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# On the infinite sums of deflated Gaussian products*

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**Abstract**

In this paper we derive the exact tail asymptotic behaviour of $S_\infty = \sum_{i=1}^{\infty} \lambda_i X_i Y_i$, where $\lambda_i, i \ge 1$ are non-negative square summable deflators (weights) and $X_i, Y_i, i \ge 1$, are independent standard Gaussian random variables. Further, we consider the tail asymptotics of $S_{\infty;p} = \sum_{i=1}^{\infty} \lambda_i X_i |Y_i|^p, p > 1$, and also discuss the influence on the asymptotic results when $\lambda_i$'s are independent random variables.

**Keywords:** Gaussian products; infinite sums; random deflation; exact tail asymptotics; max-domain of attraction; regular variation; chi-square distribution.
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## 1 Introduction and Main Result

Let $X_i, Y_i, i \ge 1$ be independent standard (zero mean and unit variance) Gaussian random variables, and let $\lambda_i, i \ge 1$ be non-negative constants. Define, for some fixed integer $n$, the weighted sum

$$S_n = \sum_{i=1}^{n} \lambda_i X_i Y_i.$$

Since a one-dimensional projection of a standard Gaussian random vector is distributed as a Gaussian random variable, we have the stochastic representation

$$S_n \stackrel{d}{=} X_1 \sqrt{\sum_{i=1}^{n} \lambda_i^2 Y_i^2} = X_1 Z_n, \quad \text{with} \quad Z_n := \sqrt{\sum_{i=1}^{n} \lambda_i^2 Y_i^2}, \tag{1.1}$$

where $\stackrel{d}{=}$ stands for the equality of the distribution functions. Note in passing that (1.1) holds for general random variables $Y_1, \ldots, Y_n$ being independent of Gaussian random variables $X_1, \ldots, X_n$. Clearly, due to the symmetry about 0 of $X_i, i \ge 1$, the following stochastic representation

$$S_n \stackrel{d}{=} \sum_{i=1}^{n} \lambda_i X_i |Y_i| \tag{1.2}$$

is also valid, and hence $S_n$ is a symmetric (about 0) random variable. In this paper, we will make use of the stochastic representations of $S_n$, following from the fact that

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$X_i, i \geq 1$, are Gaussian. For notational simplicity, we consider in the following ordered weights $\lambda_i, i \geq 1$ meaning

$$\lambda := \lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \geq \cdots \geq 0.$$

In view of [7] (see also [11] p.168 and [16]), we have (set $\prod_{m+1}^n (\cdot) =: 1$ when $m = n$)

$$\boldsymbol{P}\left\{Z_n^2 > \lambda^2 x\right\} \sim \prod_{i=m+1}^{n} (1 - \lambda_i^2/\lambda^2)^{-1/2} \frac{2^{1-m/2}}{\Gamma(m/2)} x^{m/2-1} \exp(-x/2), \ m \leq n \leq \infty \tag{1.3}$$

as $x \to \infty$. The standard notation $\sim$ stands for asymptotic equivalence as the argument tends to infinity, and $\Gamma(\cdot)$ denotes the Euler Gamma function.
In the light of Lemma 3.4 (presented in Section 3) we obtain

$$\boldsymbol{P}\left\{|S_n| > \lambda x\right\} \sim \boldsymbol{P}\left\{Z_n^2 > \lambda^2 x\right\} \exp(-x/2), \ n \geq m, \tag{1.4}$$

which is shown in the special case $\lambda_i = \lambda, i \leq n$ in the first Lemma of [8].

An interesting quantity with application in statistics, insurance and several other applied fields is the random variable

$$S_\infty = \sum_{i=1}^{\infty} \lambda_i X_i Y_i,$$

where the non-negative weights $\lambda_i, i \geq 1$ are square summable, i.e.,

$$\sum_{i=1}^{\infty} \lambda_i^2 < \infty. \tag{1.5}$$

The random variable $S_\infty$ appears as the distributional limits for various statistics; for instance as shown by [9], $S_\infty$ is essential for the characterisation of continuous, separately exchangeable processes. Several examples of statistics given as infinite weighted sums of Gaussian products appear naturally when dealing with $U$-statistics or row and column exchangeable processes.

The main result of [8] gives an upper bound for the tail probability of the random variable $S_\infty$, namely

$$\boldsymbol{P}\left\{|S_\infty| > x\right\} \leq K \boldsymbol{P}\left\{|S_m| > x\right\}, \quad x > 0, \tag{1.6}$$

with $K$ some unknown constant and $S_m = \lambda \sum_{i=1}^m X_i Y_i$. Recall that $m$ is the multiplicity of the largest weight $\lambda = \lambda_1$, and by the square summability assumption on $\lambda_i, i \geq 1$, $m$ is necessarily a finite integer.
The main goal of this contribution is to derive, instead of the bound above, the exact tail asymptotic behaviour of $|S_\infty|$. Our main result below gives such an asymptotic expansion showing further connections with the tail asymptotic behaviours of $Z_\infty$ and $S_n$ for any $n \geq m$.

**Theorem 1.1.** *Let $X_i, Y_i, i \geq 1$ be independent standard Gaussian random variables. For given constants $\lambda := \lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \geq \cdots \geq 0$ satisfying the square summability criterion* (1.5)

$$\boldsymbol{P}\left\{|S_\infty| > \lambda x\right\} \sim \Big[\prod_{i=m+1}^{\infty} (1 - \lambda_i^2/\lambda^2)^{-1/2}\Big] \frac{2^{1-m/2}}{\Gamma(m/2)} x^{m/2-1} \exp(-x) \tag{1.7}$$

*holds as $x \to \infty$. Furthermore, as $x \to \infty$, we have*

$$\boldsymbol{P}\left\{|S_\infty| > x\right\} \sim \Big[\prod_{i=n+1}^{\infty} (1-\lambda_i^2/\lambda^2)^{-1/2}\Big] \boldsymbol{P}\left\{|S_n| > x\right\}, \text{ for any } n \ge m, \tag{1.8}$$

*and*

$$\boldsymbol{P}\left\{|S_\infty| > \lambda x\right\} \sim \boldsymbol{P}\left\{Z_\infty^2 > \lambda^2 x\right\} \exp(-x/2). \tag{1.9}$$

## 2 Extensions and Discussions

In this section we discuss two directions which provide natural extensions of the main result presented in Theorem 1.1. Initially, motivated by the stochastic representation (1.2), we consider $S_{n;p}, p > 0$ defined by

$$S_{n;p} := \sum_{i=1}^{n} \lambda_i X_i \left|Y_i\right|^p .$$

The same reasoning as (1.1) yields

$$S_{n;p} \stackrel{d}{=} X_1 \sqrt{\sum_{i=1}^{n} \lambda_i^2 Y_i^{2p}} =: X_1 Z_{n;p}.$$

We show in Lemma 3.1 below that both $S_{\infty;p}$ and $Z_{\infty;p}$ exist almost surely, and moreover $S_{\infty;p} \stackrel{d}{=} X_1 Z_{\infty;p}$. Since the tail asymptotic behaviour of $Z_{\infty;p}$ is known (see [3], [10] and [11]) applying Lemma 3.3 for any $p > 1$ we obtain

$$\boldsymbol{P}\left\{|S_{\infty;p}| > \lambda x\right\} \sim \frac{2m}{\sqrt{\pi(1+p)}} p^{\frac{p}{2(1+p)}} x^{-\frac{1}{1+p}} \exp\left( -\frac{1}{2}\big(p^{\frac{1}{1+p}} + p^{-\frac{p}{1+p}}\big) x^{\frac{2}{1+p}} \right), \tag{2.1}$$

where we use the same notation and assumptions as in Theorem 1.1. We note in passing that for the cases when $p \in (0,1)$ the asymptotics can not be obtained similarly due to the complexity of the tail asymptotic behaviour of $Z_{\infty;p}$ (e.g., [10, 11, 12]).

Our second extension concerns the case of random deflators

$$\Lambda_i = \lambda_i R_i, \quad i \ge 1,$$

with $R_i \in (0,1], i \ge 1$ and $\lambda_i$'s as above. A close inspection of (1.8) indicates that the main contribution in the asymptotics of $\boldsymbol{P}\left\{|\sum_{i=1}^{\infty} \Lambda_i X_i Y_i| > x\right\}$ should be from the quantity $\boldsymbol{P}\left\{|\sum_{i=1}^{m} \Lambda_i X_i Y_i| > x\right\}$. However, the asymptotic behavior of the latter is, in general, not easy to obtain, and thus in the following we discuss a special case that

$$R_1 = \cdots = R_m \text{ and } \lim_{u\to 0} \frac{\boldsymbol{P}\left\{R_1 > 1 - su\right\}}{\boldsymbol{P}\left\{R_1 > 1-u\right\}} = s^\gamma, \ \forall s > 0, \tag{2.2}$$

for some positive constant $\gamma$. The above assumption means that the function $\boldsymbol{P}\left\{R_1 > 1-s\right\}$ is regularly varying at 0, so we have

$$\boldsymbol{P}\left\{R_1 > 1-s\right\} = L(s)s^\gamma, \tag{2.3}$$

with $L(\cdot)$ a positive slowly varying function at 0, i.e., $\lim_{u\to 0} L(us)/L(u) = 1, \forall s > 0$. For more details on the max-domains of attraction and regularly varying functions see [2], [4] or [15].

**Theorem 2.1.** *Let $R_i \in (0,1], i \geq 1$, be independent random variables, which are further independent of $X_i, Y_i, i \geq 1$, such that* (2.2) *holds. Under the conditions of Theorem 1.1, we have*

$$\boldsymbol{P}\left\{\left|\sum_{i=1}^{\infty} \Lambda_i X_i Y_i\right| > \lambda x\right\} \sim \mathcal{K}\boldsymbol{P}\left\{R_1 > 1 - 1/x\right\} x^{m/2-1} \exp(-x), \quad x \to \infty,$$

*with*

$$\mathcal{K} = \frac{2^{1-m/2}\Gamma(\gamma+1)}{\Gamma(m/2)} \prod_{i=m+1}^{\infty} \boldsymbol{E}\left\{\left(1 - \frac{\lambda_i^2}{\lambda^2} R_i^2\right)^{-1/2}\right\} \in (0,\infty).$$

Our final remark concerns the role of the deterministic weights $\lambda_i, i \geq m$. In view of our asymptotic results in the above theorems, the restriction that $\lambda_i, i > m$, are positive is not necessary since only $\lambda_i^2, i > m$, appear. Therefore this condition can be replaced by assuming instead

$$\lambda > |\lambda_i| \tag{2.4}$$

for all $i > m$.

# 3 Further Results and Proofs

We present first some lemmas and then continue with the proofs of Theorem 1.1 and Theorem 2.1.

**Lemma 3.1.** *Let $\lambda_i, X_i, Y_i, i \geq 1$, be as in Theorem 1.1. Then, for $p > 0$*

$$\boldsymbol{P}\left\{|S_{\infty;p}| < \infty\right\} = 1 = \boldsymbol{P}\left\{Z_{\infty;p} < \infty\right\}, \tag{3.1}$$

*with $Z_{\infty;p}$ being independent of $X_1$. Furthermore*

$$S_{\infty;p} \stackrel{d}{=} X_1 Z_{\infty;p}. \tag{3.2}$$

*Proof.* Since by (1.5)

$$\boldsymbol{E}\left\{Z_{\infty;p}^2\right\} \quad = \quad \boldsymbol{E}\left\{Y_1^{2p}\right\} \sum_{i=1}^{\infty} \lambda_i^2 < \infty,$$

it follows that $\boldsymbol{P}\left\{Z_{\infty;p} < \infty\right\} = 1$. Similarly, using again (1.5)

$$\begin{aligned} \boldsymbol{E}\left\{|S_{\infty;p}|\right\} &\leq \liminf_{n\to\infty} \boldsymbol{E}\left\{|S_{n;p}|\right\} = \liminf_{n\to\infty} \boldsymbol{E}\left\{|X_1|\sqrt{\sum_{i=1}^{n} \lambda_i^2 Y_i^{2p}}\right\} \\ &\leq \boldsymbol{E}\left\{|X_1|\right\} \sqrt{\boldsymbol{E}\left\{Y_1^{2p}\right\} \sum_{i=1}^{\infty} \lambda_i^2} < \infty \end{aligned}$$

establishing thus (3.1). In order to show (3.2), it is sufficient to prove that the characteristic functions of both sides coincide. For any $s \in \mathbb{R}$ we have

$$\boldsymbol{E}\left\{\exp(isX_1 Z_{\infty;p})\right\} = \boldsymbol{E}\left\{\boldsymbol{E}\left\{\exp(isX_1 Z_{\infty;p})\right\} | Z_{\infty;p}\right\} = \boldsymbol{E}\left\{\exp\left(-\frac{s^2}{2} Z_{\infty;p}^2\right)\right\}$$

and

$$\begin{aligned} \boldsymbol{E}\left\{\exp(isS_{\infty;p})\right\} &= \boldsymbol{E}\left\{\exp\left(is\sum_{j=1}^{\infty} \lambda_j X_j |Y_j|^p\right)\right\} = \prod_{j=1}^{\infty} \boldsymbol{E}\left\{\exp\left(is\lambda_j X_j |Y_j|^p\right)\right\} \\ &= \prod_{j=1}^{\infty} \boldsymbol{E}\left\{\exp\left(-\frac{s^2}{2}\lambda_j^2 |Y_j|^{2p}\right)\right\} = \boldsymbol{E}\left\{\exp\left(-\frac{s^2}{2} Z_{\infty;p}^2\right)\right\} \end{aligned}$$

implying (3.2), and thus the proof is complete. □

**Lemma 3.2.** *Let $\xi_i, i=1,2,$ be two non-negative independent random variables such that, as $x\to\infty$,*

$$\boldsymbol{P}\{\xi_i > x\} \sim C_i x^{\alpha_i}\exp(-L_i(x)x^{p_i}),\ i=1,2, \tag{3.3}$$

*with some positive constants $C_i, p_i, i=1,2,$ $\alpha_1,\alpha_2\in\mathbb{R}$, and two positive measurable functions $L_i(\cdot), i=1,2.$ If $\lim_{x\to\infty} L_i(x)=L_i>0, i=1,2,$ hold, then*

$$\boldsymbol{P}\{\xi_1\xi_2 > x\} \sim \boldsymbol{P}\{\xi_1^*\xi_2^* > x\},\quad x\to\infty, \tag{3.4}$$

*where $\xi_1^*,\xi_2^*$ are two independent non-negative random variables such that for all $x$ large enough*

$$\boldsymbol{P}\{\xi_i^* > x\} = C_i x^{\alpha_i}\exp(-L_i(x)x^{p_i}),\quad i=1,2.$$

*Proof.* The proof is similar to that of Lemma 3.2 in [5], and therefore omitted here. □

**Lemma 3.3.** *Under the assumptions of Lemma 3.2, if further*

$$L_1(x) = L_1 + o(x^{-p_1}),\quad x\to\infty, \tag{3.5}$$

*and $L_2(x)=L_2\in(0,\infty)$ for all large $x$, then we have*

$$\begin{aligned}\boldsymbol{P}\{\xi_1\xi_2 > x\} \quad\sim\quad & \Big(\frac{2\pi p_2 L_2}{p_1+p_2}\Big)^{1/2} C_1C_2A^{p_2/2+\alpha_2-\alpha_1}x^{\frac{2p_2\alpha_1+2p_1\alpha_2+p_1p_2}{2(p_1+p_2)}}\\ &\times\exp(-(L_1A^{-p_1}+L_2A^{p_2})x^{\frac{p_1p_2}{p_1+p_2}}),\quad x\to\infty,\end{aligned}$$

*where $A=[(p_1L_1)/(p_2L_2)]^{1/(p_1+p_2)}$.*

*Proof.* If $L_1(x)=L_1>0$ for all $x>0$, the claim is established by Lemma 2.1 in [1]. In the light of Lemma 3.2, we can restrict our attention to the simpler case that

$$\boldsymbol{P}\{\xi_1 > x\} = C_1x^{\alpha_1}\exp(-L_1(x)x^{p_1})\ \text{ and }\ \boldsymbol{P}\{\xi_2 > x\} = C_2x^{\alpha_2}\exp(-L_2x^{p_2})$$

hold for $x$ sufficiently large. As in the proof of Proposition 3.1 of [13], for some $0<l_1<1<l_2<\infty$ (set $z_x = Ax^{\frac{p_1}{p_1+p_2}}, A=[(p_1L_1)/(p_2L_2)]^{1/(p_1+p_2)}$), we have as $x\to\infty$

$$\begin{aligned}\boldsymbol{P}\{\xi_1\xi_2 > x\} \quad\sim\quad & C_1C_2p_2L_2x^{\alpha_1}z_x^{p_2+\alpha_2-1-\alpha_1}\int_{l_1Ax^{\frac{p_1}{p_1+p_2}}}^{l_2Ax^{\frac{p_1}{p_1+p_2}}}\exp(-L_1(x/y)x^{p_1}y^{-p_1}-L_2y^{p_2})\,dy\\ \sim\quad & C_1C_2p_2L_2x^{\alpha_1}z_x^{p_2+\alpha_2-\alpha_1}\int_{l_1}^{l_2}\exp(-x^{\frac{p_1p_2}{p_1+p_2}}[L_1(A^{-1}x^{\frac{p_2}{p_1+p_2}}y^{-1})(Ay)^{-p_1}+L_2(Ay)^{p_2}])\,dy.\end{aligned}$$

By (3.5) we can further write

$$\boldsymbol{P}\{\xi_1\xi_2 > x\} \quad\sim\quad C_1C_2p_2L_2x^{\alpha_1}z_x^{p_2+\alpha_2-\alpha_1}\int_{l_1}^{l_2}\exp(-x^{\frac{p_1p_2}{p_1+p_2}}[L_1(Ay)^{-p_1}+L_2(Ay)^{p_2}])\,dy.$$

Since the function $\psi(y)=L_1(Ay)^{-p_1}+L_2(Ay)^{p_2}$ attains its minimum in $[l_1,l_2]$ at 1, applying the Laplace approximation we obtain

$$\begin{aligned}&\int_{l_1}^{l_2}\exp(-x^{\frac{p_1p_2}{p_1+p_2}}[L_1(Ay)^{-p_1}+L_2(Ay)^{p_2}])\mathrm{d}y\\ &\sim\quad \frac{\sqrt{2\pi}}{\sqrt{x^{\frac{p_1p_2}{p_1+p_2}}\psi''(1)}}\exp(-\psi(1)x^{\frac{p_1p_2}{p_1+p_2}}),\quad x\to\infty,\end{aligned}$$

where

$$\psi(1) = L_1[(p_1L_1)/(p_2L_2)]^{-\frac{p_1}{p_1+p_2}} + L_2[(p_1L_1)/(p_2L_2)]^{\frac{p_2}{p_1+p_2}},$$

and

$$\psi'(1) = 0, \quad \psi''(1) = L_2A^{p_2}p_2(p_1+p_2) > 0,$$

hence the claim follows. □

**Lemma 3.4.** *Let $X_i, Y_i, i \le n$, be independent standard Gaussian random variables. For given weights $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \ge \cdots \ge 0$, as $x \to \infty$, we have*

$$\boldsymbol{P}\left\{|S_n| > \lambda x\right\} \sim \prod_{i=m+1}^{n}(1-\lambda_i^2/\lambda^2)^{-1/2}\frac{2^{1-m/2}}{\Gamma(m/2)}x^{m/2-1}\exp(-x), \quad n \ge m. \tag{3.6}$$

*Proof.* By the representation of $S_n$ in (1.1) we need to derive the tail asymptotic of $\boldsymbol{P}\left\{X_1^2(Z_n^2/\lambda^2) > x^2\right\}$ as $x \to \infty$. The tail asymptotic of $Z_n^2$ implies that of $X_1^2$ by taking $\lambda_2 = \cdots = \lambda_n = 0$ and $\lambda_1 = 1$. Hence (1.3) and Lemma 3.3 establish the claim. □

The following result, which is restatement of Lemma 2.1 in [14], is crucial for the proof of Theorem 1.1.

**Lemma 3.5.** *Let $G$ be a distribution function having an exponential tail with rate $\theta \ge 0$, i.e.,*

$$\lim_{x\to\infty}\frac{1-G(x+y)}{1-G(x)} = \exp(-\theta y), \quad \forall y \in \mathbb{R},$$

*and let $H$ be another distribution function satisfying $1 - H(x) = o(1-G(x))$. If further, $M_H(\beta) := \int_{-\infty}^{\infty} e^{\beta x}dH(x) < \infty$ holds for some $\beta > \theta$, then we have*

$$1 - G * H(x) \;\sim\; M_H(\theta)(1-G(x)), \quad x \to \infty,$$

*where $G * H$ denotes the convolution of distribution functions $G$ and $H$.*

Proof of Theorem *1.1* First proof: The result follows by (1.3), Lemma 3.1 and Lemma 3.3.
We present below an alternative proof. Write $S_{m,\infty} = \sum_{i=m+1}^{\infty}\lambda_iX_iY_i$, hence $S_\infty = S_m + S_{m,\infty}$. If $\lambda_{m+1} = 0$, then the proof follows by (3.6). We consider therefore below only the case $\lambda_{m+1} > 0$. By the symmetry about 0 of $S_\infty$ and $S_m$ for any $x > 0$ we have

$$\boldsymbol{P}\left\{|S_\infty| > x\right\} = 2\boldsymbol{P}\left\{S_\infty > x\right\} \quad \text{and} \quad \boldsymbol{P}\left\{|S_m| > x\right\} = 2\boldsymbol{P}\left\{S_m > x\right\}.$$

In view of (3.6), $S_m = \lambda\sum_{i=1}^{m}X_iY_i \stackrel{d}{=} \lambda\sqrt{\sum_{i=1}^{m}X_i^2}Y_1$ is in the Gumbel max-domain of attraction with constant auxiliary function $a(x) = \lambda$, i.e.,

$$\frac{\boldsymbol{P}\left\{S_m > x+y\lambda\right\}}{\boldsymbol{P}\left\{S_m > x\right\}} \sim \frac{\exp(-(x+y\lambda)/\lambda)}{\exp(-x/\lambda)} = \exp(-y), \quad \forall y \in \mathbb{R}$$

as $x \to \infty$. Since $\lambda_{m+1} \in (0,\lambda)$ (with the multiplicity denoted by $m_1$) we have

$$\begin{aligned}\boldsymbol{P}\left\{S_{m,\infty} > x\right\} &= \frac{1}{2}\boldsymbol{P}\left\{|S_{m,\infty}| > x\right\} \stackrel{(1.6)}{\le} \frac{1}{2}K\boldsymbol{P}\left\{\lambda_{m+1}\left|\sum_{i=m+1}^{m+m_1}X_iY_i\right| > x\right\}\\ &\stackrel{(1.3-1.4)}{=} o\left(\boldsymbol{P}\left\{\lambda\left|\sum_{i=1}^{m}X_iY_i\right| > x\right\}\right) = o(\boldsymbol{P}\left\{S_m > x\right\}).\end{aligned}$$

Furthermore, in view of Lemma 3.1, for any $s \in (1, \lambda/\lambda_{m+1})$,

$$\begin{aligned}
\boldsymbol{E}\left\{\exp(sS_{m,\infty}/\lambda)\right\} &= \boldsymbol{E}\left\{\exp\left(X_1\sqrt{\sum_{i=m+1}^{\infty}\left(\frac{s\lambda_i}{\lambda}\right)^2 Y_i^2}\right)\right\} = \boldsymbol{E}\left\{\exp\left(\frac{1}{2}\sum_{i=m+1}^{\infty}\left(\frac{s\lambda_i}{\lambda}\right)^2 Y_i^2\right)\right\} \\
&= \prod_{i=m+1}^{\infty}\boldsymbol{E}\left\{\exp\left(\frac{1}{2}\left(\frac{s\lambda_i}{\lambda}\right)^2 Y_i^2\right)\right\} = \prod_{i=m+1}^{\infty}(1-s^2\lambda_i^2/\lambda^2)^{-1/2} < \infty.
\end{aligned}$$

Consequently, applying Lemma 3.5 we obtain

$$\begin{aligned}
\boldsymbol{P}\left\{S_m + S_{m,\infty} > x\lambda\right\} &\sim \boldsymbol{E}\left\{\exp(S_{m,\infty}/\lambda)\right\}\boldsymbol{P}\left\{S_m/\lambda > x\right\} \\
&= \boldsymbol{E}\left\{\exp\Big(\sum_{i=m+1}^{\infty}\frac{\lambda_i X_i Y_i}{\lambda}\Big)\right\}\boldsymbol{P}\left\{S_m/\lambda > x\right\} \\
&= \left[\prod_{i=m+1}^{\infty}(1-\lambda_i^2/\lambda^2)^{-1/2}\right]\boldsymbol{P}\left\{S_m/\lambda > x\right\},
\end{aligned}$$

hence the first claim follows from (3.6). Furthermore, Eq. (1.8) follows obviously from (1.7) and (3.6). Finally, in view of (1.3) and (1.7), we establish (1.9), and thus the proof is complete. □

In the next lemma we present a result of Theorem 3.1 in [6] which will be used in the proof of Theorem 2.1.

**Lemma 3.6.** *Let $\xi \in [0,1]$ be a random variable with distribution function $G$ such that*

$$\lim_{u\to\infty}\frac{1-G(1-x/u)}{1-G(1-1/u)} = x^{\alpha}, \quad \forall x > 0$$

*for some $\alpha \geq 0$. Assume that $\eta$ is a positive random variable with distribution function $F$ in the Gumbel max-domain of attraction with some positive auxiliary function $\omega(\cdot)$, i.e.,*

$$\lim_{u\to\infty}\frac{1-F(u+x/\omega(u))}{1-F(u)} = \exp(-x), \quad \forall x \in \mathbb{R}.$$

*If both $\xi$ and $\eta$ are independent, then*

$$\boldsymbol{P}\left\{\xi\eta > x\right\} \sim \Gamma(\alpha+1)\left(1-G\left(1-\frac{1}{u\omega(u)}\right)\right)(1-F(x)), \quad x\to\infty.$$

Proof of Theorem 2.1 We use the same idea as the proof of Theorem 1.1. Set next $V_n := \sum_{i=1}^n X_i Y_i$ for $n \geq 1$. We only need to consider the case that $\lambda_{m+1} > 0$ (with the multiplicity denoted by $m_2 \geq 1$). With the aid of (1.4), (2.2) and Lemma 3.6, we obtain that, as $x \to \infty$,

$$\begin{aligned}
\frac{\boldsymbol{P}\left\{R_1 V_m > x-y\right\}}{\boldsymbol{P}\left\{R_1 V_m > x\right\}} &= \frac{\boldsymbol{P}\left\{R_1\left|V_m\right| > x-y\right\}}{\boldsymbol{P}\left\{R_1\left|V_m\right| > x\right\}} \\
&\sim \frac{\boldsymbol{P}\left\{R_1 > 1-\frac{1}{x-y}\right\}\boldsymbol{P}\left\{\left|V_m\right| > x-y\right\}}{\boldsymbol{P}\left\{R_1 > 1-\frac{1}{x}\right\}\boldsymbol{P}\left\{\left|V_m\right| > x\right\}} \sim \exp(y), \quad \forall y \in \mathbb{R}.
\end{aligned}$$

Furthermore, utilising (1.6) we have

$$\boldsymbol{P}\left\{\sum_{i=m+1}^{\infty}\Lambda_i X_i Y_i > \lambda x\right\} \leq \boldsymbol{P}\left\{\sum_{i=m+1}^{\infty}\frac{\lambda_i}{\lambda}X_i Y_i > x\right\} \leq K\boldsymbol{P}\left\{\frac{\lambda_{m+1}}{\lambda}\sum_{i=m+1}^{m+m_2}X_i Y_i > x\right\}$$

and from (1.4), (2.2) and Lemma 3.6

$$\boldsymbol{P}\{R_1V_m > x\} \sim \frac{\Gamma(\gamma+1)}{2}\boldsymbol{P}\left\{R_1 > 1-\frac{1}{x}\right\}\boldsymbol{P}\{|V_m| > x\}, \quad x \to \infty,$$

which, in the light of(1.4) and (2.3), implies

$$\boldsymbol{P}\left\{\sum_{i=m+1}^{\infty}\Lambda_i X_i Y_i > \lambda x\right\} = o\left(\boldsymbol{P}\{R_1V_m > x\}\right), \quad x \to \infty.$$

Consequently, the claim follows by using Lemma 3.5 and Lemma 3.6. □

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