DOI: 10.1214/EJP.v15-752

Electronic Journal of Probability

Vol. 15 (2010), Paper no. 11, pages 292–322.

Journal URL
http://www.math.washington.edu/~ejpecp/

Conditional limit theorems for ordered random walks

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Abstract

In a recent paper of Eichelsbacher and König (2008) the model of ordered random walks has been considered. There it has been shown that, under certain moment conditions, one can construct a k-dimensional random walk conditioned to stay in a strict order at all times. Moreover, they have shown that the rescaled random walk converges to the Dyson Brownian motion. In the present paper we find the optimal moment assumptions for the construction proposed by Eichelsbacher and König, and generalise the limit theorem for this conditional process.

Key words: Dyson's Brownian Motion, Doob h-transform, Weyl chamber.

AMS 2000 Subject Classification: Primary 60G50; Secondary: 60G40, 60F17.

Submitted to EJP on November 10, 2009, final version accepted March 18, 2010.

*The research of Denis Denisov was done while he was a research assistant in Heriot-Watt University and supported by EPSRC grant No. EP/E033717/1

1 Introduction, main results and discussion

1.1 Introduction

A number of important results have been recently proved relating the limiting distributions of random matrix theory with certain other models. These models include the longest increasing subsequence, the last passage percolation, non-colliding particles, the tandem queues, random tilings, growth models and many others. A thorough review of these results can be found in [11].

Apparently it was Dyson who first established a connection between random matrix theory and non-colliding particle systems. It was shown in his classical paper [7] that the process of eigenvalues of the Gaussian Unitary Ensemble of size $k \times k$ coincides in distribution with the k-dimensional diffusion, which can be represented as the evolution of k Brownian motions conditioned never to collide. Such conditional versions of random walks have attract a lot of attention in the recent past, see e.g. [15; 10]. The approach in these papers is based on explicit formulae for nearest-neighbour random walks. However, it turns out that the results have a more general nature, that is, they remain valid for random walks with arbitrary jumps, see [1] and [8]. The main motivation for the present work was to find minimal conditions, under which one can define multidimensional random walks conditioned never change the order.

Consider a random walk $S_n = (S_n^{(1)}, \dots, S_n^{(k)})$ on $\mathbf{R}^k$, where

$$S_n^{(j)} = \xi_1^{(j)} + \dots + \xi_n^{(j)}, \quad j = 1, \dots, k,$$

and $\{\xi_n^{(j)}, 1 \le j \le k,\ n \ge 1\}$ is a family of independent and identically distributed random variables. Let

$$W = \{x = (x^{(1)}, \dots, x^{(k)}) \in \mathbf{R}^k : x^{(1)} < \ldots < x^{(k)}\}$$

be the Weyl chamber.

In this paper we study the asymptotic behaviour of the random walk S_n conditioned to stay in W. Let τ_x be the exit time from the Weyl chamber of the random walk with starting point $x \in W$, that is,

$$\tau_x = \inf\{n \ge 1 : x + S_n \notin W\}.$$

One can attribute two different meanings to the words 'random walk conditioned to stay in W.' On the one hand, the statement could refer to the path $(S_0, S_1, \dots, S_n)$ conditioned on $\{\tau_x > n\}$. On the other hand, one can construct a new Markov process, which never leaves W. There are two different ways of defining such a conditioned processes. First, one can determine its finite dimensional distributions via the following limit

$$\mathbf{P}_x\left(\widehat{S}_i \in D_i,\, 0 \le i \le n\right) = \lim_{m\to\infty} \mathbf{P}(x + S_i \in D_i,\, 0 \le i \le n | \tau_x > m), \tag{1}$$

where D_i are some measurable sets and n is a fixed integer. Second, one can use an appropriate Doob h-transform. If there exists a function h (which is usually called an *invariant function*) such that $h(x) > 0$ for all $x \in W$ and

$$\mathbf{E}[h(x + S(1)); \tau_x > 1] = h(x), \quad x \in W, \tag{2}$$

then one can make a *change of measure*

$$\widehat{\mathbf{P}}_x^{(h)}(S_n \in dy) = \mathbf{P}(x + S_n \in dy, \tau_x > n)\frac{h(y)}{h(x)}.$$

As a result, one obtains a random walk S_n under a new measure $\widehat{\mathbf{P}}_x^{(h)}$. This transformed random walk is a Markov chain which lives on the state space W.

To realise the first approach one needs to know the asymptotic behaviour of $\mathbf{P}(\tau_x > n)$. And for the second approach one has to find a function satisfying (2). It turns out that these two problems are closely related to each other: The invariant function reflects the dependence of $\mathbf{P}(\tau_x > n)$ on the starting point x. Then both approaches give the same Markov chain. For one-dimensional random walks conditioned to stay positive this was shown by Bertoin and Doney [2]. They proved that if the random walk oscillating, then the renewal function based on the weak descending ladder heights, say V, is invariant and, moreover, $\mathbf{P}(\tau_x > n) \sim V(x)\mathbf{P}(\tau_0 > n)$. If additionally the second moment is finite, then one can show that $V(x) = x - \mathbf{E}(x + S_{\tau_x})$, which is just the mean value of the overshoot. For random walks in the Weyl chamber Eichelsbacher and König [8] have introduced the following analogue of the averaged overshoot:

$$V(x) = \Delta(x) - \mathbf{E}\Delta(x + S_{\tau_x}), \tag{3}$$

where $\Delta(x)$ denotes the Vandermonde determinant, that is,

$$\Delta(x) = \prod_{1 \le i < j \le k} (x^{(j)} - x^{(i)}), \quad x \in W.$$

Then it was shown in [8] that if $\mathbf{E}|\xi|^{r_k} < \infty$ with some $r_k > ck^3$, then it can be concluded that V is a finite and strictly positive invariant function. Moreover, the authors determined the behaviour of $\mathbf{P}(\tau_x > n)$ and studied some asymptotic properties of the conditioned random walk. They also posed a question about minimal moment assumptions under which one can construct a conditioned random walk by using V. In the present paper we answer this question. We prove that the results of [8] remain valid under the following conditions:

- *Centering assumption:* We assume that $\mathbf{E}\xi = 0$.
- *Moment assumption:* We assume that $\mathbf{E}|\xi|^\alpha < \infty$ with $\alpha = k - 1$ if $k > 3$ and some $\alpha > 2$ if $k = 3$. .

Furthermore, we assume, without loss of generality, that $\mathbf{E}\xi^2 = 1$. It is obvious, that the moment assumption is the minimal one for the finiteness of the function V defined by (3). Indeed, from the definition of Δ it is not difficult to see that the finiteness of the $(k-1)$-th moment of ξ is necessary for the finiteness of $\mathbf{E}\Delta(x + S_1)$. Thus, this moment condition is also necessary for the integrability of $\Delta(x + S_{\tau_x})$, which is equivalent to the finiteness of V. In other words, if $\mathbf{E}|\xi|^{k-1} = \infty$, then one has to define the invariant function in a different way. Moreover, we give an example, which shows that if the moment assumption does not hold, then $\mathbf{P}(\tau_x > n)$ has a different rate of divergence.

1.2 Tail distribution of τ_x

Here is our *main* result:

Theorem 1. *Assume that $k \geq 3$ and let the centering as well as the moment assumption hold. Then the function V is finite and strictly positive. Moreover, as $n \to \infty$,*

$$\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-k(k-1)/4}, \quad x \in W, \tag{4}$$

where $\varkappa$ is an absolute constant.

All the claims in the theorem have been proved in [8] under more restrictive assumptions: as we have already mentioned, the authors have assumed that $\mathbf{E}|\xi|^{r_k} < \infty$ with some r_k such that $r_k \geq ck^3$, $c > 0$. Furthermore, they needed some additional regularity conditions, which ensure the possibility to use an asymptotic expansion in the local central limit theorem. As our result shows, these regularity conditions are superfluous and one needs $k-1$ moments only.

Under the condition that $\xi^{(1)}, \ldots, \xi^{(k)}$ are identically distributed, the centering assumption does not restrict the generality since one can consider a driftless random walk $S_n - n\mathbf{E}\xi$. But if the drifts are allowed to be unequal, then the asymptotic behaviour of τ_x and that of the conditioned random walk might be different, see [16] for the case of Brownian motions.

We now turn to the discussion of the moment condition in the theorem. We start with the following example.

Example 2. Assume that $k \geq 4$ and consider the random walk, which satisfies

$$\mathbf{P}(\xi \geq u) \sim u^{-\alpha} \quad \text{as } u \to \infty, \tag{5}$$

with some $\alpha \in (k-2, k-1)$. Then,

$$\begin{aligned}\mathbf{P}(\tau_x > n) \geq \mathbf{P}&\left(\xi_1^{(k)} > n^{1/2+\varepsilon}, \min_{1\leq i\leq n} S_i^{(k)} > 0.5n^{1/2+\varepsilon}\right)\\ &\times\mathbf{P}\left(\max_{1\leq i\leq n} S_i^{(k-1)} \leq 0.5n^{1/2+\varepsilon}, \tilde{\tau}_x > n\right),\end{aligned}$$

where $\tilde{\tau}_x$ is the time of the first change of order in the random walk $(S_n^{(1)}, \ldots, S_n^{(k-1)})$. By the Central Limit Theorem,

$$\begin{aligned}\mathbf{P}&\left(\xi_1^{(k)} > n^{1/2+\varepsilon}, \min_{1\leq i\leq n} S_i^{(k)} > 0.5n^{1/2+\varepsilon}\right)\\ &\geq \mathbf{P}\left(\xi_1^{(k)} > n^{1/2+\varepsilon}\right)\mathbf{P}\left(\min_{1\leq i\leq n}(S_i^{(k)} - \xi_1^{(k)}) > -0.5n^{1/2+\varepsilon}\right) \sim n^{-\alpha(1/2+\varepsilon)}.\end{aligned}$$

The CLT is applicable because of the condition $\alpha > k-2$, which implies the finiteness of the variance.

For the second term in the product we need to analyse $(k-1)$ random walks under the condition $\mathbf{E}|\xi|^{k-2+\varepsilon} < \infty$. Using Theorem 1, we have

$$\mathbf{P}(\tilde{\tau}_x > n) \sim \tilde{V}(x) n^{-(k-1)(k-2)/4}.$$

Since S_n is of order $\sqrt{n}$ on the event $\{\tilde{\tau}_x > n\}$, we have

$$\mathbf{P}\left(\max_{1\le i\le n} S_i^{(k-1)} \le 0.5n^{1/2+\varepsilon}, \tilde{\tau}_x > n\right) \sim \mathbf{P}\left(\tilde{\tau}_x > n\right) \sim \tilde{V}(x)n^{-(k-1)(k-2)/4}.$$

As a result the following estimate holds true for sufficiently small ε,

$$\mathbf{P}(\tau_x > n) \ge C(x)n^{-(k-1)(k-2)/4}n^{-\alpha(1/2+\varepsilon)}.$$

The right hand side of this inequality decreases slower than $n^{-k(k-1)/4}$ for all sufficiently small ε.

Moreover, using the same heuristic arguments, one can find a similar lower bound when (5) holds with $\alpha \in (k-j-1, k-j)$, $j \le k-3$:

$$\mathbf{P}(\tau_x > n) \ge C(x)n^{-(k-j)(k-j-1)/4}n^{-\alpha j(1/2+\varepsilon)}.$$

We believe that the lower bounds constructed above are quite precise, and we conjecture that

$$\mathbf{P}(\tau_x > n) \sim U(x)n^{-(k-j)(k-j-1)/4-\alpha j/2}$$

when (5) holds. $\diamond$

Furthermore, the example shows that condition $\mathbf{E}|\xi|^{k-1} < \infty$ is almost necessary for the validity of (4): one can not obtain the relation $\mathbf{P}(\tau_x > n) \sim C(x)n^{-k(k-1)/4}$ when $\mathbf{E}|\xi|^{k-1} = \infty$.

If we have two random walks, i.e. $k = 2$, then τ_x is the exit time from $(0, \infty)$ of the random walk $Z_n := (x^{(2)} - x^{(1)}) + (S_n^{(2)} - S_n^{(1)})$. It is well known that, for symmetrically distributed random walks, $\mathbf{E}Z_{\tau_x} < \infty$ if and only if $\mathbf{E}(\xi_1^{(2)} - \xi_1^{(1)})^2 < \infty$. However, the existence of $\mathbf{E}Z_{\tau_x}$ is not necessary for the relation $\mathbf{P}(\tau_x > n) \sim C(x)n^{-1/2}$, which holds for all symmetric random walks. This is contary to the high-dimensional case ($k \ge 4$), where the integrability of $\Delta(x + S_{\tau_x})$ and the rate $n^{-k(k-1)/4}$ are quite close to each other.

In the case of three random walks our moment condition is not optimal. We think that the existence of the variance is sufficient for the integrability of $\Delta(x + S_{\tau_x})$. But our approach requires more than two moments. Furthermore, we conjecture that, as in the case $k = 2$, the tail of the distribution of τ_x is of order $n^{-3/2}$ for *all* random walks.

1.3 Scaling limits of conditioned random walks

Theorem 1 allows us to construct the conditioned random walk via the distributional limit (1). In fact, if (4) is used, we obtain, as $m \to \infty$,

$$\begin{aligned}\mathbf{P}(x + S_n \in D|\tau_x > m) &= \frac{1}{\mathbf{P}(\tau_x > m)}\int_D \mathbf{P}(x + S_n \in dy, \tau_x > n)\mathbf{P}(\tau_y > m - n)\\ &\to \frac{1}{V(x)}\int_D \mathbf{P}(x + S_n \in dy, \tau_x > n)V(y).\end{aligned}$$

But this means that the distribution of $\widehat{S}_n$ is given by the Doob transform with function V. (This transformation is possible, because V is well-defined, strictly positive on W and satisfies

$\mathbf{E}[V(x+S_1);\tau_x>1]=V(x)$.) In other words, both ways of construction described above give the same process.

We now turn to the asymptotic behaviour of $\widehat{S}_n$. To state our results we introduce the limit process. For the k-dimensional Brownian motion with starting point $x\in W$ one can change the measure using the Vandermonde determinant:

$$\widehat{\mathbf{P}}_x^{(\Delta)}(B_t\in dy)=\mathbf{P}(x+B_t\in dy,\tau>t)\frac{\Delta(y)}{\Delta(x)}.$$

The corresponding process is called Dyson's Brownian motion. Furthermore, one can define Dyson's Brownian motion with starting point 0 via the weak limit of $\widehat{\mathbf{P}}_x^{(\Delta)}$, for details see Section 4 of O'Connell and Yor [15]. We denote the corresponding probability measure as $\widehat{\mathbf{P}}_0^{(\Delta)}$.

Theorem 3. *If $k\geq 3$ and the centering as well as the moment assumption are valid, then*

$$\mathbf{P}\left(\frac{x+S_n}{\sqrt{n}}\in\cdot\middle|\tau_x>n\right)\to\mu\quad\textit{weakly},\tag{6}$$

where μ is the probability measure on W with density proportional to $\Delta(y)e^{-|y|^2/2}$. Furthermore, the process $X^n(t)=\frac{S_{[nt]}}{\sqrt{n}}$ under the probability measure $\widehat{\mathbf{P}}_{x\sqrt{n}}^{(V)}, x\in W$ converges weakly to Dyson's Brownian motion under the measure $\widehat{\mathbf{P}}_x^{(\Delta)}$. Finally, the process $X^n(t)=\frac{S_{[nt]}}{\sqrt{n}}$ under the probability measure $\widehat{\mathbf{P}}_x^{(V)}, x\in W$ converges weakly to the Dyson Brownian motion under the measure $\widehat{\mathbf{P}}_0^{(\Delta)}$.

Relation (6) and the convergence of the rescaled process with starting point $x\sqrt{n}$ were proven in [8] under more restrictive conditions. Convergence towards $\widehat{\mathbf{P}}_0^{(\Delta)}$ was proven for nearest-neighbour random walks, see [15] and [17]. A comprehensive treatment of the case $k=2$ can be found in [6].

One can guess that the convergence towards Dyson's Brownian motion holds even if we have finite variance only. However, it is not clear how to define an invariant function in that case.

1.4 Description of the approach

The proof of finiteness and positivity of the function V is the most difficult part of the paper. To derive these properties of V we use martingale methods. It is well known that $\Delta(x+S_n)$ is a martingale. Define the stopping time $T_x=\min\{k\geq 1:\Delta(x+S_k)\leq 0\}$. It is easy to see that $T_x\geq\tau_x$ almost surely. Furthermore, $T_x>\tau_x$ occurs iff an even number of diffrences $S_n^{(j)}-S_n^{(l)}$ change their signs at time τ_x. (Note also that the latter can not be the case for the nearest-neighbour random walk and for the Brownian motion, i.e., $\tau_x=T_x$ in that cases.) If $\{T_x>\tau_x\}$ has positive probability, then the random variable τ_x is not a stopping time with respect to the filtration $\mathcal{G}_n:=\sigma\left(\Delta(x+S_k),k\leq n\right)$. This is the reason for introducing T_x. We first show that $\Delta(x+S_{T_x})$ is integrable, which yields the integrability of $\Delta(x+S_{\tau_x})$, see Subsection 2.1. Furthermore, it follows from the integrability of $\Delta(x+S_{T_x})$ that the function $V^{(T)}(x)=\lim_{n\to\infty}\mathbf{E}\{\Delta(x+S_n),T_x>n\}$ is well defined on the set $\{x:\Delta(x)>0\}$. To show that the function V is strictly positive, we use the interesting observation that the sequence $V^{(T)}(x+S_n)\mathbf{1}\{\tau_x>n\}$ is a supermartingale, see Subsection 2.2.

It is worth mentioning that the detailed analysis of the martingale properties of the random walk S_n allows one to keep the minimal moment conditions for positivity and finiteness of V. The authors of [8] used the Hölder inequality at many places in their proof. This explains the superfluous moment condition in their paper.

To prove the asymptotic relations in our theorems we use a version of the Komlos-Major-Tusnady coupling proposed in [13], see Section 3. A similar coupling has been used in [3] and [1]. In order to have a good control over the quality of the Gaussian approximation we need more than two moments of the random walk. This fact explains partially why we required the finiteness of $\mathbf{E}|\xi|^{2+\delta} < \infty$ in the case $k = 3$.

The proposed approach uses very symmetric structure of W at many places. We believe that one can generalise our approach for random walks in other cones. A first step in this direction has been done by König and Schmid [12]. they have shown that our method works also for Weyl chambers of type C and D.

2 Finiteness and positivity of V

The main purpose of the present section is to prove the following statement.

Proposition 4. *The function V has the following properties:*

(a) $V(x) = \lim_{n\to\infty} \mathbf{E}[\Delta(x + S_n); \tau_x > n]$;

(b) V *is monotone, i.e. if* $x^{(j)} - x^{(j-1)} \le y^{(j)} - y^{(j-1)}$ *for all* $2 \le j \le k$, *then* $V(x) \le V(y)$;

(c) $V(x) \le c\Delta_1(x)$ *for all* $x \in W$, *where* $\Delta_t(x) = \prod_{1\le i<j\le k} \left(t + |x^{(j)} - x^{(i)}|\right)$;

(d) $V(x) \sim \Delta(x)$ *provided that* $\min\limits_{2\le j\le k}(x^{(j)} - x^{(j-1)}) \to \infty$;

(e) $V(x) > 0$ *for all* $x \in W$.

As it was already mentioned in the introduction our approach relies on the investigation of properties of the stopping time T_x defined by

$$T_x = T = \min\{k \ge 1 : \Delta(x + S_k) \le 0\}.$$

It is easy to see that $T_x \ge \tau_x$ for every $x \in W$.

2.1 Integrability of $\Delta(x + S_{T_x})$

We start by showing that $\mathbf{E}[\Delta(x + S_{T_x})]$ is finite under the conditions of Theorem 1. In this paragraph we omit the subscript x if there is no risk of confusion.

Lemma 5. *The sequence* $Y_n := \Delta(x + S_n)1\{T > n\}$ *is a submartingale.*

Proof. Clearly,

$$\begin{aligned}\mathbf{E}\left[Y_{n+1}-Y_n|\mathcal{F}_n\right] &= \mathbf{E}\left[(\Delta(x+S_{n+1})-\Delta(x+S_n))\,1\{T>n\}|\mathcal{F}_n\right] \\ &\quad -\mathbf{E}\left[\Delta(x+S_{n+1})1\{T=n+1\}|\mathcal{F}_n\right] \\ &= 1\{T>n\}\mathbf{E}\left[(\Delta(x+S_{n+1})-\Delta(x+S_n))\,|\mathcal{F}_n\right] \\ &\quad -\mathbf{E}\left[\Delta(x+S_{n+1})1\{T=n+1\}|\mathcal{F}_n\right].\end{aligned}$$

The statement of the lemma follows now from the facts that $\Delta(x+S_n)$ is a martingale and $\Delta(x+S_T)$ is non-positive. □

For any $\varepsilon>0$, define the following set

$$W_{n,\varepsilon}=\{x\in R^k:|x^{(j)}-x^{(i)}|>n^{1/2-\varepsilon},1\le i<j\le k\}.$$

Lemma 6. *For any sufficiently small $\varepsilon>0$ there exists $\gamma>0$ such the following inequalities hold*

$$|\mathbf{E}[\Delta(x+S_T);T\le n]|\le\frac{C}{n^\gamma}\Delta(x),\quad x\in W_{n,\varepsilon}\cap\{\Delta(x)>0\} \tag{7}$$

and

$$\mathbf{E}[\Delta_1(x+S_\tau);\tau\le n]\le\frac{C}{n^\gamma}\Delta(x),\quad x\in W_{n,\varepsilon}\cap W. \tag{8}$$

Proof. We shall prove (7) only. The proof of (8) requires some minor changes, and we omit it. For a constant $\delta>0$, which we define later, let

$$A_n=\left\{\max_{1\le i\le n,1\le j\le k}|\xi_i^{(j)}|\le n^{1/2-\delta}\right\}$$

and split the expectation into 2 parts,

$$\begin{aligned}\mathbf{E}[\Delta(x+S_T);\,T\le n] &= \mathbf{E}[\Delta(x+S_T);\,T\le n,A_n]+\mathbf{E}[\Delta(x+S_T);\,T\le n,\overline{A}_n] \\ &=: E_1(x)+E_2(x).\end{aligned} \tag{9}$$

It follows from the definition of the stopping time T that at least one of the differences $(x^{(r)}+S^{(r)}-x^{(s)}-S^{(s)})$ changes its sign at time T, i.e. one of the following events occurs

$$B_{s,r}:=\left\{(x^{(r)}+S^{(r)}_{T-1}-x^{(s)}-S^{(s)}_{T-1})(x^{(r)}+S^{(r)}_T-x^{(s)}-S^{(s)}_T)\le 0\right\},$$

$1\le s<r\le k$. Clearly,

$$|E_1(x)|\le\sum_{1\le s<r\le k}\mathbf{E}[|\Delta(x+S_T)|;\,T\le n,A_n,B_{s,r}].$$

On the event $A_n\cap B_{s,r}$,

$$\left|x^{(s)}-x^{(r)}+S^{(s)}_T-S^{(r)}_T\right|\le\left|\xi^{(s)}_T-\xi^{(r)}_T\right|\le 2n^{1/2-\delta}.$$

This implies that on the event $A_n \cap B_{s,r}$,

$$|\Delta(x+S_T)| \le 2n^{1/2-\delta}\left|\frac{\Delta(x+S_T)}{x^{(s)}-x^{(r)}+S_T^{(s)}-S_T^{(r)}}\right|.$$

Put $\mathcal{P}=\{(i,j), 1\le i<j\le k\}$. Then,

$$\begin{aligned}\frac{\Delta(x+S_T)}{x^{(s)}-x^{(r)}+S_T^{(s)}-S_T^{(r)}} &= \prod_{(i,j)\in\mathcal{P}\setminus(s,r)}\left(x^{(j)}-x^{(i)}+S_T^{(j)}-S_T^{(i)}\right)\\ &= \sum_{\mathcal{J}\subset\mathcal{P}\setminus(s,r)}\prod_{\mathcal{J}}\left(x^{(i_2)}-x^{(i_1)}\right)\prod_{\mathcal{P}\setminus(\mathcal{J}\cup(s,r))}\left(S_T^{(j_2)}-S_T^{(j_1)}\right).\end{aligned}$$

As is not difficult to see,

$$\prod_{\mathcal{P}\setminus(\mathcal{J}\cup(s,r))}(S_T^{(j_2)}-S_T^{(j_1)}) = p_{\mathcal{J}}(S_T) = \sum_{i_1,i_2,\ldots,i_k}\alpha^{\mathcal{J}}_{i_1,i_2,\ldots,i_k}(S_T^{(1)})^{i_1}\ldots(S_T^{(k)})^{i_k},$$

where the sum is taken over all $i_1,i_2,\ldots,i_k$ such that $i_1+i_2+\ldots+i_k=|\mathcal{P}|-|\mathcal{J}|-1$ and $\alpha^{\mathcal{J}}_{i_1,i_2,\ldots,i_k}$ are some absolute constants.

Put $M_n^{(j)}=\max_{0\le i\le n}|S_i^{(j)}|$. Combining Doob's and Rosenthal's inequalities, one has

$$\mathbf{E}\left(M_n^{(j)}\right)^p \le C(p)\mathbf{E}\left|S_n^{(j)}\right|^p \le C(p)\mathbf{E}[|\xi|^p]n^{p/2} \tag{10}$$

Then,

$$\begin{aligned}\mathbf{E}|p_{\mathcal{J}}(S_T)\mathbf{1}_{\{T\le n\}}| &\le \sum_{i_1,i_2,\ldots,i_k}|\alpha^{\mathcal{J}}_{i_1,i_2,\ldots,i_k}|\mathbf{E}(M_n^{(1)})^{i_1}\ldots\mathbf{E}(M_n^{(k)})^{i_k}\\ &\le \sum_{i_1,i_2,\ldots,i_k}|\alpha^{\mathcal{J}}_{i_1,i_2,\ldots,i_k}|C_{i_1}n^{i_1/2}\ldots C_{i_k}n^{i_k/2}\\ &\le C_{\mathcal{J}}(n^{1/2})^{|\mathcal{P}|-|\mathcal{J}|-1}.\end{aligned} \tag{11}$$

where $C_1, C_2, \ldots$ are universal constants. Now note that since $x\in W_{n,\varepsilon}$, we have a simple estimate

$$n^{1/2}=n^{\varepsilon}n^{1/2-\varepsilon}\le n^{\varepsilon}|x^{(j_2)}-x^{(j_1)}| \tag{12}$$

for any $j_1<j_2$. Using (11) and (12), we obtain

$$\begin{aligned}&\mathbf{E}\left[\left|\frac{\Delta(x+S_T)}{x^{(r)}-x^{(s)}+S_T^{(r)}-S_T^{(s)}}\right|; T\le n, A_n, B_{s,r}\right]\\ &\quad\le \sum_{\mathcal{J}\subset\mathcal{P}\setminus(s,r)}C_{\mathcal{J}}(n^{1/2})^{|\mathcal{P}|-|\mathcal{J}|-1}\prod_{\mathcal{J}}|x^{(i_2)}-x^{(i_1)}|\\ &\quad\le \sum_{\mathcal{J}\subset\mathcal{P}\setminus(s,r)}\mathcal{C}_J(n^{\varepsilon})^{|\mathcal{P}|-|\mathcal{J}|-1}\prod_{\mathcal{J}}|x^{(i_2)}-x^{(i_1)}|\prod_{\mathcal{P}\setminus(\mathcal{J}\cup(s,r))}|x^{(j_2)}-x^{(j_1)}|\\ &\quad\le C_k n^{\varepsilon\frac{k(k-1)-1}{2}}\frac{\Delta(x)}{|x^{(r)}-x^{(s)}|}\le C_k n^{\varepsilon\frac{k(k-1)}{2}}n^{-1/2}\Delta(x).\end{aligned}$$

Thus,

$$E_1(x) \le \sum_{1\le s<r\le k} 2n^{1/2-\delta} C_k n^{\varepsilon\frac{k(k-1)}{2}} n^{-1/2}\Delta(x) = k(k-1)C_k n^{\varepsilon\frac{k(k-1)}{2}-\delta}\Delta(x). \tag{13}$$

Now we estimate $E_2(x)$. Clearly,

$$\overline{A}_n = \bigcup_{r=1}^{k} D_r,$$

where $D_r = \{\max_{1\le i\le n}|\xi_i^{(r)}| > n^{1/2-\delta}\}$. As in the first part of the proof,

$$\Delta(x+S_T) = \sum_{\mathcal{J}\subset\mathcal{P}} \prod_{\mathcal{J}} (x^{(i_2)} - x^{(i_1)}) \prod_{\mathcal{P}\setminus\mathcal{J}} (S_T^{(j_2)} - S_T^{(j_1)})$$

and

$$\prod_{\mathcal{P}\setminus\mathcal{J}} (S_T^{(j_2)} - S_T^{(j_1)}) = \sum_{i_1,i_2,\dots,i_k} \alpha^{\mathcal{J}}_{i_1,i_2,\dots,i_k} (S_T^{(1)})^{i_1}\dots(S_T^{(k)})^{i_k}.$$

Then, using (10) once again, we get

$$\begin{aligned}
&\mathbf{E}\left[\left|\prod_{\mathcal{P}\setminus\mathcal{J}} (S_T^{(j_2)} - S_T^{(j_1)})\right|; T\le n, D_r\right] \\
&\qquad\le \sum_{i_1,i_2,\dots,i_k} \left|\alpha^{\mathcal{J}}_{i_1,i_2,\dots,i_k}\right| C_{i_1} n^{i_1/2}\dots \mathbf{E}\left[(M_n^{(r)})^{i_r}; D_r\right]\dots C_{i_k} n^{i_k/2}.
\end{aligned}$$

Applying the following estimate, which will be proved at the end of the lemma,

$$\mathbf{E}\left[(M_n^{(r)})^{i_r}; D_r\right] \le C(\delta) n^{i_r/2-\alpha/2+1+(i_r+\alpha)\delta}, \tag{14}$$

we obtain

$$\mathbf{E}\left[\left|\prod_{\mathcal{P}\setminus\mathcal{J}} (S_T^{(j_2)} - S_T^{(j_1)})\right|; T\le n, D_r\right] \le C_{\mathcal{J}} C(\delta) (n^{1/2})^{|\mathcal{P}|-|\mathcal{J}|}\ n^{-\alpha/2+1+2\alpha\delta}.$$

This implies that

$$\begin{aligned}
&\mathbf{E}\left[|\Delta(x+S_T)|;\, T\le n, D_r\right] \\
&\qquad\le C(\delta) n^{-\alpha/2+1+2\alpha\delta} \sum_{\mathcal{J}\subset\mathcal{P}} C_{\mathcal{J}} (n^{1/2})^{|\mathcal{P}|-|\mathcal{J}|} \prod_{\mathcal{J}} |x^{(i_2)} - x^{(i_1)}| \\
&\qquad\le C(\delta) n^{-\alpha/2+1+2\alpha\delta} \sum_{\mathcal{J}\subset\mathcal{P}} \mathcal{C}_J (n^{\varepsilon})^{|\mathcal{P}|-|\mathcal{J}|} \prod_{\mathcal{J}} |x^{(i_2)} - x^{(i_1)}| \prod_{\mathcal{P}\setminus\mathcal{J}} |x^{(j_2)} - x^{(j_1)}| \\
&\qquad\le C(\delta) n^{\varepsilon\frac{k(k-1)}{2}} n^{-\alpha/2+1+2\alpha\delta}\Delta(x).
\end{aligned}$$

Consequently,

$$E_2(x) \le \sum_{r=1}^{k} \mathbf{E}\left[|\Delta(x+S_T)|;\, T\le n, D_r\right] \le kC(\delta) n^{\varepsilon\frac{k(k-1)}{2}} n^{-\alpha/2+1+2\alpha\delta}\Delta(x). \tag{15}$$

Applying (13) and (15) to the right hand side of (9), and choosing ε and δ in an appropriate way, we arrive at the conclusion. Here we have also used the assumption that $\alpha > 2$.

Thus, it remains to show (14).

It is easy to see that, for any $i_r \in (0, \alpha]$,

$$\begin{aligned}\mathbf{E}\left[(M_n^{(r)})^{i_r}; D_r\right] &= i_r \int_0^\infty x^{i_r-1}\mathbf{P}(M_n^{(r)} > x, D_r)dx \\ &\le n^{i_r(1/2+\delta)}\mathbf{P}(D_r) + i_r \int_{n^{1/2+\delta}}^\infty x^{i_r-1}\mathbf{P}(M_n^{(r)} > x)dx\end{aligned}$$

Putting $y = x/p$ in Corollary 1.11 of [14], we get the inequality

$$\mathbf{P}(|S_n^{(r)}| > x) \le C(p)\Big(\frac{n}{x^2}\Big)^p + n\mathbf{P}(|\xi| > x/p).$$

As was shown in [5], this inequality remains valid for $M_n^{(r)}$, i.e.

$$\mathbf{P}(M_n^{(r)} > x) \le C(p)\Big(\frac{n}{x^2}\Big)^p + n\mathbf{P}(|\xi| > x/p).$$

Using the latter bound with $p > i_r/2$, we have

$$\begin{aligned}&\int_{n^{1/2+\delta}}^\infty x^{i_r-1}\mathbf{P}(M_n^{(r)} > x)dx \\ &\qquad \le C(p)i_r n^p \int_{n^{1/2+\delta}}^\infty x^{i_r-1-2p}dx + n\int_{n^{1/2+\delta}}^\infty x^{i_r-1}\mathbf{P}(|\xi| > x/p)dx \\ &\qquad \le C(p)\frac{i_r}{2p-i_r}n^{p-(2p-i_r)(1/2+\delta)} + p^p n\mathbf{E}[|\xi|^{i_r}, |\xi| > n^{1/2+\delta}/p] \\ &\qquad \le C(p)\Big(n^{p-(2p-i_r)(1/2+\delta)} + n^{1+(1/2+\delta)(i_r-\alpha)}\Big).\end{aligned}$$

Choosing $p > \alpha/2\delta$, we get

$$\int_{n^{1/2+\delta}}^\infty x^{i_r-1}\mathbf{P}(M_n^{(r)} > x)dx \le C(\delta)n^{i_r/2+1-\alpha/2}.$$

Note that

$$\mathbf{P}(D_r) \le n\mathbf{P}(|\xi| > n^{1/2-\delta}) \le Cn^{1-\alpha(1/2-\delta)}, \tag{16}$$

we obtain

$$\mathbf{E}\left[(M_n^{(r)})^{i_r}; D_r\right] \le C(\delta)n^{i_r/2+1-\alpha/2+(\alpha+i_r)\delta}.$$

Thus, (14) is proved for $i_r \in (0, \alpha]$. If $i_r = 0$, then $\mathbf{E}\left[(M_n^{(r)})^{i_r}; A_r\right] = \mathbf{P}(D_r)$. Therefore, (14) with $i_r = 0$ follows from (16). □

Define

$$\nu_n := \min\{k \ge 1 : x + S_k \in W_{n,\varepsilon}\}.$$

Lemma 7. *For every $\varepsilon > 0$ it holds that*

$$\mathbf{P}(\nu_n > n^{1-\varepsilon}) \le \exp\{-Cn^{\varepsilon}\}.$$

Proof. To shorten formulas in the proof we set $S_0 = x$. Also, set, for brevity, $b_n = [an^{1/2-\varepsilon}]$. The parameter a will be chosen at the end of the proof.

First note that

$$\{\nu_n > n^{1-\varepsilon}\} \subset \bigcap_{i=1}^{[n^\varepsilon/a^2]} \bigcup_{1\le j<l\le k} \{|S^{(l)}_{i\cdot b_n^2} - S^{(j)}_{i\cdot b_n^2}| \le n^{1/2-\varepsilon}\}.$$

Then there exists at least one pair $\widehat{j}, \widehat{l}$ such that for at least at $[n^\varepsilon/(a^2k^2)]$ points

$$\mathcal{I} = \{i_1, \ldots, i_{[n^\varepsilon/(a^2k^2)]}\} \subset \{b_n^2, 2b_n^2, \ldots, [n^\varepsilon/a^2]b_n^2\}$$

we have

$$|S_i^{(\widehat{l})} - S_i^{(\widehat{j})}| \le n^{1/2-\varepsilon} \text{ for } i \in \mathcal{I}.$$

Without loss of generality we may assume that $\widehat{j} = 1$ and $\widehat{l} = 2$. There must exist at least $[n^\varepsilon/(2a^2k^2)]$ points spaced atmost $2k^2b_n^2$ apart each other. To simplify notation assume that points $i_1, \ldots i_{[n^\varepsilon/(2a^2k^2)]}$ enjoy this property:

$$\max(i_2 - i_1, i_3 - i_2, \ldots, i_{[n^\varepsilon/(2a^2k^2)]} - i_{[n^\varepsilon/(2a^2k^2)]-1}) \le 2k^2b_n^2.$$

In fact this means that $i_s - i_{s-1}$ can take only values $\{jb_n^2,\ 1 \le j \le 2k^2\}$. The above considerations imply that

$$\begin{aligned}
&\mathbf{P}\left(\nu_n > n^{1-\varepsilon}\right) \\
&\le \binom{k}{2}\binom{[n^\varepsilon/a^2]}{[n^\varepsilon/(2a^2k^2)]}\mathbf{P}\left(|S_i^{(2)} - S_i^{(1)}| \le n^{1/2-\varepsilon} \text{ for all } i \in \{i_1, \ldots, i_{[n^\varepsilon/(2a^2k^2)]}\}\right) \\
&\le \binom{k}{2}\binom{[n^\varepsilon/a^2]}{[n^\varepsilon/(2a^2k^2)]} \prod_{s=2}^{[n^\varepsilon/(2a^2k^2)]} \mathbf{P}\left(\left|(S^{(2)}_{i_s} - S^{(2)}_{i_s-1}) - (S^{(1)}_{i_s} - S^{(1)}_{i_{s-1}})\right| \le 2n^{1/2-\varepsilon}\right).
\end{aligned}$$

Using the Stirling formula, we get

$$\binom{[n^\varepsilon/a^2]}{[n^\varepsilon/(2a^2k^2]} \le \frac{a}{n^\varepsilon}(2k^2)^{n^\varepsilon/a^2}.$$

By the Central Limit Theorem,

$$\lim_{n\to\infty} \mathbf{P}\left(\left|S^{(2)}_{jb_n^2} - S^{(1)}_{jb_n^2}\right| \le 2n^{1/2-\varepsilon}\right) = \int_{-\sqrt{2}/(a\sqrt{j})}^{\sqrt{2}/(a\sqrt{j})} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \le \frac{2}{a}.$$

Thus, for all sufficiently large n,

$$\prod_{s=2}^{[n^\varepsilon/(2a^2k^2)]} \mathbf{P}\left(\left|(S^{(2)}_{i_s} - S^{(2)}_{i_s-1}) - (S^{(1)}_{i_s} - S^{(1)}_{i_{s-1}})\right| \le 2n^{1/2-\varepsilon}\right) \le \left(\frac{4}{a}\right)^{n^\varepsilon/(2a^2k^2)-1}.$$

Consequently,

$$\mathbf{P}\left(\nu_n > n^{1-\varepsilon}\right) \le \left(\frac{4(2k^2)^{2k^2}}{a}\right)^{n^\varepsilon/(2a^2k^2)}$$

Choosing $a = 8(2k^2)^{2k^2}$, we complete the proof. □

Lemma 8. *For every $\varepsilon > 0$ the inequality*

$$\mathbf{E}[|\Delta_t(x+S_n)|; \nu_n > n^{1-\varepsilon}] \le c_t \Delta_1(x) \exp\{-Cn^\varepsilon\}$$

holds.

Remark 9. If $\mathbf{E}|\xi|^\alpha < \infty$ for some $\alpha > k-1$, then the claim in the lemma follows easily from the Hölder inequality and Lemma 7. But our moment assumption requires more detailed analysis. $\diamond$

Proof. We give the proof only for $t = 0$.

For $1 \le l < i \le k$ define

$$G_{l,i} = \left\{ |x^{(l)} - x^{(i)} + S^{(l)}_{jb_n^2} - S^{(i)}_{jb_n^2}| \le n^{1/2-\varepsilon} \text{ for at least } \left[\frac{n^\varepsilon}{a^2k^2}\right] \text{ values of } j \le \frac{n^\varepsilon}{a^2} \right\}.$$

Noting that $\{\nu_n > n^{1-\varepsilon}\} \subset \bigcup G_{l,i}$, we get

$$\mathbf{E}[|\Delta(x+S_n)|; \nu_n > n^{1-\varepsilon}] \le \binom{k}{2} \mathbf{E}[|\Delta(x+S_n)|; G_{1,2}].$$

Therefore, we need to derive an upper bound for $\mathbf{E}[|\Delta(x+S_n)|; G_{1,2}]$.

Let $\mu = \mu_{1,2}$ be the moment when $|x^{(2)} - x^{(1)} + S^{(2)}_{jb_n^2} - S^{(1)}_{jb_n^2}| \le n^{1/2-\varepsilon}$ for the $[n^\varepsilon/(a^2k^2)]$ time. Then it follows from the proof of the previous lemma that

$$\mathbf{P}(\mu \le n^{1-\varepsilon}) = \mathbf{P}(G_{1,2}) \le \exp\{-Cn^\varepsilon\}. \tag{17}$$

Using the inequality $|a+b| \le (1+|a|)(1+|b|)$ one can see that

$$\begin{aligned}
\mathbf{E}[|\Delta(x+S_n)|; G_{1,2}] &\le \mathbf{E}[|\Delta(x+S_n)|; \mu \le n^{1-\varepsilon}] = \sum_{m=1}^{n^{1-\varepsilon}} \mathbf{E}[|\Delta(x+S_n)|; \mu = m] \\
&\le \sum_{m=1}^{n^{1-\varepsilon}} \mathbf{E}[\Delta_1(S_n - S_m)] \mathbf{E}[\Delta_1(x+S_m); \mu = m] \\
&\le \max_{m \le n^{1-\varepsilon}} \mathbf{E}[\Delta_1(S_n - S_m)] \mathbf{E}[\Delta_1(x+S_\mu); \mu \le n^{1-\varepsilon}]. \qquad (18)
\end{aligned}$$

Making use of (10), one can verify that

$$\max_{m \le n^{1-\varepsilon}} \mathbf{E}[\Delta_1(S_n - S_m)] \le Cn^{k(k-1)/4}. \tag{19}$$

Recall that by the definition of μ we have $|x^{(2)} - x^{(1)} + S^{(2)}_\mu - S^{(1)}_\mu| \le n^{1/2-\varepsilon}$. Therefore,

$$\begin{aligned}
\Delta_1(x+S_\mu) &\le n^{1/2-\varepsilon} \frac{\Delta_1(x+S_\mu)}{1 + |x^{(2)} - x^{(1)} + S^{(2)}_\mu - S^{(1)}_\mu|} \\
&\le n^{1/2-\varepsilon} \frac{\Delta_2(x)}{2 + |x^{(2)} - x^{(1)}|} \frac{\Delta_2(S_\mu)}{2 + |S^{(2)}_\mu - S^{(1)}_\mu|}
\end{aligned}$$

It is easy to see that

$$\frac{\Delta_2(S_\mu)}{2+|S_\mu^{(2)}-S_\mu^{(1)}|}\le\sum_{i_1,\ldots,i_k}C_{(i_1,\ldots,i_k)}\prod\left(|S_\mu^{(r)}|\right)^{i_r},$$

where the sum is taken over all $i_1,\ldots,i_k$ such that all $i_1,i_2\le k-2, i_3,\ldots i_k\le k-1$, there is at most one $i_j=k-1$, and the sum $\sum i_r$ does not exceed $k(k-1)/2$. Thus,

$$\begin{aligned}\mathbf{E}\left[\left|\frac{\Delta_2(S_\mu)}{2+|S_\mu^{(2)}-S_\mu^{(1)}|}\right|;\mu\le n^{1-\varepsilon}\right]&\le\sum_{i_1,\ldots,i_k}C_{(i_1,\ldots,i_k)}\mathbf{E}\left[\prod_{r=1}^k\left(|S_\mu^{(r)}|\right)^{i_r};\mu\le n^{1-\varepsilon}\right]\\&\le\sum_{i_1,\ldots,i_k}C_{(i_1,\ldots,i_k)}\mathbf{E}\left[\left(|S_\mu^{(1)}|\right)^{i_1}\left(|S_\mu^{(2)}|\right)^{i_2};\mu\le n^{1-\varepsilon}\right]\prod_{r=3}^k\mathbf{E}\left(M_n^{(r)}\right)^{i_r}.\end{aligned}$$

Since $i_1\le k-2$ and $i_2\le k-2$, we can apply the Hölder inequality, which gives

$$\mathbf{E}\left[\left(|S_\mu^{(1)}|\right)^{i_1}\left(|S_\mu^{(2)}|\right)^{i_2};\mu\le n^{1-\varepsilon}\right]\le n^{(i_1+i_2)/2}\exp\{-Cn^\varepsilon\}.$$

Consequently,

$$\mathbf{E}[|\Delta(x+S_\mu)|;\mu\le n^{1-\varepsilon}]\le c\Delta_2(x)n^{k(k-1)/2}\exp\{-Cn^\varepsilon\}.\qquad(20)$$

Plugging (19) and (20) into (18), we get

$$\mathbf{E}[|\Delta(x+S_n)|;\nu_n>n^{1-\varepsilon}]\le C\Delta_2(x)\exp\{-Cn^\varepsilon\}.$$

Noting that $(2+|x^{(j)}-x^{(i)}|)\le 2(1+|x^{(j)}-x^{(i)}|)$ yields $\Delta_2(x)\le 2^k\Delta_1(x)$, we arrived at the conclusion. □

Lemma 10. *There exists a constant C such that*

$$\mathbf{E}[\Delta(x+S_n);T>n]\le C\Delta_1(x)$$

for all $n\ge1$ and all $x\in W$.

Proof. We first split the expectation into 2 parts,

$$\begin{aligned}\mathbf{E}[\Delta(x+S_n);T>n]&=E_1(x)+E_2(x)\\&=\mathbf{E}\left[\Delta(x+S_n);T>n,\nu_n\le n^{1-\varepsilon}\right]+\mathbf{E}\left[\Delta(x+S_n);T>n,\nu_n>n^{1-\varepsilon}\right].\end{aligned}$$

By Lemma 8, the second term on the right hand side is bounded by

$$E_2(x)\le c\Delta_1(x)\exp\{-Cn^\varepsilon\}.$$

Using Lemma 5, we have

$$\begin{aligned}E_1(x)&\le\sum_{i=1}^{n^{1-\varepsilon}}\int_{W_{n,\varepsilon}}\mathbf{P}\{\nu_n=k,T>k,x+S_k\in dy\}\mathbf{E}[\Delta(y+S_{n-k});T>n-k]\\&\le\sum_{i=1}^{n^{1-\varepsilon}}\int_{W_{n,\varepsilon}}\mathbf{P}\{\nu_n=k,T>k,x+S_k\in dy\}\mathbf{E}[\Delta(y+S_n);T>n]\\&=\sum_{i=1}^{n^{1-\varepsilon}}\int_{W_{n,\varepsilon}}\mathbf{P}\{\nu_n=k,T>k,x+S_k\in dy\}\left(\Delta(y)-\mathbf{E}[\Delta(y+S_T);T\le n]\right),\end{aligned}$$

in the last step we used the fact that $\Delta(x+S_n)$ is a martingale. Then, by Lemma 6,

$$\begin{aligned}
E_1(x) &\le \left(1+\frac{C}{n^\gamma}\right)\sum_{i=1}^{n^{1-\varepsilon}}\int_{W_{n,\varepsilon}}\mathbf{P}\{\nu_n=k,T>k,x+S_k\in dy\}\Delta(y)\\
&\le \left(1+\frac{C}{n^\gamma}\right)\mathbf{E}[\Delta(x+S_{\nu_n});\nu_n\le n^{1-\varepsilon},T>\nu_n].
\end{aligned}$$

Using Lemma 5 once again, we arrive at the bound

$$E_1(x)\le\left(1+\frac{C}{n^\gamma}\right)\mathbf{E}[\Delta(x+S_{n^{1-\varepsilon}});T>n^{1-\varepsilon}].$$

As a result we have

$$\begin{aligned}
&\mathbf{E}[\Delta(x+S_n);T>n]\\
&\qquad\le\left(1+\frac{C}{n^\gamma}\right)\mathbf{E}[\Delta(x+S_{n^{1-\varepsilon}});T>n^{1-\varepsilon}]+c\Delta_1(x)\exp\{-Cn^\varepsilon\}.
\end{aligned}\tag{21}$$

Iterating this procedure m times, we obtain

$$\begin{aligned}
\mathbf{E}[\Delta(x+S_n);T>n]\le\prod_{j=0}^{m}\left(1+\frac{C}{n^{\gamma(1-\varepsilon)^j}}\right)\times\\
\left(\mathbf{E}[\Delta(x+S_{n^{(1-\varepsilon)^{m+1}}});T>n^{(1-\varepsilon)^{m+1}}]+c\Delta_1(x)\sum_{j=0}^{m}\exp\{-Cn^{\varepsilon(1-\varepsilon)^j}\}\right).
\end{aligned}\tag{22}$$

Choosing $m=m(n)$ such that $n^{(1-\varepsilon)^{m+1}}\le 10$ and noting that the product and the sum remain uniformly bounded, we finish the proof of the lemma. $\square$

Lemma 11. *The function $V^{(T)}(x):=\lim_{n\to\infty}\mathbf{E}[\Delta(x+S_n);T>n]$ has the following properties:*

$$\Delta(x)\le V^{(T)}(x)\le C\Delta_1(x)\tag{23}$$

and

$$V^{(T)}(x)\sim\Delta(x)\quad \textit{if}\quad \min_{j<k}(x^{(j+1)}-x^{(j)})\to\infty.\tag{24}$$

Proof. Since $\Delta(x+S_n)\mathbf{1}\{T_x>n\}$ is a submartingale, the limit $\lim_{n\to\infty}\mathbf{E}[\Delta(x+S_n);T>n]$ exists, and the function $V^{(T)}$ satisfies $V^{(T)}(x)\ge\Delta(x),\quad x\in\{y:\Delta(y)>0\}$. The upper bound in (23) follows immediately from Lemma 10.

To show (24) it suffices to obtain an upper bound of the form $(1+o(1))\Delta(x)$. Furthermore, because of monotonicity of $\mathbf{E}[\Delta(x+S_n);T>n]$, we can get such a bound for a specially chosen subsequence $\{n_m\}$. Choose ε so that (22) is valid, and set $n_m=(n_0)^{(1-\varepsilon)^{-m}}$. Then we can rewrite (22) in the following form

$$\begin{aligned}
&\mathbf{E}[\Delta(x+S_{n_m});T>n_m]\le\\
&\quad\prod_{j=0}^{m-1}\left(1+\frac{C}{n_j^\gamma}\right)\times\left(\mathbf{E}[\Delta(x+S_{n_0});T>n_0]+c\Delta_1(x)\sum_{j=0}^{m-1}\exp\{-Cn_j^\varepsilon\}\right).
\end{aligned}$$

It is clear that for every $\delta > 0$ we can choose n_0 such that

$$\prod_{j=0}^{m-1}\left(1+\frac{C}{n_j^\gamma}\right) \le 1+\delta \quad \text{and} \quad \sum_{j=0}^{m-1} \exp\{-Cn_j^\varepsilon\} \le \delta$$

for all $m \ge 1$. Consequently,

$$V^{(T)}(x) = \lim_{m\to\infty} \mathbf{E}[\Delta(x+S_{n_m}); T > n_m] \le (1+\delta)\mathbf{E}[\Delta(x+S_{n_0}); T > n_0] + C\delta\Delta_1(x).$$

It remains to note that $E[\Delta(x+S_{n_0}); T > n_0] \sim \Delta(x)$ and that $\Delta_1(x) \sim \Delta(x)$ as $\min_{j<k}(x^{(j+1)} - x^{(j)}) \to \infty$. $\square$

2.2 Proof of Proposition 4

We start by showing that Lemma 10 implies the integrability of $\Delta(x + S_{\tau_x})$. Indeed, setting $\tau_x(n) := \min\{\tau_x, n\}$ and $T_x(n) := \min\{T_x, n\}$, and using the fact that $|\Delta(x + S_n)|$ is a submartingale, we have

$$\begin{aligned}\mathbf{E}|\Delta(x+S_{\tau_x(n)})| &\le \mathbf{E}|\Delta(x+S_{T_x(n)})| \\ &= \mathbf{E}[\Delta(x+S_n)1\{T_x(n) > n\}] - \mathbf{E}[\Delta(x+S_{T_x})1\{T_x \le n\}].\end{aligned}$$

Since $\Delta(x + S_n)$ is a martingale, we have

$$\mathbf{E}[\Delta(x+S_T)1\{T \le n\}] = \mathbf{E}[\Delta(x+S_n)1\{T \le n\}] = \Delta(x) - \mathbf{E}[\Delta(x+S_n)1\{T > n\}].$$

Therefore, we get

$$\mathbf{E}|\Delta(x+S_{\tau_n})| \le 2\mathbf{E}[\Delta(x+S_n)1\{T > n\}] - \Delta(x).$$

This, together with Lemma 10, implies that the sequence $\mathbf{E}[|\Delta(x + S_\tau)|\mathbf{1}\{\tau \le n\}]$ is uniformly bounded. Then, the finiteness of the expectation $\mathbf{E}|\Delta(x + S_\tau)|$ follows from the monotone convergence.

To prove (a) note that since $\Delta(x + S_n)$ is a martingale, we have an equality

$$\mathbf{E}[\Delta(x+S_n); \tau_x > n] = \Delta(x) - \mathbf{E}[\Delta(x+S_n); \tau_x \le n] = \Delta(x) - \mathbf{E}[\Delta(x+S_{\tau_x}); \tau_x \le n].$$

Letting n to infinity we obtain (a) by the dominated convergence theorem.

For (b) note that

$$\Delta(x+S_n)\mathbf{1}\{\tau_x > n\} \le \Delta(y+S_n)\mathbf{1}\{\tau_x > n\} \le \Delta(y+S_n)\mathbf{1}\{\tau_y > n\}.$$

Then letting n to infinity and applying (a) we obtain (b).

(c) follows directly from Lemma 10.

We now turn to the proof of (d). It follows from (24) and the inequality $\tau_x \le T_x$ that

$$V(x) \le V^{(T)}(x) \le (1+o(1))\Delta(x).$$

Thus, we need to get a lower bound of the form $(1 + o(1))\Delta(x)$. We first note that

$$V(x) = \Delta(x) - \mathbf{E}[\Delta(x+S_{\tau_x})] \ge \Delta(x) - \mathbf{E}[\Delta(x+S_{\tau_x}); T_x > \tau_x].$$

Therefore, it is sufficient to show that

$$\mathbf{E}[\Delta(x+S_{\tau_x}); T_x > \tau_x] = o(\Delta(x)) \tag{25}$$

under the condition $\min_{j<k}(x^{(j+1)} - x^{(j)}) \to \infty$.

The sequence $Z_n := V^{(T)}(x+S_n)\mathbf{1}\{T_x > n\}$ is a non-negative martingale. Indeed, using martingale property of $\Delta(x+S_n)$, one gets easily

$$\begin{aligned}\mathbf{E}[\Delta(x+S_n); T > n] &= \mathbf{E}\Delta(x+S_n) - \mathbf{E}[\Delta(x+S_n); T \le n] \\ &= \Delta(x) - \mathbf{E}[\Delta(x+S_T); T \le n]\end{aligned}$$

Consequently,

$$V^{(T)}(x) = \lim_{n\to\infty} \mathbf{E}[\Delta(x+S_n); T > n] = \Delta(x) - \mathbf{E}[\Delta(x+S_T)].$$

Then,

$$\begin{aligned}&\mathbf{E}[V^{(T)}(x+S_1)]\mathbf{1}\{T_x > 1\}] \\ &\qquad = \mathbf{E}[\Delta(x+S_1)]\mathbf{1}\{T_x > 1\}] - \mathbf{E}[\mathbf{E}[\Delta(x+S_T)|S_1]\mathbf{1}\{T_x > 1\}] \\ &\qquad = \mathbf{E}[\Delta(x+S_1)]\mathbf{1}\{T_x > 1\}] - \mathbf{E}[[\Delta(x+S_T)]\mathbf{1}\{T_x > 1\}] \\ &\qquad = \Delta(x) - \mathbf{E}[\Delta(x+S_1)]\mathbf{1}\{T_x = 1\}] - \mathbf{E}[[\Delta(x+S_T)]\mathbf{1}\{T_x > 1\}] \\ &\qquad = \Delta(x) - \mathbf{E}[\Delta(x+S_T)] = V^{(T)}(x).\end{aligned}$$

This implies the desired martingale property of $V^{(T)}(x+S_n)\mathbf{1}\{T_x > n\}$. Furthermore, recalling that $T_x \ge \tau_x$ and arguing as in Lemma 5, one can easily get

$$\begin{aligned}&\mathbf{E}[V^{(T)}(x+S_n)\mathbf{1}\{\tau_x > n\} - V^{(T)}(x+S_{n-1})\mathbf{1}\{\tau_x > n-1\}|\mathcal{F}_{n-1}] \\ &\qquad = \mathbf{E}[Z_n\mathbf{1}\{\tau_x > n\} - Z_{n-1}\mathbf{1}\{\tau_x > n-1\}|\mathcal{F}_{n-1}] \\ &\qquad = \mathbf{1}\{\tau_x > n-1\}\mathbf{E}[Z_n - Z_{n-1}|\mathcal{F}_{n-1}] - \mathbf{E}[Z_n\mathbf{1}\{\tau_x = n\}|\mathcal{F}_{n-1}] \\ &\qquad = -\mathbf{E}[Z_n\mathbf{1}\{\tau_x = n\}|\mathcal{F}_{n-1}] \le 0,\end{aligned}$$

i.e., the sequence $V^{(T)}(x+S_n)\mathbf{1}\{\tau_x > n\}$ is a supermartingale.

We bound $\mathbf{E}[V^{(T)}(x+S_n)\mathbf{1}\{\tau_x > n\}]$ from below using its supermartingale property. This is similar to the Lemma 10, where an upper bound has been obtained using submartingale properties of $\Delta(x+S_n)\mathbf{1}\{T_x > n\}$. We have

$$\begin{aligned}&\mathbf{E}[V^{(T)}(x+S_n); \tau_x > n] \\ &\ge \sum_{i=1}^{n^{1-\varepsilon}} \int_{W_{n,\varepsilon}} \mathbf{P}\{\nu_n = k, \tau_x > k, x+S_k \in dy\}\mathbf{E}[V^{(T)}(y+S_{n-k}); \tau_y > n-k] \\ &\ge \sum_{i=1}^{n^{1-\varepsilon}} \int_{W_{n,\varepsilon}} \mathbf{P}\{\nu_n = k, \tau_x > k, x+S_k \in dy\}\mathbf{E}[V^{(T)}(y+S_n); \tau_y > n] \\ &= \sum_{i=1}^{n^{1-\varepsilon}} \int_{W_{n,\varepsilon}} \mathbf{P}\{\nu_n = k, \tau_x > k, x+S_k \in dy\} \left(V^{(T)}(y) - \mathbf{E}[V^{(T)}(y+S_{\tau_y}); \tau_y \le n]\right).\end{aligned}$$

Then, applying (23) and (8), we obtain

$$\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n] \ge \left(1-\frac{C}{n^\gamma}\right)\mathbf{E}[V^{(T)}(x+S_{n^{1-\varepsilon}});\tau_x>n^{1-\varepsilon},\nu_n\le n^{1-\varepsilon}].$$

Using now Lemma 8, we have

$$\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n] \ge \left(1-\frac{C}{n^\gamma}\right)\mathbf{E}[V^{(T)}(x+S_{n^{1-\varepsilon}});\tau_x>n^{1-\varepsilon}] - C\Delta_1(x)e^{-Cn^\varepsilon}.$$

Starting from n_0 and iterating this procedure, we obtain for the sequence $n_m=(n_0)^{(1-\varepsilon)^{-m}}$ the inequality

$$\mathbf{E}[V^{(T)}(x+S_{n_m});\tau_x>n_m] \ge \prod_{j=1}^m\left(1-\frac{C}{n_0^{\gamma(1-\varepsilon)^{-j}}}\right)\mathbf{E}[V^{(T)}(x+S_{n_0});\tau_x>n_0]$$
$$-c\Delta_1(x)\sum_{j=1}^m \exp\{-Cn_0^{\varepsilon(1-\varepsilon)^{-j}}\}.$$

Next we fix a constant $\delta>0$ and pick n_0 such that

$$\prod_{j=1}^\infty\left(1-\frac{C}{n_0^{\gamma(1-\varepsilon)^{-j}}}\right)\ge(1-\delta),\quad c\sum_{j=1}^\infty \exp\{-Cn_0^{\varepsilon(1-\varepsilon)^{-j}}\}\le\delta.$$

This is possible since both the series and the product converge. Together with the fact that $V^{(T)}(x+S_n)\mathbf{1}\{\tau_x>n\}$ is a supermartingale and the with lower bound in (23) this gives us,

$$\lim_{n\to\infty}\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n]\ge(1-\delta)\mathbf{E}[\Delta(x+S_{n_0});\tau_x>n_0]-\delta\Delta_1(x).$$

As is not difficult to see $\mathbf{E}[\Delta(x+S_{n_0});\tau_x>n_0]\sim\Delta(x)$ and $\Delta_1(x)\sim\Delta(x)$ as $\min_{2\le j\le k}(x^{(j)}-x^{(j-1)})\to\infty$. Therefore, since $\delta>0$ is arbitrary we have a lower asymptotic bound

$$\lim_{n\to\infty}\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n]\ge(1-o(1))\Delta(x), \tag{26}$$

provided that $\min_{2\le j\le k}(x^{(j)}-x^{(j-1)})\to\infty$.
Using the martingale property of $V^{(T)}(x+S_n)\mathbf{1}\{T_x>n\}$ and noting that

$$\{T_x>n\}=\{\tau_x>n\}\bigcup\left(\bigcup_{k=1}^n\{T_x>n,\,\tau_x=k\}\right),$$

we get

$$\begin{aligned}V^{(T)}(x)&=\mathbf{E}[V^{(T)}(x+S_n)\mathbf{1}\{T_x>n\}]\\&=\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n]+\sum_{k=1}^n\mathbf{E}[V^{(T)}(x+S_n)\mathbf{1}\{T_x>n\};\tau_x=k]\\&=\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n]+\sum_{k=1}^n\mathbf{E}[V^{(T)}(x+S_k)\mathbf{1}\{T_x>k\};\tau_x=k]\\&=\mathbf{E}[V^{(T)}(x+S_n);\tau_x>n]+\mathbf{E}[V^{(T)}(x+S_{\tau_x})\mathbf{1}\{T_x>\tau_x\};\tau_x\le n].\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbf{E}[V^{(T)}(x+S_n); \tau_x > n] = V^{(T)}(x) - \mathbf{E}[V^{(T)}(x+S_{\tau_x}); T_x > \tau_x]. \tag{27}$$

Combining (24), (26) and (27), we have $E[V^{(T)}(x+S_{\tau_x}); T_x > \tau_x] = o(\Delta(x))$. Now (25) follows from the obvious bound

$$\mathbf{E}[\Delta(x+S_{\tau_x}); T_x > \tau_x] \le \mathbf{E}[V^{(T)}(x+S_{\tau_x}); T_x > \tau_x].$$

Thus, the proof of (d) is finished.

To prove (e) note that it follows from (d) that there exists R and $\delta > 0$ such that $V(x) \ge \delta$ on the set $S_R = \{x : \min_{2\le j\le k}(x^{(j)} - x^{(j-1)}) > R\}$. Then, with a positive probability p the random walk can reach this set after N steps if N is sufficiently large. Therefore,

$$\begin{aligned} V(x) = \sup_{n\ge1} \mathbf{E}[\Delta(x+S_n); \tau_x > n] &\ge \int_{S_R} \mathbf{P}\{x+S_N \in dy\} \sup_{n\ge1} \mathbf{E}[\Delta(y+S_n); \tau_y > n] \\ &= \int_{S_R} \mathbf{P}\{x+S_N \in dy\} V(y) \ge \delta p > 0. \end{aligned}$$

This completes the proof of the proposition.

3 Coupling

We start by formulating a classical result on the normal approximation of random walks.

Lemma 12. *If $\mathbf{E}\xi^{2+\delta} < \infty$ for some $\delta \in (0,1)$, then one can define a Brownian motion B_t on the same probability space such that, for any a satisfying $0 < a < \frac{\delta}{2(2+\delta)}$,*

$$\mathbf{P}\left(\sup_{u\le n} |S_{[u]} - B_u| \ge n^{1/2-a}\right) = o\left(n^{2a+a\delta-\delta/2}\right). \tag{28}$$

This statement easily follows from Theorem 2 of [13], see also Theorem 2 of [4].

Lemma 13. *There exists a finite constant C such that*

$$\mathbf{P}(\tau_y^{bm} > n) \le C\frac{\Delta(y)}{n^{k(k-1)/4}}, \quad y \in W. \tag{29}$$

Moreover,

$$\mathbf{P}(\tau_y^{bm} > n) \sim \varkappa \frac{\Delta(y)}{n^{k(k-1)/4}}, \tag{30}$$

uniformly in $y \in W$ satisfying $|y| \le \theta_n\sqrt{n}$ with some $\theta_n \to 0$. Finally, the density $b_t(y,z)$ of the probability $\mathbf{P}(\tau_y^{bm} > t, B_t \in dz)$ is

$$b_t(y,z) \sim K t^{-k/2} e^{-|z|^2/(2t)} \Delta(y)\Delta(z) t^{-\frac{k(k-1)}{2}} \tag{31}$$

uniformly in $y, z \in W$ satisfying $|y| \le \theta_n\sqrt{n}$ and $|z| \le \sqrt{n/\theta_n}$ with some $\theta_n \to 0$. Here,

$$K = (2\pi)^{-k/2} \prod_{l=0}^{k-1} \frac{1}{l!}; \;\; \varkappa = K\frac{1}{k!}\int_{\mathbf{R}^k} e^{-|x|^2/2} |\Delta(x)| dx = K\frac{1}{k!} 2^{3k/2} \prod_{j=1}^{k} \Gamma(1+j/2).$$

Proof. (29) has been proved by Varopoulos [18], see Theorem 1 and formula (0.4.1) there. The proof of (30) and (31) can be found in Sections 5.1-5.2 of [9]. $\square$

Using the coupling we can translate the results of Lemma 13 to the random walks setting when $y \in W_{n,\varepsilon}$.

Lemma 14. *For all sufficiently small $\varepsilon > 0$,*

$$\mathbf{P}(\tau_y > n) = \varkappa \Delta(y) n^{-k(k-1)/4}(1 + o(1)), \quad \text{as } n \to \infty \tag{32}$$

uniformly in $y \in W_{n,\varepsilon}$ such that $|y| \le \theta_n \sqrt{n}$ for some $\theta_n \to 0$. Moreover, there exists a constant C such that

$$\mathbf{P}(\tau_y > n) \le C \Delta(y) n^{-k(k-1)/4}, \tag{33}$$

uniformly in $y \in W_{n,\varepsilon}, n \ge 1$. Finally, for any bounded open set $D \subset W$,

$$\mathbf{P}(\tau_y > n, y + S_n \in \sqrt{n} D) \sim K \Delta(y) n^{-k(k-1)/4} \int_D dz e^{-|z|^2/2} \Delta(z), \tag{34}$$

uniformly in $y \in W_{n,\varepsilon}$.

Proof. For every $y \in W_{n,\varepsilon}$ denote

$$y^{\pm} = (y_i \pm 2(i-1) n^{1/2 - 2\varepsilon}, 1 \le i \le k).$$

Define $A = \left\{ \sup_{u \le n} |S^{(r)}_{[u]} - B^{(r)}_u| \le n^{1/2 - 2\varepsilon} \text{ for all } r \le k \right\}$, where $B^{(r)}$ are as in Lemma 12. Then, using (28) with $a = 2\varepsilon$, we obtain

$$\begin{aligned} \mathbf{P}(\tau_y > n) &= \mathbf{P}(\tau_y > n, A) + o\left(n^{-r}\right) \\ &= \mathbf{P}(\tau_y > n, \tau^{bm}_{y^+} > n, A) + o\left(n^{-r}\right) \\ &\le \mathbf{P}(\tau^{bm}_{y^+} > n, A) + o\left(n^{-r}\right) \\ &= \mathbf{P}(\tau^{bm}_{y^+} > n) + o\left(n^{-r}\right), \end{aligned} \tag{35}$$

where $r = r(\delta, \varepsilon) = \delta/2 - 4\varepsilon - 2\varepsilon\delta$. In the same way one can get

$$\mathbf{P}(\tau^{bm}_{y^-} > n) \le \mathbf{P}(\tau_y > n) + o\left(n^{-r}\right). \tag{36}$$

By Lemma 13,

$$\mathbf{P}(\tau^{bm}_{y^{\pm}} > n) \sim \varkappa \Delta(y^{\pm}) n^{-k(k-1)/4}.$$

Next, since $y \in W_{n\varepsilon}$,

$$\Delta(y^{\pm}) = \Delta(y)(1 + O(n^{-\varepsilon}))$$

Therefore, we conclude that

$$\mathbf{P}(\tau^{bm}_{y^{\pm}} > n) = \varkappa \Delta(y) n^{-k(k-1)/4}(1 + O(n^{-\varepsilon})).$$

From this relation and bounds (35) and (36) we obtain

$$\mathbf{P}(\tau_y > n) = \varkappa \Delta(y) n^{-k(k-1)/4}(1 + O(n^{-\varepsilon})) + o\left(n^{-r}\right).$$

Thus, it remains to show that

$$n^{-r} = o(\Delta(y)n^{-k(k-1)/4}) \tag{37}$$

for all sufficiently small $\varepsilon > 0$ and all $y \in W_{n,\varepsilon}$. For that note that for $y \in W_{n,\varepsilon}$,

$$\Delta(y)n^{-k(k-1)/4} \geq \prod_{i<j}(j-i)n^{-\varepsilon\frac{k(k-1)}{2}}.$$

Therefore, (37) will be valid for all ε satisfying

$$r = 4\varepsilon + 2\delta\varepsilon - \delta/2 < \varepsilon\frac{k(k-1)}{2}.$$

This proves (32). To prove (33) it is sufficient to substitute (29) in (35).

The proof of (34) is similar. Define two sets,

$$D^+ = \{z \in W : dist(z, D) \leq 3kn^{-2\varepsilon}\},\ D^- = \{z \in D : dist(z, \partial D) \geq 3kn^{-2\varepsilon}\}.$$

Clearly $D^- \subset D \subset D^+$. Then, arguing as above, we get

$$\begin{aligned}
\mathbf{P}(\tau_y > n, y + S_n \in \sqrt{n}D) &\leq \mathbf{P}(\tau_y > n, y + S_n \in \sqrt{n}D, A) + o\left(n^{-r}\right) \\
&\leq \mathbf{P}(\tau_{y^+}^{bm} > n, y^+ + B_n \in \sqrt{n}D^+, A) + o\left(n^{-r}\right) \\
&\leq \mathbf{P}(\tau_{y^+}^{bm} > n, y^+ + B_n \in \sqrt{n}D^+) + o\left(n^{-r}\right). \qquad (38)
\end{aligned}$$

Similarly,

$$\mathbf{P}(\tau_y > n, y + S_n \in \sqrt{n}D) \geq \mathbf{P}(\tau_{y^-}^{bm} > n, y^- + B_n \in \sqrt{n}D^-) + o\left(n^{-r}\right). \tag{39}$$

Now we apply (31) and obtain

$$\begin{aligned}
\mathbf{P}(\tau_{y^\pm}^{bm} > n, y^\pm + B_n \in \sqrt{n}D^\pm) &\sim K\Delta(y^\pm)\int_{\sqrt{n}D^\pm} dz e^{-|z|^2/(2n)}\Delta(z)n^{-\frac{k}{2}}n^{-\frac{k(k-1)}{4}} \\
&= K\Delta(y^\pm)\int_{D^\pm} dz e^{-|z|^2/2}\Delta(z)n^{-\frac{k(k-1)}{4}}.
\end{aligned}$$

It is sufficient to note now that

$$\Delta(y^\pm) \sim \Delta(y) \text{ and } \int_{D^\pm} dz e^{-|z|^2/2}\Delta(z) \to \int_D dz e^{-|z|^2/2}\Delta(z)$$

as $n \to \infty$. From these relations and bounds (38) and (39) we obtain

$$\mathbf{P}(\tau_y > n, y + S_n \in \sqrt{n}D) = (K + o(1))\Delta(y)\int_D dz e^{-|z|^2/2}\Delta(z)n^{-\frac{k(k-1)}{4}} + o\left(n^{-r}\right).$$

Recalling (37) we arrive at the conclusion. $\square$

4 Asymptotics for $\mathbf{P}\{\tau_x > n\}$

We first note that, in view of Lemma 7,

$$\begin{aligned}\mathbf{P}(\tau_x > n) &= \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) + \mathbf{P}(\tau_x > n, \nu_n > n^{1-\varepsilon}) \\ &= \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) + O\left(e^{-Cn^{\varepsilon}}\right). \end{aligned} \tag{40}$$

Using the strong Markov property, we get for the first term the following estimates

$$\begin{aligned}\int_{W_{n,\varepsilon}} \mathbf{P}\left(S_{\nu_n} \in dy, \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}\right) \mathbf{P}(\tau_y > n) &\le \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) \\ \le \int_{W_{n,\varepsilon}} \mathbf{P}\left(S_{\nu_n} \in dy, \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}\right) &\mathbf{P}(\tau_y > n - n^{1-\varepsilon}). \end{aligned} \tag{41}$$

Applying now Lemma 14, we obtain

$$\begin{aligned}&\mathbf{P}(\tau_x > n; \nu_n \le n^{1-\varepsilon}) \\ &= \frac{\varkappa + o(1)}{n^{k(k-1)/4}} \mathbf{E}\left[\Delta(x + S_{\nu_n}); \tau_x > \nu_n, |S_{\nu_n}| \le \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\right] \\ &\quad + O\left(\frac{1}{n^{k(k-1)/4}} \mathbf{E}\left[\Delta(x + S_{\nu_n}); \tau_x > \nu_n, |S_{\nu_n}| > \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\right]\right) \\ &= \frac{\varkappa + o(1)}{n^{k(k-1)/4}} \mathbf{E}\left[\Delta(x + S_{\nu_n}); \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}\right] \\ &\quad + O\left(\frac{1}{n^{k(k-1)/4}} \mathbf{E}\left[\Delta(x + S_{\nu_n}); \tau_x > \nu_n, |S_{\nu_n}| > \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\right]\right). \end{aligned} \tag{42}$$

We now show that the first expectation converges to $V(x)$ and that the second expectation is negligibly small.

Lemma 15. *Under the assumptions of Theorem 1,*

$$\lim_{n\to\infty} \mathbf{E}\left[\Delta(x + S_{\nu_n}) \mathbf{1}\{\tau_x > \nu_n\}; \nu_n \le n^{1-\varepsilon}\right] = V(x).$$

Proof. Rearranging, we have

$$\begin{aligned}&\mathbf{E}\left[\Delta(x + S_{\nu_n}) \mathbf{1}\{\tau_x > \nu_n\}; \nu_n \le n^{1-\varepsilon}\right] \\ &\quad = \mathbf{E}\left[\Delta(x + S_{\nu_n \wedge n^{1-\varepsilon}}) \mathbf{1}\{\tau_x > \nu_n \wedge n^{1-\varepsilon}\}; \nu_n \le n^{1-\varepsilon}\right] \\ &\quad = \mathbf{E}\left[\Delta(x + S_{\nu_n \wedge n^{1-\varepsilon}}) \mathbf{1}\{\tau_x > \nu_n \wedge n^{1-\varepsilon}\}\right] \\ &\qquad - \mathbf{E}\left[\Delta(x + S_{n^{1-\varepsilon}}) \mathbf{1}\{\tau_x > n^{1-\varepsilon}\}; \nu_n > n^{1-\varepsilon}\right]. \end{aligned} \tag{43}$$

According to Lemma 8,

$$\left|\mathbf{E}\left[\Delta(x + S_{n^{1-\varepsilon}}) \mathbf{1}\{\tau_x > n^{1-\varepsilon}\}; \nu_n > n^{1-\varepsilon}\right]\right| \le C(x) \exp\{-Cn^{\varepsilon}\}. \tag{44}$$

Further,

$$\begin{aligned}&\mathbf{E}\left[\Delta(x + S_{\nu_n \wedge n^{1-\varepsilon}}) \mathbf{1}\{\tau_x > \nu_n \wedge n^{1-\varepsilon}\}\right] \\ &\quad = \mathbf{E}\left[\Delta(x + S_{\nu_n \wedge n^{1-\varepsilon}})\right] - \mathbf{E}\left[\Delta(x + S_{\nu_n \wedge n^{1-\varepsilon}}) \mathbf{1}\{\tau_x \le \nu_n \wedge n^{1-\varepsilon}\}\right] \\ &\quad = \Delta(x) - \mathbf{E}\left[\Delta(x + S_{\nu_n \wedge n^{1-\varepsilon}}) \mathbf{1}\{\tau_x \le \nu_n \wedge n^{1-\varepsilon}\}\right] \\ &\quad = \Delta(x) - \mathbf{E}\left[\Delta(x + S_{\tau_x}) \mathbf{1}\{\tau_x \le \nu_n \wedge n^{1-\varepsilon}\}\right], \end{aligned}$$

here we have used the martingale property of $\Delta(x+S_n)$. Noting that $\nu_n \wedge n^{1-\varepsilon} \to \infty$ almost surely, we have

$$\Delta(x+S_{\tau_x})\mathbf{1}\{\tau_x \le \nu_n \wedge n^{1-\varepsilon}\} \to \Delta(x+S_{\tau_x}).$$

Then, using the integrability of $\Delta(x+S_{\tau_x})$ and the dominated convergence, we obtain

$$\mathbf{E}\left[\Delta(x+S_{\tau_x})\mathbf{1}\{\tau_x \le \nu_n \wedge n^{1-\varepsilon}\}\right] \to \mathbf{E}\left[\Delta(x+S_{\tau_x})\right]. \tag{45}$$

Combining (43)–(45), we finish the proof of the lemma. □

Lemma 16. *Under the assumptions of Theorem 1,*

$$\lim_{n\to\infty} \mathbf{E}\left[\Delta(x+S_{\nu_n}); \tau_x > \nu_n, |S_{\nu_n}| > \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}\right] = 0.$$

Proof. We first note that

$$\begin{aligned}
&\mathbf{E}\left[\Delta(x+S_{\nu_n}); \tau_x > \nu_n, |S_{\nu_n}| > \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}\right] \\
&\le \mathbf{E}\left[\Delta(x+S_{\nu_n}); T_x > \nu_n, |S_{\nu_n}| > \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}\right] \\
&\le \mathbf{E}\left[\Delta(x+S_{n^{1-\varepsilon}}); T_x > n^{1-\varepsilon}, M_{n^{1-\varepsilon}} > \theta_n\sqrt{n}/k\right],
\end{aligned}$$

where we used the submartingale property of $\Delta(x+S_j)\mathbf{1}\{T_x > j\}$, see Lemma 5. (Recall that $M_j = \max_{i\le j, r\le k} |S_i^{(r)}|$.) Therefore, it is sufficient to show that

$$\mathbf{E}\left[\Delta(x+S_n); T_x > n, M_n > n^{1/2+2\delta}\right] \to 0 \tag{46}$$

for any positive δ.

Define

$$A_n = \left\{\max_{1\le i\le n, 1\le j\le k} |\xi_i^{(j)}| \le n^{1/2+\delta}\right\}.$$

Then

$$\mathbf{E}\left[\Delta(x+S_n); T_x > n, M_n > n^{1/2+2\delta}, A_n\right] \le \mathbf{E}\left[|\Delta(x+S_n)|; M_n > n^{1/2+2\delta}, A_n\right].$$

Since $|S_n^{(j)}| \le n \max_{i\le n} |\xi_i^{(j)}| \le n^{3/2+\delta}$ on the event A_n, we arrive at the following upper bound

$$\begin{aligned}
&\mathbf{E}\left[\Delta(x+S_n); T_x > n, M_n > n^{1/2+2\delta}, A_n\right] \\
&\le C(x)\left(n^{3/2+\delta}\right)^{k(k-1)/2} \mathbf{P}(M_n > n^{1/2+2\delta}, A_n).
\end{aligned}$$

Applying now one of the Fuk-Nagaev inequalities, see Corollary 1.11 in [14], we have

$$\mathbf{P}(M_n > n^{1/2+2\delta}, A_n) \le \exp\{-Cn^{\delta}\}.$$

As a result,

$$\lim_{n\to\infty} \mathbf{E}\left[\Delta(x+S_n); T_x > n, M_n > n^{1/2+2\delta}, A_n\right] = 0 \tag{47}$$

Define

$$\Sigma_l := \sum_{i=1}^{l} \sum_{j=1}^{k} \mathbf{1}\{|\xi_i^{(j)}| > n^{1/2+\delta}\}, \quad l \le n$$

and

$$\Sigma_{l,n} := \sum_{i=l+1}^{n} \sum_{j=1}^{k} \mathbf{1}\{|\xi_i^{(j)}| > n^{1/2+\delta}\}, \quad l < n.$$

We note that

$$\begin{aligned}\mathbf{E}\left[\Delta(x+S_n); T_x > n, M_n > n^{1/2+2\delta}, \overline{A_n}\right] &\le \mathbf{E}\left[\Delta(x+S_n)\Sigma_n; T_x > n\right] \\ &= \mathbf{E}\left[\Delta(x+S_n)\Sigma_n\right] - \mathbf{E}\left[\Delta(x+S_n)\Sigma_n; T_x \le n\right]. \end{aligned} \tag{48}$$

Since the conditioned distribution of S_n given Σ is exchangeable, we may apply Theorem 2.1 of [10], which says that

$$\mathbf{E}[\Delta(x+S_l)|\Sigma_l] = \Delta(x), \quad l \le n.$$

Therefore,

$$\mathbf{E}[\Delta(x+S_l)\Sigma_l] = \Delta(x)\mathbf{E}[\Sigma_l] = k\Delta(x)l\mathbf{P}(|\xi| > n^{1/2+\delta}), \quad l \le n. \tag{49}$$

Using this equality and conditioning on $\mathcal{F}_l$, we have

$$\begin{aligned}&\mathbf{E}\left[\Delta(x+S_n)\Sigma_n; T_x = l\right] = \mathbf{E}\left[\Delta(x+S_n)\Sigma_l; T_x = l\right] + \mathbf{E}\left[\Delta(x+S_n)\Sigma_{l,n}; T_x = l\right] \\ &= \mathbf{E}\left[\Delta(x+S_l)\Sigma_l; T_x = l\right] + \mathbf{E}\left[\mathbf{E}[\Delta(x+S_n)\Sigma_{l,n}|\mathcal{F}_l]; T_x = l\right] \\ &= \mathbf{E}\left[\Delta(x+S_l)\Sigma_l; T_x = l\right] + \mathbf{E}\left[\Delta(x+S_l); T_x = l\right]\mathbf{E}\Sigma_{l,n},\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{E}\left[\Delta(x+S_n)\Sigma_n; T_x \le n\right] &= \mathbf{E}\left[\Delta(x+S_T)\Sigma_T; T_x \le n\right] \\ &\quad + O\left(n\mathbf{P}(|\xi| > n^{1/2+\delta})\mathbf{E}\left[\Delta(x+S_T); T_x \le n\right]\right) \\ &= \mathbf{E}\left[\Delta(x+S_T)\Sigma_T; T_x \le n\right] + o(1).\end{aligned}$$

Finally,

$$|\mathbf{E}\left[\Delta(x+S_T)\Sigma_T; T_x \le n\right]| \le \mathbf{E}\left[|\Delta(x+S_T)|\Sigma_n\right] = o(1),$$

by the dominated convergence, since $\Sigma_n \to 0$. This implies that

$$\mathbf{E}\left[\Delta(x+S_n)\Sigma; T_x \le n\right] = o(1). \tag{50}$$

Combining (48)–(50), we see that the left hand side of (48) converges to zero. Then, taking into account (47), we get (46). Thus, the proof is finished. □

Now we are in position to complete the proof of Theorem 1. It follows from the lemmas and (40) and (42) that

$$\mathbf{P}(\tau_x > n) = \frac{\varkappa V(x)}{n^{k(k-1)/4}}(1 + o(1)).$$

5 Weak convergence results

Lemma 17. *For any $x \in W$, the distribution $\mathbf{P}\left(\frac{x+S_n}{\sqrt{n}} \in \cdot | \tau_x > n\right)$ weakly converges to the distribution with the density $\frac{1}{Z_1} e^{-|y|^2/2}\Delta(y)$, where Z_1 is the norming constant.*

Proof. We need to show that

$$\frac{\mathbf{P}(x + S_n \in \sqrt{n}A, \tau_x > n)}{\mathbf{P}(\tau_x > n)} \to Z_1^{-1} \int_A e^{-|y|^2/2}\Delta(y)dy. \tag{51}$$

First note that, as in (40) and (42),

$$\begin{aligned}\mathbf{P}(x + S_n \in \sqrt{n}A, \tau_x > n) &= \mathbf{P}(\tau_x > n, x + S_n \in \sqrt{n}A, \nu_n \le n^{1-\varepsilon}) + O\left(e^{-Cn^{\varepsilon}}\right) \\ &= \mathbf{P}(\tau_x > n, x + S_n \in \sqrt{n}A, |S_{\nu_n}| \le \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}) + o(\mathbf{P}(\tau_x > n)).\end{aligned}$$

Next,

$$\begin{aligned}&\mathbf{P}(\tau_x > n, x + S_n \in \sqrt{n}A, |S_{\nu_n}| \le \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}) \\ &= \sum_{j=1}^{n^{1-\varepsilon}} \int_{W_{n,\varepsilon}\cap\{|y|\le\theta_n\sqrt{n}\}} \mathbf{P}(\tau_x > j, x + S_j \in \sqrt{n}A, \nu_n = j) \\ &\qquad\qquad \times \mathbf{P}(\tau_y > n - j, y + S_{n-j} \in \sqrt{n}A).\end{aligned}$$

Using the coupling and arguing as in Lemma 14, one can show that

$$\mathbf{P}(\tau_y > n - k, y + S_{n-k} \in \sqrt{n}A) \sim \mathbf{P}(\tau_y^{bm} > n, y + B_n \in \sqrt{n}A)$$

uniformly in $k \le n^{1-\varepsilon}$ and $y \in W_{n,\varepsilon}$. Next we apply asymptotics (31) and obtain that

$$\mathbf{P}(\tau_y > n - k, y + S_{n-k} \in \sqrt{n}A) \sim K \int_A dz e^{-|z|^2/2}\Delta(y)\Delta(z)n^{-k(k-1)/4}$$

uniformly in $y \in W_{n,\varepsilon}$, $|y| \le \theta_n\sqrt{n}$. As a result we obtain

$$\begin{aligned}\mathbf{P}(x + S_n \in \sqrt{n}A, \tau_x > n) &\sim \int_A dz e^{-|z|^2/2}\Delta(z)n^{-k(k-1)/4} \\ &\quad \times K\mathbf{E}[\Delta(x + S_{\nu_n}), \tau_x > \nu_n, x + S_n \in \sqrt{n}A, |S_{\nu_n}| \le \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}] \\ &\sim K \int_A dz e^{-|z|^2/2}\Delta(z)n^{-k(k-1)/4}V(x),\end{aligned}$$

where the latter equivalence holds due to Lemma 15. Substituting the latter equivalence in (51) and using the asymptotics for $\mathbf{P}(\tau_x > n)$, we arrive at the conclusion. □

Now we change slightly notation. Let

$$\mathbf{P}_x(S_n \in A) = \mathbf{P}(x + S_n \in A).$$

Lemma 18. *Let $X^n(t) = \frac{S_{[nt]}}{\sqrt{n}}$ be the family of processes with the probability measure $\widehat{\mathbf{P}}^{(V)}_{x\sqrt{n}}, x \in W$. Then X^n weakly converges in $C(0,\infty)$ to the Dyson Brownian motion with starting point x, i.e. to the process distributed according to the probability measure $\widehat{\mathbf{P}}^{(\Delta)}_x$.*

Proof. The proof is given via coupling from Lemma 12. To prove the claim we need to show that the convergence take place in $C[0,l]$ for every l. The proof is identical for l, so we let $l = 1$ to simplify notation. Thus it sufficient to show that for every function $f : 0 \le f \le 1$ uniformly continuous on $C[0,1]$,

$$\widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}} f(X^n) \to \widehat{\mathbf{E}}^{(\Delta)}_x f(B) \quad \text{as } n \to \infty.$$

By Lemma 12 one can define B_n and S_n on the same probability in such a way that the complement of the event

$$A_n = \{\sup_{u\le n} |S_{[u]} - B_u| \le n^{1/2-a}\}$$

is negligible:

$$\mathbf{P}(\overline{A}_n) = o(n^{-\gamma})$$

for some $a > 0$ and $\gamma > 0$. Let $B^n_t = B_{nt}/\sqrt{n}$. By the scaling property of the Brownian motion $\widehat{\mathbf{E}}^{(\Delta)}_x f(B) = \widehat{\mathbf{E}}^{(\Delta)}_{x\sqrt{n}} f(B^n)$.

Split the expectation into two parts,

$$\widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}} f(X^n) = \widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}}[f(X^n); A_n] + \widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}}[f(X^n); \overline{A}_n] \equiv E_1 + E_2.$$

Since the function f is uniformly continuous,

$$|f(X^n) - f(B^n)| \le C \sup_{0\le u\le 1} |X^n_u - B^n_u| \le C n^{-a}$$

on the event A_n. Then,

$$\begin{aligned}
&\frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[(f(X^n) - f(B^n))V(S_n); \tau > n, A_n] \\
&\le C n^{-a}\frac{\mathbf{E}_{x\sqrt{n}}[V(S_n); \tau > n, A_n]}{V(x\sqrt{n})} \le C n^{-a}\frac{\mathbf{E}_{x\sqrt{n}}[V(S_n); \tau > n]}{V(x\sqrt{n})} = C n^{-a}
\end{aligned}$$

tends to 0 as $n \to \infty$. Therefore,

$$\begin{aligned}
E_1 &= \frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[f(X^n)V(S_n); \tau > n, A_n] \\
&= o(1) + \frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[f(B^n)V(S_n); \tau > n, A_n].
\end{aligned}$$

Moreover, on the event A_n the following inequalities hold

$$B^{(j)}_i - B^{(j-1)}_i - 2n^{1/2-a} \le S^{(j)}_i - S^{(j-1)}_i \le B^{(j)}_i - B^{(j-1)}_i + 2n^{1/2-a}$$

for $1 \le i \le n$ and $2 \le j \le k$. Let $x_n^{\pm} = (x\sqrt{n} \pm 2(j-1)n^{1/2-a})$. Arguing as in Lemma 14 and using monotonicity of V, we obtain

$$\begin{aligned}
&\frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[f(B^n)V(S_n); \tau > n, A_n] \\
&\quad \le \frac{(1+o(1))}{V(x\sqrt{n})}\mathbf{E}_{x_n^+}[f(B^n)V(B_n); \tau^{bm} > n, A_n] \\
&\quad \le \frac{(1+o(1))}{V(x\sqrt{n})}\mathbf{E}_{x_n^+}[f(B^n)\Delta(B_n); \tau^{bm} > n, A_n] \\
&\quad = (1+o(1))\frac{\Delta(x_n^+)}{V(x\sqrt{n})}\widehat{\mathbf{E}}^{\Delta}_{x_n^+}[f(B^n); A_n] = (1+o(1))\widehat{\mathbf{E}}^{\Delta}_{x_n^+}[f(B^n); A_n],
\end{aligned}$$

where we used (d) of Proposition 4 in the second and the third lines. Replacing x^+ with x^-, one can easily obtain the following lower bound

$$\frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[f(B^n)V(S_n); \tau > n, A_n] \ge (1+o(1))\widehat{\mathbf{E}}^{\Delta}_{x_n^-}[f(B^n); A_n].$$

Note also that

$$\begin{aligned}
\widehat{\mathbf{E}}^{\Delta}_{x_n^{\pm}}[f(B^n); A_n] &= \widehat{\mathbf{E}}^{\Delta}_{x_n^{\pm}}[f(B^n)] - \widehat{\mathbf{E}}^{\Delta}_{x_n^{\pm}}[f(B^n); \overline{A}_n] \\
&= (1+o(1))\widehat{\mathbf{E}}^{\Delta}_{x\sqrt{n}}[f(B^n)] - \widehat{\mathbf{E}}^{\Delta}_{x_n^{\pm}}[f(B^n); \overline{A}_n]
\end{aligned}$$

Therefore,

$$|E_1 - \widehat{\mathbf{E}}^{\Delta}_{x\sqrt{n}}[f(B^n)]| \le o(1) + \widehat{\mathbf{E}}^{\Delta}_{x_n^+}[f(B^n); \overline{A}_n] + \widehat{\mathbf{E}}^{\Delta}_{x_n^-}[f(B^n); \overline{A}_n].$$

Thus, if we show that

$$\widehat{\mathbf{E}}^{\Delta}_{x_n^{\pm}}[f(B^n); \overline{A}_n] = o(1), \quad \text{and} \quad E_2 = o(1),$$

we are done. Since the proofs of these statements are almost identical we concentrate on showing that $E_2 = o(1)$. We have, since $f \le 1$,

$$\begin{aligned}
E_2 \le &\frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[V(S_n); |S_n| \le n^{1/2+\delta}, \overline{A_n}, \tau_{x\sqrt{n}} > n] \\
&+ \frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[V(S_n); |S_n| > n^{1/2+\delta}, \tau_{x\sqrt{n}} > n].
\end{aligned}$$

Put $y_n = (2n^{1/2+\delta}, \dots, 2kn^{1/2+\delta})$. Then,

$$\begin{aligned}
&\frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[V(S_n); |S_n| \le n^{1/2+\delta}, \overline{A_n}, \tau_{x\sqrt{n}} > n] \\
&\qquad \le \frac{V(x_n + y_n)}{V(x\sqrt{n})}\mathbf{P}_{x\sqrt{n}}(|S_n| \le n^{1/2+\delta}, \overline{A_n}) \\
&\qquad \le C\frac{\Delta_1(x_n + y_n)}{\Delta(x\sqrt{n})}\mathbf{P}_{x\sqrt{n}}(\overline{A_n}) \le Cn^{\delta k(k-1)/2}n^{-\gamma} \to 0, \qquad (52)
\end{aligned}$$

if we pick δ sufficiently small. Next, using the bounds $V(x) \le V^{(T)}(x) \le \Delta_1(x)$, we get

$$\mathbf{E}_{x\sqrt{n}}[V(S_n); |S_n| > n^{1/2+\delta}, \tau_{x\sqrt{n}} > n] \le \sum_{j=1}^{k}\mathbf{E}[\Delta_1(x\sqrt{n} + S_n); |S_n^{(j)}| > n^{1/2+\delta}/k].$$

Arguing similarly to the second part of Lemma 6, one can see that

$$\mathbf{E}[\Delta_1(x\sqrt{n}+S_n); |S_n^{(j)}| > n^{1/2+\delta}/k] \\ \le C(x) \sum_{\mathcal{J}\subset\mathcal{P}} n^{|\mathcal{J}|/2} \prod_{\mathcal{P}\setminus\mathcal{J}} E[|S_n^{(j_2)} - S_n^{(j_1)}|; |S_n^{(j)}| > n^{1/2+\delta}/k].$$

The expectation of the product can be estimated exactly as in Lemma 6 using the Fuk-Nagaev inequality. This gives us

$$\frac{1}{V(x\sqrt{n})}\mathbf{E}_{x\sqrt{n}}[V(S_n); |S_n| > n^{1/2+\delta}] = \frac{o(n^{\frac{k(k-1)}{4}})}{\Delta(x\sqrt{n})} = o(1).$$

Thus, the proof is finished. □

Now we consider start from a fixed point x.

Lemma 19. *Let $X^n(t) = \frac{S_{[nt]}}{\sqrt{n}}$ be the family of processes with the probability measure $\widehat{\mathbf{P}}_x^{(V)}, x \in W$. Then X^n converges weakly to the Dyson Brownian motion with starting point 0.*

Proof. As in the proof of the previous lemma, wee show the convergence on $C[0,1]$ only. It sufficient to show that for every function $f : 0 \le f \le 1$ uniformly continuous on $C[0,1]$,

$$\widehat{\mathbf{E}}_x^{(V)} f(X^n) \to \widehat{\mathbf{E}}_0 f(B) \quad \text{as } n \to \infty.$$

First,

$$\widehat{\mathbf{E}}_x^{(V)}[f(X^n)] = \widehat{\mathbf{E}}_x^{(V)}[f(X^n), \nu_n \le n^{1-\varepsilon}] + \widehat{\mathbf{E}}_x^{(V)}[f(X^n), \nu_n > n^{1-\varepsilon}].$$

The second term

$$\widehat{\mathbf{E}}_x^{(V)}[f(X^n), \nu_n > n^{1-\varepsilon}] \le \widehat{\mathbf{P}}_x^{(V)}(\nu_n > n^{1-\varepsilon}) = \frac{\mathbf{E}[V(x+S_n); \tau_x > \nu_n, \nu_n > n^{1-\varepsilon}]}{V(x)} \\ \le C\frac{\mathbf{E}[\Delta_1(x+S_n); \tau_x > \nu_n, \nu_n > n^{1-\varepsilon}]}{V(x)} \to 0,$$

where the latter convergence follows from Lemma 8. Next,

$$\widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}] = \widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}] \\ + \widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}].$$

Then,

$$\widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}] \le \widehat{\mathbf{P}}_x^{(V)}(\nu_n \le n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}) \\ = \frac{\mathbf{E}(V(x+S_{\nu_n}); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}, \tau_x > \nu_n)}{V(x)} \\ \le (1+o(1))\frac{\mathbf{E}(\Delta(x+S_{\nu_n}); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}, \tau_x > \nu_n)}{V(x)} \to 0,$$

by (46). These preliminary estimates give us

$$\widehat{\mathbf{E}}_x^{(V)}[f(X^n)] = \widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}] + o(1). \tag{53}$$

Next let

$$f(y,k,X^n) = f\left(\frac{y}{\sqrt{n}}\mathbf{1}_{\{t\le k/n\}} + X^n(t)\mathbf{1}_{\{t>k/n\}}\right).$$

It is not difficult to see that on the event $\{x+S_{\nu_n}\in dy, M_{\nu_n}\le\theta_n\sqrt{n}\}$, the following holds

$$f(y,k,X^n) - f(X^n) = o(1)$$

uniformly in $|y|\le\theta_n\sqrt{n}$ and $k\le n^{1-\varepsilon}$. Therefore,

$$\begin{aligned}
&\widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}]\\
&\sim \widehat{\mathbf{E}}_x^{(V)}[f(S_{\nu_n},\nu_n,X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}]\\
&= \sum_{k\le n^{1-\varepsilon}}\int_{W_{n,\varepsilon}} \mathbf{P}\left(x+S_k\in dy, \tau_x>k, \nu_n=k, M_{\nu_n}\le\theta_n\sqrt{n}\right)\frac{V(y)}{V(x)}\\
&\qquad\qquad \times \widehat{\mathbf{E}}_y^{(V)} f\left(\frac{y}{\sqrt{n}}\mathbf{1}_{\{t\le k/n\}} + X^n(t-k/n)\mathbf{1}_{\{t>k/n\}}\right).
\end{aligned}$$

Using coupling arguments from Lemma 18, one can easily get

$$\begin{aligned}
&\widehat{\mathbf{E}}_y^{(V)} f\left(\frac{y}{\sqrt{n}}\mathbf{1}_{\{t\le k/n\}} + X^n(t-k/n)\mathbf{1}_{\{t>k/n\}}\right)\\
&\qquad\qquad \sim \widehat{\mathbf{E}}_y^{(\Delta)} f\left(\frac{y}{\sqrt{n}}\mathbf{1}_{\{t\le k/n\}} + B^n(t-k/n)\mathbf{1}_{\{t>k/n\}}\right).
\end{aligned}$$

Using results of Section 4 of [15], one has

$$\widehat{\mathbf{E}}_y^{(\Delta)} f\left(\frac{y}{\sqrt{n}}\mathbf{1}_{\{t\le k/n\}} + B^n(t-k/n)\mathbf{1}_{\{t>k/n\}}\right) \sim \widehat{\mathbf{E}}_0^{(\Delta)}[f(B)].$$

Consequently,

$$\begin{aligned}
&\widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}]\\
&\sim \widehat{\mathbf{E}}_0^{(\Delta)}[f(B)]\frac{\mathbf{E}[V(x+S_{\nu_n}); \tau_x>\nu_n, \nu_n\le n^{1-\varepsilon}, M_{\nu_n}\le\theta_n\sqrt{n}]}{V(x)}\\
&\sim \widehat{\mathbf{E}}_0^{(\Delta)}[f(B)]\frac{\mathbf{E}[\Delta(x+S_{\nu_n}); \tau_x>\nu_n, \nu_n\le n^{1-\varepsilon}, M_{\nu_n}\le\theta_n\sqrt{n}]}{V(x)}.
\end{aligned}$$

Using now Lemma 15 and relation (46), we get finally

$$\widehat{\mathbf{E}}_x^{(V)}[f(X^n); \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}] \sim \widehat{\mathbf{E}}_0^{(\Delta)}[f(B)].$$

Combining this with (53), we complete the proof of the lemma. □

Acknowledgement The research of Denis Denisov was done while he was a research assistant in Heriot-Watt University and supported by EPSRC grant No. EP/E033717/1. This work was carried out during the visits of the first author to Technische Universität München and Ludwig-Maximilians-Universität München and the visit of the second author to Heriot-Watt University. Both authors are grateful to the above-mentioned institutions for their hospitality.

We thank two anonymous referees for their remarks, which led to a better representaion of our results.

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