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Noncommutative characterization of free Meixner processes*

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Abstract

In this article we give a purely noncommutative criterion for the characterization of free Meixner random variables. We prove that some families of free Meixner distributions can be described in terms of the conditional expectation, which has no classical analogue. We also show a generalization of Speicher's formula (relating moments and free cumulants) and establish a new relation in free probability.

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1 Introduction

Classical Meixner distributions appeared in the work of Meixner [22] on the theory of orthogonal polynomials. In free probability the Meixner systems of polynomials were introduced by Anshelevich [2] and Saitoh and Yoshida [24]. The study of free Meixner distributions has been an active field of research during the last decade - see works [3, 4, 9, 10, 11, 14, 20]. It is common in free probability, that their properties, to a large extent, are analogous to those of the classical Meixner distributions. This especially regards their characterizations: in both cases it is achieved in terms of generating functions of the polynomials and the quadratic regression property. The main aim of this paper is to produce a new characterization of the free Meixner laws which is close to the quadratic regression property, but with no analog to classical Meixner distributions. The quadratic regression property for free Meixner distributions has been established by Bożejko and Bryc [9] - see also Ejsmont [17] for the reverse part. Shortly this says that the first conditional moment is a linear regression and conditional variance is quadratic if and only if corresponding variables have free Meixner distributions. As an example we can give two random variables which have the same distribution (because the result is more transparent with this assumption). Suppose that $\mathbb{X}$,$\mathbb{Y}$ are free, self-adjoint, non-degenerate, centered and have the same distribution. Then the $\mathbb{X}$ and $\mathbb{Y}$ have free Meixner laws (with respect to some state τ, which we will discuss later in section 2) if and only if there exist constant a, b such that

$$\tau\big((\mathbb{X}-\mathbb{Y})^2|\mathbb{X}+\mathbb{Y}\big)=\frac{1}{b+1}\big(\mathbb{I}+a(\mathbb{X}+\mathbb{Y})+b(\mathbb{X}+\mathbb{Y})^2\big).$$

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The key result of the present paper says that the above condition can be replaced by

$$\left(\frac{b}{b+1}(\mathbb{X}+\mathbb{Y})+\frac{a}{b+1}\mathbb{I}\right)\tau\big((\mathbb{X}-\mathbb{Y})^2|\mathbb{X}+\mathbb{Y}\big)=\tau\big((\mathbb{X}-\mathbb{Y})\mathbf{(}\mathbb{X}+\mathbb{Y}\mathbf{)}(\mathbb{X}-\mathbb{Y})|\mathbb{X}+\mathbb{Y}\big). \tag{1.1}$$

Loosely speaking, the formula (1.1) says that we can discard $\mathbf{(}\mathbb{X}+\mathbb{Y}\mathbf{)}$ linearly from the right side of equation (1.1) in the front of the conditional expectation if and only if $\mathbb{X}$ and $\mathbb{Y}$ have free Meixner distribution. This result is unexpected because in commutative probability equation (1.1) takes the form

$$\mathbf{(}\mathbb{X}+\mathbb{Y}\mathbf{)}\tau\big((\mathbb{X}-\mathbb{Y})^2|\mathbb{X}+\mathbb{Y}\big)=\tau\big((\mathbb{X}-\mathbb{Y})\mathbf{(}\mathbb{X}+\mathbb{Y}\mathbf{)}(\mathbb{X}-\mathbb{Y})|\mathbb{X}+\mathbb{Y}\big)$$

for any classical variables $\mathbb{X}$ and $\mathbb{Y}$. In particular, we apply fact (1.1) to prove the main result of this paper about characterization of free Lévy processes. It is worth mentioning here that a Laha-Lukacs type characterizations of random variables in free probability are also studied by Szpojankowski, Wesołowski [27] and Bryc [13]. The first authors give a characterization of noncommutative free-Poisson and free-Binomial variables by properties of the first two conditional moments, which mimics Lukacs type assumptions known from classical probability. Bryc in [13] proved that q-Gaussian processes have linear regressions and quadratic conditional variances.

The paper is organized as follows. In section 2 we review basic free probability, free Meixner laws and the statement of the main result. In the third section we look more closely at non-crossing partitions with first two elements in the same block. In this section we also give a new characterization of free Meixner systems and we generalize the Speicher's identity. Finally, in Section 4 we prove our main results.

2 Free Meixner laws and statement of the main result

2.1 Free probability and Meixner laws

We assume that our probability space is a von Neumann algebra $\mathcal{A}$ with a normal faithful tracial state $\tau:\mathcal{A}\to\mathbb{C}$, i.e. $\tau(\cdot)$ is linear, continuous in weak* topology, $\tau(\mathbb{X}\mathbb{Y})=\tau(\mathbb{Y}\mathbb{X})$, $\tau(\mathbb{I})=1$, $\tau(\mathbb{X}\mathbb{X}^*)\geq 0$ and $\tau(\mathbb{X}\mathbb{X}^*)=0$ implies $\mathbb{X}=0$ for all $\mathbb{X},\mathbb{Y}\in\mathcal{A}$. A (noncommutative) bounded random variable $\mathbb{X}$ is a self-adjoint (i.e. $\mathbb{X}=\mathbb{X}^*$) element of $\mathcal{A}$. We are interested in the two-parameter family of compactly supported probability measures (so that their moments do not grow faster than exponentially) $\{\mu_{a,b}: a\in\mathbb{R}, b\geq -1\}$ with the moment generating function given by the formula

$$M(z)=\sum_{i=0}^{\infty}\tau(\mathbb{X}^i)z^i=\frac{1+2b+az-\sqrt{(1-za)^2-4z^2(1+b)}}{2(z^2+az+b)}, \tag{2.1}$$

for $|z|$ small enough (the branch of the analytic square root should be determined by the condition that $\Im(z)>0\Rightarrow\Im(G_\mu(z))\leqslant 0$ (see [24]). Equation (2.1) describes the distribution with mean zero and variance one (see [24]). For particular values a,b we have six types of distribution: the Wigner semicircle, the free Poisson, the free Pascal (free negative binomial), the free Gamma, a law that we will call pure free Meixner and the free binomial law (see [9, 17] for more details).

Let $\mathbb{C}\langle\mathbb{X}_1,\dots,\mathbb{X}_n\rangle$ denote the non-commutative ring of polynomials in variables $\mathbb{X}_1,\dots,\mathbb{X}_n$. The free (non-crossing) cumulants are the k-linear maps $R_k:\mathbb{C}\langle\mathbb{X}_1,\dots,\mathbb{X}_k\rangle\to\mathbb{C}$ defined by the recursive formula (connecting them with mixed moments)

$$\tau(\mathbb{X}_1\mathbb{X}_2\dots\mathbb{X}_n)=\sum_{\nu\in NC(n)}R_\nu(\mathbb{X}_1,\mathbb{X}_2,\dots,\mathbb{X}_n), \tag{2.2}$$

where

$$R_\nu(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n) := \Pi_{B\in\nu} R_{|B|}(\mathbb{X}_i : i \in B) \tag{2.3}$$

and $NC(n)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$ (see [23, 26]). Sometimes we will write $R_k(\mathbb{X}) = R_k(\mathbb{X}, \ldots, \mathbb{X})$.

The $\mathcal{R}$-transform of a random variable $\mathbb{X}$ is defined as $\mathcal{R}_{\mathbb{X}}(z) = \sum_{i=0}^{\infty} R_{i+1}(\mathbb{X})z^i$, where $R_i(\mathbb{X})$ is a sequences defined by (2.2) (see [7, 23] for more details). If $\mathbb{X}$ has the distribution μ, then sometimes we will write $\mathcal{R}_\mu$ for the $\mathcal{R}$-transform of $\mathbb{X}$.

Random variables $\mathbb{X}_1, \ldots, \mathbb{X}_n$ are freely independent (free) if, for every $k \geq 2$ and every non-constant choice of $\mathbb{Y}_i \in \{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$, where $i \in \{1, \ldots, k\}$ (for some positive integer k) we get $R_k(\mathbb{Y}_1, \ldots, \mathbb{Y}_k) = 0$.

The $\mathcal{R}$-transform linearizes the free convolution, i.e. if μ and ν are (compactly supported) probability measures on $\mathbf{R}$, then we have

$$\mathcal{R}_{\mu\boxplus\nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z), \tag{2.4}$$

where $\boxplus$ denotes the free convolution (the free convolution $\boxplus$ of measures μ, ν is the law of $\mathbb{X} + \mathbb{Y}$ where $\mathbb{X}$, $\mathbb{Y}$ are free and have laws μ, ν respectively).

If $\mathcal{B} \subset \mathcal{A}$ is a von Neumann subalgebra and $\mathcal{A}$ has a trace τ, then there exists a unique conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ with respect to τ, which we denote by $\tau(\cdot|\mathcal{B})$. This map is a weakly continuous, completely positive, identity preserving, contraction and it is characterized by the property that, for any $\mathbb{X} \in \mathcal{A}$ and $\mathbb{Y} \in \mathcal{B}$, $\tau(\mathbb{X}\mathbb{Y}) = \tau(\tau(\mathbb{X}|\mathcal{B})Y)$ (see [8, 28]). For a fixed $\mathbb{X} \in \mathcal{A}$ by $\tau(\cdot|\mathbb{X})$ we denote the conditional expectation corresponding to the von Neumann algebra $\mathcal{B}$ generated by $\mathbb{X}$. The conditional variance is defined as usual

$$Var(\mathbb{X}|\mathcal{B}) = \tau((\mathbb{X} - \tau(\mathbb{X}|\mathcal{B}))^2|\mathcal{B}). \tag{2.5}$$

A non-commutative stochastic process $(\mathbb{X}_t)$ is a free Lévy process if it has free additive and stationary increments. For a more detailed discussion of free and classical Lévy processes with finite moments of all orders we refer to [6, 21]. Let us first recall some properties of free Lévy processes which follow from [9]. If $(\mathbb{X}_t)$ is a free Lévy process such as $\tau(\mathbb{X}_t) = 0$ and $\tau(\mathbb{X}_t^2) = t$ for all $t > 0$ then

$$\tau(\mathbb{X}_s|\mathbb{X}_u) = \frac{s}{u}\mathbb{X}_u \tag{2.6}$$

for all $0 < s < u$. The conditional variance for free Lévy process is equal (for $0 < s < u$)

$$Var(\mathbb{X}_s|\mathbb{X}_u) = \tau((\mathbb{X}_s - \tau(\mathbb{X}_s|\mathbb{X}_u))^2|\mathbb{X}_u) = \frac{1}{u^2}\tau((u\mathbb{X}_s - s\mathbb{X}_u)^2|\mathbb{X}_u). \tag{2.7}$$

For more details about free convolutions and free probability theory, the reader can consult [19, 23, 29].

2.2 The main result

The main result of this paper is the following characterization of free Meixner processes in terms of the conditional expectation. The proof of this theorem is given in Section 4.

Theorem 2.1. *Suppose that $(\mathbb{X}_{t\geq 0})$ is a free Lévy process such that $\tau(\mathbb{X}_t) = 0$ and $\tau(\mathbb{X}_t^2) = t$ for all $t > 0$. Then the increment $(\mathbb{X}_{t+s} - \mathbb{X}_t)/\sqrt{s}$ $(t, s > 0)$ has the free Meixner law $\mu_{a/\sqrt{s},b/s}$ (for some $b \geqslant 0$) if and only if for all $t < s$*

$$\tau(\mathbb{X}_t\mathbb{X}_s\mathbb{X}_t|\mathbb{X}_s) = \left(\frac{b}{b+s}\mathbb{X}_s + \frac{as}{b+s}\mathbb{I}\right)Var(\mathbb{X}_t|\mathbb{X}_s) + \frac{t^2}{s^2}\mathbb{X}_s^3. \tag{2.8}$$

Remark 2.2. *The existence of a free Lévy process was demonstrated by Biane [8] who proved that from every infinitely divisible distribution we can construct a free Lévy process. We assume that $b \geqslant 0$ in Theorem 2.1 because a free Meixner variable is infinitely divisible if and only if $b \geqslant 0$ (see [5, 9]).*

The proof of Theorem 2.1 is based on the following fact.

Theorem 2.3. *Suppose that $\mathbb{X}$,$\mathbb{Y}$ are free, self-adjoint, non-degenerate, centered ($\tau(\mathbb{X}) = \tau(\mathbb{Y}) = 0$) and $\tau(\mathbb{X}^2 + \mathbb{Y}^2) = 1$. Then $\mathbb{X}/\sqrt{\alpha}$ and $\mathbb{Y}/\sqrt{\beta}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$, respectively, where $a \in \mathbb{R}, b \geq -1$ if and only if*

$$\tau(\mathbb{X}|\mathbb{X}+\mathbb{Y}) = \alpha(\mathbb{X}+\mathbb{Y}) \tag{2.9}$$

and

$$\big(b(\mathbb{X}+\mathbb{Y}) + a\mathbb{I}\big)Var(\mathbb{X}|\mathbb{X}+\mathbb{Y}) = (b+1)\tau\big((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})|\mathbb{X}+\mathbb{Y}\big) \tag{2.10}$$

for some $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Additionally, we assume that $b \geq max\{-\alpha, -\beta\}$ if $b < 0$ (free binomial case).

Corollary 2.4. *For particular values a and b we get that $\mathbb{X}$ and $\mathbb{Y}$ have (after simple computations and under the assumption of Theorem 2.3)*

- *the free Poisson law if and only if ($b = 0$ and $a \neq 0$)*

$$\tau\big((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})|(\mathbb{X}+\mathbb{Y})\big) = a\tau\big((\beta\mathbb{X}-\alpha\mathbb{Y})^2|(\mathbb{X}+\mathbb{Y})\big),$$

- *the normalized Kesten law if and only if ($b \neq 0$ and $a = 0$)*

$$\tau\big((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})|(\mathbb{X}+\mathbb{Y})\big) = \left(\frac{b(\mathbb{X}+\mathbb{Y})}{b+1}\right)\tau\big((\beta\mathbb{X}-\alpha\mathbb{Y})^2|(\mathbb{X}+\mathbb{Y})\big).$$

2.3 Complementary facts and indications

We need the following lemmas on conditional expectations to prove the main result. The lemmas 2.5 and 2.6 were proved in [9] and [18], respectively.

Lemma 2.5. *If $\tau(\mathbb{U}_1\mathbb{V}^n) = \tau(\mathbb{U}_2\mathbb{V}^n)$ for all $n \geq 1$, then $\tau(\mathbb{U}_1|\mathbb{V}) = \tau(\mathbb{U}_2|\mathbb{V})$.*

Lemma 2.6. *If $\mathbb{X}$ and $\mathbb{Y}$ are free independent and centered, then the condition $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ for $\beta, \alpha > 0$ and all integers k is equivalent to*

$$\tau(\mathbb{X}|\mathbb{X}+\mathbb{Y}) = \frac{\alpha}{\alpha+\beta}(\mathbb{X}+\mathbb{Y}). \tag{2.11}$$

Let $NC^k(n+k)$ denote the set of all non-crossing partitions of $\{1,2,\ldots,n+k\}$ (where $n \geq 0$) which have first k elements in the same block. For example for $k=3$ and $n=2$, see Figure 1.

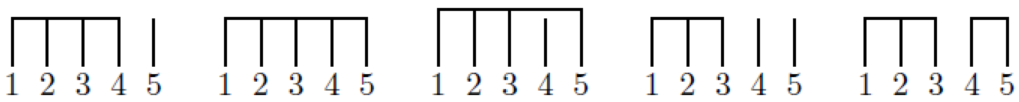

Figure 1: Non-crossing partitions of $\{1,2,3,4,5\}$ with the first 3 elements in the same block.

Definition 2.7. *Let $\mathbb{Z}$ be a self-adjoint element of the von Neumann algebra $\mathcal{A}$. We define*

$$c_n^{(k)} = c_n^{(k)}(\mathbb{Z}) = \sum_{\nu \in NC^k(n+k)} R_\nu(\mathbb{Z}),$$

and the following functions (power series):

$$C^{(k)}(z) = \sum_{n=0}^{\infty} c_n^{(k)} z^{k+n} \text{ where } k \geq 1 \tag{2.12}$$

for sufficiently small $|z| < \epsilon$ and $z \in \mathbb{C}$.

Remark 2.8. *This series is convergent because we consider compactly supported probability measures, so moments and cumulants do not grow faster than exponentially (see [7]). This implies that $c_n^{(k)}$ also does not grow faster than exponentially.*

Now we introduce a lemma which we will use in the proof of the theorems 3.2, 3.6 and 2.3 (for the proof see [18]).

Lemma 2.9. *Let $\mathbb{Z}$ be a (self-adjoint) element of the von Neumann algebra $\mathcal{A}$ then*

$$C^{(k)}(z) = M(z)C^{(k+1)}(z) + R_k(\mathbb{Z})z^k M(z). \tag{2.13}$$

Below we recall some results of [9], which we will apply in the proof of the main theorem to calculate the moment generating function of free convolution.

Lemma 2.10. *(A). Suppose that $\mathbb{X}$, $\mathbb{Y}$ are free, self-adjoint and $\mathbb{X}/\sqrt{\alpha}$,$\mathbb{Y}/\sqrt{\beta}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$ respectively, where $\alpha,\beta > 0$, $\alpha+\beta = 1$ and $a \in \mathbb{R}, b \geq -1$. Then $\mathbb{X}+\mathbb{Y}$ has the law $\mu_{a,b}$ and the moment generating function $M(z)$ of $\mathbb{X}+\mathbb{Y}$ satisfying quadratic equation*

$$(z^2+az+b)M^2(z) - (1+az+2b)M(z) + 1 + b = 0. \tag{2.14}$$

(B). Suppose that $\mathbb{X}$, $\mathbb{Y}$ are free, self-adjoint, non-degenerate, $\tau(\mathbb{X}|\mathbb{X}+\mathbb{Y}) = \alpha(\mathbb{X}+\mathbb{Y})$ and the moment generating function $M(z)$ of $\mathbb{X}+\mathbb{Y}$ satisfying quadratic equation (2.14) *where $\alpha,\beta > 0$, $\alpha+\beta=1$ and $a \in \mathbb{R}, b \geq -1$. Then $\mathbb{X}$ and $\mathbb{Y}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$, respectively.*

3 A new relation in free probability

3.1 A generalization of Speicher's identity

By the main result of [25], we have the following relation

$$M(z)\big(1 - z\mathcal{R}_{\mathbb{X}}(zM(z))\big) = 1. \tag{3.1}$$

The relation (3.1) can be generalized as following:

Proposition 3.1. *Suppose that $\mathbb{X}$ is a self-adjoint element of the algebra $\mathcal{A}$ and denote by μ the distribution of $\mathbb{X}$, then*

$$C_{\mu}^{(k)}(z) = \mathcal{R}_{\mathbb{X}}^{(k)}(zM(z))z^k M(z) \tag{3.2}$$

where $\mathcal{R}_{\mathbb{X}}^{(k)}(z) = \sum_{i=k}^{\infty} R_i(\mathbb{X})z^{i-k}$.

Proof. We prove this by the induction on k. The case $k = 1$ is clear because $C_{\mu}^{(1)}(z) = M(z) - 1$. The induction step $k \Rightarrow k + 1$ (for $k > 1$) follows immediately by using Lemma 2.9 which gives

$$C^{(k+1)}(z) = \frac{C^{(k)}(z)}{M(z)} - R_k(\mathbb{X})z^k = \mathcal{R}_{\mathbb{X}}^{(k)}(zM(z))z^k - R_k(\mathbb{X})z^k = \mathcal{R}_{\mathbb{X}}^{(k+1)}(zM(z))z^{k+1}M(z).$$

□

3.2 A new relation between moments of free Meixner laws

Any probability measure μ on the real line, whose all moments are finite, has two associated sequences of Jacobi parameters α_i, β_i for example, μ is the spectral measure of the tridiagonal matrix (see [5, 16])

$$\begin{pmatrix} \alpha_0, & \beta_0, & 0, & 0, & \ddots \\ 1, & \alpha_1, & \beta_1, & 0, & \ddots \\ 0, & 1, & \alpha_2, & \beta_2, & \ddots \\ 0, & 0, & 1, & \alpha_3, & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.3}$$

We will denote this fact by

$$J(\mu) = \begin{pmatrix} \alpha_0, & \alpha_1, & \alpha_2, & \dots \\ \beta_0, & \beta_1, & \beta_2, & \dots \end{pmatrix} \tag{3.4}$$

with $\alpha_n(\mu) := \alpha_n, \beta_n(\mu) := \beta_n$. If the measure μ has all finite moments, then by a theorem of Stieltjes (see [1]), its Cauchy-Stieltjes transform can be expressed as a continued fraction:

$$G_\mu(z) = \int_{\mathbf{R}} \frac{1}{z-y}\mu(dy) = \cfrac{1}{z - \alpha_0 - \cfrac{\beta_0}{z - \alpha_1 - \cfrac{\beta_1}{z - \alpha_2 - \cfrac{\beta_2}{\ddots}}}} \tag{3.5}$$

If some $\beta_i = 0$, the continued fraction terminates, in which case the subsequent α and β coefficients can be defined arbitrarily. See [12, 16] for more details. The monic orthogonal polynomials P_n for μ satisfy a recursion relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \tag{3.6}$$

with $P_{-1}(x) = 0$.
Thus it is natural to ask about the relation between measures whose Jacobi parameters are described by (3.4) and some other measures whose Jacobi parameter are equal

$$\begin{pmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \dots \\ \beta_1, & \beta_2, & \beta_3, & \dots \end{pmatrix}. \tag{3.7}$$

Theorem 3.2. *Suppose that $\mathbb{X}$ is a self-adjoint element of the algebra $\mathcal{A}$ and denote by μ the distribution of $\mathbb{X}$. If the measure μ has Jacobi parameters described by* (3.4) *where $\beta_1 > 0$, then the relation between the measure ρ of the variable $\mathbb{Y}$ described by parameter* (3.7) *is given by*

$$c_n^{(2)}(\mu) = \beta_0 m_n(\rho), \tag{3.8}$$

for all $n \geqslant 0$.

Proof. From (3.5) we have

$$G_\mu(z) = \frac{1}{z - \alpha_0 - \beta_0 G_\rho(z)}. \tag{3.9}$$

Using the relations $M_\mu(z) = \frac{1}{z}G_\mu(\frac{1}{z})$ and $M_\rho(z) = \frac{1}{z}G_\rho(\frac{1}{z})$ we see that

$$M_\mu(z)(1 - z\alpha_0 - \beta_0 z^2 M_\rho(z)) = 1. \tag{3.10}$$

Applying Lemma 2.9 to $k = 1$ we get

$$M_\mu(z) - 1 = M_\mu(z)C_\mu^{(2)}(z) + \alpha_0 z M_\mu(z), \tag{3.11}$$

where $C_\mu^{(2)}(z)$ is function for $\mathbb{X}$. Now we apply (3.11) to the equation (3.10) and after a simple computation, we obtain

$$C_\mu^{(2)}(z) = \beta_0 z^2 M_\rho(z), \tag{3.12}$$

which is equivalent to (3.8) and this completes the proof. □

Corollary 3.3. *If $\beta_0 = 1$ then $c_n^{(2)}(\mu)$ is the moment of the variable described by Jacobi parameters* (3.7)*.*

Remark 3.4. *The normalized free Meixner distributions $\mu_{a,b}$ have Jacobi parameter*

$$J(\mu_{a,b}) = \begin{pmatrix} 0, & a, & a, & a, & \dots \\ 1, & b+1, & b+1, & b+1, & \dots \end{pmatrix}, \tag{3.13}$$

in other words their Jacobi parameters are independent of n for $n \geq 1$ (see also [5]).

Proposition 3.5. *Suppose that $\mathbb{X}$ is a self-adjoint element of the algebra $\mathcal{A}$ and have the free normalized Meixner law $\mu_{a,b}$ where $b > -1$. Then*

$$c_n^{(2)}(\mu_{a,b}) = \int x^n dw_{a,b+1}(dx), \tag{3.14}$$

for all $n \geqslant 0$ and $w_{a,b}$ is the Wigner semicircle law with mean a and variance b.

Proof. Under the assumption that $\mathbb{X}$ is a free Meixner random variable we have that Jacobi parameter given in (3.13). By Theorem 3.2 we get that $c_n^{(2)}(\mu)$ is the moment of the measure described by Jacobi parameters are

$$\begin{pmatrix} a, & a, & a, & a, & \dots \\ b+1, & b+1, & b+1, & b+1, & \dots \end{pmatrix}, \tag{3.15}$$

so $c_n^{(2)}(\mu)$ is the moment of the Wigner semicircle law with mean a and variance $b+1$ (see also [5, 9]). □

Theorem 3.6. *Suppose that $\mathbb{X}$ is a self-adjoint, non-degenerate such that $\tau(\mathbb{X}) = 0$ and $\tau(\mathbb{X}^2) = 1$. Then $\mathbb{X}$ has the free Meixner laws $\mu_{a,b}$ where $a \in \mathbb{R}, b > -1$ if and only if its moments $m_n = \tau(\mathbb{X}^n)$ satisfy the recursion*

$$(b+1)\int x^n dw_{a,b+1}(dx) = bm_{n+2} + am_{n+1} + m_n, \tag{3.16}$$

for all integers $n \geq 0$ and $w_{a,b}$ is the Wigner semicircle law with mean a and variance b.

Proof. $\Rightarrow$: Suppose that $\mathbb{X}$ have the free Meixner laws $\mu_{a,b}$. Then, from Lemma 2.10 the moment generating functions satisfy equation (2.14). If in (2.14) we multiply the both sides by $(1 - C^{(2)}(z))$ and use Lemma 2.9 with $k = 1$ ($R_1(\mathbb{X}) = 0$), we get

$$M(z)(b + za + z^2) - (2b + 1 + za) + (b+1)(1 - C^{(2)}(z)) = 0, \tag{3.17}$$

where $C^{(2)}(z)$ is function for $\mathbb{X}$. Expanding $M(z)$ in a power series ($M(z) = 1 + \sum_{i=1}^{\infty} z^i m_i$), we get

$$bm_{n+2} + am_{n+1} + m_n = (b+1)c_n^{(2)}. \tag{3.18}$$

Now we apply Proposition 3.5 and get $c_n^{(2)}$ equal to (3.14).
$\Leftarrow$: Let's suppose, that equality (3.16) holds. Multiplying (3.16) by z^{n+2} for $n \geqslant 0$ we obtain ($m_1 = 0$ and $m_2 = 1$)

$$M(z)(b + za + z^2) - (2b + 1 + za) + (b+1) = (b+1)z^2 M'(z), \tag{3.19}$$

where $M'(z)$ is the moment generation function for the Wigner semicircle law with mean a and variance $b+1$. The above equation is equivalent to

$$M(z) = \frac{(b+1)z^2 M'(z) + b + za}{b + za + z^2}. \tag{3.20}$$

It is well known (see [9, 23]) that

$$M'(z) = \frac{1 - az - \sqrt{(1-za)^2 - 4z^2(1+b)}}{2z^2(1+b)}. \tag{3.21}$$

Thus after simple computation, we see that $M(z)$ is equal to (2.1), which proves the theorem. □

4 Proof of the main theorem

Bellow we prove Theorem 2.3.
Proof of Theorem 2.3. $\Rightarrow$: Suppose that $\mathbb{X}/\sqrt{\alpha}$ and ,$\mathbb{Y}/\sqrt{\beta}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$, respectively. The condition (2.9) holds because we can use Theorem 3.1 from [17]. From this theorem we also have $\alpha R_k(\mathbb{Y}) = \beta R_k(\mathbb{X})$. From

Lemma 2.10 the moment generating functions M of $\mathbb{X}+\mathbb{Y}$ satisfy equation (2.14). If in (2.14) we multiply the both sides by $(1-C^{(2)}(z))$ and use Lemma 2.9 with $k=1$ ($R_1(\mathbb{X}+\mathbb{Y})=0$), we get

$$M(z)(b+za+z^2)-(2b+1+za)+(b+1)(1-C^{(2)}(z))=0, \tag{4.1}$$

where $C^{(2)}(z)$ is function for $\mathbb{X}+\mathbb{Y}$. Now we apply Lemma 2.9 with $k=2$ to equation (4.1) (using the assumption $R_2(\mathbb{X}+\mathbb{Y})=1$) and after simple computations, we see that

$$M(z)(b+za+z^2)-(b+za)=(b+1)(M(z)C^{(3)}(z)+z^2M(z)) \tag{4.2}$$

or equivalently

$$b+za-z^2b-\frac{(b+za)}{M(z)}=(b+1)C^{(3)}(z) \tag{4.3}$$

for $|z|$ small enough. Then we again use Lemma 2.9 with $k=1$ to get

$$-z^2b+(b+za)C^{(2)}(z)=(b+1)C^{(3)}(z). \tag{4.4}$$

Expanding the above equation in power series, we get

$$bc_{n+1}^{(2)}+ac_n^{(2)}=c_n^{(3)}(b+1) \text{ for all } n\geqslant 0. \tag{4.5}$$

From the assumption of Theorem 2.3 and Lemma 2.6 we get

$$R_k(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y})=\beta R_k(\mathbb{X})-\alpha R_k(\mathbb{Y})=0 \tag{4.6}$$

and similarly for $k\geqslant 3$

$$\begin{aligned} R_k(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) &= \beta^2R_k(\mathbb{X})+\alpha^2R_k(\mathbb{Y}) \\ &= \beta\alpha R_k(\mathbb{X}+\mathbb{Y}). \end{aligned} \tag{4.7}$$

Now we use the moment-cumulant formula (2.2)

$$\begin{aligned} &\tau((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})^n) \\ &= \sum_{\nu\in NC(n+3)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\underbrace{\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}}_{n\text{-times}}) \\ &= \sum_{\nu\in NC^3(n+3)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) \\ &+ \sum_{\nu\in NC(n+3)\setminus NC^3(n+3)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}). \end{aligned}$$

Let us look more closely at the second summand from the last equation. We have that either the first and the third elements are in different blocks, or they are in the same block. In the first case, the second sum (from the last equation) vanishes by (4.6). On the other hand, if they are in the same block, the sum disappears by $\tau(\mathbb{X}+\mathbb{Y})=0$. So, by (4.7) we have

$$\begin{aligned} &\tau((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})^n) \\ &= \alpha\beta\sum_{\nu\in NC^3(n+3)} R_\nu(\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\underbrace{\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}}_{n\text{-times}})=\alpha\beta c_n^{(3)} \end{aligned} \tag{4.8}$$

and by the same method

$$\tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2(\mathbb{X}+\mathbb{Y})^n)$$
$$=\alpha\beta\sum_{\nu\in NC^2(n+2)}R_\nu(\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\underbrace{\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}}_{n\text{-times}})=\alpha\beta c_n^{(2)}. \tag{4.9}$$

Therefore the equation (4.5) is equivalent to

$$b\tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2(\mathbb{X}+\mathbb{Y})^{n+1})+a\tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2(\mathbb{X}+\mathbb{Y})^n)$$
$$=\tau((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})^n)(b+1), \tag{4.10}$$

for all $n\geqslant 1$ or equivalently

$$\tau((b(\mathbb{X}+\mathbb{Y})+a\mathbb{I})(\beta\mathbb{X}-\alpha\mathbb{Y})^2(\mathbb{X}+\mathbb{Y})^n)$$
$$=\tau((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})^n)(b+1), \tag{4.11}$$

for all $n\geqslant 0$. Now we use Lemma 2.5 which essentially shows

$$(b(\mathbb{X}+\mathbb{Y})+a\mathbb{I})\tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2|\mathbb{X}+\mathbb{Y})$$
$$=(b+1)\tau((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})|\mathbb{X}+\mathbb{Y}) \tag{4.12}$$

because $b(\mathbb{X}+\mathbb{Y})+a\mathbb{I}$ is in the algebra generated by $\mathbb{X}+\mathbb{Y}$.
$\Leftarrow$: Let's suppose now, that equalities (2.9) and (2.10) hold. Then, in particular, we have equation (4.12). Multiplying (4.12) by $(\mathbb{X}+\mathbb{Y})^n$ for $n\geqslant 0$ and applying $\tau(\cdot)$ we obtain (4.10). As it can be seen in the above proof "$\Rightarrow$", each of the above steps are equivalent, so from (4.10) we get equation (2.14). Lemma 2.10 (part B) says that $\mathbb{X}$ and $\mathbb{Y}$ have the Meixner laws, which completes the proof of Theorem 2.3.

□

Now we are ready to a prove the main theorem.
Proof of Theorem 2.1. Let's rewrite Theorem 2.3 for the variables (non-degenerate) $\mathbb{X}$ and $\mathbb{Y}$ such that $\tau(\mathbb{X}^2)=\alpha$, $\tau(\mathbb{Y}^2)=\beta$ and $\tau(\mathbb{Y})=\tau(\mathbb{X})=0$. After a simple parameter normalization (α by $\frac{\alpha}{\alpha+\beta}$, β by $\frac{\beta}{\alpha+\beta}$, a by $\frac{a}{\sqrt{\alpha+\beta}}$, b by $\frac{b}{\alpha+\beta}$) we get that $\mathbb{X}/\sqrt{\alpha}=\frac{\mathbb{X}}{\sqrt{\alpha+\beta}}/\frac{\sqrt{\alpha}}{\sqrt{\alpha+\beta}}$ and $\mathbb{Y}/\sqrt{\beta}=\frac{\mathbb{Y}}{\sqrt{\alpha+\beta}}/\frac{\sqrt{\beta}}{\sqrt{\alpha+\beta}}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$, respectively, if and only if (after a simple computation)

$$(b+\alpha+\beta)\tau((\beta\mathbb{X}-\alpha\mathbb{Y})(\mathbb{X}+\mathbb{Y})(\beta\mathbb{X}-\alpha\mathbb{Y})|\mathbb{X}+\mathbb{Y})$$
$$=(b(\mathbb{X}+\mathbb{Y})+a(\alpha+\beta)\mathbb{I})\tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2|\mathbb{X}+\mathbb{Y}), \tag{4.13}$$

i.e. we apply Theorem 2.3 with $\mathbb{X}$ equal to $\frac{\mathbb{X}}{\sqrt{\alpha+\beta}}$ and $\mathbb{Y}$ equal to $\frac{\mathbb{Y}}{\sqrt{\alpha+\beta}}$ and the parameters mentioned above in the brackets. Now we consider two variables $\mathbb{X}_t/\sqrt{t}$ and $(\mathbb{X}_s-\mathbb{X}_t)/\sqrt{s-t}$, which are free and centered. Thus, the formula (4.13) tells us that $\mathbb{X}_t/\sqrt{t}$ and $(\mathbb{X}_s-\mathbb{X}_t)/\sqrt{s-t}$ ($\mathbb{X}=\mathbb{X}_t$, $\mathbb{Y}=\mathbb{Y}_t$, $\alpha=t$, $\beta=s-t$), have the free Meixner laws $\mu_{a/\sqrt{t},b/t}$ and $\mu_{a/\sqrt{s-t},b/(s-t)}$, respectively, if and only if

$$\tau((t\mathbb{X}_s-s\mathbb{X}_t)\mathbb{X}_s(t\mathbb{X}_s-s\mathbb{X}_t)|\mathbb{X}_s)$$
$$\overset{(2.6)}{=}t^2\mathbb{X}_s^3-t^2\mathbb{X}_s^3+s^2\tau(\mathbb{X}_t\mathbb{X}_s\mathbb{X}_t|\mathbb{X}_s)-t^2\mathbb{X}_s^3$$
$$=\frac{(b\mathbb{X}_s+as\mathbb{I})}{(b+s)}\tau((t\mathbb{X}_s-s\mathbb{X}_t)^2|\mathbb{X}_s). \tag{4.14}$$

Thus Theorem 2.1 holds.

□

Open problems and remarks. In Theorem 2.1 of this paper we assume that random variables are bounded that is $X_t \in \mathcal{A}$. It would be interesting to show if this assumption can be replaced by $X_t \in L^2(\mathcal{A}, \tau)$. It would be also worth to investigate if Theorem 3.6 is related to the main result of [15].

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