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A connection of the Brascamp-Lieb inequality with Skorokhod embedding*

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Abstract

We reveal a connection of the Brascamp-Lieb inequality with Skorokhod embedding. Error bounds for the inequality in terms of the variance are also provided.

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1 Introduction

The Brascamp-Lieb moment inequality plays an important role in statistical mechanics, such as in the analysis of gradient interface models; see, e.g., [10, 8, 12]. It asserts that centered moments of a distribution with log-concave density relative to a Gaussian distribution do not exceed those of that Gaussian's; it is used to derive the tightness of finite-volume Gibbs measures describing the static interface, strict convexity of the associated surface tension, and so on.

The Skorokhod embedding problem is to find a stopping time T for one-dimensional Brownian motion B such that $B(T)$ is distributed as a given probability measure on $\mathbb{R}$. The problem was proposed by Skorokhod [17] and a number of solutions have been constructed since then ([15]); they are applied to the proof of Donsker's invariance principle, robust pricings of options in mathematical finance (see, e.g., [13]), and so on.

In this paper, we reveal a connection between the Brascamp-Lieb inequality and the Skorokhod embedding of Bass [1]; as a by-product, we also provide error bounds for the inequality in terms of the variance by applying the Itô-Tanaka formula. Let Y be an n-dimensional Gaussian random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ with law ν. Let X be an n-dimensional random variable on $(\Omega, \mathcal{F}, P)$, whose law μ is given in the form

$$\mu(dx) = \frac{1}{Z} e^{-V(x)}\, \nu(dx) \tag{1.1}$$

with $V : \mathbb{R}^n \to \mathbb{R}$ a convex function, where

$$Z := \int_{\mathbb{R}^n} e^{-V(x)}\, \nu(dx) \in (0, \infty).$$

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In what follows, we fix $v \in \mathbb{R}^n$ ($v \neq 0$) arbitrarily. For a one-dimensional random variable ξ, we denote its variance by $\mathrm{var}(\xi)$: $\mathrm{var}(\xi) = E[(\xi - E[\xi])^2]$. We set $\mathsf{a} := \mathrm{var}(v \cdot Y)$. Here $a \cdot b$ denotes the inner product of $a, b \in \mathbb{R}^n$. We also set

$$\mathsf{p}(t; x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad t > 0,\ x \in \mathbb{R}.$$

The result of this paper is stated as follows:

Theorem 1.1. *For every convex function ψ on $\mathbb{R}$, we have the following.*

(i) It holds that

$$E\left[\psi\left(v \cdot Y - E\left[v \cdot Y\right]\right)\right] \geq E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right]. \tag{1.2}$$

More precisely, we have

$$\begin{aligned} E\left[\psi\left(v \cdot Y - E\left[v \cdot Y\right]\right)\right] \geq & E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right] \\ & + \frac{1}{2}\int_{\mathbb{R}} \psi''(dx) \int_0^{\mathsf{a}^{-1}(\mathsf{a} - \mathrm{var}(v \cdot X))^2} ds\, \mathsf{p}\left(s; \sqrt{x^2 + \mathsf{a}}\right), \end{aligned} \tag{1.3}$$

where $\psi''(dx)$ denotes the second derivative of ψ in the sense of distribution.

(ii) For every $p > 1$, it holds that

$$\begin{aligned} E\left[\psi\left(v \cdot Y - E\left[v \cdot Y\right]\right)\right] \leq & E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right] \\ & + C(\mathsf{a}, \psi, q)\left(\mathsf{a} - \mathrm{var}(v \cdot X)\right)^{\frac{1}{2p}}. \end{aligned} \tag{1.4}$$

Here $C(\mathsf{a}, \psi, q) \in [0, \infty]$ is given by

$$C(\mathsf{a}, \psi, q) = (\mathsf{a}(1+q))^{\frac{1}{2q}} \int_{\mathbb{R}} \psi''(dx)\, \mathsf{p}\left(1; \frac{x}{\sqrt{\mathsf{a}(1+q)}}\right)$$

with q the conjugate of p: $p^{-1} + q^{-1} = 1$. Note that $\mathsf{a} - \mathrm{var}(v \cdot X) \geq 0$ by (1.2)*.*

The above inequalities (1.2)–(1.4) *are understood to hold also in the case that both sides are infinity.*

Remark 1.2. (1) *The inequality* (1.2) *is called the Brascamp-Lieb inequality. It was originally proved by Brascamp and Lieb [4, Theorem 5.1] in the case $\psi(x) = |x|^p$, $p \geq 1$; it was then extended to general convex ψ's by Caffarelli [5, Corollary 6] based on a deep understanding of optimal transportation between μ and ν, and the related Monge-Ampère equation.*
(2) *In the case $\psi''(\mathbb{R}) < \infty$, letting $p \to 1$ in* (1.4) *yields*

$$E\left[\psi\left(v \cdot Y - E\left[v \cdot Y\right]\right)\right] - E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right] \leq \frac{1}{\sqrt{2\pi}} \psi''(\mathbb{R}) \left(\mathsf{a} - \mathrm{var}(v \cdot X)\right)^{\frac{1}{2}}.$$

Taking $\psi(x) = |x|$, after some manipulation we see that

$$\frac{E\left[\left|v \cdot X - E\left[v \cdot X\right]\right|\right]}{\mathrm{var}(v \cdot X)} \geq \frac{1}{\sqrt{2\pi \mathrm{var}(v \cdot Y)}}$$

for any convex V.
(3) *In the case $\psi(x) = x^2$, inequalities* (1.3) *and* (1.4) *hold merely in the obvious manner; in other words, they do not give any information on $\mathrm{var}(v \cdot X)$, other than $\mathrm{var}(v \cdot X) \leq \mathsf{a}$.*

By [4, Theorem 4.1], we remark that if $V \in C^2(\mathbb{R}^n)$, then $\mathrm{var}(v \cdot X)$ admits the upper bound

$$\int_{\mathbb{R}^n} \mu(dx)\, v \cdot \left(\Sigma^{-1} + D^2V(x)\right)^{-1} v,$$

which is less than or equal to $\mathsf{a} \equiv v \cdot \Sigma v$. Here Σ is the covariance matrix of the Gaussian ν and D^2V is the Hessian of V.

The rest of the paper is organized as follows: In Section 2 we prove Theorem 1.1. The Brascamp-Lieb inequality (1.2) is proved in Subsection 2.1; we devote Subsection 2.2 to the proof of (1.3) and (1.4); in Subsection 2.3 we prove Lemma 2.1, which plays an essential role in the proof of Theorem 1.1. In the appendix we discuss an extension of the Brascamp-Lieb inequality to the case with V not necessarily convex.

For every real-valued function f on $\mathbb{R}$ and for every $x \in \mathbb{R}$, we denote respectively by $f'_+(x)$ and $f'_-(x)$ the right- and left-derivatives of f at x if they exist. For each $x, y \in \mathbb{R}$, we write $x \wedge y = \min\{x, y\}$ and $x^+ = \max\{x, 0\}$. Other notation will be introduced as needed.

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Without loss of generality, we may assume that ν is centered: $E[Y] = 0$. Moreover, Theorem 4.3 of [4] reduces the proof to the case $n = 1$; that is, the density of the law $P \circ (v \cdot X)^{-1}$ relative to the one-dimensional Gaussian measure $P \circ (v \cdot Y)^{-1}$ is log-concave. Therefore in what follows, we take the Gaussian measure ν in (1.1) as

$$\nu(dx) = \frac{1}{\sqrt{2\pi \mathsf{a}}} \exp\left(-\frac{x^2}{2\mathsf{a}}\right) dx, \quad x \in \mathbb{R},$$

and V as a convex function on $\mathbb{R}$. We accordingly write X and Y for $v \cdot X$ and $v \cdot Y$, respectively; that is, X is distributed as μ and Y as ν.

2.1 Proof of (1.2)

In this subsection we prove the inequality (1.2) in Theorem 1.1. We denote by F_μ the distribution function of μ:

$$F_\mu(x) := \frac{1}{Z} \int_{-\infty}^{x} e^{-V(y)}\, \nu(dy), \quad x \in \mathbb{R}.$$

We also set

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2} y^2\right) dy, \quad x \in \mathbb{R},$$

and

$$g := F_\mu^{-1} \circ \Phi. \tag{2.1}$$

Here $F_\mu^{-1} : (0,1) \to \mathbb{R}$ is the inverse function of F_μ. Apparently g is differentiable and strictly increasing. By convexity of V we have moreover

Lemma 2.1. *It holds that $g'(x) \leq \sqrt{\mathsf{a}}$ for all $x \in \mathbb{R}$.*

We postpone the proof of this lemma to Subsection 2.3. Once this lemma is shown, the proof of (1.2) is straightforward from the Skorokhod embedding of Bass [1]; for other types of embeddings, we refer the reader to the detailed survey [15] by Obłój. Let $\{W_t\}_{t \geq 0}$ be a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$.

Proof of (1.2). Note that $g(W_1)$ is distributed as μ. Applying Clark's formula (see, e.g., [14, Appendix E]) to $g(W_1)$ yields

$$g(W_1) - E\left[g(W_1)\right] = \int_0^1 a(s, W_s)\, dW_s \quad P\text{-a.s.},$$

where for $0 \le s \le 1$ and $y \in \mathbb{R}$,

$$\begin{aligned} a(s,y) :=& \frac{\partial}{\partial y} E\left[g(y + W_{1-s})\right] \\ =& E\left[g'(y + W_{1-s})\right]. \end{aligned} \tag{2.2}$$

By the Dambis-Dubins-Schwarz theorem (see, e.g., [16, Theorem V.1.6]), there exists a Brownian motion $\{B(t)\}_{t\ge 0}$ on $(\Omega, \mathcal{F}, P)$ such that

$$\int_0^t a(s, W_s)\, dW_s = B\left(\int_0^t a(s, W_s)^2\, ds\right) \quad \text{for all } 0 \le t \le 1,\ P\text{-a.s.}$$

We know from [1] that $T := \int_0^1 a(s, W_s)^2\, ds$ is a stopping time in the natural filtration of B. Moreover, by (2.2) and Lemma 2.1, we have $T \le \mathsf{a}$ P-a.s. We denote by $\{L_t^x\}_{t\ge 0, x\in\mathbb{R}}$ the local time process of B. For every $x \in \mathbb{R}$, Tanaka's formula yields

$$E\left[(B(\mathsf{a}) - x)^+\right] = E\left[(B(T) - x)^+\right] + \frac{1}{2} E\left[L_{\mathsf{a}}^x - L_T^x\right], \tag{2.3}$$

$$E\left[(x - B(\mathsf{a}))^+\right] = E\left[(x - B(T))^+\right] + \frac{1}{2} E\left[L_{\mathsf{a}}^x - L_T^x\right]. \tag{2.4}$$

From (2.3) and (2.4), it follows that for every convex ψ,

$$E\left[\psi\left(B(\mathsf{a})\right)\right] = E\left[\psi\left(B(T)\right)\right] + \frac{1}{2}\int_{\mathbb{R}} \psi''(dx)\, E\left[L_{\mathsf{a}}^x - L_T^x\right]. \tag{2.5}$$

Indeed, by Fubini's theorem,

$$\begin{aligned} &\int_{[0,\infty)} \psi''(dx)\, E\left[(B(\mathsf{a}) - x)^+\right] + \int_{(-\infty,0)} \psi''(dx)\, E\left[(x - B(\mathsf{a}))^+\right] \\ &= E\left[\psi\left(B(\mathsf{a})\right) - \psi'_-(0) B(\mathsf{a}) - \psi(0)\right] \\ &= E\left[\psi\left(B(\mathsf{a})\right)\right] - \psi(0), \end{aligned}$$

which is equal, by (2.3), (2.4) and $E\left[B(T)\right] = 0$, to the right-hand side of (2.5) with $\psi(0)$ subtracted. Hence (2.5) holds. As ψ'' is a nonnegative measure and $T \le \mathsf{a}$ a.s., it is immediate from (2.5) that

$$E\left[\psi\left(B(\mathsf{a})\right)\right] \ge E\left[\psi\left(B(T)\right)\right], \tag{2.6}$$

which is nothing but (1.2) since

$$B(T) = g(W_1) - E\left[g(W_1)\right] \overset{(d)}{=} X - E\left[X\right] \tag{2.7}$$

and $B(\mathsf{a}) \overset{(d)}{=} Y$. The proof is complete. □

Remark 2.2. (1) *For any convex ψ such that $\int_0^{\cdot} \psi'_-\left(B(s)\right) dB(s)$ is a martingale, the identity* (2.5) *is immediate from the Itô-Tanaka formula.*
(2) *For any convex ψ such that $E\left[\left|\psi\left(B(\mathsf{a})\right)\right|\right] < \infty$ (i.e., $E\left[\psi\left(B(\mathsf{a})\right)\right] < \infty$), the inequality* (2.6) *follows readily from the optional sampling theorem applied to the submartingale $\{\psi(B(t))\}_{0\le t\le \mathsf{a}}$.*

(3) *Let* $\{\beta(t)\}_{t\geq 0}$ *be a standard one-dimensional Brownian motion and* τ *denote Root's solution that embeds the law of* $X - E[X]$ *in* β*:* $\beta(\tau) \stackrel{(d)}{=} X - E[X]$*. Since* τ *is of* minimal residual expectation*, it follows that* τ *is also bounded from above by* a*, which indicates that the Brascamp-Lieb inequality* (1.2) *can also be proved by using Root's solution. For the construction of embedding due to D.H. Root and the notion of minimal residual expectation, see [13, Section 5.1] and references therein. In addition, the boundedness of Root's solution as noted above in the Brascamp-Lieb framework gives an answer to the question raised in [7, Section 7] as to when Root's barrier is bounded; see also the proof of Proposition A.1 below.*
(4) *The author is informed by one of the referees that another proof of* (1.2) *is possible by using the Helffer-Sjöstrand random walk representation; for the representation and its connection with the Brascamp-Lieb and other related inequalities, we refer the reader to [9, Section 4], where a proof of* (1.2) *with* $\psi(x) = x^2$ *by means of the Helffer-Sjöstrand representation is presented.*

2.2 Proof of (1.3) and (1.4)

In this subsection we prove the inequalities (1.3) and (1.4) in Theorem 1.1. We keep the notation in the previous subsection. By (2.5), the proof is reduced to showing the following proposition.

Proposition 2.3. (1) *It holds that for all* $x \in \mathbb{R}$*,*

$$E\left[L_{\mathsf{a}}^x - L_T^x\right] \geq \int_0^{\mathsf{a}^{-1}(\mathsf{a}-\mathrm{var}(X))^2} ds\, \mathsf{p}\left(s; \sqrt{x^2+\mathsf{a}}\right). \tag{2.8}$$

(2) *For every* $p > 1$*, it holds that for all* $x \in \mathbb{R}$*,*

$$E\left[L_{\mathsf{a}}^x - L_T^x\right] \leq 2\left(\mathsf{a}(1+q)\right)^{\frac{1}{2q}} \mathsf{p}\left(1; \frac{x}{\sqrt{\mathsf{a}(1+q)}}\right)\left(\mathsf{a} - \mathrm{var}(X)\right)^{\frac{1}{2p}}. \tag{2.9}$$

To prove these estimates, we prepare a lemma.

Lemma 2.4. *For every* $t > 0$ *and* $x \in \mathbb{R}$*, we have*

$$E\left[L_t^x\right] = \int_0^t ds\, \mathsf{p}(s;x) \tag{2.10}$$

$$= 2\int_0^\infty dy\, (y - |x|)^+\, \mathsf{p}(t;y) \tag{2.11}$$

$$= 2\int_0^\infty dy\, \left(\sqrt{t}\, y - |x|\right)^+ \mathsf{p}(1;y). \tag{2.12}$$

Proof. The first equality is seen from the occupation time formula. The second is due to the identity

$$\{L_t^x\}_{t\geq 0} \stackrel{(d)}{=} \left\{\left(\max_{0\leq s\leq t} B(s) - |x|\right)^+\right\}_{t\geq 0}$$

for every $x \in \mathbb{R}$, which is deduced from Lévy's theorem for Brownian local time. The third one follows from change of variables. □

The proof of the proposition then proceeds as follows. Recall $T \leq \mathsf{a}$ a.s.

Proof of Proposition 2.3. (1) By the strong Markov property of Brownian motion,

$$E\left[L_{\mathsf{a}}^x - L_T^x\right] = E\left[E\left[L_{\mathsf{a}-t}^{x-z}\right]\Big|_{(t,z)=(T,B(T))}\right]. \tag{2.13}$$

By (2.12), this is rewritten as

$$2E\left[\int_0^\infty dy\left(\sqrt{\mathsf{a}-T}\,y-|x-B(T)|\right)^+\mathsf{p}(1;y)\right].$$

Using Fubini's theorem and Jensen's inequality, we bound the above expression from below by

$$2\int_0^\infty dy\left(E\Big[\sqrt{\mathsf{a}-T}\Big]y-E\left[|x-B(T)|\right]\right)^+\mathsf{p}(1;y). \tag{2.14}$$

By the optional sampling theorem and Schwarz's inequality,

$$\begin{aligned} E\left[|x-B(T)|\right] &\le E\left[|x-B(\mathsf{a})|\right] \\ &\le \sqrt{x^2+\mathsf{a}}. \end{aligned}$$

Plugging this estimate into (2.14) and using the identity between (2.12) and (2.10) lead to

$$E\left[L_{\mathsf{a}}^x-L_T^x\right]\ge\int_0^{E\left[\sqrt{\mathsf{a}-T}\right]^2}ds\,\mathsf{p}\left(s;\sqrt{x^2+\mathsf{a}}\right).$$

Since $\sqrt{\mathsf{a}-t}\ge\mathsf{a}^{-1/2}(\mathsf{a}-t)$ for $0\le t\le\mathsf{a}$, we see that

$$\begin{aligned} E\left[\sqrt{\mathsf{a}-T}\right]^2 &\ge \mathsf{a}^{-1}\left(\mathsf{a}-E[T]\right)^2 \\ &= \mathsf{a}^{-1}\left(\mathsf{a}-\mathrm{var}(X)\right)^2, \end{aligned}$$

where the equality follows from Wald's identity

$$E[T]=E\left[B(T)^2\right] \tag{2.15}$$

and from (2.7). This proves (2.8).
(2) First we show that for every $t>0$ and $x\in\mathbb{R}$,

$$E\left[\int_0^t ds\,\mathsf{p}\left(s;|x-B(T)|\right)\right]\le\int_0^{\mathsf{a}+t}ds\,\mathsf{p}(s;x). \tag{2.16}$$

By the identity between (2.10) and (2.11), and by Fubini's theorem, the left-hand side is equal to

$$2\int_0^\infty dy\,E\left[(y-|x-B(T)|)^+\right]\mathsf{p}(t;y). \tag{2.17}$$

We note the identity $(y-|x-z|)^+=(z-x+y)^+\wedge(x+y-z)^+$ for $z\in\mathbb{R}$, to bound the expectation in the integrand from above by

$$\begin{aligned} &E\left[(B(T)-x+y)^+\right]\wedge E\left[(x+y-B(T))^+\right] \\ &\le E\left[(B(\mathsf{a})-x+y)^+\right]\wedge E\left[(x+y-B(\mathsf{a}))^+\right] \\ &= E\left[(B(\mathsf{a})+y-|x|)^+\right]. \end{aligned}$$

Here for the inequality, we used the optional sampling theorem; the equality follows from the monotonicity of $E\left[(B(\mathsf{a})-x+y)^+\right]$ in x and the symmetry in the sense that

$E\left[(B(\mathsf{a})-(-x)+y)^+\right]=E\left[(x+y-B(\mathsf{a}))^+\right]$. Therefore (2.17) is dominated by

$$\begin{aligned}
&2\int_0^\infty dy\int_{\mathbb{R}} dz\,(z+y-|x|)^+\,\mathsf{p}(\mathsf{a};z)\mathsf{p}(t;y)\\
&=2\int_{\mathbb{R}} du\left(\sqrt{\mathsf{a}+t}\,u-|x|\right)^+\mathsf{p}(1;u)\int_{-\infty}^{\sqrt{\mathsf{a}^{-1}t}\,u} dv\,\mathsf{p}(1;v)\\
&\le 2\int_0^\infty du\left(\sqrt{\mathsf{a}+t}\,u-|x|\right)^+\mathsf{p}(1;u),
\end{aligned}$$

where we changed variables with $u=\frac{z+y}{\sqrt{\mathsf{a}+t}}$ and $v=\frac{tz-\mathsf{a}y}{\sqrt{\mathsf{a}t(\mathsf{a}+t)}}$ for the equality. Now (2.16) follows from the identity between (2.12) and (2.10).

By (2.13), (2.10) and Hölder's inequality,

$$\begin{aligned}
E\left[L_{\mathsf{a}}^x-L_T^x\right]&\le\left(\frac{1}{2\pi}\right)^{\frac{1}{2p}}E\left[\int_0^{\mathsf{a}-T}\frac{ds}{\sqrt{s}}\right]^{\frac{1}{p}}E\left[\int_0^{\mathsf{a}} ds\,\mathsf{p}\left(s;\sqrt{q}|x-B(T)|\right)\right]^{\frac{1}{q}}\\
&=\left(\frac{2}{\pi}\right)^{\frac{1}{2p}}q^{\frac{1}{2q}}E\left[\sqrt{\mathsf{a}-T}\right]^{\frac{1}{p}}E\left[\int_0^{\mathsf{a}q^{-1}} ds\,\mathsf{p}\left(s;|x-B(T)|\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

By Jensen's inequality, and by (2.15) and (2.7),

$$E\left[\sqrt{\mathsf{a}-T}\right]\le(\mathsf{a}-E[T])^{\frac{1}{2}}=(\mathsf{a}-\mathrm{var}(X))^{\frac{1}{2}}.$$

Moreover, by (2.16),

$$\begin{aligned}
E\left[\int_0^{\mathsf{a}q^{-1}} ds\,\mathsf{p}\left(s;|x-B(T)|\right)\right]&\le\int_0^{\mathsf{a}(1+q^{-1})} ds\,\mathsf{p}(s;x)\\
&\le\left(\frac{2\mathsf{a}(1+q)}{\pi q}\right)^{\frac{1}{2}}\exp\left\{-\frac{qx^2}{2\mathsf{a}(1+q)}\right\}.
\end{aligned}$$

Combining these leads to (2.9) and ends the proof of Proposition 2.3. □

Proof of (1.3) *and* (1.4). They are immediate from (2.5) and Proposition 2.3. □

2.3 Proof of Lemma 2.1

We conclude this section with the proof of Lemma 2.1; the assertion itself is corresponding to that of [5, Theorem 11] in the case of one dimension, where we can give a more straightforward proof which we think is worthy of presentation. To begin with, note that we only need to consider the case $\mathsf{a}=1$; indeed, setting

$$\widetilde{V}(x):=V\left(\sqrt{\mathsf{a}}x\right),\qquad \widetilde{F}_\mu(x):=\frac{\sqrt{\mathsf{a}}}{Z}\int_{-\infty}^x\exp\left(-\frac{1}{2}y^2-\widetilde{V}(y)\right)dy,$$

we have $F_\mu(x)=\widetilde{F}_\mu\left(x/\sqrt{\mathsf{a}}\right)$, from which it follows that

$$F_\mu^{-1}\circ\Phi(x)=\sqrt{\mathsf{a}}\widetilde{F}_\mu^{-1}\circ\Phi(x).$$

Therefore the assertion of Lemma 2.1 is equivalent to

$$\left(\widetilde{F}_\mu^{-1}\circ\Phi\right)'\le 1.$$

Note that $\widetilde{V}$ remains convex.

From now on we let $\mathsf{a}=1$. We utilize the following:

Lemma 2.5. *It holds that for all $x \in \mathbb{R}$,*

$$F_\mu'(x) \geq \Phi'\left(x + V_-'(x)\right).$$

Proof. Since $V(y) - V(x) \geq V_-'(x)(y-x)$ for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\frac{1}{F_\mu'(x)} &= \int_{\mathbb{R}} \exp\left(-\frac{1}{2}y^2 - V(y)\right) dy \times \exp\left(\frac{1}{2}x^2 + V(x)\right) \\ &\leq \exp\left(\frac{1}{2}x^2\right) \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}y^2 - V_-'(x)(y-x)\right\} dy \\ &= \exp\left\{\frac{1}{2}\left(x + V_-'(x)\right)^2\right\} \times \sqrt{2\pi},\end{aligned}$$

which is the desired inequality. □

The proof of Lemma 2.1 follows readily from the above lemma.

Proof of Lemma 2.1. Since

$$g'(x) = \frac{\Phi'(x)}{F_\mu' \circ F_\mu^{-1}\left(\Phi(x)\right)},$$

the assertion of the lemma with $\mathsf{a} = 1$ is equivalent to

$$G(\xi) := F_\mu' \circ F_\mu^{-1}(\xi) - \Phi' \circ \Phi^{-1}(\xi) \geq 0 \quad \text{for all } \xi \in (0,1). \tag{2.18}$$

First note that

$$G(0+) = G(1-) = 0 \tag{2.19}$$

because both $F_\mu' \circ F_\mu^{-1}$ and $\Phi' \circ \Phi^{-1}$ are zero at $\xi = 0+$ and $\xi = 1-$. Next, G is both right- and left-differentiable since F_μ' is and since F_μ^{-1} is monotone. Suppose now that G has a local minimum at some $\xi_0 \in (0,1)$. Then $G_-'(\xi_0) \leq 0$ and $G_+'(\xi_0) \geq 0$. Since

$$\begin{aligned}G_\pm'(\xi) &= \frac{\left(F_\mu'\right)_\pm'}{F_\mu'} \circ F_\mu^{-1}(\xi) + \Phi^{-1}(\xi) \\ &= -\left(x + V_\pm'(x)\right)\big|_{x = F_\mu^{-1}(\xi)} + \Phi^{-1}(\xi),\end{aligned}$$

we have

$$\left(x + V_+'(x)\right)\big|_{x = F_\mu^{-1}(\xi_0)} \leq \Phi^{-1}(\xi_0) \leq \left(x + V_-'(x)\right)\big|_{x = F_\mu^{-1}(\xi_0)},$$

which entails that

$$\Phi^{-1}(\xi_0) = \left(x + V_-'(x)\right)\big|_{x = F_\mu^{-1}(\xi_0)}$$

because $V_-'(x) \leq V_+'(x)$ for all $x \in \mathbb{R}$ by convexity. Hence by Lemma 2.5

$$G(\xi_0) = \left\{F_\mu'(x) - \Phi'\left(x + V_-'(x)\right)\right\}\big|_{x = F_\mu^{-1}(\xi_0)} \geq 0.$$

Combining this observation with (2.19), we conclude (2.18). This finishes the proof. □

Appendix

In this appendix we discuss an extension of the Brascamp-Lieb inequality (1.2) to the case with potential function V not necessarily convex. To avoid complexity, we restrict ourselves to one dimension; multidimensional generalizations may be done by considering a one-dimensional marginal, namely the law of $v \cdot X$ for every $v \in \mathbb{R}^n$, with X being an n-dimensional random variable. Recently, gradient interface models with nonconvex potentials have been studied with great interest, see, e.g., [2, 6, 3]; we expect that the result presented here has a contribution to that study. A type of Brascamp-Lieb inequalities with nonconvex potentials is also discussed by Funaki and Toukairin [11, Section 4] with some restriction on convex ψ.

For a given $\alpha > 0$, suppose that the function $k \in C^1(\mathbb{R})$ satisfies

$$k'(x) \geq \sqrt{\alpha} \quad \text{for all } x \in \mathbb{R}. \tag{A.1}$$

Set

$$U(x) = \frac{1}{2}|k(x)|^2 - \log k'(x), \quad x \in \mathbb{R}, \tag{A.2}$$

and let the distribution μ on $\mathbb{R}$ be given in the form

$$\mu(dx) = \frac{1}{Z'} e^{-U(x)}\, dx,$$

where the normalizing factor $Z' = \int_{\mathbb{R}} e^{-U(x)}\, dx$ is equal to $\sqrt{2\pi}$. Let X be a random variable distributed as μ, and Y a centered Gaussian random variable with variance $1/\alpha$. Under the above assumption, we have

Proposition A.1. *For every convex function ψ on $\mathbb{R}$, it holds that*

$$E\left[\psi\left(X - E[X]\right)\right] \leq E\left[\psi\left(Y\right)\right]. \tag{A.3}$$

Proof. Since the distribution function F_μ of μ is written as

$$\begin{aligned} F_\mu(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} k'(y) \exp\left\{-\frac{1}{2}|k(y)|^2\right\} dy \\ &= \Phi\left(k(x)\right), \end{aligned}$$

the function g defined by (2.1) is equal to k^{-1}, the inverse function of k. Therefore by assumption (A.1), we have $g'(x) \leq 1/\sqrt{\alpha}$ for all $x \in \mathbb{R}$, hence the same proof as that of (1.2) applies. □

Remark A.2. (1) *In the case that the Gaussian measure ν in* (1.1) *has mean 0 and variance $1/\sqrt{\alpha}$, we may write* (1.1) *as*

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\alpha x^2 - V(x) - C\right) dx$$

by suitably picking a constant C. Lemma 2.1 indicates that the function $\mathbb{R} \ni x \mapsto \frac{1}{2}\alpha x^2 + V(x) + C$ with V convex can be expressed as (A.2) *for some k satisfying* (A.1).
(2) *In addition to* (A.1), *if we assume that*

$$k'(x) \leq \sqrt{\beta} \quad \textit{for all } x \in \mathbb{R},$$

with some $\beta > \alpha$, then we also have the reverse inequality

$$E\left[\psi(Y')\right] \leq E\left[\psi\left(X - E[X]\right)\right] \tag{A.4}$$

for every convex ψ. Here Y' is a centered Gaussian random variable with variance $1/\beta$.

We conclude this paper with two examples of U.

Example A.3 (double-well type). *Take $\alpha = 1$ and $k(x) = x + x^3$. Then*

$$U(x) = \frac{1}{2}x^2 + x^4 + \frac{1}{2}x^6 - \log\left(1 + 3x^2\right).$$

This potential U has a double-well near the origin.

Example A.4 (log-mixture of centered Gaussians). *For given $p, q > 0$ and $0 < a < b$ such that*

$$\frac{p}{\sqrt{a}} + \frac{q}{\sqrt{b}} = 1, \tag{A.5}$$

we take

$$k(x) = \Phi^{-1}\left(\frac{p}{\sqrt{a}}\Phi\left(\sqrt{a}x\right) + \frac{q}{\sqrt{b}}\Phi\left(\sqrt{b}x\right)\right).$$

Then the corresponding U is expressed as

$$U(x) = -\log\left(pe^{-\frac{1}{2}ax^2} + qe^{-\frac{1}{2}bx^2}\right). \tag{A.6}$$

This type of potentials is dealt with in [2, 6, 3]. The function k satisfies

$$p \le k'(x) \le \sqrt{b} \quad \text{for all } x \in \mathbb{R}, \tag{A.7}$$

hence we have (A.3) *with $\alpha = p^2$ and* (A.4) *with $\beta = b$. To verify* (A.7)*, we start with the expression*

$$k'(x) = \frac{p\Phi'\left(\sqrt{a}x\right) + q\Phi'\left(\sqrt{b}x\right)}{\Phi' \circ \Phi^{-1}\left(\frac{p}{\sqrt{a}}\Phi\left(\sqrt{a}x\right) + \frac{q}{\sqrt{b}}\Phi\left(\sqrt{b}x\right)\right)}. \tag{A.8}$$

To prove the lower bound, it is sufficient to take $x \le 0$ by symmetry. Then, as

$$\frac{p}{\sqrt{a}}\Phi\left(\sqrt{a}x\right) + \frac{q}{\sqrt{b}}\Phi\left(\sqrt{b}x\right) \le \Phi\left(\sqrt{a}x\right) \le \frac{1}{2},$$

the denominator of (A.8) *is dominated by*

$$\Phi' \circ \Phi^{-1}\left(\Phi\left(\sqrt{a}x\right)\right) = \Phi'\left(\sqrt{a}x\right)$$

because $\Phi' \circ \Phi^{-1}$ is increasing on $(0, 1/2]$. Therefore

$$k'(x) \ge p + qe^{-\frac{1}{2}(b-a)x^2}$$

and the lower bound follows. For the upper bound, we use the concavity of $\Phi' \circ \Phi^{-1}$: $\left(\Phi' \circ \Phi^{-1}\right)'' = -\left(\Phi^{-1}\right)' < 0$. Noting the relation (A.5)*, we apply Jensen's inequality to see that the denominator of* (A.8) *is bounded from below by*

$$\frac{p}{\sqrt{a}}\Phi' \circ \Phi^{-1}\left(\Phi\left(\sqrt{a}x\right)\right) + \frac{q}{\sqrt{b}}\Phi' \circ \Phi^{-1}\left(\Phi\left(\sqrt{b}x\right)\right) \ge \frac{1}{\sqrt{b}}\left\{p\Phi'\left(\sqrt{a}x\right) + q\Phi'\left(\sqrt{b}x\right)\right\},$$

from which we obtain the upper bound in (A.7)*. We end this example with a remark that this upper bound also holds true in a general situation where k is given by*

$$k(x) = \Phi^{-1}\left(\int_0^\infty \frac{\rho(d\kappa)}{\sqrt{\kappa}}\Phi\left(\sqrt{\kappa}x\right)\right)$$

for a positive measure ρ *on* $(0, \infty)$ *such that its support is included in* $(0, b]$ *and*

$$\int_0^\infty \frac{\rho(d\kappa)}{\sqrt{\kappa}} = 1.$$

The potential U *corresponding to this* k *is given in the form*

$$U(x) = -\log \int_0^\infty \rho(d\kappa)\, e^{-\frac{1}{2}\kappa x^2},$$

which is referred to as a log-mixture of centered Gaussians *in [3].*

Remark A.5. *For* U *given by* (A.6)*, a concrete calculation shows that in fact* (A.3) *holds with* $\alpha = a$*, which gives a better bound than the one discussed above because* $p^2 < a$ *by the relation* (A.5)*.*

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