alpha)}$ or $|x-y| \ge a^{-\alpha/(2-\alpha)}$.

Recall that a domain (a connected open set) D in $\mathbb{R}^d$ (when $d\ge 2$) is said to be $C^{1,1}$ if there exist a localization radius $R_0>0$ and a constant $\Lambda_0>0$ such that for every $z\in\partial D$, there exist a $C^{1,1}$ function $\psi=\psi_z:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\psi(0)=0$, $\nabla\psi(0)=(0,\dots,0)$, $\|\nabla\psi\|_\infty\le\Lambda_0$, $|\nabla\psi(x)-\nabla\psi(w)|\le\Lambda_0|x-w|$, and an orthonormal coordinate system CS_z: $y=(y_1,\cdots,y_{d-1},y_d):=(\widetilde{y},\, y_d)$ with origin at z such that $B(z,R_0)\cap D=\{y=(\tilde{y},y_d)\in B(0,R_0) \text{ in } CS_z : y_d>\psi(\widetilde{y})\}$. The pair (R_0,Λ_0) will be called the $C^{1,1}$ characteristics of the domain D. A $C^{1,1}$ domain in $\mathbb{R}$ is simply a possibly unbounded open interval.

For a domain $D\subset\mathbb{R}^d$ and $\lambda_0\ge 1$, we say *the path distance in D is comparable to the Euclidean distance with characteristic* λ_0 if for every $x,y\in D$, there is a rectifiable curve l in D connecting x to y so that the length of l is no larger than $\lambda_0|x-y|$. Clearly, such a property holds for all bounded $C^{1,1}$ domains, $C^{1,1}$ domains with compact complements and domains above the graphs of bounded $C^{1,1}$ functions.

For any open subset $D\subset\mathbb{R}^d$, we use τ^a_D to denote the first time the process X^a exits D. We define the process $X^{a,D}$ by $X^{a,D}_t = X^a_t$ for $t<\tau^a_D$ and $X^{a,D}_t=\partial$ for $t\ge\tau^a_D$, where ∂ is a cemetery point. $X^{a,D}$ is called the subprocess of X^a in D. The generator of $X^{a,D}$ is denoted by $\mathcal{L}^a|_D$. It follows from [11] that $X^{a,D}$ has a continuous transition density $p^a_D(t,x,y)$ with respect to the Lebesgue measure. One can easily see that, when D is bounded, the operator $-\mathcal{L}^a|_D$ has discrete spectrum. In this case, we use $\lambda^{a,D}_1>0$ to denote the smallest eigenvalue of $-\mathcal{L}^a|_D$.

For an open set $D\subset\mathbb{R}^d$ and $x\in D$, we will use $\delta_D(x)$ to denote the Euclidean distance between x and D^c. The following is a particular case of a more general result proved in [6, Theorem 1.3] (cf. Proposition 1.2 above).

Theorem 1.3. *Suppose that D is a $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0,Λ_0) such that the path distance in D is comparable to the Euclidean distance with characteristic λ_0.*

(i) *For every $M>0$ and $T>0$, there are constants $c_1=c_1(R_0,\Lambda_0,\lambda_0,M,\alpha,T)\ge 1$ and $C_2=C_2(R_0,\Lambda_0,\lambda_0,M,\alpha,T)\ge 1$ such that for all $a\in(0,M]$ and $(t,x,y)\in(0,T]\times D\times D$,*

$$\begin{aligned} &c_1^{-1}\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)h^a_{C_2}(t,x,y)\\ &\qquad \le p^a_D(t,x,y)\le c_1\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)h^a_{1/C_2}(t,x,y). \end{aligned}$$

(ii) *Suppose in addition that D is bounded. For every $M > 0$ and $T > 0$, there is a constant $c_2 = c_2(D, M, \alpha, T) \ge 1$ so that for all $a \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,*

$$c_2^{-1}\, e^{-t\,\lambda_1^{a,D}}\, \delta_D(x)\, \delta_D(y) \;\le\; p_D^a(t,x,y) \;\le\; c_2\, e^{-t\,\lambda_1^{a,D}}\, \delta_D(x)\, \delta_D(y).$$

Note that Theorem 1.3 does not give large time estimates for $p_D^a(t,x,y)$ when D is unbounded. The goal of this paper is to establish large time two-sided estimates on $p_D^a(t,x,y)$ for a large class of unbounded $C^{1,1}$ domains, namely half-space-like $C^{1,1}$ domains. A domain D is said to be half-space-like if, after isometry, there exist two real numbers $b_1 \le b_2$ such that $\mathbb{H}_{b_2} \subset D \subset \mathbb{H}_{b_1}$. Here and throughout this paper, $\mathbb{H}_b$ stands for the set $\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > b\}$. We will denote $\mathbb{H}_0$ by $\mathbb{H}$.

We are now in a position to state the main result of this paper. For $a > 0$, define $\phi_a(r) := r \wedge (r/a)^{\alpha/2}$.

Theorem 1.4. *Suppose D is a half-space-like $C^{1,1}$ domain with $C^{1,1}$ characteristics (R_0, Λ_0) and $\mathbb{H}_b \subset D \subset \mathbb{H}$ for some $b > 0$ such that the path distance in D is comparable to the Euclidean distance with characteristic λ_0. Then for any $M > 0$, there exist constants $c_i = c_i(R_0, \Lambda_0, \lambda_0, M, \alpha, b) \ge 1$, $i = 1, 2$, such that for all $a \in (0, M]$ and $(t, x, y) \in (0, \infty) \times D \times D$,*

$$\begin{aligned} & c_1^{-1}\left(1 \wedge \frac{\phi_a(\delta_D(x))}{\sqrt{t}}\right)\left(1 \wedge \frac{\phi_a(\delta_D(y))}{\sqrt{t}}\right) h^a_{c_2}(t,x,y) \\ \le & p_D^a(t,x,y) \le c_1\left(1 \wedge \frac{\phi_a(\delta_D(x))}{\sqrt{t}}\right)\left(1 \wedge \frac{\phi_a(\delta_D(y))}{\sqrt{t}}\right) h^a_{1/c_2}(t,x,y). \end{aligned} \tag{1.4}$$

Remark 1.5. (i) The Lévy process X^a is uniquely determined by its characteristic function

$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t^a - X_0^a)}\right] \;=\; e^{-t(|\xi|^2 + a^\alpha |\xi|^\alpha)} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

Hence the Lévy exponent for X^a is $\Phi_a(|\xi|)$ with $\Phi_a(r) := r^2 + a^\alpha r^\alpha$. The function $\phi_a(r)$ is related to $\Phi_a(r)$ as follows:

$$\frac{1}{\Phi_a(1/r)} = \frac{1}{r^{-2} + a^\alpha r^{-\alpha}} \asymp \frac{1}{r^{-2}} \wedge \frac{1}{(a/r)^\alpha} = r^2 \wedge (r/a)^\alpha = \phi_a(r)^2.$$

Here for two non-negative functions f and g, the notation $f \asymp g$ means that there is a positive constant $c \ge 1$ so that $g(x)/c \le f(x) \le c g(x)$ in the common domain of definition for f and g. Hence in view of Theorem 1.1, the estimates (1.4) can be restated as follows. For every $M > 0$, there are constants $c_1, c_2 \ge 1$ so that for every $a \in (0, M]$ and $(t, x, y) \in (0, \infty) \times D \times D$,

$$c_1^{-1} C p^a(t, c_2 x, c_2 y) \le p_D^a(t,x,y) \le c_1 C p^a(t, x/c_2, y/c_2)$$

where

$$C = \left(1 \wedge \frac{1}{t\Phi_a(1/\delta_D(x))}\right)^{1/2}\left(1 \wedge \frac{1}{t\Phi_a(1/\delta_D(y))}\right)^{1/2}$$

We conjecture that the above Dirichlet heat kernel estimates hold for a large class of rotationally symmetric Lévy processes in $\mathbb{R}^d$; see [7, Conjecture].

(ii) Note that $t \le a^{2\alpha/(\alpha-2)}$ if and only if $(a^\alpha t)^{-d/\alpha} \ge t^{-d/2}$. If $(\delta_D(x)/a)^{\alpha/2} < \delta_D(x)$, then $\delta_D(x) \ge a^{\alpha/(\alpha-2)}$ and so $\delta_D(x) \wedge (\delta_D(x)/a)^{\alpha/2} \ge a^{\alpha/(\alpha-2)}$. Thus when $t \le a^{2\alpha/(\alpha-2)}$ and $(\delta_D(x)/a)^{\alpha/2} < \delta_D(x)$, we have $\frac{(\delta_D(x)/a)^{\alpha/2}}{\sqrt{t}} \ge \frac{a^{\alpha/(\alpha-2)}}{a^{\alpha/(\alpha-2)}} = 1$, and consequently

$$1 \wedge \frac{\delta_D(x) \wedge (\delta_D(x)/a)^{\alpha/2}}{\sqrt{t}} = 1 = 1 \wedge \frac{\delta_D(x)}{\sqrt{t}}.$$

Hence in view of Theorem 1.1 and Proposition 1.2, the statement of Theorem 1.4 can be restated as follows. For all $a \in (0, M]$ and $(t, x, y) \in (0, a^{2\alpha/(\alpha-2)}] \times D \times D$,

$$\begin{aligned} &c_1^{-1} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-c_2|x-y|^2/t} + t^{-d/2} \wedge \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}}\right)\right) \\ &\le p_D^a(t,x,y) \le c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-|x-y|^2/(c_2 t)} + t^{-d/2} \wedge \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}}\right)\right) \end{aligned} \tag{1.5}$$

and for all $a \in (0, M]$ and $(t, x, y) \in [a^{2\alpha/(\alpha-2)}, \infty) \times D \times D$,

$$\begin{aligned} &c_1^{-1} \left(1 \wedge \frac{\delta_D(x) \wedge (a^{-1}\delta_D(x))^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y) \wedge (a^{-1}\delta_D(y))^{\alpha/2}}{\sqrt{t}}\right) \left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right) \\ &\le p_D^a(t,x,y) \le \\ &c_1 \left(1 \wedge \frac{\delta_D(x) \wedge (a^{-1}\delta_D(x))^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y) \wedge (a^{-1}\delta_D(y))^{\alpha/2}}{\sqrt{t}}\right) \left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right). \end{aligned} \tag{1.6}$$

In fact, Theorem 1.4 will be proved in this form. □

Remark 1.6. Unlike [7, 12], there are dramatic differences between the behaviors of the heat kernel $p_D^a(x,y)$ on half-space-like $C^{1,1}$ domains and disconnected half-space-like $C^{1,1}$ open sets even if x and y are in the same connected component. For example, if D is $\mathbb{H} \cup B(x_0, 1)$ where $x_0 = (0, \dots, 0, -2)$ and $x, y \in B(x_0, 1)$, then, as $a \to 0$, $p_D^a(x,y)$ converges to $p^0_{B(x_0,1)}(x,y)$, the Dirichlet heat kernel for Brownian motion on $B(x_0, 1)$. Thus, in this case, the heat kernel estimates for $p_D^a(t,x,y)$ when t is large cannot be of the form (1.4) even if x and y are in the same connected component. Furthermore, as one can see from [6, Theorem 1.3], when D is a disconnected half-space-like $C^{1,1}$ open set (containing bounded connected component), we can not expect that the heat kernel estimates for $p_D^a(x,y)$ to be written in a simple form as the one in (1.4). To keep our exposition as transparent as possible, we are content with establishing the heat kernel estimates for half-space-like $C^{1,1}$ domains. □

Integrating the heat kernel estimates in Theorem 1.4 with respect to t, we get sharp two-sided estimates on the Green function $G_D^a(x,y) := \int_0^\infty p_D^a(t,x,y)dt$ for X^a in half-space-like $C^{1,1}$ domains D.

Define for $d \ge 1$ and $a > 0$,

$$f_D^a(x,y) = \begin{cases} \frac{1}{|x-y|^{d-\alpha}} \left(a^{-\alpha/2} \wedge \frac{\phi_a(\delta_D(x))}{|x-y|^{\alpha/2}}\right) \left(a^{-\alpha/2} \wedge \frac{\phi_a(\delta_D(y))}{|x-y|^{\alpha/2}}\right) & \text{when } d > \alpha, \\ \log\left(\left(1 + a\, \frac{\phi_a(\delta_D(x))\phi_a(\delta_D(y))}{|x-y|}\right)^{1/a}\right) & \text{when } d = 1 = \alpha, \\ \frac{\phi_a(\delta_D(x))\phi_a(\delta_D(y))}{|x-y|} \wedge \left(a^{-1}\left(\phi_a(\delta_D(x))\phi_a(\delta_D(y))\right)^{(\alpha-1)/\alpha}\right) & \text{when } d = 1 < \alpha. \end{cases} \tag{1.7}$$

For $d \ge 2$ and $a > 0$, define

$$g_D^a(x,y) = \begin{cases} \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d \ge 3, \\ \log\left(1 + \frac{a^{2\alpha/(\alpha-2)} \wedge (\delta_D(x)\delta_D(y))}{|x-y|^2}\right) & \text{when } d = 2, \end{cases}$$

for $d=1$ and $a>0$, define

$$g_D^a(x,y)=\begin{cases}(\delta_D(x)\delta_D(y))^{1/2}\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|}\wedge\left(a^{-\alpha}(\delta_D(x)\delta_D(y))^{(\alpha-1)/2}\right) & \text{when } \alpha\in(1,2),\\ \frac{\delta_D(x)\delta_D(y)}{|x-y|}\wedge\log\left(1+a\,(\delta_D(x)\delta_D(y))^{1/2}\right)^{1/a} & \text{when } \alpha=1,\\ (\delta_D(x)\delta_D(y))^{1/2}\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|}\wedge a^{\alpha/(\alpha-2)} & \text{when } \alpha\in(0,1).\end{cases}$$

Theorem 1.7. *Suppose D is a half-space-like $C^{1,1}$ domain with $C^{1,1}$ characteristics (R_0,Λ_0) and $\mathbb{H}_b\subset D\subset\mathbb{H}$ for some $b>0$ such that the path distance in D is comparable to the Euclidean distance with characteristic λ_0. Then for any $M>0$, there exists a constant $c=c(M,R_0,\Lambda_0,\lambda_0,b,\alpha)\geq 1$ such that for all $a\in(0,M]$ and $(x,y)\in D\times D$,*

$$c^{-1}g_D^a(x,y)\leq G_D^a(x,y)\leq cg_D^a(x,y)\qquad \text{when } |x-y|\leq a^{-\alpha/(2-\alpha)}, \tag{1.8}$$
$$c^{-1}f_D^a(x,y)\leq G_D^a(x,y)\leq cf_D^a(x,y)\qquad \text{when } |x-y|\geq a^{-\alpha/(2-\alpha)}. \tag{1.9}$$

Remark 1.8. (i) Note that, when $d\geq 3$, $g_D^a(x,y)$ is independent of a and is comparable to the Green function of Brownian motion in a bounded $C^{1,1}$ domain or in a domain above the graph of a bounded $C^{1,1}$ function. On the other hand, when $d=1$ or 2, $g_D^a(x,y)$ depends on a, which is due to recurrent nature of one- and two-dimensional Brownian motions.

(ii) Observe that if $(X_t^{a,D},t\geq 0)$ is the subprocess in D of the independent sum of a Brownian motion and a symmetric α-stable process in $\mathbb{R}^d$ with weight a, then $(\lambda X^{a,D}_{\lambda^{-2}t},t\geq 0)$ is the subprocess in λD of the independent sum of a Brownian motion and a symmetric α-stable process in $\mathbb{R}^d$ with weight $a\lambda^{(\alpha-2)/\alpha}$ (see the second paragraph of [6, Section 2]). Consequently for any $\lambda>0$, we have

$$p_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(t,x,y)=\lambda^{-d}p_D^a(\lambda^{-2}t,\lambda^{-1}x,\lambda^{-1}y)\qquad \text{for } t>0 \text{ and } x,y\in\lambda D. \tag{1.10}$$

When D is a half space, we see from (1.10) that Theorems 1.4 and 1.7 hold with $M=\infty$.

(iii) The estimates in Theorems 1.4 and 1.7 are uniform in $a\in(0,M]$ in the sense that the constants c_1, c_2 and c in the estimates are independent of $a\in(0,M]$. Since X^a converges weakly to X^0, by taking $a\to 0$ these estimates yield the following estimates for the heat kernel $p_D^0(t,x,y)$ and Green function $G^0(x,y)$ of Brownian motion in half-space-like domains D in which the path distance is comparable to the Euclidean distance:

$$\begin{aligned}&c_1^{-1}\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)t^{-d/2}e^{-c_2|x-y|^2/t}\\ &\leq p_D^0(t,x,y)\leq c_1\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)t^{-d/2}e^{-|x-y|^2/(c_2t)}\end{aligned} \tag{1.11}$$

for every $(t,x,y)\in(0,\infty)\times D\times D$, and

$$c_2^{-1}\,g_D^0(x,y)\;\leq\;G_D^0(x,y)\;\leq\;c_2\,g_D^0(x,y)\qquad \text{for } x,y\in D. \tag{1.12}$$

The estimates (1.11) and (1.12) extend the main results in [21], where the corresponding estimates were established for domains in $\mathbb{R}^d$ with $d\geq 3$ that are above the graphs of bounded $C^{1,1}$ functions.

(iv) By Theorem 1.4, for each fixed $t>0$, when $x\in D$ is near the boundary ∂D of D relative to time t in the sense that $\delta_D(x)\wedge\delta_D(x)^{\alpha/2}\leq\sqrt{t}$, the boundary decay rate of the Dirichlet heat kernel of $\mathcal{L}^1$ is given by $\delta_D(x)\wedge\delta_D(x)^{\alpha/2}/\sqrt{t}$. This indicates that the Dirichlet heat kernel estimates for $\mathcal{L}^1=\Delta+\Delta^{\alpha/2}$ in half-space-like $C^{1,1}$ domains cannot be obtained by a “simple" perturbation argument from Δ nor from $\Delta^{\alpha/2}$.

Since the Lévy process X^a contains a discontinuous component aY, its Lévy system plays an important role in our approach. As

$$a^\alpha|\xi|^\alpha = \int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))\,\frac{a^\alpha \mathcal{A}(d,\alpha)}{|y|^{d+\alpha}}dy,$$

X^a has Lévy intensity function

$$J^a(x,y) = j^a(|x-y|) := a^\alpha \mathcal{A}(d,\alpha)|x-y|^{-(d+\alpha)}.$$

The function $J^a(x,y)$ determines a Lévy system for X^a, which describes the jumps of the process X^a: for any stopping time T (with respect to the filtration of X^a), any $x \in \mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s,y,y)=0$ for all $y \in \mathbb{R}^d$ and $s>0$,

$$\mathbb{E}_x\left[\sum_{s\le T} f(s, X^a_{s-}, X^a_s)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^a_s, y) J^a(X^a_s, y)dy\right) ds\right] \tag{1.13}$$

(see, for example, [9, Proof of Lemma 4.7] and [10, Appendix A]).

Throughout this paper, the constants C_1, C_2, C_3, R_0, R_1, R_2, R_3 will be fixed. The lower case constants $c_1, c_2, \dots$ will denote generic constants whose exact values are not important and can change from one appearance to another. The dependence of the lower case constants on the dimension d will not be mentioned explicitly. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial)=0$. We will use dx or $m(dx)$ to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its d-dimensional Lebesgue measure. For every function f, let $f^+ := f \vee 0$. We now present the

Proof of Proposition 1.2. We first deal with the case $a=1$. For $t \ge c_1$ and $r \ge 0$,

$$t^{-d/2}e^{-cr^2/t} \le t^{-d/2}\frac{c_2}{(cr^2/t)^{(d+\alpha)/2}} \le c_3\frac{t^{\alpha/2}}{r^{d+\alpha}} \le c_4\frac{t}{r^{d+\alpha}}.$$

Hence for $t \ge c_1$,

$$t^{-d/\alpha} \wedge \left(t^{-d/2}e^{-c|x-y|^2/t} + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}. \tag{1.14}$$

Thus $h^1_c(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}$ on $[c_1,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. On the other hand, for $r \ge 1$,

$$t^{-d/2}e^{-cr^2/t} \le t^{-d/2}\frac{c_5}{(cr^2/t)^{(d/2)+1}} = \frac{c_6 t}{r^{d+2}} \le \frac{c_6 t}{r^{d+\alpha}}.$$

So for $t \in (0, c_1]$ and $r \ge 1$,

$$t^{-d/2}e^{-cr^2/t} + \left(t^{-d/2} \wedge \frac{t}{r^{d+\alpha}}\right) \asymp t^{-d/2} \wedge \frac{t}{r^{d+\alpha}} \asymp \frac{t}{r^{d+\alpha}} \asymp t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}}.$$

Thus this and (1.14) prove the proposition for $a=1$. For $a>0$, with $\lambda = a^{\alpha/(2-\alpha)}$,

$$\begin{aligned} h^a_c(t,x,y) &= \lambda^d h^1_c(\lambda^2 t, \lambda x, \lambda y) \\ &\asymp \lambda^d \left(\left(\lambda^2 t\right)^{-d/\alpha} \wedge \frac{\lambda^2 t}{\lambda^{d+\alpha}|x-y|^{d+\alpha}}\right) = (a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}, \end{aligned}$$

provided either $\lambda^2 t \ge c_1$ or $\lambda|x-y| \ge 1$. This completes the proof of the proposition. □

The rest of the paper is organized as follows. In Section 2, we derive some preliminary exit probability estimates that will be used later to obtain large time two-sided estimates on $p_{\mathbb{H}}$. These estimates are derived through some detailed estimates of $\mathcal{L}^1\phi$ for some testing functions ϕ. The upper bound estimate on $p_{\mathbb{H}}$ is established in Section 3, while the lower bound estimate is derived in Section 4. The two-sided Dirichlet heat kernel estimates in half-space-like $C^{1,1}$ domains are then established in Section 5 from that of $p_{\mathbb{H}}$ by a "push in" method (see Lemma 3.7 below) that is originally employed in [12]. Integrating the estimates for $p_D(t, x, y)$ given by Theorem 1.4 yields the uniform sharp estimates of the Green function $G^a_D(x, y)$ of $\mathcal{L}^a$ in D. However it is far from trivial and requires considerable amount of effort. This is done in Section 6.

2 Preliminary estimates

We will focus on the case $D = \mathbb{H}$ in Sections 2–4. In this section we will prove some preliminary estimates that will be used to establish our heat kernel estimates in $\mathbb{H}$. We start with some one-dimensional results.

Let S be the sum of a unit drift and an $\alpha/2$-stable subordinator and let W be an independent one-dimensional Brownian motion. Define a process Z by $Z_t = W_{S_t}$. The process Z is simply the process X^1 in the case of dimension 1 defined in the previous section. We will use the fact that S is a complete subordinator, that is, the Lévy measure of S has a completely monotone density (for more details see [18] or [22]). Let $\overline{Z}_t := \sup\{0 \vee Z_s : 0 \le s \le t\}$ and let L_t be a local time of $\overline{Z} - Z$ at 0. L is also called a local time of the process Z reflected at the supremum. Then the right continuous inverse L^{-1}_t of L is a subordinator and is called the ladder time process of Z. The process $\overline{Z}_{L^{-1}_t}$ is also a subordinator and is called the ladder height process of Z. (For the basic properties of the ladder time and ladder height processes, we refer our readers to [1, Chapter 6].) Let $V(dr)$ denote the potential measure of the ladder height process $\overline{Z}_{L^{-1}_t}$ of Z and $v(r)$ its density, which is a decreasing function on $[0, \infty)$. We know by [16, (5.1)] that

$$v(r) \asymp 1 \wedge r^{\alpha/2-1} \quad \text{for } r > 0. \tag{2.1}$$

Let $G_{(0,\infty)}$ be the Green function of $Z^{(0,\infty)}$, the subprocess of Z in $(0, \infty)$. By using [1, Theorem 20, p. 176] which was originally proved in [19], the following formula for $G_{(0,\infty)}$ was shown in [15, Proposition 2.8]:

$$G_{(0,\infty)}(x, y) = \int_0^{x\wedge y} v(z)v(z + |x - y|)dz. \tag{2.2}$$

For any $r > 0$, let $G_{(0,r)}$ be the Green function of $Z^{(0,r)}$, the subprocess of Z in $(0, r)$. Then we have the following result.

Proposition 2.1. *There exists $c = c(\alpha) > 0$ such that for every $r \in (0, \infty)$,*

$$\int_0^r G_{(0,r)}(x, y)dy \le c(r \wedge r^{\alpha/2}) \left((x \wedge x^{\alpha/2}) \wedge ((r - x) \wedge (r - x)^{\alpha/2}) \right), \quad x \in (0, r).$$

Proof. For every $r > 0$ and every $x \in (0, r)$, we have by (2.2) and (2.1) that

$$\begin{aligned}
\int_0^r G_{(0,r)}(x, y)dy &\le \int_0^r G_{(0,\infty)}(x, y)dy \\
&= \int_0^x \int_{x-y}^x v(z)v(y + z - x)dzdy + \int_x^r \int_0^x v(z)v(y + z - x)dzdy \\
&= \int_0^x v(z) \int_{x-z}^x v(y + z - x)dydz + \int_0^x v(z) \int_x^r v(y + z - x)dydz \\
&\le 2\, V((0, r))\, V((0, x)) \le c(r \wedge r^{\alpha/2})(x \wedge x^{\alpha/2}).
\end{aligned}$$

This together with the property that $G_{(0,r)}(x,y) = G_{(0,r)}(r-x, r-y)$ establishes the proposition. $\square$

Now we return to the process X^1 in $\mathbb{R}^d$. Recall that $C_c^\infty(\mathbb{R}^d)$ is contained in the domain of the L_2-generator $\mathcal{L}^1$ of X^1 and

$$\mathcal{L}^1\phi(x) = \Delta\phi(x) + \int_{\mathbb{R}^d}(\phi(x+y) - \phi(x) - (\nabla\phi(x)\cdot y)1_{B(0,\varepsilon)}(y))j^1(|y|)dy, \quad \forall \phi \in C_c^\infty(\mathbb{R}^d)$$

(see [20, Section 4.1]). Using the argument in [14, page 152], one can easily see that the last formula on [14, page 152] is valid for X^1 for all $d \geq 1$. Thus we have the following generalization of Dynkin's formula: for every ϕ in $C_c^\infty(\mathbb{R}^d)$ and $x \in U$,

$$\mathbb{E}_x\left[\phi\left(X^1_{\tau^1_U}\right)\right] - \phi(x) = \int_U G^1_U(x,y)\mathcal{L}^1\phi(y)dy = \mathbb{E}_x\int_0^{\tau^1_U}\mathcal{L}^1\phi(X^1_s)ds. \tag{2.3}$$

The following estimates on harmonic measures will play a crucial role in Section 3.

Theorem 2.2. *For any $R > 0$, there exists a constant $c = c(\alpha, R) > 0$ such that for every $r \geq R$ and open set $U \subset B(0,r)$,*

$$\mathbb{P}_x\left(X^1_{\tau^1_U} \in B(0,r)^c\right) \;\leq\; c\, r^{-\alpha}\int_U G^1_U(x,y)dy, \quad \textit{for every } x \in U\cap B(0,r/2).$$

Proof. Without loss of generality, we assume that $R \in (0,1)$. Take a sequence of radial functions ϕ_k in $C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \phi_k \leq 1$,

$$\phi_k(y) = \begin{cases} 0, & \text{if } |y| < 1/2 \\ 1, & \text{if } 1 \leq |y| \leq k+1 \\ 0, & \text{if } |y| > k+2, \end{cases}$$

and that $\sum_{i,j}|\frac{\partial^2}{\partial y_i\partial y_j}\phi_k|$ is uniformly bounded. Define $\phi_{k,r}(y) = \phi_k(\frac{y}{r})$. Then we have $0 \leq \phi_{k,r} \leq 1$,

$$\phi_{k,r}(y) = \begin{cases} 0, & \text{if } |y| < r/2 \\ 1, & \text{if } r \leq |y| \leq r(k+1) \\ 0, & \text{if } |y| > r(k+2), \end{cases} \qquad \text{and} \qquad \sup_{y\in\mathbb{R}^d}\sum_{i,j}\left|\frac{\partial^2}{\partial y_i\partial y_j}\phi_{k,r}(y)\right| \;<\; c_1\, r^{-2}.$$

Using this inequality, we have for $r \geq R$

$$\begin{aligned}
&\left|\mathcal{L}^1\phi_{k,r}(z)\right| \\
&\leq c_1\, r^{-2} + \sup_{k\geq 1}\sup_{z\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(\phi_{k,r}(z+y) - \phi_{k,r}(z) - (\nabla\phi_{k,r}(z)\cdot y)1_{B(0,r)}(y))j^1(|y|)dy\right| \\
&\leq c_1\, r^{-2} + c_2\sup_{k\geq 1}\sup_{z\in\mathbb{R}^d}\left(\int_{\{|y|\leq r\}}\left|\frac{\phi_{k,r}(z+y) - \phi_{k,r}(z) - (\nabla\phi_{k,r}(z)\cdot y)}{|y|^{d+\alpha}}\right|dy + \int_{\{r<|y|\}}|y|^{-d-\alpha}dy\right) \\
&\leq c_1\, r^{-2} + c_3\left(\frac{1}{r^2}\int_{\{|y|\leq r\}}\frac{|y|^2}{|y|^{d+\alpha}}dy + \int_{\{r<|y|\}}|y|^{-d-\alpha}dy\right) \;\leq\; c_1\, r^{-2} + c_4 r^{-\alpha}.
\end{aligned} \tag{2.4}$$

When $U \subset B(0,r)$ for some $r \geq R$, we get, by combining (2.3) and (2.4), that for any $x \in U\cap B(0,r/2)$,

$$\mathbb{P}_x\left(X^1_{\tau^1_U} \in B(0,r)^c\right) \leq \lim_{k\to\infty}\mathbb{E}_x\left[\phi_{k,r}\left(X^1_{\tau^1_U}\right)\right] \leq c_5 r^{-\alpha}\int_U G^1_U(x,y)dy.$$

□

In the remainder of this section we will establish a result (Lemma 2.4) that will be crucial for our heat kernel estimates in Section 4.

Recall that the operator $\widehat{\Delta}^{\alpha/2}$ is defined in (1.1) and that $\widehat{\Delta}^{\alpha/2} = \Delta^{\alpha/2}$ on $C_c^\infty(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ and $p > 0$, set $w_p(x) := (x_d^+)^p$. For $0 < p < \alpha < 2$, let

$$\Lambda = \Lambda(\alpha, p) = \frac{p\mathcal{A}(d, -\alpha)}{\alpha} \int_0^1 \frac{t^{\alpha-p-1} - t^{p-1}}{(1-t)^\alpha} dt \int_{|y|=1, y_d \geq 0} y_d^\alpha \, m(dy), \tag{2.5}$$

with the convention that $m(dy)$ is the Dirac measure when $d = 1$. Then it follows from [13, Lemma 6.1] that

$$\widehat{\Delta}^{\alpha/2} w_p(x) = \Lambda(\alpha, p) w_{p-\alpha}(x), \quad x \in \mathbb{H}. \tag{2.6}$$

In particular, on $\mathbb{H}$ we have

$$\widehat{\Delta}^{\alpha/2} w_p < 0, \;\; 0 < p < \alpha/2; \;\; \widehat{\Delta}^{\alpha/2} w_p = 0, \;\; p = \alpha/2; \;\; \widehat{\Delta}^{\alpha/2} w_p > 0, \;\; \alpha/2 < p < \alpha. \tag{2.7}$$

For any $x \in \mathbb{R}^d$ and $a, b > 0$, we define

$$Q_x(a, b) := \{y \in \mathbb{H} : |\widetilde{y} - \widetilde{x}| < a, y_d < b\} \tag{2.8}$$

and $Q_0(a, b)$ will simply be denoted as $Q(a, b)$. Note that, when $d = 1$, $Q_x(a, b) = Q(a, b)$ is simply the open interval (a, b).

Lemma 2.3. *Suppose* $0 < p \leq \frac{\alpha}{2}$ *and* $R > 8$. *Let*

$$h_p(y) := w_p(y) \mathbf{1}_{Q(R,R)}(y), \quad y \in \mathbb{H}. \tag{2.9}$$

There exist constants $c_1, c_2 > 0$ *such that for every* $R > 8$ *and* $x \in Q(2R/3, 2R/3)$,

$$- c_1 (x_d)^{p-\alpha} \leq \widehat{\Delta}^{\alpha/2} h_p(x) \leq \Lambda (x_d)^{p-\alpha} \qquad \text{when } 0 < p < \frac{\alpha}{2} \tag{2.10}$$

and

$$- c_1 R^{-\alpha/2} \leq \widehat{\Delta}^{\alpha/2} h_{\alpha/2}(x) \leq -c_2 R^{-\alpha/2} \qquad \text{when } p = \frac{\alpha}{2}, \tag{2.11}$$

where $\Lambda = \Lambda(\alpha, p) < 0$ *is the constant defined in* (2.5).

Proof. Since $h_p(y) = w_p(y)$ for $y \in Q(R, R)$, by (2.7), we have for any $x \in Q(2R/3, 2R/3)$,

$$\begin{aligned}\widehat{\Delta}^{\alpha/2} h_p(x) &= \widehat{\Delta}^{\alpha/2}(h_p - w_p)(x) + \widehat{\Delta}^{\alpha/2} w_p(x) \\ &= - \int_{Q(R,R)^c} (y_d^+)^p \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} \, dy + \widehat{\Delta}^{\alpha/2} w_p(x).\end{aligned}$$

Observe that for $x \in Q(2R/3, 2R/3)$ and $y \in Q(R, R)^c$, $|y - x| \geq |y|/3$. Thus for $x \in Q(2R/3, 2R/3)$, by the change of variable $z = R^{-1} y$,

$$\begin{aligned}\int_{Q(R,R)^c} \frac{(y_d^+)^p}{|x - y|^{d+\alpha}} dy &\leq c_1 \int_{\{y \in \mathbb{R}^d : |y| > R\}} \frac{1}{|y|^{d+\alpha-p}} dy \leq c_2 R^{p-\alpha} \int_{\{z \in \mathbb{R}^d : |z| > 1\}} \frac{1}{|z|^{d+\alpha-p}} \, dz \\ &\leq c_3 R^{p-\alpha}.\end{aligned}$$

On the other hand, since $|x| \leq 2\sqrt{2}R/3 \leq 2\sqrt{2}(|\widetilde{y}| \vee |y_d|)/3 \leq 2\sqrt{2}|y|/3$ on $Q(R, R)^c$, we have $|x - y| \leq (1 + 2\sqrt{2}/3)|y|$ on $Q(R, R)^c$. Moreover, $y_d^p \geq c_4 |y|^p$ on $\{y_d \geq R, |\widetilde{y}| \leq R\} \subset Q(R, R)^c$. Thus

$$\begin{aligned}&\int_{Q(R,R)^c} \frac{(y_d^+)^p}{|x - y|^{d+\alpha}} dy \geq c_4 \int_{\{y \in \mathbb{R}^d : y_d \geq R, |\widetilde{y}| \leq R\}} \frac{1}{|y|^{d+\alpha-p}} dy \\ &\geq c_5 R^{p-\alpha} \int_{\{z \in \mathbb{R}^d : z_d \geq 1, |\widetilde{z}| \leq 1\}} \frac{1}{|z|^{d+\alpha-p}} \, dz \geq c_6 R^{p-\alpha}.\end{aligned}$$

The conclusion of the lemma now follows from the above three displays and (2.6)–(2.7). □

Lemma 2.4. *There exist $c = c(\alpha) > 0$ and $R_1 = R_1(\alpha) > 2$ such that for every $R > 8R_1$ and $x \in Q(R/4, R/2) \setminus Q(R/4, 2R_1)$, we have*

$$\mathbb{P}_x \left(X^1_{\tau^1_{V_R}} \in Q(R,R) \setminus Q(R, R/2) \right) \geq c \frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{R^{\alpha/2}},$$

where $V_R := Q(R/2, R/2) \setminus Q(R/2, R_1)$.

Proof. Recall that h_p is defined in (2.9). We fix $p := (\alpha/4) \vee (\alpha - 1)$. We choose $R_1 > 2$ large such that

$$\frac{\alpha}{2}(1 - \frac{\alpha}{2})(R_1/2)^{\alpha-2} \leq |\Lambda|, \tag{2.12}$$

where Λ is the constant defined in (2.5). Obviously, with the above value of p, $\Lambda < 0$. For $R > 8R_1$ and $y \in Q(2R/3, 2R/3) \setminus Q(2R/3, R_1/2)$, by Lemma 2.3 and using the fact that $0 \vee (\frac{3\alpha}{2} - 2) < p < \frac{\alpha}{2} < 1$, we obtain

$$\begin{aligned}
& (\Delta + \widehat{\Delta}^{\alpha/2})\Big(h_{\alpha/2}(y) - R_1^{\alpha/2-p} h_p(y)\Big) \\
\geq \;& -\frac{\alpha}{2}(1 - \frac{\alpha}{2})(y_d)^{\frac{\alpha}{2}-2} - c_1 R^{-\alpha/2} - R_1^{\alpha/2-p}\, p(p-1)(y_d)^{p-2} + |\Lambda| R_1^{\alpha/2-p}(y_d)^{p-\alpha} \\
= \;& (y_d)^{p-\alpha}\left(|\Lambda| R_1^{\alpha/2-p} + p(1-p)R_1^{\alpha/2-p}(y_d)^{\alpha-2} - \frac{\alpha}{2}(1-\frac{\alpha}{2})(y_d)^{\frac{3\alpha}{2}-2-p}\right) - c_1 R^{-\alpha/2} \\
\geq \;& (y_d)^{p-\alpha}\left(|\Lambda| R_1^{\alpha/2-p} - \frac{\alpha}{2}(1-\frac{\alpha}{2})(R_1/2)^{\frac{3\alpha}{2}-2-p}\right) - c_1 R^{-\alpha/2}.
\end{aligned}$$

Now, using (2.12), we have, for $y \in Q(2R/3, 2R/3) \setminus Q(2R/3, R_1/2)$,

$$(\Delta + \widehat{\Delta}^{\alpha/2})\Big(h_{\alpha/2}(y) - R_1^{\alpha/2-p} h_p(y)\Big) \geq -c_1 R^{-\alpha/2}. \tag{2.13}$$

Moreover, for $y \in Q(R, R_1)$,

$$(h_{\alpha/2} - R_1^{\alpha/2-p} h_p)(y) = y_d^{\alpha/2}(1 - (R_1/y_d)^{\alpha/2-p}) \leq 0. \tag{2.14}$$

Let g be a nonnegative smooth radial function with compact support in $\mathbb{R}^d$ such that $g(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}^d} g(x)dx = 1$. For $k \geq 1$, define $g_k(x) = 2^{kd} g(2^k x)$. Define

$$u_k(z) := g_k * \Big(h_{\alpha/2} - R_1^{\alpha/2-p} h_p\Big)(z) := \int_{\mathbb{R}^d} g_k(y)(h_{\alpha/2} - R_1^{\alpha/2-p} h_p)(z-y)dy \in C_c^\infty(\mathbb{R}^d).$$

Let $Q_{R,k} := \{z \in \mathbb{H} : \mathrm{dist}(z, Q(R,R)) < 2^{-k}\}$ and $A_k = \{x \in \mathbb{H} : x_d \in (R_1 - 2^{-k}, R_1]\}$. Note that $u_k = 0$ on $Q^c_{R,k}$ and by (2.14), $u_k(z) \leq 0$ for every $k \geq 1$ and $z_d \leq R_1 - 2^{-k}$. Moreover, for $z \in V_R$, by (2.13),

$$\mathcal{L}^1 u_k(z) = (\Delta + \widehat{\Delta}^{\alpha/2}) u_k(z) = g_k * (\Delta + \widehat{\Delta}^{\alpha/2})(h_{\alpha/2} - R_1^{\alpha/2-p} h_p)(z) \geq -c_1 R^{-\alpha/2}.$$

Therefore, using these observations, (2.3) and (2.14), we have for every $x\in V_R$,

$$\begin{aligned}
u_k(x) &= -\mathbb{E}_x\left[\int_0^{\tau^1_{V_R}}\mathcal{L}^1 u_k(X^1_t)dt\right]+\mathbb{E}_x\left[u_k\left(X^1_{\tau^1_{V_R}}\right)\right]\\
&\le c_1R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}]+\mathbb{E}_x\left[u_k\left(X^1_{\tau^1_{V_R}}\right):X^1_{\tau^1_{V_R}}\in Q_{R,k}\setminus Q(R,R_1)\right]\\
&\quad +\mathbb{E}_x\left[u_k\left(X^1_{\tau^1_{V_R}}\right):X^1_{\tau^1_{V_R}}\in A_k\right]\\
&\le c_1R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}]+\sup_{z\in A_k}|u_k(z)|\,\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in A_k\right)\\
&\quad +\left(\sup_{z\in Q_{R,k}\setminus Q(R,R_1)}u_k(z)\right)\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q_{R,k}\setminus Q(R,R_1)\right)\\
&\le c_1R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}]+\sup_{z\in A_k}|u_k(z)|+\left(\sup_{z\in Q_{R,k}}h_{\alpha/2}(z)\right)\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q_{R,k}\setminus Q(R,R_1)\right)\\
&\le c_1R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}]+\sup_{z\in A_k}|u_k(z)|+R^{\alpha/2}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q_{R,k}\setminus Q(R,R_1)\right).
\end{aligned}$$

Since $h_{\alpha/2}(z)-R_1^{\alpha/2-p}h_p(z)=0$ when $z_d=R_1$, $\lim_{k\to\infty}\sup_{z\in A_k}|u_k(z)|=0$. Observe that $Q_{R,k}(R,R)\setminus Q(R,R_1))$ decreases to $\overline{Q(R,R)}\setminus Q(R,R_1)$ as $k\to\infty$. We have

$$\begin{aligned}
\lim_{k\to\infty}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q_{R,k}\setminus Q_1(R,R_1)\right) &= \mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in \overline{Q(R,R)}\setminus Q(R,R_1)\right)\\
&= \mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q(R,R)\setminus Q(R,R_1)\right),
\end{aligned}$$

where the last equality is due to an application of the Lévy system (1.13) and the fact that $\partial Q(R,R)$ has zero Lebesgue measure. Therefore for $x\in Q(R/2,R/2)\setminus Q(R/2,2R_1)$, since $x_d\ge 2R_1$,

$$\begin{aligned}
(1-2^{p-\alpha/2})(x_d)^{\alpha/2} &\le (x_d)^{\alpha/2}(1-(R_1/x_d)^{\alpha/2-p})=\lim_{k\to\infty}u_k(x)\\
&\le c_1R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}]+R^{\alpha/2}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q(R,R)\setminus Q(R,R_1)\right),
\end{aligned}$$

which implies

$$(x_d)^{\alpha/2}\le c_1\frac{R^{-\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{E}_x[\tau^1_{V_R}]+\frac{R^{\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}}\in Q(R,R)\setminus Q(R,R_1)\right). \tag{2.15}$$

Now take a non-negative function ϕ in $C_c^\infty(\mathbb{R}^d)$ such that $0\le\phi\le1$,

$$\phi(y)=\begin{cases}0 & \text{if } |\widetilde{y}|<1/4 \quad\text{or}\quad |y_d|>2,\\ 1 & \text{if } 1/2\le|\widetilde{y}|\le 2 \quad\text{and}\quad |y_d|<1,\\ 0 & \text{if } |\widetilde{y}|>3,\end{cases}$$

and that $\sum_{i,j}|\frac{\partial^2}{\partial y_i\partial y_j}\phi|$ is uniformly bounded. Define $\phi_R(y)=\phi(\frac{y}{R})$. Then we have $0\le\phi_R\le1$,

$$\phi_R(y)=\begin{cases}0 & \text{if } |\widetilde{y}|<R/4 \text{ or } |y_d|>2R,\\ 1 & \text{if } R/2\le|\widetilde{y}|\le 2R \text{ and } |y_d|<R,\\ 0 & \text{if } |\widetilde{y}|>3R,\end{cases}\quad\text{and}\quad \sup_{y\in\mathbb{R}^d}\sum_{i,j}\left|\frac{\partial^2}{\partial y_i\partial y_j}\phi_R(y)\right|<c_2\,R^{-2}. \tag{2.16}$$

Using this inequality, by the argument leading to (2.4), we get $\sup_{z\in\mathbb{R}^d}|\mathcal{L}^1\phi_R(z)|\le c_3\,R^{-\alpha}$ for every $k\ge1$. Thus, by this and the fact that $\widehat{\Delta}^{\alpha/2}h_{\alpha/2}\le0$ on $Q(2R/3,2R/3)$

by Lemma 2.3, we obtain that for $R > 8R_1$ and $y \in Q(2R/3, 2R/3)$,

$$(\Delta + \widehat{\Delta}^{\alpha/2})\Big(h_{\alpha/2}(y) + \frac{2R^{\alpha/2}}{1-2^{p-\alpha/2}}\phi_R(y)\Big) \le -\frac{\alpha}{2}(1-\frac{\alpha}{2})(y_d)^{\frac{\alpha}{2}-2} + c_4 R^{\alpha/2}R^{-\alpha} \le c_4 R^{-\alpha/2}. \tag{2.17}$$

For any $k \ge 1$, define

$$v_k(z) := g_k * \Big(h_{\alpha/2} + \frac{2R^{\alpha/2}}{1-2^{p-\alpha/2}}\phi_R\Big)(z) \in C_c^\infty(\mathbb{R}^d).$$

Put $\Omega_R := Q(R, R/2)\setminus(Q(R, R_1)\cup Q(R/2, R/2))$. By (2.17), we have $\mathcal{L}^1 v_k(y) \le c_4 R^{-\alpha/2}$ for all $y \in V_R$. Thus, using this and (2.3), we have that for any $k \ge 1$ and $x \in Q(R/4, R/2) \setminus Q(R/4, 2R_1)$,

$$\begin{aligned} v_k(x) &= -\mathbb{E}_x\left[\int_0^{\tau^1_{V_R}} \mathcal{L}^1 v_k(X^1_t)dt\right] + \mathbb{E}_x\left[v_k\Big(X^1_{\tau^1_{V_R}}\Big)\right] \\ &\ge -c_4 R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}] + \mathbb{E}_x\left[v_k\Big(X^1_{\tau^1_{V_R}}\Big) : X^1_{\tau^1_{V_R}} \in \Omega_R\right]. \end{aligned}$$

Letting $k \to \infty$ and using (2.16), we get that for any $x \in Q(R/4, R/2) \setminus Q(R/4, 2R_1)$ (where $\phi_R(x) = 0$),

$$\begin{aligned} &(x_d)^{\alpha/2} = \Big(h_{\alpha/2} + \frac{2R^{\alpha/2}}{1-2^{p-\alpha/2}}\phi_R\Big)(x) = \lim_{k\to\infty} v_k(x) \\ &\ge -c_4 R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}] + \mathbb{E}_x\left[\Big(h_{\alpha/2} + \frac{2R^{\alpha/2}}{1-2^{p-\alpha/2}}\phi_R\Big)\Big(X^1_{\tau^1_{V_R}}\Big) : X^1_{\tau^1_{V_R}} \in \Omega_R\right] \\ &\ge -c_4 R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}] + \frac{2R^{\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}} \in \Omega_R\right). \end{aligned} \tag{2.18}$$

Combining (2.15) and (2.18), we get

$$\begin{aligned} (x_d)^{\alpha/2} &\le \frac{c_1 R^{-\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{E}_x[\tau^1_{V_R}] + \frac{R^{\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}} \in Q(R,R)\setminus Q(R,R/2)\right) \\ &\quad + \frac{R^{\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{P}_x\Big(X^1_{\tau^1_{V_R}} \in \Omega_R\Big) \\ &\le \frac{c_1 R^{-\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{E}_x[\tau^1_{V_R}] + \frac{R^{\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}} \in Q(R,R)\setminus Q(R,R/2)\right) \\ &\quad + \frac{1}{2}\left(c_4 R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}] + (x_d)^{\alpha/2}\right). \end{aligned}$$

Therefore, we conclude that

$$(x_d)^{\alpha/2} \le \left(\frac{2c_1}{1-2^{p-\alpha/2}} + c_4\right) R^{-\alpha/2}\mathbb{E}_x[\tau^1_{V_R}] + \frac{2R^{\alpha/2}}{1-2^{p-\alpha/2}}\mathbb{P}_x\left(X^1_{\tau^1_{V_R}} \in Q(R,R)\setminus Q(R,R/2)\right). \tag{2.19}$$

On the other hand, by the Lévy system of X^1,

$$\begin{aligned} &\mathbb{P}_x\left(X^1_{\tau^1_{V_R}} \in Q(R,R)\setminus Q(R,R/2)\right) \ge \mathbb{P}_x\left(X^1_{\tau^1_{V_R}} \in Q(R,R)\setminus Q(R,3R/4)\right) \\ &= \mathbb{E}_x\left[\int_0^{\tau^1_{V_R}}\left(\int_{Q(R,R)\setminus Q(R,3R/4)} J^1(X^1_s, z)dz\right)ds\right] \ge c_5 R^{-\alpha}\,\mathbb{E}_x[\tau^1_{V_R}]. \end{aligned}$$

This together with (2.19) establishes the lemma. □

3 Upper bound heat kernel estimates on half-space

In this section we will establish the desired large time upper bound for $p^1_{\mathbb{H}}(t,x,y)$.

Lemma 3.1. *For any $t_0 > 0$ and $R > 0$, there exists $c = c(\alpha, t_0, R) > 1$ such that for $t \ge t_0$ and $x \in \mathbb{H}$ with $\delta_{\mathbb{H}}(x) = x_d \ge R$, we have*

$$\mathbb{P}_x(\tau^1_{\mathbb{H}} > t) \le c\left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right).$$

Proof. Clearly, we can assume $R \le t_0^{1/\alpha}$ and we only need to show the lemma for $R \le \delta_{\mathbb{H}}(x) < t^{1/\alpha}$. Let $u(x) = (x_d^+)^{\alpha/2} + 1$ and $U(r) := \{x \in \mathbb{H}; x_d < r\}$. By (2.7), for every $x \in \mathbb{H}$ with $\delta_{\mathbb{H}}(x) \ge R$,

$$(\Delta + \widehat{\Delta}^{\alpha/2})u(x) = -\frac{\alpha}{2}(1 - \frac{\alpha}{2})(x_d)^{\alpha/2-2} < 0.$$

Using the same approximation argument as in the proof of Lemma 2.4 with $u_k(z) := (g_k * u)(z)$ where g_k is the function defined in the proof of Lemma 2.4 and letting $k \to \infty$, we see that for $x \in \mathbb{H}$ with $r > \delta_{\mathbb{H}}(x) = x_d > R$,

$$(1 + R^{-\alpha/2})x_d^{\alpha/2} \ge x_d^{\alpha/2} + 1 = u(x) \ge \mathbb{E}_x\left[u\left(X^1_{\tau^1_{U(r)}}\right)\right] \ge r^{\alpha/2}\mathbb{P}_x\left(X^1_{\tau^1_{U(r)}} \in \mathbb{H} \setminus U(r)\right).$$

Applying this and Proposition 2.1, we get that for $R < \delta_{\mathbb{H}}(x) < t^{1/\alpha}$,

$$\begin{aligned}
\mathbb{P}_x\left(\tau^1_{\mathbb{H}} > t\right) &\le \mathbb{P}_x\left(\tau^1_{U(t^{1/\alpha})} > t\right) + \mathbb{P}_x\left(X^1_{\tau^1_{U(t^{1/\alpha})}} \in \mathbb{H} \setminus U(t^{1/\alpha})\right) \\
&\le \frac{1}{t}\mathbb{E}_x\left[\tau^1_{U(t^{1/\alpha})}\right] + (1 + R^{-\alpha/2})\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \\
&\le c_1\frac{1}{t}(t^{1/\alpha} \wedge t^{1/2})(\delta_{\mathbb{H}}(x)^{\alpha/2} \wedge \delta_{\mathbb{H}}(x)) + (1 + R^{-\alpha/2})\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \le c_2\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

□

Lemma 3.2. *For every t_0 and $R > 0$, there exists $c = c(\alpha, t_0, R) > 1$ such that for every $(t, x, y) \in [t_0, \infty) \times \mathbb{H} \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) \ge R$,*

$$p^1_{\mathbb{H}}(t,x,y) \le ct^{-d/\alpha}\left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right).$$

Proof. Let $C(t) := \sup_{z,w \in \mathbb{R}^d} p^1(t/3, z, w)$. By the semigroup property and symmetry,

$$\begin{aligned}
p^1_{\mathbb{H}}(t,x,y) &= \int_{\mathbb{H}}\int_{\mathbb{H}} p^1_{\mathbb{H}}(t/3,x,z)p^1_{\mathbb{H}}(t/3,z,w)p^1_{\mathbb{H}}(t/3,w,y)dzdw \\
&\le C(t)\,\mathbb{P}_x(\tau^1_{\mathbb{H}} > t/3)\mathbb{P}_y(\tau^1_{\mathbb{H}} > t/3).
\end{aligned}$$

Now the lemma follows from Theorem 1.1 and Lemma 3.1. □

The next lemma and its proof are given in [6] (also see [4, Lemma 2] and [5, Lemma 2.2]).

Lemma 3.3. *Suppose that U_1, U_3, E are open subsets of $\mathbb{R}^d$ with $U_1, U_3 \subset E$ and $\mathrm{dist}(U_1, U_3) > 0$. Let $U_2 := E \setminus (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for all $t > 0$,*

$$p^1_E(t,x,y) \le \mathbb{P}_x\left(X^1_{\tau^1_{U_1}} \in U_2\right)\left(\sup_{s<t,\, z\in U_2} p^1_E(s,z,y)\right) + \mathbb{E}_x\left[\tau^1_{U_1}\right]\left(\sup_{u\in U_1,\, z\in U_3} J^1(u,z)\right). \tag{3.1}$$

Lemma 3.4. *Suppose that $t_0, R > 0$. There exists $c = c(\alpha, t_0, R) > 0$ such that for every $(t, x, y) \in [t_0, \infty) \times \mathbb{H} \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) \geq R$,*

$$p^1_{\mathbb{H}}(t, x, y) \leq c \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).$$

Proof. By Theorem 1.1, Proposition 1.2 and Lemma 3.2, without loss of generality we can assume $R = t_0^{1/\alpha}$ and it is enough to prove the lemma for $t_0^{1/\alpha} \leq \delta_{\mathbb{H}}(x) \leq (16)^{-1} t^{1/\alpha}$ and $|x-y| \geq t^{1/\alpha}$. Let $x_0 = (\widetilde{x}, 0)$, $U_1 := B(x_0, 8^{-1}t^{1/\alpha}) \cap \mathbb{H}$, $U_3 := \{z \in \mathbb{H} : |z - x| > |x-y|/2\}$ and $U_2 := \mathbb{H} \setminus (U_1 \cup U_3)$.

Let $X^1 = (X^{1,1}, \dots, X^{1,d})$ and, for any open interval (β, γ) in $\mathbb{R}$, let $\widehat{\tau}_{(\beta,\gamma)} := \inf\{t > 0 : X^{1,d} \notin (\beta, \gamma)\}$. Note that, by Proposition 2.1 and the assumption that $16^{-1}t^{1/\alpha} \geq \delta_{\mathbb{H}}(x) = x_d \geq t_0^{1/\alpha}$, we have

$$\mathbb{E}_x[\tau^1_{U_1}] \;\leq\; \mathbb{E}_{x_d}[\widehat{\tau}_{(0,t^{1/\alpha})}] \;\leq\; c_1 \sqrt{t}\, x_d^{\alpha/2} \;=\; c_1 \sqrt{t}\, \delta_{\mathbb{H}}(x)^{\alpha/2}. \tag{3.2}$$

Since $U_1 \cap U_3 = \emptyset$ and

$$|z - x| > \frac{|x-y|}{2} \geq \frac{1}{2} t^{1/\alpha} \quad \text{for } z \in U_3,$$

we have for $u \in U_1$ and $z \in U_3$,

$$|u - z| \geq |z - x| - |x_0 - x| - |x_0 - u| \geq |z-x| - 4^{-1}t^{1/\alpha} \geq \frac{1}{2}|z - x| \geq \frac{1}{4}|x-y|. \tag{3.3}$$

Thus,

$$\sup_{u \in U_1,\, z \in U_3} J^1(u, z) \leq \sup_{(u,z): |u-z| \geq \frac{1}{4}|x-y|} J^1(u,z) \;\leq\; c_3 |x-y|^{-d-\alpha}. \tag{3.4}$$

If $z \in U_2$,

$$\frac{3}{2}|x-y| \geq |x-y| + |x-z| \geq |z-y| \geq |x-y| - |x-z| \geq \frac{|x-y|}{2} \geq 2^{-1} t^{1/\alpha}. \tag{3.5}$$

By Theorem 1.1 and (3.5),

$$\begin{aligned}
\sup_{s \leq t,\, z \in U_2} p^1(s, z, y) &\leq c_4 \sup_{\substack{s \leq t \\ |x-y|/2 \leq |z-y|}} \big(sJ^1(z,y)\big) + c_4 \sup_{|x-y|/2 \leq |z-y| \leq \sqrt{s} \leq \sqrt{t}} s^{-d/2} \\
&\quad + c_4 \sup_{\substack{s \leq t \\ \sqrt{s} \wedge (|x-y|/2) \leq |z-y| \leq 1}} s^{-d/2} e^{-c_5 |z-y|^2/s} \\
&\leq c_6 t |x-y|^{-d-\alpha} + 2^{d+\alpha} c_4 \left(\sup_{s \leq t} \frac{s^{\alpha/2}}{|x-y|^{d+\alpha}} \right) + c_4 \left(\sup_{a \geq 1} a^{-d/2} e^{-c_5 a} \right) \sup_{|x-y|/2 \leq |z-y| \leq 1} |z-y|^{-d} \\
&\leq c_7 t |x-y|^{-d-\alpha} + c_8 \sup_{|x-y|/2 \leq |z-y| \leq 1} \frac{|z-y|^{\alpha}}{|x-y|^{d+\alpha}} \leq c_9 t |x-y|^{-d-\alpha}.
\end{aligned} \tag{3.6}$$

Applying Lemma 3.3, (3.2), (3.4) and (3.6), we obtain,

$$\begin{aligned}
p^1_{\mathbb{H}}(t, x, y) &\leq c_{10} \mathbb{E}_x[\tau^1_{U_1}] |x-y|^{-d-\alpha} + c_{11} \mathbb{P}_x\Big(X^1_{\tau^1_{U_1}} \in U_2 \Big) t |x-y|^{-d-\alpha} \\
&\leq c_{12} \sqrt{t}\, \delta_{\mathbb{H}}(x)^{\alpha/2} |x-y|^{-d-\alpha} + c_{11} \mathbb{P}_x\Big(X^1_{\tau^1_{U_1}} \in U_2 \Big) t |x-y|^{-d-\alpha}.
\end{aligned}$$

Finally, applying Theorem 2.2 with $U = U_1$ and $r = 8^{-1}t^{1/\alpha} \geq 2{t_0}^{1/\alpha}$, we have

$$\mathbb{P}_x\Big(X^1_{\tau^1_{U_1}} \in U_2 \Big) \leq \mathbb{P}_x\Big(X^1_{\tau^1_{U_1}} \in B(x_0, 8^{-1}t^{1/\alpha})^c \Big) \;\leq\; c_{14} \frac{1}{t} \int_{U_1} G^1_{U_1}(x, y) dy \;=\; c_{14} \frac{1}{t} \mathbb{E}_x[\tau^1_{U_1}].$$

Now applying (3.2), we have proved the lemma. □

Lemma 3.5. *For every $R > 0$ and $t_0 > 0$, there exists a constant $c = c(R, \alpha, t_0)$ such that for all $(t, x, y) \in [t_0, \infty) \times \mathbb{H} \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \geq R$.*

$$p^1_{\mathbb{H}}(t, x, y) \quad \leq \quad c \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).$$

Proof. By Lemma 3.4 and Theorem 1.1, we only need to prove the lemma for $\delta_{\mathbb{H}}(x) \vee \delta_{\mathbb{H}}(y) \leq t^{1/\alpha}$. Denote by $q(t, x, y)$ the transition density of the α-stable process Y in $\mathbb{R}^d$. It is well-known (see, e.g., [2, 9]) that

$$q(t, x, y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.7}$$

By Lemma 3.4 and the lower bound estimate of $q(t, x, y)$ in (3.7), there is a constant $c_1 > 0$ so that

$$\begin{aligned} p^1_{\mathbb{H}}(t/2, x, z) \quad &\leq \quad c_1 \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) q(t/2, x, z) \quad \text{and} \quad p^1_{\mathbb{H}}(t/2, z, y) \\ &\leq \quad c_1 \left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) q(t/2, y, z). \end{aligned}$$

Thus, by semigroup property and the upper bound estimate of $q(t, x, y)$ in (3.7),

$$\begin{aligned} p^1_{\mathbb{H}}(t, x, y) &= \int_{\mathbb{H}} p^1_{\mathbb{H}}(t/2, x, z) p^1_{\mathbb{H}}(t/2, z, y) dz \\ &\leq c_2^2 \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \int_{\mathbb{H}} q(t/2, x, z) q(t/2, y, z) dz \\ &\leq c_2^2 \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) q(t, x, y) \\ &\leq c_3 \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \end{aligned}$$

□

To get the sharp upper bound estimate on $p_{\mathbb{H}}$, we need two results that will be used several times in this paper. Let e_d denote the unit vector in the positive direction of the x_d-axis in $\mathbb{R}^d$.

Lemma 3.6. *Let D be an open set in $\mathbb{R}^d$ so that $\mathbb{H}_b \subset D \subset \mathbb{H}$ for some $b > 0$. For any $t_0 \geq b^2$ and $M > 0$, there exists a constant $c = c(\alpha, M, t_0, b) > 1$ such that for any $a \in (0, M]$ and $(t, x) \in [t_0, \infty) \times D$,*

$$\begin{aligned} (1 \wedge \delta_D(x)) \left(1 \wedge \frac{\delta_{\mathbb{H}}(x_0) \wedge (a^{-1}\delta_{\mathbb{H}}(x_0))^{\alpha/2}}{\sqrt{t}} \right) \quad &\leq \quad c \left(1 \wedge \frac{\delta_D(x) \wedge (a^{-1}\delta_D(x))^{\alpha/2}}{\sqrt{t}} \right), \\ (1 \wedge \delta_D(x)) \left(1 \wedge \frac{\delta_{\mathbb{H}_b}(x_0) \wedge (a^{-1}\delta_{\mathbb{H}_b}(x_0))^{\alpha/2}}{\sqrt{t}} \right) \quad &\geq \quad c^{-1} \left(1 \wedge \frac{\delta_D(x) \wedge (a^{-1}\delta_D(x))^{\alpha/2}}{\sqrt{t}} \right) \end{aligned}$$

where $x_0 := x + 2t_0^{1/2} e_d$.

Proof. Note that $\delta_D(x) + t_0^{1/2} \leq \delta_{\mathbb{H}_b}(x_0) \leq \delta_D(x) + 2t_0^{1/2}$ and $\delta_D(x) + 2t_0^{1/2} \leq \delta_{\mathbb{H}}(x_0) \leq \delta_D(x) + 3t_0^{1/2}$. When $\delta_D(x) > t_0^{1/2}$, we have $\delta_D(x) \leq \delta_{\mathbb{H}_b}(x_0) < \delta_{\mathbb{H}}(x_0) \leq 4\delta_D(x)$. Thus in this case, the conclusion of the lemma is trivial. When $\delta_D(x) \le t_0^{1/2}$, using the fact $t \ge t_0$ and $a \in (0, M]$, we have

$$(1 \wedge \delta_D(x)) \left(1 \wedge \frac{\delta_{\mathbb{H}}(x_0) \wedge (a^{-1}\delta_{\mathbb{H}}(x_0))^{\alpha/2}}{\sqrt{t}}\right) \asymp (1 \wedge \delta_D(x)) \left(1 \wedge \frac{\delta_{\mathbb{H}_b}(x_0) \wedge (a^{-1}\delta_{\mathbb{H}_b}(x_0))^{\alpha/2}}{\sqrt{t}}\right)$$
$$\asymp \delta_D(x) \left(1 \wedge \frac{1}{\sqrt{t}}\right) \asymp 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \asymp 1 \wedge \frac{\delta_D(x) \wedge (a^{-1}\delta_D(x))^{\alpha/2}}{\sqrt{t}}.$$

The proof is now complete. □

The next result will allow us to "push" points in D a fixed distance away from D when doing heat kernel estimates. Such a strategy has been previously used in [12], where global Dirichlet heat kernel estimates are obtained for symmetric α-stable processes in half-space-like $C^{1,1}$-open sets as well as in $C^{1,1}$ exterior open sets.

Lemma 3.7. *Suppose that D is a half-space-like $C^{1,1}$ domain with $C^{1,1}$ characteristics (R_0, Λ_0) and $\mathbb{H}_b \subset D \subset \mathbb{H}$ for some $b > 0$ such that the path distance in D is comparable to the Euclidean distance with characteristic λ_0. Fix $t_0 > b^2$ and define for $x \in D$, $x_0 := x + 2t_0^{1/2} e_d$. Then there exists $c = c(b, t_0, R_0, \Lambda_0, \alpha, \lambda_0) \ge 1$ such that for all $x, z \in D$,*

$$c^{-1} (1 \wedge \delta_D(x)) \le \frac{p_D^1(t_0, x, z)}{p_D^1(t_0, x_0, z)} \le c (1 \wedge \delta_D(x)). \tag{3.8}$$

Proof. First observe that

$$\delta_D(x_0) \ge \delta_{\mathbb{H}}(x_0) > t_0^{1/2}, \tag{3.9}$$

and $|x - x_0| = 2t_0^{1/2}$. Let C_2 be the constant in Theorem 1.3 (i) with $T = t_0$. By Theorem 1.3(i) and (3.9), we see that

$$c_1^{-1} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t_0}}\right) \left(\frac{h^1_{C_2}(t_0, x, z)}{h^1_{1/C_2}(t_0, x_0, z)}\right) \le \frac{p_D^1(t_0, x, z)}{p_D^1(t_0, x_0, z)} \le c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t_0}}\right) \left(\frac{h^1_{1/C_2}(t_0, x, z)}{h^1_{C_2}(t_0, x_0, z)}\right). \tag{3.10}$$

For $z \in B(x_0, 2^{-1}t_0^{1/2})$ we have

$$\frac{3}{2} t_0^{1/2} \le |x_0 - x| - |z - x_0| \le |x - z| \le |z - x_0| + |x_0 - x| = |z - x_0| + 2t_0^{1/2} \le \frac{5}{2} t_0^{1/2}.$$

Similarly, for $z \in B(x, 2^{-1}t_0^{1/2})$ we have $\frac{3}{2} t_0^{1/2} \le |x - z| \le \frac{5}{2} t_0^{1/2}$. Thus in these cases, (3.8) follows from (3.10) and Proposition 1.2.

In the case $z \notin B(x, 2^{-1}t_0^{1/2}) \cup B(x_0, 2^{-1}t_0^{1/2})$, we have $|x - z| \le |z - x_0| + |x_0 - x| = |z - x_0| + 2t_0^{1/2} \le 5|z - x_0|$ and $|x_0 - z| \le |z - x| + |x_0 - x| = |z - x| + 2t_0^{1/2} \le 5|z - x|$. So $5^{-1}|x_0 - z| \le |z - x| \le 5|x_0 - z|$. Therefore using this and Proposition 1.2, we have

$$\frac{h^1_{1/C_2}(t_0, x, z)}{h^1_{C_2}(t_0, x_0, z)} \le c_2 \quad \text{and} \quad \frac{h^1_{C_2}(t_0, x, z)}{h^1_{1/C_2}(t_0, x_0, z)} \ge c_3.$$

□

Theorem 3.8. *Let t_0 be a positive constant. Then there exists a constant $c = c(\alpha, t_0) > 0$ such that for all $t \in [t_0, \infty)$ and $x, y \in \mathbb{H}$,*

$$p^1_{\mathbb{H}}(t, x, y) \le c \left(\frac{\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_{\mathbb{H}}(y) \wedge \delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

Proof. Define for x and y in D,

$$x_0 := x + 2t_0^{1/2} e_d \qquad \text{and} \qquad y_0 := y + 2t_0^{1/2} e_d\,. \tag{3.11}$$

By the semigroup property and (3.8), we have

$$\begin{aligned} p^1_{\mathbb{H}}(t,x,y) &= \int_{\mathbb{H}}\int_{\mathbb{H}} p^1_{\mathbb{H}}(t_0,x,z)p^1_{\mathbb{H}}(t-2t_0,z,w)p^1_{\mathbb{H}}(t_0,w,y)dzdw \\ &\asymp (1\wedge\delta_{\mathbb{H}}(x))\,(1\wedge\delta_{\mathbb{H}}(y)) \int_{\mathbb{H}}\int_{\mathbb{H}} p^1_{\mathbb{H}}(t_0,x_0,z)p^1_{\mathbb{H}}(t-2t_0,z,w)p_{\mathbb{H}}(t_0,w,y_0)dzdw \\ &= (1\wedge\delta_{\mathbb{H}}(x))\,(1\wedge\delta_{\mathbb{H}}(y))\, p^1_{\mathbb{H}}(t,x_0,y_0). \end{aligned} \tag{3.12}$$

By Lemma 3.5 and the fact $|x_0-y_0| = |x-y|$, we have

$$p^1_{\mathbb{H}}(t,x_0,y_0) \le c_1 \left(\frac{\delta_{\mathbb{H}}(x_0)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_{\mathbb{H}}(y_0)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

This together with Lemma 3.6 (with $a=1$ there) and (3.12) proves the theorem. □

4 Lower bound heat kernel estimates on half-space

In this section we establish the desired sharp large time lower bound on $p^1_{\mathbb{H}}(t,x,y)$. We will use some ideas from [4, 6].

Lemma 4.1. *For any positive constant t_0, there exists $c = c(t_0,\alpha) > 0$ such that for any $t \ge t_0$ and $y \in \mathbb{R}^d$,*

$$\mathbb{P}_y\left(\tau^1_{B(y,8^{-1}t^{1/\alpha})} > t/3\right) \;\ge\; c.$$

Proof. By [11, Proposition 6.2], there exists $\varepsilon = \varepsilon(t_0,\alpha) > 0$ such that for every $t \ge t_0$,

$$\inf_{y\in\mathbb{R}^d} \mathbb{P}_y\left(\tau^1_{B(y,16^{-1}t^{1/\alpha})} > \varepsilon t\right) \ge \frac{1}{2}.$$

Suppose $\varepsilon < \frac{1}{3}$, then by the parabolic Harnack inequality in [11, 23],

$$c_1\, p^1_{B(y,8^{-1}t^{1/\alpha})}(\varepsilon t, y, w) \;\le\; p^1_{B(y,8^{-1}t^{1/\alpha})}(t/3,y,w) \qquad \text{for } w \in B(y,16^{-1}t^{1/\alpha}),$$

where the constant $c_1 = c_1(t_0,\alpha) > 0$ is independent of $y \in \mathbb{R}^d$. Thus

$$\begin{aligned} \mathbb{P}_y\left(\tau^1_{B(y,8^{-1}t^{1/\alpha})} > t/3\right) &= \int_{B(y,8^{-1}t^{1/\alpha})} p^1_{B(y,8^{-1}t^{1/\alpha})}(t/3,y,w)dw \\ &\ge c_1 \int_{B(y,16^{-1}t^{1/\alpha})} p^1_{B(y,8^{-1}t^{1/\alpha})}(\varepsilon t,y,w)dw \ge \frac{c_1}{2}. \end{aligned}$$

□

The next result holds for any symmetric discontinuous Hunt process that possesses a transition density and whose Lévy system admits a jumping density kernel. The proof is the same as that of [7, Lemma 3.3] and so it is omitted here.

Lemma 4.2. *Suppose that U_1, U_2, U are open subsets of $\mathbb{R}^d$ with $U_1, U_2 \subset U$ and $\mathrm{dist}(U_1,U_2) > 0$. If $x \in U_1$ and $y \in U_2$, then for all $t > 0$,*

$$p^1_U(t,x,y) \;\ge\; t\,\mathbb{P}_x(\tau^1_{U_1} > t)\,\mathbb{P}_y(\tau^1_{U_2} > t) \inf_{u\in U_1,\, z\in U_2} J^1(u,z)\,. \tag{4.1}$$

Lemma 4.3. *Suppose that $t_0 > 0$. There exists $c = c(t_0, \alpha) > 0$ such that for all $t \geq t_0$ and $u, v \in \mathbb{R}^d$ with $|u - v| \geq t^{1/\alpha}/2$,*

$$p^1_{B(u,t^{1/\alpha}) \cup B(v,t^{1/\alpha})}(t/3, u, v) \ \geq \ c\, t\, |u - v|^{-d-\alpha}.$$

Proof. Let $U = B(u, t^{1/\alpha}) \cup B(v, t^{1/\alpha})$, $U_1 = B(u, t^{1/\alpha}/8)$, $U_2 = B(v, t^{1/\alpha}/8)$ and $K = \inf_{w \in U_1,\, z \in U_2} j^1(|w - z|)$. We have by Lemma 4.2 that

$$p^1_U(t/3, u, v) \geq \frac{Kt}{3}\, \mathbb{P}_u(\tau^1_{U_1} > t/3)\, \mathbb{P}_v(\tau^1_{U_2} > t/3)\,.$$

Moreover, for $(w, z) \in U_1 \times U_2$, $|w-z| \leq |u-v| + |w-u| + |z-v| \leq |u-v| + t^{1/\alpha}/4 \leq \frac{3}{2}|u-v|$. Hence $K \geq c_1 |u - v|^{-d-\alpha}$. Thus by Lemma 4.1,

$$\frac{Kt}{3}\, \left(\mathbb{P}_0(\tau^1_{B(0,t^{1/\alpha}/8)} > t/3)\right)^2 \geq c_2\, t\, |u - v|^{-d-\alpha}.$$

□

The next result follows from [23, Proposition 3.4].

Lemma 4.4. *There exist $R_2 = R_2(\alpha) > 1$ and $c = c(\alpha) > 0$ such that for all $t \geq R_2^\alpha$,*

$$\inf_{x,y \in B(0,6t^{1/\alpha})} p^1_{B(0,12t^{1/\alpha})}(t/3, x, y) \ \geq \ c\, t^{-d/\alpha}.$$

For the remainder of this section, we define $R_3 := R_1 \vee R_2$, where $R_1 > 2$ is the constant in Lemma 2.4. Recall that $Q_x(a, b)$ is defined in (2.8).

Lemma 4.5. *There is a positive constant $c = c(\alpha)$ such that for all $(t, x) \in ((4R_1)^\alpha, \infty) \times \mathbb{H}$ with $2R_1 < \delta_{\mathbb{H}}(x) < t^{1/\alpha}/2$,*

$$\mathbb{P}_x(\tau^1_{Q_x(2t^{1/\alpha}, 2t^{1/\alpha})} > t/3) \geq c\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}.$$

Proof. Without loss of generality we assume that $\widetilde{x} = \widetilde{0}$. Recall that $Q(a, b) = Q_0(a, b)$. Let $V(t) := Q(t^{1/\alpha}/2, t^{1/\alpha}/2) \setminus Q(t^{1/\alpha}/2, R_1)$. By Lemma 2.4, Lemma 4.1 and the strong Markov property,

$$\begin{aligned}
& \mathbb{P}_x\Big(\tau^1_{Q(2t^{1/\alpha}, 2t^{1/\alpha})} > t/3\Big) \\
\geq\ & \mathbb{P}_x\Big(\tau^1_{Q(2t^{1/\alpha}, 2t^{1/\alpha})} > t/3,\ X^1_{\tau^1_{V(t)}} \in Q(t^{1/\alpha}, t^{1/\alpha}) \setminus Q(t^{1/\alpha}, t^{1/\alpha}/2)\Big) \\
=\ & \mathbb{E}_x\Big[\mathbb{P}_{X^1_{\tau^1_{V(t)}}}\Big(\tau^1_{Q(2t^{1/\alpha}, 2t^{1/\alpha})} > t/3\Big) : X^1_{\tau^1_{V(t)}} \in Q(t^{1/\alpha}, t^{1/\alpha}) \setminus Q(t^{1/\alpha}, t^{1/\alpha}/2)\Big] \\
\geq\ & \mathbb{E}_x\Big[\mathbb{P}_{X^1_{\tau^1_{V(t)}}}\Big(\tau^1_{B(X^1_{\tau^1_{V(t)}}, 4^{-1}t^{1/\alpha})} > t/3\Big) : X^1_{\tau^1_{V(t)}} \in Q(t^{1/\alpha}, t^{1/\alpha}) \setminus Q(t^{1/\alpha}, t^{1/\alpha}/2)\Big] \\
\geq\ & c_1 \mathbb{P}_x\Big(X^1_{\tau^1_{V(t)}} \in Q(t^{1/\alpha}, t^{1/\alpha}) \setminus Q(t^{1/\alpha}, 2^{-1}t^{1/\alpha})\Big) \ \geq \ c_2 \frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

This proves the Lemma. □

Recall that e_d denote the unit vector in the positive direction of the x_d-axis in $\mathbb{R}^d$.

Lemma 4.6. *There is a positive constant $c = c(\alpha)$ such that for all $(t, x, y) \in [(4R_3)^\alpha, \infty) \times \mathbb{H} \times \mathbb{H}$ with $\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(y) \geq 2R_3$,*

$$p^1_{\mathbb{H}}(t, x, y) \geq c \left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

Proof. Fix $x, y \in \mathbb{H}$. Let $x_0 = (\widetilde{x}, 0)$, $y_0 = (\widetilde{y}, 0)$, $\xi_x := x + 32t^{1/\alpha}e_d$ and $\xi_y := y + 32t^{1/\alpha}e_d$. If $2R_3 \le \delta_{\mathbb{H}}(x) < t^{1/\alpha}/2$, by Lemmas 4.1, 4.2 and 4.5,

$$\begin{aligned}
&\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)du \\
\ge& t\,\mathbb{P}_x\left(\tau^1_{Q_x(2t^{1/\alpha},2t^{1/\alpha})} > t/3\right)\left(\inf_{\substack{v\in Q_x(2t^{1/\alpha},2t^{1/\alpha})\\ w\in B(\xi_x,4t^{1/\alpha})}} J^1(v,w)\right)\int_{B(\xi_x,2t^{1/\alpha})}\mathbb{P}_u\left(\tau^1_{B(\xi_x,4t^{1/\alpha})} > t/3\right)du \\
\ge& c_1 t\,\mathbb{P}_x\left(\tau^1_{Q_x(2t^{1/\alpha},2t^{1/\alpha})} > t/3\right)t^{-d/\alpha-1}\,\mathbb{P}_0\left(\tau^1_{B(0,t^{1/\alpha}/8)} > t/3\right)|B(\xi_x,2t^{1/\alpha})| \\
\ge& c_2\mathbb{P}_x\left(\tau^1_{Q_x(2t^{1/\alpha},2t^{1/\alpha})} > t/3\right) \;\ge\; c_3\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

On the other hand, if $\delta_{\mathbb{H}}(x) \ge t^{1/\alpha}/2 \ge 2R_3$, by Lemmas 4.1 and 4.2,

$$\begin{aligned}
&\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)du \\
\ge& t\,\mathbb{P}_x\left(\tau^1_{B(x,8^{-1}t^{1/\alpha})\cap\mathbb{H}} > t/3\right)\left(\inf_{\substack{v\in B(x_0,2t^{1/\alpha})\cap\mathbb{H}\\ w\in B(\xi_x,4t^{1/\alpha})}} J^1(v,w)\right)\int_{B(\xi_x,2t^{1/\alpha})}\mathbb{P}_u\left(\tau^1_{B(\xi_x,4t^{1/\alpha})} > t/3\right)du \\
\ge& c_4 t\,\mathbb{P}_x\left(\tau^1_{B(x,8^{-1}t^{1/\alpha})} > t/3\right)t^{-d/\alpha-1}\,\mathbb{P}_0\left(\tau^1_{B(0,t^{1/\alpha}/8)} > t/3\right)|B(\xi_x,2t^{1/\alpha})| \\
\ge& c_5\mathbb{P}_x\left(\tau^1_{B(x,8^{-1}t^{1/\alpha})} > t/3\right) \ge c_6.
\end{aligned}$$

Thus

$$\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)du \ge c_7\left(1\wedge\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}\right), \tag{4.2}$$

and similarly,

$$\int_{B(\xi_y,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,y,u)du \ge c_7\left(1\wedge\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}}\right). \tag{4.3}$$

Now we deal with the cases $|x-y| \ge 5t^{1/\alpha}$ and $|x-y| < 5t^{1/\alpha}$ separately.

Case 1: Suppose that $|x-y| \ge 5t^{1/\alpha}$. Note that by the semigroup property and Lemma 4.3,

$$\begin{aligned}
&p^1_{\mathbb{H}}(t,x,y) \\
\ge& \int_{B(\xi_y,2t^{1/\alpha})}\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)p^1_{\mathbb{H}}(t/3,u,v)p^1_{\mathbb{H}}(t/3,v,y)dudv \\
\ge& \int_{B(\xi_y,2t^{1/\alpha})}\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)p^1_{B(u,t^{1/\alpha})\cup B(v,t^{1/\alpha})}(t/3,u,v)p^1_{\mathbb{H}}(t/3,v,y)dudv \\
\ge& c_8 t\left(\inf_{(u,v)\in B(\xi_x,2t^{1/\alpha})\times B(\xi_y,2t^{1/\alpha})}|u-v|^{-d-\alpha}\right)\int_{B(\xi_y,2t^{1/\alpha})}\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)p^1_{\mathbb{H}}(t/3,v,y)dudv.
\end{aligned}$$

It then follows from (4.2)–(4.3) that

$$p^1_{\mathbb{H}}(t,x,y) \ge c_9 t\left(\inf_{(u,v)\in B(\xi_x,2t^{1/\alpha})\times B(\xi_y,2t^{1/\alpha})}|u-v|^{-d-\alpha}\right)\left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}}\wedge 1\right). \tag{4.4}$$

Using the assumption $|x-y| \geq 5t^{1/\alpha}$ we get that, for $u \in B(\xi_x, 2t^{1/\alpha})$ and $v \in B(\xi_y, 2t^{1/\alpha})$, $|u-v| \leq 4t^{1/\alpha} + |x-y| \leq 2|x-y|$. Hence

$$\inf_{(u,v)\in B(\xi_x,2t^{1/\alpha})\times B(\xi_y,2t^{1/\alpha})} |u-v|^{-d-\alpha} \;\geq\; c_{10}|x-y|^{-d-\alpha}. \tag{4.5}$$

By (4.4) and (4.5), we conclude that for $|x-y| \geq 5t^{1/\alpha}$

$$p^1_{\mathbb{H}}(t,x,y) \geq c_{11}\left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) t|x-y|^{-d-\alpha}.$$

Case 2: Suppose $|x-y| < 5t^{1/\alpha}$. In this case, for every $(u,v) \in B(\xi_x, 2t^{1/\alpha}) \times B(\xi_y, 2t^{1/\alpha})$, $|u-v| \leq 9t^{1/\alpha}$. Thus, using the fact that $\delta_{\mathbb{H}}(\xi_x) \wedge \delta_{\mathbb{H}}(\xi_y) \geq 32t^{1/\alpha}$, there exists $w_0 \in \mathbb{H}$ such that

$$B(\xi_x, 2t^{1/\alpha}) \cup B(\xi_y, 2t^{1/\alpha}) \subset B(w_0, 6t^{1/\alpha}) \subset B(w_0, 12t^{1/\alpha}) \subset \mathbb{H}. \tag{4.6}$$

Now, by the semigroup property and (4.6), we get

$$\begin{aligned}
&p^1_{\mathbb{H}}(t,x,y)\\
&\geq \int_{B(\xi_y,2t^{1/\alpha})}\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)p^1_{B(w_0,8t^{1/\alpha})}(t/3,u,v)p^1_{\mathbb{H}}(t/3,v,y)dudv\\
&\geq \left(\inf_{u,v\in B(w_0,6t^{1/\alpha})} p^1_{B(w_0,12t^{1/\alpha})}(t/3,u,v)\right)\int_{B(\xi_y,2t^{1/\alpha})}\int_{B(\xi_x,2t^{1/\alpha})} p^1_{\mathbb{H}}(t/3,x,u)p^1_{\mathbb{H}}(t/3,v,y)dudv.
\end{aligned}$$

It then follows from (4.2)–(4.3) and Lemma 4.4 that

$$p^1_{\mathbb{H}}(t,x,y) \geq c_{12}\left(\frac{\delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) t^{-d/\alpha}.$$

Combining these two cases, we have proved the lemma. □

Theorem 4.7. *There exists a positive constant $c = c(\alpha)$ such that for all $t \in [(4R_3)^\alpha, \infty)$ and $x, y \in \mathbb{H}$,*

$$p^1_{\mathbb{H}}(t,x,y) \geq c\left(1 \wedge \frac{\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_{\mathbb{H}}(y) \wedge \delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

Proof. Let $t_0 = (4R_3)^2 > (4R_3)^\alpha$ and let x_0 and y_0 be as in (3.11). By the semigroup property and (3.8) we have

$$\begin{aligned}
p^1_{\mathbb{H}}(t,x,y) &= \int_{\mathbb{H}}\int_{\mathbb{H}} p^1_{\mathbb{H}}(t_0,x,z)p^1_{\mathbb{H}}(t-2t_0,z,w)p^1_{\mathbb{H}}(t_0,w,y)dzdw\\
&\asymp (1 \wedge \delta_{\mathbb{H}}(x))\,(1 \wedge \delta_{\mathbb{H}}(y))\int_{\mathbb{H}}\int_{\mathbb{H}} p^1_{\mathbb{H}}(t_0,x_0,z)p^1_{\mathbb{H}}(t-2t_0,z,w)p^1_{\mathbb{H}}(t_0,w,y_0)dzdw\\
&= (1 \wedge \delta_{\mathbb{H}}(x))\,(1 \wedge \delta_{\mathbb{H}}(y))\,p^1_{\mathbb{H}}(t,x_0,y_0).
\end{aligned} \tag{4.7}$$

Since, $\delta_{\mathbb{H}}(x_0) \wedge \delta_{\mathbb{H}}(y_0) > t_0^{1/2} = 4R_3$, by Lemma 4.6 and the fact $|x_0 - y_0| = |x-y|$,

$$\begin{aligned}
p^1_{\mathbb{H}}(t,x_0,y_0) &\geq c_1\left(\frac{\delta_{\mathbb{H}}(x_0)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_{\mathbb{H}}(y_0)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\\
&\geq c_1\left(\frac{\delta_{\mathbb{H}_{\sqrt{t_0}}}(x_0)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_{\mathbb{H}_{\sqrt{t_0}}}(y_0)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).
\end{aligned}$$

The conclusion of the theorem now follows from the above inequality, Lemma 3.6 and (4.7). □

5 Heat kernel estimates on half-space-like domains

In this section, we will establish the main result of this paper. In the remainder of this paper we will always assume that D is a half-space-like $C^{1,1}$ domain with $C^{1,1}$ characteristics (R_0, Λ_0) and $\mathbb{H}_b \subset D \subset \mathbb{H}$ for some $b > 0$ such that the path distance in D is comparable to the Euclidean distance with characteristic λ_0. Fix $t_0 := 1 \vee b^2$ and for $x, y \in D$, let x_0 and y_0 be defined as in (3.11). Using Theorem 1.3(i), the following result can be proved in a similar way as that for Lemma 3.7.

Lemma 5.1. *For any $M > 0$, there exists $c = c(b, R_0, \Lambda_0, \alpha, \lambda_0) \geq 1$ such that for all $a \in (0, M]$ and $x, z \in D$,*

$$\begin{aligned} &c^{-1} (1 \wedge \delta_D(x)) (1 \wedge \delta_D(z)) \, h^a_{25C_2}(t_0, x_0, z) \\ &\quad \leq p^a_D(t_0, x, z) \leq c \, (1 \wedge \delta_D(x)) (1 \wedge \delta_D(z)) \, h^a_{1/(25C_2)}(t_0, x_0, z), \end{aligned} \tag{5.1}$$

where C_2 is the constant in Theorem 1.3(i) with $T = t_0$.

Combining Theorem 1.3(i), Theorems 3.8 and 4.7, we get that for every $T > 0$, there exist constants $c_i = c_i(\alpha, T) \geq 1$, $i = 1, 2$, such that for all $(t, x, y) \in (0, T] \times \mathbb{H} \times \mathbb{H}$,

$$\begin{aligned} &c_1^{-1} \left(1 \wedge \frac{\delta_{\mathbb{H}}(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-c_2|x-y|^2/t} + \left(\frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) \\ &\leq p^1_{\mathbb{H}}(t, x, y) \leq c_1 \left(1 \wedge \frac{\delta_{\mathbb{H}}(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-|x-y|^2/(c_2 t)} + \left(\frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) \end{aligned}$$

and for all $t \in [T, \infty)$ and x, y in H,

$$\begin{aligned} &c_1^{-1} \left(1 \wedge \frac{\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y) \wedge \delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ &\leq p^1_{\mathbb{H}}(t, x, y) \leq c_1 \left(1 \wedge \frac{\delta_{\mathbb{H}}(x) \wedge \delta_{\mathbb{H}}(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y) \wedge \delta_{\mathbb{H}}(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

Now using (1.10), we established Theorem 1.4 for $D = \mathbb{H}$ in the form of (1.5)–(1.6).

Theorem 5.2. *For every $T > 0$, there exist $c = c(\alpha, T) \geq 1$ and $C_3 = C_3(\alpha, T) \geq 1$ such that for all $a > 0$ and $(t, x, y) \in (0, a^{2\alpha/(\alpha-2)}T] \times \mathbb{H} \times \mathbb{H}$,*

$$\begin{aligned} &c^{-1} \left(1 \wedge \frac{\delta_{\mathbb{H}}(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-C_3|x-y|^2/t} + \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) \\ &\leq p^a_{\mathbb{H}}(t, x, y) \leq c \left(1 \wedge \frac{\delta_{\mathbb{H}}(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-|x-y|^2/(C_3 t)} + \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) \end{aligned}$$

and for all $t \in [a^{2\alpha/(\alpha-2)}T, \infty)$ and x, y in $\mathbb{H}$,

$$\begin{aligned} &c^{-1} \left(1 \wedge \frac{\delta_{\mathbb{H}}(x) \wedge (a^{-1}\delta_{\mathbb{H}}(x))^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y) \wedge (a^{-1}\delta_{\mathbb{H}}(y))^{\alpha/2}}{\sqrt{t}}\right) \left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right) \\ &\leq p^a_{\mathbb{H}}(t, x, y) \\ &\leq c \left(1 \wedge \frac{\delta_{\mathbb{H}}(x) \wedge (a^{-1}\delta_{\mathbb{H}}(x))^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_{\mathbb{H}}(y) \wedge (a^{-1}\delta_{\mathbb{H}}(y))^{\alpha/2}}{\sqrt{t}}\right) \left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

Now we are in a position to establish the main result of this paper.

Proof of Theorem 1.4. We first observe the following trivial inequalities

$$p^a_{\mathbb{H}_b}(t, x, y) \leq p^a_D(t, x, y) \leq p^a_{\mathbb{H}}(t, x, y), \quad a > 0, (t, x, y) \in (0, \infty) \times \mathbb{H}_b \times \mathbb{H}_b. \tag{5.2}$$

Recall that $t_0 = 1 \vee b^2$. It follows from Theorem 1.3 that we only need to prove the theorem for $t > 3t_0$. Now we suppose $t > 3t_0$. For any $x, y \in D$, we define x_0 and y_0 as in (3.11). By the semigroup property and Lemma 5.1, we have

$$\begin{aligned}
p^a_D(t,x,y) &= \int_{D\times D} p^a_D(t_0,x,z)p^a_D(t-2t_0,z,w)p^a_D(t_0,w,y)dzdw \\
&\le c_1\,(1\wedge\delta_D(x))(1\wedge\delta_D(y))\int_{D\times D} h^a_{1/(25C_2)}(t_0,x_0,z)p^a_D(t-2t_0,z,w)h^a_{1/(25C_2)}(t_0,w,y_0)dzdw.
\end{aligned}$$

It follows from Theorem 5.2 with $T = 1$ and (5.2),

$$\begin{aligned}
&p^a_D(t-2t_0,z,w) \le p^a_{\mathbb{H}}(t-2t_0,z,w) \\
&\le c_2 \begin{cases}
\left(1\wedge\frac{\delta_{\mathbb{H}}(z)}{\sqrt{t-2t_0}}\right)\left(1\wedge\frac{\delta_{\mathbb{H}}(w)}{\sqrt{t-2t_0}}\right)\left((t-2t_0)^{-d/2}e^{-|z-w|^2/(C_3(t-2t_0))}+\left(\frac{a^\alpha(t-2t_0)}{|z-w|^{d+\alpha}}\wedge(t-2t_0)^{-d/2}\right)\right), \\
\qquad\qquad \text{for } t\in(2t_0, 2t_0+a^{2\alpha/(\alpha-2)}]; \\
\left(1\wedge\frac{\delta_{\mathbb{H}}(z)\wedge(a^{-1}\delta_{\mathbb{H}}(z))^{\alpha/2}}{\sqrt{t-2t_0}}\right)\left(1\wedge\frac{\delta_{\mathbb{H}}(w)\wedge(a^{-1}\delta_{\mathbb{H}}(w))^{\alpha/2}}{\sqrt{t-2t_0}}\right)\left((a^\alpha(t-2t_0))^{-d/\alpha}\wedge\frac{a^\alpha(t-2t_0)}{|z-w|^{d+\alpha}}\right), \\
\qquad\qquad \text{for } t\ge 2t_0+a^{2\alpha/(\alpha-2)},
\end{cases}
\end{aligned}$$

where C_3 is the constant in Theorem 5.2 with $T = 1$. Put $A = (C_3 \vee (25C_2))$ where C_2 is the constant in Theorem 1.3 with $T = t_0$. Applying Theorem 5.2 with $T = 1$ again, we get $p^a_D(t-2t_0,z,w) \le c_3 p^a_{\mathbb{H}}(t-2t_0, A^{-2}z, A^{-2}w)$ and so, by Theorem 1.3

$$\begin{aligned}
&p^a_D(t,x,y) \\
\le& c_4\,(1\wedge\delta_D(x))(1\wedge\delta_D(y))\int_{D\times D} h^a_{1/A}(t_0,x_0,z)p^a_{\mathbb{H}}(t-2t_0,A^{-2}z,A^{-2}w)h^a_{1/A}(t_0,w,y_0)dzdw \\
\le& c_4\,(1\wedge\delta_D(x))(1\wedge\delta_D(y))\int_{\mathbb{H}\times\mathbb{H}} h^a_{1/A}(t_0,x_0,z)p^a_{\mathbb{H}}(t-2t_0,A^{-2}z,A^{-2}w)h^a_{1/A}(t_0,w,y_0)dzdw \\
\le& c_5\,(1\wedge\delta_D(x))(1\wedge\delta_D(y))\int_{\mathbb{H}\times\mathbb{H}}(1\wedge\delta_{\mathbb{H}_{-b/2}}(z))(1\wedge\delta_{\mathbb{H}_{-b/2}}(x_0))h^a_{1/A}(t_0,x_0,z) \\
&\quad\times p^a_{\mathbb{H}}(t-2t_0,A^{-2}z,A^{-2}w)(1\wedge\delta_{\mathbb{H}_{-b/2}}(y_0))(1\wedge\delta_{\mathbb{H}_{-b/2}}(w))h^a_{1/A}(t_0,w,y_0)dzdw \\
\le& c_6\,(1\wedge\delta_D(x))(1\wedge\delta_D(y))\int_{\mathbb{H}_{-b/2}\times\mathbb{H}_{-b/2}}(1\wedge\delta_{\mathbb{H}_{-b/2}}(z))(1\wedge\delta_{\mathbb{H}_{-b/2}}(x_0)) \\
&\quad\times\left(t_0^{-d/2}e^{-|x_0-z|^2/(At_0)}+\left(\frac{a^\alpha t_0}{|x_0-z|^{d+\alpha}}\wedge t_0^{-d/2}\right)\right)p^a_{\mathbb{H}}(t-2t_0,A^{-2}z,A^{-2}w) \\
&\quad\times(1\wedge\delta_{\mathbb{H}_{-b/2}}(y_0))(1\wedge\delta_{\mathbb{H}_{-b/2}}(w))\left(t_0^{-d/2}e^{-|w-y_0|^2/(At_0)}+\left(\frac{a^\alpha t_0}{|w-y_0|^{d+\alpha}}\wedge t_0^{-d/2}\right)\right)dzdw.
\end{aligned}$$

Thus, by a change of variable $\widehat{z} = A^{-2}z$, $\widehat{w} = A^{-2}w$, and using (5.2) and Theorem 1.3, the above is less than or equal to $(1\wedge\delta_D(x))(1\wedge\delta_D(y))$ times

$$\begin{aligned}
&c_7\int_{\mathbb{H}_{-b/(2A^2)}\times\mathbb{H}_{-b/(2A^2)}}(1\wedge\delta_{\mathbb{H}_{-b/(2A^2)}}(\widehat{z}))(1\wedge\delta_{\mathbb{H}_{-b/(2A^2)}}(A^{-2}x_0)) \\
&\quad\times\left(t_0^{-d/2}e^{-C_2|A^{-2}x_0-\widehat{z}|^2/t_0}+\left(\frac{a^\alpha t_0}{|A^{-2}x_0-\widehat{z}|^{d+\alpha}}\wedge t_0^{-d/2}\right)\right) \\
&\quad\times p^a_{\mathbb{H}_{-b/(2A^2)}}(t-2t_0,\widehat{z},\widehat{w})(1\wedge\delta_{\mathbb{H}_{-b/(2A^2)}}(A^{-2}y_0)) \\
&\quad\times(1\wedge\delta_{\mathbb{H}_{-b/(2A^2)}}(\widehat{w}))\left(t_0^{-d/2}e^{-C_2|\widehat{w}-A^2y_0|^2/t_0}+\left(\frac{a^\alpha t_0}{|\widehat{w}-A^{-2}y_0|^{d+\alpha}}\wedge t_0^{-d/2}\right)\right)d\widehat{z}d\widehat{w} \\
\le& c_8\int_{\mathbb{H}_{-b/(2A^2)}\times\mathbb{H}_{-b/(2A^2)}} p^a_{\mathbb{H}_{-b/(2A^2)}}(t_0,A^{-2}x_0,\widehat{z})p^a_{\mathbb{H}_{-b/(2A^2)}}(t-2t_0,\widehat{z},\widehat{w})p^a_{\mathbb{H}_{-b/(2A^2)}}(t_0,\widehat{w},A^{-2}y_0)d\widehat{z}d\widehat{w} \\
=& c_8\,p^a_{\mathbb{H}_{-b/(2A^2)}}(t,A^{-2}x_0,A^{-2}y_0).
\end{aligned}$$

Now using (1.10) and Theorem 5.2 with $T = A^{-4}(1 \wedge M^{2\alpha/(2-\alpha)})t_0$, we get

$$p_D^a(t,x,y) \le c_9(1 \wedge \delta_D(x))(1 \wedge \delta_D(y)) p_{\mathbb{H}_{-b/2}}^{A^{2(\alpha-2)/\alpha}a}(A^4 t, x_0, y_0)$$

$$\le c_{10}\begin{cases} (1 \wedge \delta_D(x))\left(\frac{\delta_{\mathbb{H}_{-b/2}}(x_0)}{\sqrt{t}} \wedge 1\right)(1 \wedge \delta_D(y))\left(\frac{\delta_{\mathbb{H}_{-b/2}}(y_0)}{\sqrt{t}} \wedge 1\right) & \\ \quad \times \left(t^{-d/2}e^{-|x-y|^2/(c_{11}t)} + \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) & \text{for } t \in (3t_0, t_0 a^{-2\alpha/(2-\alpha)}], \\ (1 \wedge \delta_D(x))\left(\frac{\delta_{\mathbb{H}_{-b/2}}(x_0) \wedge (a^{-1}\delta_{\mathbb{H}_{-b/2}}(x_0))^{\alpha/2}}{\sqrt{t}} \wedge 1\right)(1 \wedge \delta_D(y)) & \\ \quad \times \left(\frac{\delta_{\mathbb{H}_{-b/2}}(y_0) \wedge (a^{-1}\delta_{\mathbb{H}_{-b/2}}(y_0))^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right) & \text{for } t > t_0/a^{2\alpha/(2-\alpha)} \end{cases}$$

$$\le c_{12}\begin{cases} (1 \wedge \delta_D(x))\left(\frac{\delta_{\mathbb{H}}(x_0)}{\sqrt{t}} \wedge 1\right)(1 \wedge \delta_D(y))\left(\frac{\delta_{\mathbb{H}}(y_0)}{\sqrt{t}} \wedge 1\right) & \\ \quad \times \left(t^{-d/2}e^{-|x-y|^2/(c_{11}t)} + \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) & \text{for } t \in (3t_0, t_0 a^{-2\alpha/(2-\alpha)}]; \\ (1 \wedge \delta_D(x))\left(\frac{\delta_{\mathbb{H}}(x_0) \wedge (a^{-1}\delta_{\mathbb{H}}(x_0))^{\alpha/2}}{\sqrt{t}} \wedge 1\right)(1 \wedge \delta_D(y)) & \\ \quad \times \left(\frac{\delta_{\mathbb{H}}(y_0) \wedge (a^{-1}\delta_{\mathbb{H}}(y_0))^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right) & \text{for } t > t_0/a^{2\alpha/(2-\alpha)}. \end{cases}$$

In the case when $t > M^{2\alpha/(2-\alpha)} t_0 a^{2\alpha/(\alpha-2)}$, since $M^{2\alpha/(2-\alpha)} t_0 a^{2\alpha/(\alpha-2)} \ge t_0$, the desired result follows from (5.2), Lemma 3.6, Theorem 5.2 and Remark 1.5(ii). In the case when $3t_0 < t \le M^{2\alpha/(2-\alpha)} t_0 a^{2\alpha/(\alpha-2)}$, the desired upper bound follows from (5.2), Theorem 5.2, Remark 1.5(ii) and [12, Lemma 2.2] (with α there replaced by 2).

The lower bound can be proved similarly. We omit the details. □

6 Green function estimates

In this section, we give the full proof of Theorem 1.7. Recall that D is a fixed half-space-like $C^{1,1}$ domain with $C^{1,1}$ characteristics (R_0, Λ_0) and $\mathbb{H}_b \subset D \subset \mathbb{H}$ for some $b > 0$ such that the path distance in D is comparable to the Euclidean distance with characteristic λ_0. We first establish a few lemmas.

Recall that $\phi_a(r) = r \wedge (r/a)^{\alpha/2}$. When $a = 1$, we simply denote ϕ_1 by ϕ; that is, $\phi(r) = r \wedge r^{\alpha/2}$.

Lemma 6.1. *For every $r \in (0,1]$ and every open subset U of $\mathbb{R}^d$,*

$$\frac{1}{2}\left(1 \wedge \frac{r^2\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha}\right) \le \left(1 \wedge \frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1 \wedge \frac{r\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right) \le 1 \wedge \frac{r^2\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha}. \tag{6.1}$$

Proof. The second inequality holds trivially. Without loss of generality, we assume $\delta_U(x) \le \delta_U(y)$. If both $\frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}}$ and $\frac{r\phi(\delta_U(y))}{|x-y|^{\alpha/2}}$ are less than 1 or if both are larger than one,

$$\left(1 \wedge \frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1 \wedge \frac{r\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right) = 1 \wedge \frac{r^2\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha}.$$

So we only need to consider the case when $\frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}} \le 1 < \frac{r\phi(\delta_U(y))}{|x-y|^{\alpha/2}}$. Note that $\phi(\delta_U(y)) \le \phi(\delta_U(x) + |x-y|)$. If $\delta_U(x) \ge |x-y|$, then $\phi(\delta_U(y)) \le \phi(2\delta_U(x)) \le 2\phi(\delta_U(x))$ and so

$$1 \wedge \frac{r^2\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha} \le 1 \wedge 2\left(\frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)^2 \le 2\left(1 \wedge \frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right).$$

When $\delta_U(x) < |x-y|$, then $\phi(\delta_U(y)) \le \phi(2|x-y|) \le 2|x-y|^{\alpha/2}$ and so

$$1 \wedge \frac{r^2\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha} \le 1 \wedge \frac{2r^2\phi(\delta_U(x))|x-y|^{\alpha/2}}{|x-y|^\alpha} \le 2\left(1 \wedge \frac{r\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)$$

where the assumption $r \le 1$ is used in the last inequality. This establishes the first inequality of (6.1). □

For every open subset U of $\mathbb{R}^d$ and $a > 0$, let

$$q^a_U(t,x,y) := \left(1 \wedge \frac{\phi_a(\delta_U(x))}{\sqrt{t}}\right)\left(1 \wedge \frac{\phi_a(\delta_U(y))}{\sqrt{t}}\right)\left((a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}\right). \tag{6.2}$$

The following lemma is a direct consequence of (the proof of) Proposition 1.2, Theorem 1.4 and Remark 1.5(ii).

Lemma 6.2. *For every positive constants c_1, c_2, there exists $c_3 = c_3(c_1,c_2) > 1$ such that for every $a > 0$, $t \le c_1 a^{-2\alpha/(2-\alpha)}$, every open subset U of $\mathbb{R}^d$ and $x, y \in U$ with $|x-y| \ge a^{-\alpha/(2-\alpha)}$,*

$$c_3^{-1}\left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right) h^a_{c_2}(t,x,y) \le q^a_U(t,x,y) \le c_3\left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right) h^a_{c_2}(t,x,y). \tag{6.3}$$

Under the assumption of Theorem 1.4, there is a constant $c = c(M, R_0, \Lambda_0, \lambda_0, \alpha, b) \ge 1$ such that

$$c^{-1} q^a_D(t,x,y) \le p^a_D(t,x,y) \le c q^a_D(t,x,y)$$

holds for every $a \in (0, M]$, $t < \infty$, $x, y \in D$ with $|x-y| \ge a^{-\alpha/(2-\alpha)}$.

Observe that

$$\phi_a(\delta_D(\lambda x)) = \left(\lambda \delta_{\lambda^{-1}D}(x)\right) \wedge \left(\lambda^{\alpha/2} a^{-\alpha/2} \delta_{\lambda^{-1}D}(x)^{\alpha/2}\right) \qquad \text{for every } \lambda > 0. \tag{6.4}$$

Let $x_a := a^{\alpha/(2-\alpha)}x$, $y_a := a^{\alpha/(2-\alpha)}y$ and $D_a := a^{\alpha/(2-\alpha)}D$. By (6.4),

$$\phi_a(\delta_D(x)) = \phi_a(\delta_D(a^{-\alpha/(2-\alpha)}x_a)) = a^{-\alpha/(2-\alpha)}\phi(\delta_{D_a}(x_a)) \tag{6.5}$$

and so, for every $s > 0$,

$$q^a_D(a^{-2\alpha/(2-\alpha)}s, x, y) = q^a_D(a^{-2\alpha/(2-\alpha)}s, a^{-\alpha/(2-\alpha)}x_a, a^{-\alpha/(2-\alpha)}y_a) = a^{\alpha d/(2-\alpha)} q^1_{D_a}(s, x_a, y_a). \tag{6.6}$$

We recall that $f^a_D(x,y)$ is defined in (1.7).

Lemma 6.3. *For every $d \ge 1$ and $x, y \in D$, $\int_0^\infty q^a_D(t,x,y)dt \asymp f^a_D(x,y)$, where the implicit constants are independent of D.*

Proof. Let U be an arbitrary open subset of $\mathbb{R}^d$. We first consider the case $a = 1$ and prove the lemma for U. By a change of variable $u = \frac{|x-y|^\alpha}{t}$, we have

$$\begin{aligned}
&\int_0^\infty q^1_U(t,x,y)dt \\
&= \frac{1}{|x-y|^{d-\alpha}}\left(\int_0^1 + \int_1^\infty\right)\left(u^{(d/\alpha)-2} \wedge u^{-3}\right)\left(1 \wedge \frac{\sqrt{u}\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1 \wedge \frac{\sqrt{u}\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right) du \\
&=: I + II.
\end{aligned} \tag{6.7}$$

Note that

$$\begin{aligned}
& \frac{1}{2|x-y|^{d-\alpha}}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right) \\
= & \frac{1}{|x-y|^{d-\alpha}}\int_1^\infty u^{-3}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right)du \\
\le & \; II = \frac{1}{|x-y|^{d-\alpha}}\int_1^\infty u^{-2}\left(u^{-1/2}\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(u^{-1/2}\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right)du \\
\le & \frac{1}{|x-y|^{d-\alpha}}\int_1^\infty u^{-2}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right)du \\
= & \frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right). \qquad (6.8)
\end{aligned}$$

(i) Assume $d > \alpha$. Observe that

$$\begin{aligned}
I & \le \frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right)\int_0^1 u^{(d/\alpha)-2}du \\
& \le \frac{\alpha}{d-\alpha}\frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right). \qquad (6.9)
\end{aligned}$$

So by (6.7)–(6.9),

$$\int_0^\infty q_U^1(t,x,y)dt \asymp \frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\phi(\delta_U(x))}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_U(y))}{|x-y|^{\alpha/2}}\right). \qquad (6.10)$$

For the rest of the proof, we assume without loss of generality that $\delta_U(x) \le \delta_U(y)$ and define

$$u_0 := \frac{\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha}.$$

(ii) Now assume $d = \alpha = 1$. We have by Lemma 6.1,

$$\begin{aligned}
I & \asymp \int_0^1 u^{-1}\mathbf{1}_{\{u\ge 1/u_0\}}du + \int_0^1 u_0\mathbf{1}_{\{u<1/u_0\}}du \\
& = \log(u_0\vee 1) + u_0\left((1/u_0)\wedge 1\right) = \log(u_0\vee 1) + (u_0\wedge 1). \qquad (6.11)
\end{aligned}$$

Now by Lemma 6.1, (6.7)-(6.8) and (6.11), we have

$$\int_0^\infty q_U^1(t,x,y)dt \asymp \log(u_0\vee 1) + 1\wedge u_0 \asymp \log(1+u_0).$$

(iii) Lastly we consider the case $d = 1 < \alpha < 2$. By Lemma 6.1,

$$\begin{aligned}
I & \asymp \frac{1}{|x-y|^{1-\alpha}}\left(\int_0^1 u^{(1/\alpha)-2}\mathbf{1}_{\{u\ge 1/u_0\}}du + \int_0^1 u_0 u^{(1/\alpha)-1}\mathbf{1}_{\{u<1/u_0\}}du\right) \\
& = \frac{1}{|x-y|^{1-\alpha}}\left(\frac{\alpha}{\alpha-1}\left((u_0\vee 1)^{1-(1/\alpha)}-1\right) + \alpha u_0(u_0\vee 1)^{-1/\alpha}\right).
\end{aligned}$$

Hence by (6.7)-(6.8), Lemma 6.1 and the last display we have

$$\begin{aligned}
& \int_0^\infty q_U^1(t,x,y)dt \\
& \asymp \frac{1}{|x-y|^{1-\alpha}}(1\wedge u_0) + \frac{1}{|x-y|^{1-\alpha}}\left(\left((u_0\vee 1)^{1-(1/\alpha)}-1\right) + u_0(u_0\vee 1)^{-1/\alpha}\right) \\
& \asymp \frac{1}{|x-y|^{1-\alpha}}\left(u_0\wedge u_0^{1-(1/\alpha)}\right) = \frac{\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|}\wedge\left(\phi(\delta_U(x))\phi(\delta_U(y))\right)^{(\alpha-1)/\alpha}.
\end{aligned}$$

Thus we have proved the lemma for any open set U and $a = 1$. For general $a > 0$, we have by (6.5) and (6.6) that

$$\int_0^\infty q_D^a(t,x,y)dt \;=\; a^{-2\alpha/(2-\alpha)}\int_0^\infty q_D^a(a^{-2\alpha/(2-\alpha)}s,x,y)ds \;=\; a^{\alpha(d-2)/(2-\alpha)}\int_0^\infty q^1_{D_a}(s,x_a,y_a)ds$$

$$\asymp a^{\alpha(d-2)/(2-\alpha)}\begin{cases}\frac{1}{|x_a-y_a|^{d-\alpha}}\left(1\wedge\frac{\phi(\delta_{D_a}(x_a))}{|x_a-y_a|^{\alpha/2}}\right)\left(1\wedge\frac{\phi(\delta_{D_a}(y_a))}{|x_a-y_a|^{\alpha/2}}\right) & \text{when } d>\alpha,\\ \log\left(1+\frac{\phi(\delta_{D_a}(x_a))\phi(\delta_{D_a}(y_a))}{|x_a-y_a|^{\alpha}}\right) & \text{when } d=1=\alpha,\\ \frac{\phi(\delta_{D_a}(x_a))\phi(\delta_{D_a}(y_a))}{|x_a-y_a|}\wedge(\phi(\delta_{D_a}(x_a))\phi(\delta_{D_a}(y_a)))^{(\alpha-1)/\alpha} & \text{when } d=1<\alpha.\end{cases}$$

$$= a^{\alpha(d-2)/(2-\alpha)}\begin{cases}\frac{a^{-(d-\alpha)\alpha/(2-\alpha)}}{|x-y|^{d-\alpha}}\left(1\wedge\frac{a^{\alpha/(2-\alpha)}\phi_a(\delta_D(x))}{a^{\alpha^2/2(2-\alpha)}|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{a^{\alpha/(2-\alpha)}\phi_a(\delta_D(y))}{a^{\alpha^2/2(2-\alpha)}|x-y|^{\alpha/2}}\right) & \text{when } d>\alpha,\\ \log\left(1+\frac{a^2\phi_a(\delta_D(x))\phi_a(\delta_D(y))}{a|x-y|}\right) & \text{when } d=1=\alpha,\\ \frac{a^{2\alpha/(2-\alpha)}\phi_a(\delta_D(x))\phi_a(\delta_D(y))}{a^{\alpha/(2-\alpha)}|x-y|}\wedge\left(a^{2\alpha/(2-\alpha)}\phi_a(\delta_D(x))\phi_a(\delta_D(y))\right)^{(\alpha-1)/\alpha} & \text{when } d=1<\alpha\end{cases}$$

$$= f_D^a(x,y).$$

□

Lemma 6.4. *For every $c > 0$, when $|x-y| \le a^{-\alpha/(2-\alpha)}$,*

$$\int_0^{a^{-2\alpha/(2-\alpha)}}\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)\left[t^{-d/2}e^{-c\frac{|x-y|^2}{t}}+\left(\frac{a^\alpha t}{|x-y|^{d+\alpha}}\wedge t^{-d/2}\right)\right]dt$$
$$\asymp \begin{cases}|x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d\ge 3,\\ \log(1+\frac{a^{2\alpha/(\alpha-2)}\wedge(\delta_D(x)\delta_D(y))}{|x-y|^2}) & \text{when } d=2,\\ a^{\alpha/(\alpha-2)}\wedge(\delta_D(x)\delta_D(y))^{1/2}\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|} & \text{when } d=1,\end{cases}$$

where the implicit constants depend only on c, α and d.

Proof. We first consider the case $a = 1$ and assume U is an arbitrary open set and $x, y \in U$ with $|x-y| \le 1$. Using the change of variables $u = \frac{|x-y|^2}{t}$, we have

$$\int_0^1\left(1\wedge\frac{\delta_U(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_U(y)}{\sqrt{t}}\right)\left[t^{-d/2}e^{-c_1\frac{|x-y|^2}{t}}+\left(\frac{t}{|x-y|^{d+\alpha}}\wedge t^{-d/2}\right)\right]dt$$
$$=|x-y|^{2-d}\left(\int_{|x-y|^2}^2+\int_2^\infty\right)\left(1\wedge\frac{\sqrt{u}\delta_U(x)}{|x-y|}\right)\left(1\wedge\frac{\sqrt{u}\delta_U(y)}{|x-y|}\right)\left[u^{d/2}e^{-c_1u}+\left(\frac{|x-y|^{2-\alpha}}{u}\wedge u^{d/2}\right)\right]\frac{du}{u^2}$$
$$=:I_1+I_2\,.$$

Note that since $|x-y|^{2-\alpha} \le 1$, for $u \ge 2$, $\frac{|x-y|^{2-\alpha}}{u}\wedge u^{d/2} = \frac{|x-y|^{2-\alpha}}{u}$. Thus for any $d \ge 1$,

$$\begin{aligned} I_2 &= |x-y|^{2-d}\int_2^\infty\left(u^{-1/2}\wedge\frac{\delta_U(x)}{|x-y|}\right)\left(u^{-1/2}\wedge\frac{\delta_U(y)}{|x-y|}\right)\left[u^{d/2}e^{-c_1u}+\frac{|x-y|^{2-\alpha}}{u}\right]\frac{du}{u}\\ &\le |x-y|^{2-d}\left(1\wedge\frac{\delta_U(x)}{|x-y|}\right)\left(1\wedge\frac{\delta_U(y)}{|x-y|}\right)\int_2^\infty\left(u^{d/2-1}e^{-c_1u}+u^{-2}\right)du\\ &\le c_2|x-y|^{2-d}\left(1\wedge\frac{\delta_U(x)}{|x-y|}\right)\left(1\wedge\frac{\delta_U(y)}{|x-y|}\right)\end{aligned}$$

and

$$\begin{aligned} I_2 &\geq |x-y|^{2-d} \int_2^\infty \left(1 \wedge \frac{\delta_U(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_U(y)}{|x-y|}\right) \left[u^{d/2} e^{-c_1 u} + \frac{|x-y|^{2-\alpha}}{u}\right] \frac{du}{u^2} \\ &\geq |x-y|^{2-d} \left(1 \wedge \frac{\delta_U(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_U(y)}{|x-y|}\right) \int_2^\infty u^{d/2-2} e^{-c_1 u}\, du \\ &\geq c_3 |x-y|^{2-d} \left(1 \wedge \frac{\delta_U(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_U(y)}{|x-y|}\right). \end{aligned}$$

One the other hand, since $|x-y|^{2-\alpha} \leq 1$, if $u \leq 2$, then

$$u^{-2} \left[u^{d/2} e^{-c_1 u} + \left(\frac{|x-y|^{2-\alpha}}{u} \wedge u^{d/2}\right)\right] \asymp u^{d/2-2}.$$

Using this and the fact that for every $r \in (0, 2]$,

$$\left(1 \wedge \frac{r\delta_U(x)}{|x-y|}\right) \left(1 \wedge \frac{r\delta_U(y)}{|x-y|}\right) \leq 1 \wedge \frac{r^2 \delta_U(x)\delta_U(y)}{|x-y|^2} \leq 4 \left(1 \wedge \frac{r\delta_U(x)}{|x-y|}\right) \left(1 \wedge \frac{r\delta_U(y)}{|x-y|}\right), \tag{6.12}$$

we have

$$I_1 \asymp |x-y|^{2-d} \int_{|x-y|^2}^2 \left(1 \wedge \frac{u\delta_U(x)\delta_U(y)}{|x-y|^2}\right) u^{d/2-2}\, du.$$

Let $u_0 := \frac{\delta_U(x)\delta_U(y)}{|x-y|^2}$.

(i) When $d \geq 3$, it is easy to see that $I_1 \leq |x-y|^{2-d} (1 \wedge u_0)$.

(ii) Assume $d = 2$. We deal with three cases separately.

(a) $u_0 \leq 1$: In this case, since $|x-y| \leq 1$, we have $\delta_U(x)\delta_U(y) \leq 1$ and $I_1 \asymp \int_{|x-y|^2}^2 u_0 du \asymp u_0 \asymp \log(1+u_0)$.

(b) $u_0 > 1$ and $|x-y|^2 \leq 1/u_0$: In this case we have $\delta_U(x)\delta_U(y) \leq 1$ and

$$\begin{aligned} I_1 &\asymp \int_{|x-y|^2}^{u_0^{-1}} u_0 du + \int_{u_0^{-1}}^2 u^{-1} du = u_0(u_0^{-1} - |x-y|^2) + \log 2 + \log u_0 \\ &= (1 - u_0|x-y|^2) + \log 2 + \log u_0 \asymp \log(1+u_0). \end{aligned}$$

(c) $u_0 > 1$ and $|x-y|^2 > 1/u_0$: In this case we have $\delta_U(x)\delta_U(y) \geq 1$ and

$$I_1 \asymp \int_{|x-y|^2}^2 u^{-1} du = \log 2 + \log|x-y|^{-2} \asymp \log(1 + |x-y|^{-2}) = \log\left(1 + \frac{1 \wedge (\delta_U(x)\delta_U(y))}{|x-y|^2}\right).$$

(iii) Now we consider the case $d = 1$. We again deal with three cases separately.

(a) $u_0 \leq 1$. In this case we have

$$I_1 \asymp |x-y| \int_{|x-y|^2}^2 u_0 u^{-1/2} du \asymp |x-y| u_0 (\sqrt{2} - |x-y|) \asymp |x-y| u_0.$$

(b) $u_0 > 1$ and $|x-y|^2 \leq 1/u_0$. In this case we have

$$\begin{aligned} I_1 &\asymp |x-y| \int_{|x-y|^2}^{u_0^{-1}} u_0 u^{-1/2} du + |x-y| \int_{u_0^{-1}}^2 u^{-3/2} du \\ &\asymp u_0 |x-y| (u_0^{-1/2} - |x-y|) + |x-y| (u_0^{1/2} - 2^{-1/2}) \asymp |x-y| u_0^{1/2}. \end{aligned}$$

(c) $u_0 > 1$ and $|x-y|^2 > 1/u_0$. In this case we have

$$I_1 \asymp |x-y| \int_{|x-y|^2}^2 u^{-3/2} du \asymp |x-y| (|x-y|^{-1} - 2^{-1/2}) \asymp 1 - 2^{-1/2}|x-y| \asymp 1.$$

So we have

$$I_1+I_2 \asymp \begin{cases} |x-y|^{2-d}\left(1\wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|^2}\right) & \text{when } d\geq 3,\\ \log(1+\frac{1\wedge(\delta_U(x)\delta_U(y))}{|x-y|^2}) & \text{when } d=2,\\ 1\wedge(\delta_U(x)\delta_U(y))^{1/2}\ \wedge\ \frac{\delta_U(x)\delta_U(y)}{|x-y|} & \text{when } d=1. \end{cases} \tag{6.13}$$

Thus we have proved the lemma for any open set U and $a=1$. For general $a>0$, we have by (6.5), (6.6) and (6.13),

$$\begin{aligned}
&\int_0^{a^{-2\alpha/(2-\alpha)}}\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)\left[t^{-d/2}e^{-c_1\frac{|x-y|^2}{t}}+\left(\frac{a^\alpha t}{|x-y|^{d+\alpha}}\wedge t^{-d/2}\right)\right]dt\\
=&a^{-2\alpha/(2-\alpha)}\int_0^1\left(1\wedge\frac{\delta_D(x)}{a^{-\alpha/(2-\alpha)}\sqrt{s}}\right)\left(1\wedge\frac{\delta_D(y)}{a^{-\alpha/(2-\alpha)}\sqrt{s}}\right)\\
&\times\left[(a^{-2\alpha/(2-\alpha)}s)^{-d/2}e^{-c_1\frac{|x-y|^2}{a^{-2\alpha/(2-\alpha)}s}}+\left(\frac{a^\alpha a^{-2\alpha/(2-\alpha)}s}{|x-y|^{d+\alpha}}\wedge(a^{-2\alpha/(2-\alpha)}s)^{-d/2}\right)\right]ds\\
=&a^{\alpha(d-2)/(2-\alpha)}\int_0^1\left(1\wedge\frac{\delta_{D_a}(x_a)}{\sqrt{s}}\right)\left(1\wedge\frac{\delta_{D_a}(y_a)}{\sqrt{s}}\right)\left[s^{-d/2}e^{-c_1\frac{|x_a-y_a|^2}{s}}+\left(\frac{s}{|x_a-y_a|^{d+\alpha}}\wedge s^{-d/2}\right)\right]ds\\
\asymp&a^{\alpha(d-2)/(2-\alpha)}\begin{cases} |x_a-y_a|^{2-d}\left(1\wedge\frac{\delta_{D_a}(x_a)\delta_{D_a}(y_a)}{|x_a-y_a|^2}\right) & \text{when } d\geq 3,\\ \log(1+\frac{1\wedge(\delta_{D_a}(x_a)\delta_{D_a}(y_a))}{|x_a-y_a|^2}) & \text{when } d=2,\\ 1\wedge(\delta_{D_a}(x_a)\delta_{D_a}(y_a))^{1/2}\ \wedge\ \frac{\delta_{D_a}(x_a)\delta_{D_a}(y_a)}{|x_a-y_a|} & \text{when } d=1\end{cases}\\
=&\begin{cases} |x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d\geq 3,\\ \log(1+\frac{a^{2\alpha/(\alpha-2)}\wedge(\delta_D(x)\delta_D(y))}{|x-y|^2}) & \text{when } d=2,\\ a^{\alpha/(\alpha-2)}\wedge(\delta_D(x)\delta_D(y))^{1/2}\ \wedge\ \frac{\delta_D(x)\delta_D(y)}{|x-y|} & \text{when } d=1.\end{cases}
\end{aligned}$$

□

Lemma 6.5. *For every $d\geq 2$, there exists $c=c(\alpha,d)>1$ such that, for every $a>0$, when $|x-y|\leq a^{-\alpha/(2-\alpha)}$,*

$$\int_{a^{-2\alpha/(2-\alpha)}}^\infty q_D^a(t,x,y)\,dt\leq ca^{\alpha(d-2)/(2-\alpha)}\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right).$$

Proof. We first consider the case $a=1$ and assume U is an arbitrary open set and $x,y\in U$ with $|x-y|\leq 1$. Let $J:=\int_1^\infty q_U^1(t,x,y)\,dt$. By a change of variables $u=\frac{|x-y|^\alpha}{t}$,

$$\begin{aligned} J &= |x-y|^{\alpha-d}\int_0^{|x-y|^\alpha}\left(1\wedge\frac{\sqrt{u}(\delta_U(x)\wedge\delta_U(x)^{\alpha/2})}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\sqrt{u}(\delta_U(y)\wedge\delta_U(y)^{\alpha/2})}{|x-y|^{\alpha/2}}\right)\\ &\quad\times\left(u^{d/\alpha}\wedge u^{-1}\right)\frac{du}{u^2}\,. \end{aligned} \tag{6.14}$$

Since $|x-y|\leq 1$, for $u\in[0,\,|x-y|^\alpha]$, $u^{d/\alpha}\wedge u^{-1}=u^{d/\alpha}$. Hence

$$\begin{aligned} J &\leq |x-y|^{\alpha-d}\left(1\wedge\frac{\delta_U(x)\wedge\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\delta_U(y)\wedge\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\int_0^{|x-y|^\alpha}u^{d/\alpha-2}\,du\\ &= c_1\left(1\wedge\frac{\delta_U(x)\wedge\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\delta_U(y)\wedge\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right). \end{aligned}$$

Since $|x-y| \le |x-y|^{\alpha/2} \le 1$, we have that $\frac{1}{|x-y|^{\alpha/2}} \le \frac{1}{|x-y|}$ and so $1 \wedge \frac{\delta_U(x)\wedge\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}} \le 1 \wedge \frac{\delta_U(x)}{|x-y|}$. Consequently, it follows from Lemma 6.1 by taking $a=1$ and $\alpha=2$ there that

$$J \le c_1 \left(1 \wedge \frac{\delta_U(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_U(y)}{|x-y|}\right) \le 2c_1 \left(1 \wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|^2}\right). \tag{6.15}$$

Thus we have proved the lemma for any open set U and $a=1$. For general $a>0$, by (6.5), (6.6) and (6.15), we have for every $a>0$,

$$\begin{aligned}&\int_{a^{-2\alpha/(2-\alpha)}}^{\infty} q_D^a(t,x,y)dt = a^{\alpha(d-2)/(2-\alpha)} \int_1^\infty q^1_{D_a}(s,x_a,y_a)ds \\ &\le 2c_1 a^{\alpha(d-2)/(2-\alpha)} \left(1 \wedge \frac{\delta_{D_a}(x_a)\delta_{D_a}(y_a)}{|x_a-y_a|^2}\right).\end{aligned}$$

□

Lemma 6.6. *For every $c>0$, when $d=1$ and $|x-y| \le a^{-\alpha/(2-\alpha)}$,*

$$\begin{aligned}&\int_0^{a^{-2\alpha/(2-\alpha)}} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\left(t^{-d/2}e^{-c\frac{|x-y|^2}{t}} + \left(\frac{a^\alpha t}{|x-y|^{d+\alpha}} \wedge t^{-d/2}\right)\right) dt \\ &+ \int_{a^{-2\alpha/(2-\alpha)}}^\infty q_D^a(t,x,y)\, dt \asymp g_D^a(x,y)\end{aligned}$$

where the implicit constants depend only on c and α.

Proof. We first consider the case $a=1$ and assume U is an arbitrary open set and $x,y \in U$ with $|x-y| \le 1$. Let $J := \int_1^\infty q_U^1(t,x,y)\,dt$ and

$$I := \int_0^1 \left(1 \wedge \frac{\delta_U(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)}{\sqrt{t}}\right)\left(t^{-1/2}e^{-c_1\frac{|x-y|^2}{t}} + \left(\frac{t}{|x-y|^{1+\alpha}} \wedge t^{-1/2}\right)\right) dt.$$

By Lemma 6.4, $I \asymp 1 \wedge (\delta_D(x)\delta_D(y))^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|}$. Using Lemma 6.1 and (6.14), we get that

$$\int_1^\infty q_U^1(t,x,y)\,dt \asymp |x-y|^{\alpha-1}\int_0^{|x-y|^\alpha}\left(1 \wedge \frac{u\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha}\right)u^{1/\alpha-2}\,du.$$

Put $u_0 := \frac{\phi(\delta_U(x))\phi(\delta_U(y))}{|x-y|^\alpha}$. Then we have

$$J \asymp |x-y|^{\alpha-1}\left(u_0\int_0^{|x-y|^\alpha\wedge u_0^{-1}} u^{1/\alpha-1}\,du + \int_{|x-y|^\alpha\wedge u_0^{-1}}^{|x-y|^\alpha} u^{1/\alpha-2}\,du\right).$$

Without loss of generality, we assume $\delta_U(x) \le \delta_U(y)$. Note that, since $|x-y|\le 1$, if $\delta_U(x) \le 1$ then $\delta_U(y) \le 2$, and if $\delta_U(x) > 1$ then $1 < \delta_U(x) \le \delta_U(y) \le 2\delta_U(x)$ and $\delta_U(x)\delta_U(y) \ge |x-y|^2$.

Now we look at three separate cases.

(i) $\alpha \in (1,2)$: In this case we have

$$\begin{aligned}J &\asymp |x-y|^{\alpha-1}\left(\alpha u_0\left(|x-y| \wedge u_0^{-1/\alpha}\right) + \frac{\alpha}{\alpha-1}\left(|x-y|^\alpha \wedge u_0^{-1}\right)^{(1-\alpha)/\alpha} - \frac{\alpha}{\alpha-1}|x-y|^{1-\alpha}\right) \\ &\asymp \phi(\delta_U(x))\phi(\delta_U(y)) \wedge (\phi(\delta_U(x))\phi(\delta_U(y)))^{(\alpha-1)/\alpha}.\end{aligned}$$

Thus

$$I + J \asymp \begin{cases} (\delta_U(x)\delta_U(y))^{1/2} & \text{when } \delta_U(x) \le 1, \delta_U(x)\delta_U(y) \ge |x-y|^2, \\ \frac{\delta_U(x)\delta_U(y)}{|x-y|} & \text{when } \delta_U(x) \le 1, \delta_U(x)\delta_U(y) \le |x-y|^2, \\ (\delta_U(x)\delta_U(y))^{(\alpha-1)/2} & \text{when } \delta_U(x) > 1 \end{cases}$$
$$= (\delta_U(x)\delta_U(y))^{1/2} \wedge (\delta_U(x)\delta_U(y))^{(\alpha-1)/2} \wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|}.$$

(ii) $\alpha = 1$: In this case we have

$$\begin{aligned} J &\asymp \left(u_0(|x-y| \wedge u_0^{-1}) + \log \frac{|x-y|^\alpha}{|x-y|^\alpha \wedge u_0^{-1}} \right) \\ &\asymp \phi(\delta_U(x))\phi(\delta_U(y)) \wedge 1 + \log\left(1 \vee \phi(\delta_U(x))\phi(\delta_U(y))\right) \\ &\asymp \log\left(1 + \phi(\delta_U(x))\phi(\delta_U(y))\right). \end{aligned}$$

Thus

$$I + J \asymp \begin{cases} (\delta_U(x)\delta_U(y))^{1/2} & \text{when } \delta_U(x) \le 1, \delta_U(x)\delta_U(y) \ge |x-y|^2, \\ \frac{\delta_U(x)\delta_U(y)}{|x-y|} & \text{when } \delta_U(x) \le 1, \delta_U(x)\delta_U(y) \le |x-y|^2, \\ \log(1 + \delta_U(x)\delta_U(y)) & \text{when } \delta_U(x) > 1 \end{cases}$$
$$\asymp \frac{\delta_U(x)\delta_U(y)}{|x-y|} \wedge \log\left(1 + (\delta_U(x)\delta_U(y))^{1/2}\right).$$

(iii) $\alpha \in (0,1)$: In this case (note that $1 - 1/\alpha$ is negative) we have

$$\begin{aligned} J &\asymp |x-y|^{\alpha-1} \left(\alpha u_0(|x-y| \wedge u_0^{-1/\alpha}) + \frac{\alpha}{1-\alpha}|x-y|^{1-\alpha} - \frac{\alpha}{1-\alpha}(|x-y|^\alpha \wedge u_0^{-1})^{(1-\alpha)/\alpha} \right) \\ &\asymp \phi(\delta_U(x))\phi(\delta_U(y)) \wedge 1. \end{aligned}$$

Thus

$$I + J \asymp \begin{cases} (\delta_U(x)\delta_U(y))^{1/2} & \text{when } \delta_U(x) \le 1, \delta_U(x)\delta_U(y) \ge |x-y|^2, \\ \frac{\delta_U(x)\delta_U(y)}{|x-y|} & \text{when } \delta_U(x) \le 1, \delta_U(x)\delta_U(y) \le |x-y|^2, \\ 1 & \text{when } \delta_U(x) > 1 \end{cases}$$
$$= (\delta_U(x)\delta_U(y))^{1/2} \wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|} \wedge 1.$$

Therefore we have proved the lemma for any arbitrary open set U and $a = 1$. The general case $a > 0$ now follows from the same scaling arguments as in the proofs for Lemmas 6.3 and 6.4. □

Proof of Theorem 1.7. Without loss of generality, we assume $M = b = 1$. Estimates (1.8) follow from Theorem 1.4, Remark 1.5(ii) and Lemmas 6.4–6.6. Estimates (1.9) follow from Theorem 1.4 and Lemmas 6.2 and 6.3. □

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