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Moments of Wiener integrals for subordinators

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Abstract

Moments formulae for Wiener integrals of a subordinator with exponential moments are obtained in terms of the general Bell polynomials and the moments of the Lévy measure of this subordinator. We also express the Appell and Scheffer polynomials associated to a random variable in terms of the Bell polynomials.

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1 Introduction

In this paper we establish moments formulae for the random variables:

$$I_t^{(\nu)}(\phi) \stackrel{def}{=} \int_0^t \phi(s) d\gamma_s^{(\nu)},$$

where $\left(\gamma_t^{(\nu)}, t \geq 0\right)$ denotes a subordinator with Lévy measure $\nu(dx)$, and no drift, so that

$$\begin{aligned} E\left[\exp\left(-\lambda\gamma_t^{(\nu)}\right)\right] &= \exp\left(-t\psi^{(\nu)}(\lambda)\right), \\ with\ \psi^{(\nu)}(\lambda) &= \int \nu(dx)\left(1-\exp\left(-\lambda x\right)\right) \end{aligned}$$

and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic Borel, bounded function. We call $I_t^{(\nu)}(\phi)$ the Wiener integral of ϕ with respect to $\gamma^{(\nu)}$. There are many instances where the variables $I_t^{(\nu)}(\phi)$ appear in stochastic analysis, in particular when $\phi(s) = e^{-as}$. For example if S is a self decomposable random variable taking values in $\mathbb{R}_+$, then it may be represented as $S \stackrel{(d)}{=} \int_0^\infty e^{-ps} d\gamma_s^{(\nu)}$ for some background driving subordinator, and some given p (Jurek-Vervaat (1983)). Furthermore, this family of random variables obtained from $I_t^{(\nu)}(\phi)$, for a generic function ϕ, integrated with respect to the gamma process $(\gamma_t, t \geq 0)$ with Lévy measure

$$\nu(dx) = \frac{dx}{x} e^{-x},\ x > 0$$

plays a fundamental role in Eberlein, Madan, Pistorius and Yor (2011) to which we refer the reader in view of some motivation for this work. Also recall that, for $\gamma^{(\nu)} \equiv \gamma$ the

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variables $I_t^{(\nu)}(\phi)$ are precisely the generalized gamma variables (abbreviated GGC) for which the reader may consult James, Roynette and Yor (2008). In the present paper our hypothesis on ν is that for any integer $p \geq 1$,

$$\nu_p = \int_0^\infty x^p \nu(dx) < \infty.$$

Our hypothesis on $\phi : [0,t] \to \mathbb{R}_+$ is that for any integer $p \geq 1$

$$\int_0^t (\phi(s))^p \, ds < \infty.$$

Our paper is organized as follows. In section 2 we recall basic facts about the family of Bell polynomials $B_{n,k}(w.)$ attached to sequences $(w_j : j = 1, 2, \cdots)$. This information and the corresponding notation are taken from Pitman (2006)(Chapter 1 of 2002 St Flour course). The main formula we are concerned with here is

$$\exp\left(\sum_{p=1}^{\infty} \frac{\lambda^p}{p!} w_p\right) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} b_n(w_1, \cdots, w_n) \tag{1.1}$$

where the polynomial $b_n(w_1, \cdots, w_n)$ is given by

$$b_n(w_1, \cdots, w_n) = \sum_{k=1}^{n} B_{n,k}(w_1, \cdots, w_n) \tag{1.2}$$

with

$$B_{n,k}(w_1, \cdots, w_n) = \frac{n!}{k!} \sum_{\substack{p_1+p_2+\cdots p_k = n \\ p_i \geq 1}} \prod_{i=1}^{k} \frac{w_{p_i}}{(p_i)!} \tag{1.3}$$

We also recall explicit formulae for the $B_{n,k}$, hence for the $b_n's$ for $n = 1, 2, 3, 4$ and 5.

In Section 3 we prove the formula

$$E\left[\left(I_t^{(\nu)}(\phi)\right)^n\right] = b_n(\nu_1\Phi_1(t), \nu_2\Phi_2(t), \cdots, \nu_n\Phi_n(t)) \tag{1.4}$$

where

$$\Phi_k(t) = \int_0^t ds \, (\phi(s))^k.$$

Note that formula (1.4) is unchanged as ϕ is changed into $\widetilde{\phi}(s) = \phi(t-s)$ (time reversal on $[0,t]$), which reflects the identity in law

$$\left(\gamma_s^{(\nu)}, s \leq t\right) \overset{(law)}{=} \left(\gamma_t^{(\nu)} - \gamma_{(t-s)-}^{(\nu)}, s \leq t\right)$$

In the same section we particularize formula (1.4) for

$a)$ The Poisson Process

$b)$ The Gamma Process

$c)$ The compound Poisson process with exponential jumps

In Section 4, we start with a direct recursive approach for the moments:

$$m_n^{(\phi)}(t) \overset{def}{=} E\left[\left(I_t^{(\nu)}(\phi)\right)^n\right]$$

and we discuss how this approach is related to our first approach.

In Section 5 we extend the previous formulae to multivariate cross moments such as

$$E\left[\left(I_t^{(\nu)}(\phi)\right)^n\left(I_t^{(\nu)}(\psi)\right)^m\right]$$

and more generally

$$E\left[\prod_{j=1}^{K}\left(I_t^{(\nu)}(\phi_j)\right)^{n_j}\right]. \tag{1.5}$$

For these more general formulae, the Bell polynomials to be used are defined via

$$\begin{aligned}&\exp\left\{\sum_{\substack{p_1,\cdots p_K\\ p_1+\cdots p_K\geq 1}}\frac{a_1^{p_1}\cdots a_K^{p_K}}{p_1!\cdots p_K!}w_{p_1,p_2,\cdots p_K}\right\} &(1.6)\\ = &\sum_{n_1,\cdots n_K=0}^{\infty}\frac{a_1^{n_1}}{n_1!}\cdots\frac{a_K^{n_K}}{n_K!}b_{n_1,\cdots n_K}\left(w_{p_1,p_2,\cdots,p_K};p_i\leq n_i,i=1,\cdots,K\right)\end{aligned}$$

and finally we obtain:

$$\begin{aligned}&E\left[\prod_{j=1}^{K}\left(I_t^{(\nu)}(\phi_j)\right)^{n_j}\right]\\ &= b_{n_1,\cdots n_K}\left(\nu_{p_1+\cdots+p_K}\int_0^t ds\ \left(\phi_1(s)\right)^{p_1}\cdots\left(\phi_K(s)\right)^{p_K}\right); &(1.7)\\ p_i &\leq n_i;\ i=1,2,\cdots,K.\end{aligned}$$

Multivariate examples will also be provided, e.g. for the gamma process $(\gamma_u, u\geq 0)$, for which we give an explicit formula for

$$E\left[\prod_{j=1}^{K}\left(\int_0^{\infty}\exp\left(-a_j s\right)d\gamma_s\right)^{n_j}\right]. \tag{1.8}$$

2 Basic facts about Bell polynomials

As we noted in the introduction, the formulae (1.1), (1.2) and (1.3) announced there are lifted from Pitman's St. Flour course [2002]. Clearly one obtains them by developing the products of exponentials:

$$\exp\left(\frac{\lambda^p}{p!}w_p\right)$$

as a series of powers of λ^p, and then gathering the powers of λ.

In Pitman's chapter 1, one shall find the origins of such formulae related to composite structures, and more general Bell polynomials associated with 2 sequences (v_j) and (w_k) instead of only one (w_k) which is enough for our discussion. To be explicit we reproduce Pitman's Table 2 for the Bell polynomials $(B_{n,k}(w.)\,, k\leq n\leq 5)$

n	$B_{n,1}(w.)$	$B_{n,2}(w.)$	$B_{n,3}(w.)$	$B_{n,4}(w.)$	$B_{n,5}(w.)$
1	w_1				
2	w_2	w_1^2			
3	w_3	$3w_1w_2$	w_1^3		
4	w_4	$4w_1w_3+3w_2^2$	$6w_1^2w_2$	w_1^4	
5	w_5	$5w_1w_4+10w_2w_3$	$10w_1^2w_3+15w_1w_2^2$	$10w_1^3w_2$	w_1^5

Consequently we obtain the 5 polynomials $b_n(w.)$, $1 \leq n \leq 5$ as

$$\begin{aligned} b_1(w_1) &= w_1 \\ b_2(w_1, w_2) &= w_2 + w_1^2 \\ b_3(w_1, w_2, w_3) &= w_3 + 3w_1 w_2 + w_1^3 \\ b_4(w_1, w_2, w_3, w_4) &= w_4 + 4w_1 w_3 + 3w_2^2 + 6w_1^2 w_2 + w_1^4 \\ b_5(w_1, w_2, w_3, w_4, w_5) &= w_5 + 5w_1 w_4 + 10 w_2 w_3 + 15 w_1 w_2^4 + 10 w_1^3 w_2 + w_1^5 \end{aligned}$$

At this point, we suggest that our reader looks at the 3 exercises on page 22 of Pitman (2006) as they are intimately linked with our discussion. In particular, formula (78) in Pitman (2006), which reads, in our notation

$$E\left[\left(\gamma_t^{(\nu)}\right)^n\right] = b_n\left(\nu_1 t, \cdots, \nu_n t\right)$$

is the particular case of our formula (1.4), where the function ϕ is taken identical to 1. We also point out the connection between the Bell polynomials and the Appell polynomials $\left\{Q_k^{(A)}\right\}$ associated in Salminen (2011) to a random variable X with some exponential moments:

$$\begin{aligned} \frac{\exp(\lambda x)}{E\left[\exp(\lambda X)\right]} &= \exp\left(\lambda x - \log E\left[\exp(\lambda X)\right]\right) \\ &= \exp\left(\lambda(x - E[X]) - \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} c_k\right) \\ &= 1 + \sum_{j=1}^{\infty} \frac{\lambda^j}{j!} Q_j^{(A)}(x), \end{aligned} \tag{2.1}$$

where on the second line the coefficients $(c_k, k \geq 2)$ are the cumulants of X. Now it is immediate, by comparison of formula (1.1) above and (2.1) here, that

$$Q_k^{(A)}(x) = b_k(x - E[X], -c_2, -c_3, \cdots, -c_k)$$

We also find it interesting to compare this formula for the family of Appell polynomials with the polynomials discussed in Schoutens (2000) which he calls Scheffer polynomials. We borrow the following notation from Schoutens (2000) page 45, and also adapt them to our discussion: W. Schoutens is interested in the following generating function expansions

$$\exp\left(xu(z)\right) f(z) = \sum_{m=0}^{\infty} Q_m^{(S)}(x) \frac{z^m}{m!} \tag{2.2}$$

under certain conditions on u and f. We write $Q_m^{(S)}$ in reference to Scheffer and/or Schoutens. In the particular case where

$$f(z) = \frac{1}{E\left[\exp(\lambda X)\right]}$$

with

$$u(z) = \lambda$$

so that

$$z = u^{-1}(\lambda) = \tau(\lambda)$$

(in Schoutens notation) formula (2.2) becomes

$$\frac{\exp(\lambda x)}{E\left[\exp(\lambda X)\right]} = \sum_{m=0}^{\infty} Q_m^{(S)}(x) \frac{(\tau(\lambda))^m}{m!} \tag{2.3}$$

However, assuming

$$\tau(\lambda) = \sum_{j=1}^{\infty} t_j \frac{\lambda^j}{j!}$$

we get the power formulae:

$$(\tau(\lambda))^m = \sum_{j=m}^{\infty} t_j^{(m)} \frac{\lambda^j}{j!}$$

for some coefficients $(t_j^{(m)})$ obtained from the (t_j) sequence, and m; this is yet another instance of the formulae in Chapter 1 of Pitman (2006). Now formula (2.3) becomes

$$\frac{\exp(\lambda x)}{E\left[\exp\left(\lambda X\right)\right]} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \left(\sum_{m \le j} t_j^{(m)} Q_m^{(S)}(x) \frac{1}{m!} \right) \tag{2.4}$$

so that comparison with the preceding Appell-Salminen formula:

$$\frac{\exp(\lambda x)}{E\left[\exp\left(\lambda X\right)\right]} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} Q_j^{(A)}(x)$$

we get that for every j,

$$Q_j^{(A)}(x) = \sum_{m \le j} \frac{t_j^{(m)}}{m!} Q_m^{(S)}(x) \tag{2.5}$$

This ends our discussion of the three different approaches.

3 Proof of the moments formula (1.4)

We start from the identity

$$E\left[\exp\left(\lambda \int_0^t \phi(s) d\gamma_s^{(\nu)}\right)\right] = \exp\left(\int_0^t ds \int \nu(dx) \left[\exp\left(\lambda \phi(s) x\right) - 1\right]\right)$$

On the left hand side we get

$$1 + \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E\left[\left(I_t^{(\nu)}(\phi)\right)^n\right],$$

whereas on the right we get

$$\begin{aligned} & \exp\left(\sum_{p=1}^{\infty} \frac{\lambda^p}{p!} \nu_p \Phi_p(t)\right) \\ = \quad & 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} b_n\left(\nu_1 \Phi_1(t), \cdots, \nu_n \Phi_n(t)\right) \end{aligned}$$

Formula (1.4) is thus proven, by identification of the coefficients of λ^n.

In the particular cases of :

the Poisson process, then $\nu(dx) = \varepsilon_1(dx)$, so that $\nu_p = 1$;

the Gamma process, then $\nu(dx) = \frac{e^{-x} dx}{x}$, so that $\nu_p = \Gamma(p) = (p-1)!$;

the compound Poisson process with exponential jumps, then $\nu(dx) = c \exp(-ax)$, $\nu_p = \frac{cp!}{a^{p+1}}$.

For the case of Poisson and Gamma, one shall find further remarkable formulae such as Dobinski's formula

$$b_n(1, \cdots, 1) = e^{-1} \sum_{m=1}^{\infty} \frac{m^n}{m!}$$

in Pitman's Chapter 1.

4 A Recursion formula for $m_n^{(\phi)}(t) = E\left[\left(I_t^{(\nu)}(\phi)\right)^n\right]$

In this section we obtain a recursion formula for $(m_n^{(\phi)}(t), n \geq 1)$ by simply using the compensation formula for our subordinator $\left(\gamma_t^{(\nu)}, t \geq 0\right)$, that is, for any predictable process $(H_s, s \geq 0)$ and any Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$E\left[\sum_{s\leq t} H_s f\left(\Delta\gamma_s^{(\nu)}\right) \mathbf{1}_{\Delta\gamma_s^{(\nu)}\neq 0}\right] = E\left[\int_0^t H_s ds\right] \int_0^\infty \nu(dx) f(x) \tag{4.1}$$

We then write

$$\begin{aligned} m_n^{(\phi)}(t) &= E\left[\left(I_t^{(\nu)}(\phi)\right)^n\right] \\ &= E\left[\sum_{s\leq t}\left(\left(I_s^{(\nu)}(\phi)\right)^n - \left(I_{s_-}^{(\nu)}(\phi)\right)^n\right)\mathbf{1}_{\Delta\gamma_s^{(\nu)}>0}\right] \end{aligned}$$

Then on the set $\left(\Delta\gamma_s^{(\nu)} > 0\right)$, we write

$$\begin{aligned} \left(I_s^{(\nu)}(\phi)\right)^n &= \left(I_{s_-}^{(\nu)}(\phi) + \phi(s)\Delta\gamma_s^{(\nu)}\right)^n \\ &= \sum_{p=0}^{n} \binom{n}{p} \left(I_{s_-}^{(\nu)}(\phi)\right)^p \left(\phi(s)\Delta\gamma_s^{(\nu)}\right)^{n-p} \end{aligned}$$

so that

$$\left(I_s^{(\nu)}(\phi)\right)^n - \left(I_{s_-}^{(\nu)}(\phi)\right)^n = \sum_{p=0}^{n-1} \binom{n}{p} \left(I_{s_-}^{(\nu)}(\phi)\right)^p (\phi(s))^{n-p} \left(\Delta\gamma_s^{(\nu)}\right)^{n-p}$$

and we thus obtain using (4.1) that

$$\begin{aligned} m_n^{(\phi)}(t) &= \sum_{p=0}^{n-1} \binom{n}{p} \int_0^t ds E\left[\left(I_s^{(\nu)}(\phi)\right)^p\right] (\phi(s))^{n-p}\, \nu_{n-p} \\ &= \sum_{p=0}^{n-1} \binom{n}{p} \int_0^t ds\; m_p^{(\phi)}(s)(\phi(s))^{n-p}\nu_{n-p} \end{aligned}$$

Define

$$\widetilde{m}_n^{(\phi)}(t) = \frac{1}{n!} m_n^{(\phi)}(t)$$

then we obtain the formula

$$\begin{aligned} \widetilde{m}_n^{(\phi)}(t) &= \sum_{p=0}^{n-1} \int_0^t ds\; (\phi(s))^{n-p} \widetilde{m}_p^{(\phi)}(s) \widetilde{\nu}_{n-p} \qquad (4.2) \\ \text{where } \widetilde{\nu}_k &= \frac{\nu_k}{k!}. \end{aligned}$$

It is left to the reader to show that formulae (4.2) and (1.4) agree.

5 Multivariate cross moments formulae

The reader who followed us so far should have no difficulty to derive (1.7) from (1.6) as this is really the same argument as of section 3. It may be of interest to write explicit expressions then obtained for say the gamma process, $t = \infty$, $\phi_j(s) = \exp(-a_j s)$. Then

$$\begin{aligned}\int_0^\infty ds \prod_j \left(\phi_j(s)\right)^{p_j} &= \int_0^\infty ds \ \exp\left(-\sum_j a_j p_j s\right) \\ &= \frac{1}{\sum_j a_j p_j}\end{aligned}$$

and formula (1.7) now reads

$$E\left[\prod_{j=1}^K \left(I_\infty\left(e^{-a_j \cdot}\right)\right)^{n_j}\right] = b_{n_1,\cdots,n_K}\left(\Gamma\left(p_1 + \cdots p_K\right)\frac{1}{\sum_{j=1}^K a_j p_j},\ p_i \le n_i,\ i \le K\right)$$

thus providing an explicit answer for (1.8)

To conclude, we simply add a little discussion about formulae (1.6), in which we connect (easily) the polynomials $b_{n_1,n_2,\cdots,n_K}$ to the polynomials b_n in formula (1.1).

We proceed as follows.

$$\begin{aligned}&\exp\left(\sum_{\substack{p_1,\cdots p_K \\ p_1+\cdots p_K \ge 1}} \frac{a_1^{p_1}\cdots a_K^{p_K}}{p_1!\cdots p_K!} w_{p_1,p_2,\cdots p_K}\right) \\ = \ &\exp\left(\sum_{p_1}\frac{a_1^{p_1}}{p_1!}W_{p_1}\right), \\ = \ &1 + \sum_{n_1}\frac{a_1^{n_1}}{n_1!} b_{n_1}(W_1,\cdots,W_{n_1})\end{aligned}$$

where

$$W_p = \sum_{p_2+\cdots+p_K \ge 1-p} \frac{a_2^{p_2}}{p_2!}\cdots\frac{a_K^{p_K}}{p_K!} w_{p,p_2,\cdots,p_K}$$

Consequently

$$b_n(W_1,\cdots,W_n) = \sum_{n_2,\cdots,n_K=0}^\infty \frac{a_2^{n_2}}{n_2!}\cdots\frac{a_K^{n_K}}{n_K!} b_{n,n_2,\cdots n_K}(w_{p_1,\cdots p_K}; p_i \le n_i, i \le K).$$

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