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GAUSSIAN APPROXIMATIONS OF MULTIPLE INTEGRALS

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Abstract
Fix $k \geq 1$, and let $I(l)$, $l \geq 1$, be a sequence of k-dimensional vectors of multiple Wiener-Itô integrals with respect to a general Gaussian process. We establish necessary and sufficient conditions to have that, as $l \to +\infty$, the law of $I(l)$ is asymptotically close (for example, in the sense of Prokhorov's distance) to the law of a k-dimensional Gaussian vector having the same covariance matrix as $I(l)$. The main feature of our results is that they require minimal assumptions (basically, boundedness of variances) on the asymptotic behaviour of the variances and covariances of the elements of $I(l)$. In particular, we will not assume that the covariance matrix of $I(l)$ is convergent. This generalizes the results proved in Nualart and Peccati (2005), Peccati and Tudor (2005) and Nualart and Ortiz-Latorre (2007). As shown in Marinucci and Peccati (2007b), the criteria established in this paper are crucial in the study of the high-frequency behaviour of stationary fields defined on homogeneous spaces.

1 Introduction

Let $\mathbf{U}(l) = (U_1(l), ..., U_k(l))$, $l \geq 1$, be a sequence of centered random observations (not necessarily independent) with values in $\mathbb{R}^k$. Suppose that the application $l \mapsto \mathbb{E}U_i(l)^2$ is bounded for every i, and also that the sequence of covariances $c_l(i,j) = \mathbb{E}U_i(l)U_j(l)$ *does not* converge as $l \to +\infty$ (that is, for some fixed $i \neq j$, the limit $\lim_{l\to\infty} c_l(i,j)$ does not exist). Then, a natural question is the following: *is it possible to establish criteria ensuring that, for large l, the law of $\mathbf{U}(l)$ is close (in the sense of some distance between probability measures) to the law of a Gaussian vector $\mathbf{N}(l) = (N_1(l), ..., N_k(l))$ such that $\mathbb{E}N_i(l)N_j(l) = \mathbb{E}U_i(l)U_j(l) = c_l(i,j)$?* Note that the question is not trivial, since the asymptotic irregularity of the covariance matrix $c_l(\cdot,\cdot)$ may in general prevent $\mathbf{U}(l)$ from converging in law toward a k-dimensional Gaussian distribution.
In this paper, we shall provide an exhaustive answer to the problem above in the special case

where the sequence $\mathbf{U}(l)$ has the form

$$\mathbf{U}(l) = \mathbf{I}(l) = \left(I_{d_1}\left(f_l^{(1)}\right), ..., I_{d_k}\left(f_l^{(k)}\right)\right), \quad l \geq 1, \tag{1}$$

where the integers $d_1, ..., d_k \geq 1$ do not depend on l, I_{d_j} indicates a multiple stochastic integral of order d_j (with respect to some isonormal Gaussian process X over a Hilbert space $\mathfrak{H}$ – see Section 2 below for definitions), and each $f_l^{(j)} \in \mathfrak{H}^{\odot d_j}$, $j = 1, ..., k$, is a symmetric kernel. In particular, we shall prove that, whenever the elements of the vectors $\mathbf{I}(l)$ have bounded variances (and without any further requirements on the covariance matrix of $\mathbf{I}(l)$), the following three conditions are equivalent as $l \to +\infty$:

(i) $\gamma(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) \to 0$, where $\mathcal{L}(\cdot)$ indicates the law of a given random vector, $\mathbf{N}(l)$ is a Gaussian vector having the same covariance matrix as $\mathbf{I}(l)$, and γ is some appropriate metric on the space of probability measures on $\mathbb{R}^k$;

(ii) for every $j = 1, ..., k$, $\mathbb{E}\left(I_{d_j}\left(f_l^{(j)}\right)^4\right) - 3\mathbb{E}\left(I_{d_j}\left(f_l^{(j)}\right)^2\right)^2 \to 0$;

(iii) for every $j = 1, ..., k$ and every $p = 1, ..., d_j - 1$, the sequence of contractions (to be formally defined in Section 2) $f_l^{(j)} \otimes_p f_l^{(j)}$, $l \geq 1$, is such that

$$f_l^{(j)} \otimes_p f_l^{(j)} \to 0 \quad \text{in} \quad \mathfrak{H}^{\otimes 2(d_j - p)}. \tag{2}$$

Some other conditions, involving for instance Malliavin operators, are derived in the subsequent sections. As discussed in Section 5, our results are motivated by the derivation of high-frequency Gaussian approximations of stationary fields defined on homogeneous spaces – a problem tackled in [9] and [10].
Note that the results of this paper are a generalization of the following theorem, which combines results proved in [13], [14] and [15].

Theorem 0. *Suppose that the vector $\mathbf{I}(l)$ in (1) is such that, as $l \to +\infty$,*

$$\mathbb{E} I_{d_i}\left(f_l^{(i)}\right) I_{d_j}\left(f_l^{(j)}\right) \to \mathbf{C}(i,j), \quad 1 \leq i, j \leq k,$$

where $\mathbf{C} = \{\mathbf{C}(i,j)\}$ is some positive definite matrix. Then, the following four conditions are equivalent, as $l \to +\infty$:

1. $\mathbf{I}(l) \overset{Law}{\to} \mathbf{N}(0, \mathbf{C})$, *where $\mathbf{N}(0, \mathbf{C})$ is a k-dimensional centered Gaussian vector with covariance matrix $\mathbf{C}$;*
2. *relation (2) takes place for every $j = 1, ..., k$ and every $p = 1, ..., d_j - 1$;*
3. *for every $j = 1, ...k$, $\mathbb{E}\left(I_{d_j}\left(f_l^{(j)}\right)^4\right) \to 3\mathbf{C}(j,j)^2$;*
4. *for every $j = 1, ...k$, $\left\|D\left[I_{d_j}\left(f_l^{(j)}\right)\right]\right\|_{\mathfrak{H}}^2 \to d_j$ in L^2, where $D\left[I_{d_j}\left(f_l^{(j)}\right)\right]$ denotes the Malliavin derivative of $I_{d_j}\left(f_l^{(j)}\right)$ (see the next section).*

The equivalence of Points 1.-3. in the case $k = 1$ has been first proved in [14] by means of the Dambis-Dubins-Schwarz (DDS) Theorem (see [16, Ch. V]), whereas the proof in the case $k \geq 2$ has been achieved (by similar techniques) in [15]; the fact that Point 4. is also necessary and sufficient for the CLT at Point 1. has been recently proved in [13], by means of a Malliavin calculus approach. For some applications of Theorem 0 (in quite different frameworks), see e.g. [2], [3], [5], [9] or [11].
The techniques we use to achieve our main results are once again the DDS Theorem, combined with Burkholder-Davis-Gundy inequalities and some results (taken from [4, Section 11.7]) concerning 'uniformities' over classes of probability measures.
The paper is organized as follows. In Section 2 we discuss some preliminary notions concerning Gaussian fields, multiple integrals and metrics on probabilities. Section 3 contains the statements of the main results of the paper. The proof of Theorem 1 (one of the crucial results of this note) is achieved in Section 4. Section 5 is devoted to applications.

2 Preliminaries

We present a brief review of the main notions and results that are needed in the subsequent sections. The reader is referred to [6] or [12, Ch. 1] for any unexplained definition.

Hilbert spaces. In what follows, the symbol $\mathfrak{H}$ indicates a real separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and norm $\|\cdot\|_{\mathfrak{H}}$. For every $d \geq 2$, we denote by $\mathfrak{H}^{\otimes 2}$ and $\mathfrak{H}^{\odot 2}$, respectively, the nth *tensor product* of $\mathfrak{H}$ and the nth *symmetric tensor product* of $\mathfrak{H}$. We also write $\mathfrak{H}^{\otimes 1} = \mathfrak{H}^{\odot 1} = \mathfrak{H}$.

Isonormal Gaussian processes. We write $X = \{X(h) : h \in \mathfrak{H}\}$ to indicate an *isonormal Gaussian process* over $\mathfrak{H}$. This means that X is a collection of real-valued, centered and (jointly) Gaussian random variables indexed by the elements of $\mathfrak{H}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that, for every $h, h' \in \mathfrak{H}$,

$$\mathbb{E}\left[X(h)\, X(h')\right] = \langle h, h' \rangle_{\mathfrak{H}} .$$

We denote by $L^2(X)$ the (Hilbert) space of the real-valued and square-integrable functionals of X.

Isometry, chaoses and multiple integrals. For every $d \geq 1$ we will denote by I_d the isometry between $\mathfrak{H}^{\odot d}$ equipped with the norm $\sqrt{d!}\, \|\cdot\|_{\mathfrak{H}^{\otimes d}}$ and the dth Wiener chaos of X. In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, $(A, \mathcal{A})$ is a measurable space, and μ is a σ-finite and non-atomic measure, then $\mathfrak{H}^{\odot d} = L^2_s\left(A^d, \mathcal{A}^{\otimes d}, \mu^{\otimes d}\right)$ is the space of symmetric and square integrable functions on A^d and for every $f \in \mathfrak{H}^{\odot d}$, $I_d(f)$ is the *multiple Wiener-Itô integral* (of order d) of f with respect to X, as defined e.g. in [12, Ch. 1]. It is well-known that a random variable of the type $I_d(f)$, where $d \geq 2$ and $f \neq 0$, *cannot be Gaussian*. Moreover, every $F \in L^2(X)$ admits a unique *Wiener chaotic decomposition* of the type $F = \mathbb{E}(F) + \sum_{d=1}^{\infty} I_d(f_d)$, where $f_d \in \mathfrak{H}^{\odot d}$, $d \geq 1$, and the convergence of the series is in $L^2(X)$.

Malliavin derivatives. We will use Malliavin derivatives in Section 3, where we generalize some of the results proved in [13]. The class $\mathcal{S}$ of *smooth* random variables is defined as the collection of all functionals of the type

$$F = f\left(X(h_1), ..., X(h_m)\right), \tag{3}$$

where $h_1, ..., h_m \in \mathfrak{H}$ and f is bounded and has bounded derivatives of all order. The operator D, called the *Malliavin derivative operator*, is defined on $\mathcal{S}$ by the relation

$$DF = \sum_{i=1}^{M} \frac{\partial}{\partial x_i} f(h_1, ..., h_m) h_i,$$

where F has the form (3). Note that DF is an element of $L^2(\Omega; \mathfrak{H})$. As usual, we define the domain of D, noted $\mathbb{D}^{1,2}$, to be the closure of $\mathcal{S}$ with respect to the norm $\|F\|_{1,2} \triangleq \mathbb{E}(F^2) + \mathbb{E}\|DF\|_{\mathfrak{H}}^2$. When $F \in \mathbb{D}^{1,2}$, we may sometimes write $DF = D[F]$, depending on the notational convenience. Note that any finite sum of multiple Wiener-Itô integrals is an element of $\mathbb{D}^{1,2}$.

Contractions. Let $\{e_k : k \geq 1\}$ be a complete orthonormal system of $\mathfrak{H}$. For any fixed $f \in \mathfrak{H}^{\odot n}$, $g \in \mathfrak{H}^{\odot m}$ and $p \in \{0, ..., n \wedge m\}$, we define the pth *contraction* of f and g to be the element of $\mathfrak{H}^{\otimes n+m-2p}$ given by

$$f \otimes_p g = \sum_{i_1,...,i_p=1}^{\infty} \left\langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathfrak{H}^{\otimes p}} \otimes \left\langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathfrak{H}^{\otimes p}}.$$

We stress that $f \otimes_p g$ need not be an element of $\mathfrak{H}^{\odot n+m-2p}$. We denote by $f \widetilde{\otimes}_p g$ the symmetrization of $f \otimes_p g$. Note that $f \otimes_0 g$ is just the tensor product $f \otimes g$ of f and g. If $n = m$, then $f \otimes_n g = \langle f, g \rangle_{\mathfrak{H}^{\otimes n}}$.

Metrics on probabilities. For $k \geq 1$ we define $\mathbf{P}(\mathbb{R}^k)$ to be the class of all probability measures on $\mathbb{R}^k$. Given a metric $\gamma(\cdot, \cdot)$ on $\mathbf{P}(\mathbb{R}^k)$, we say that γ *metrizes the weak convergence on* $\mathbf{P}(\mathbb{R}^k)$ whenever the following double implication holds for every $Q \in \mathbf{P}(\mathbb{R}^k)$ and every $\{Q_l : l \geq 1\} \subset \mathbf{P}(\mathbb{R}^k)$ (as $l \to +\infty$): $\gamma(Q_l, Q) \to 0$ if, and only if, Q_l converges weakly to Q. Some examples of metrizing γ are the *Prokhorov metric* (usually noted ρ) or the *Fortet-Mounier metric* (usually noted β). Recall that

$$\rho(P, Q) = \inf\{\epsilon > 0 : P(A) \leq Q(A^\epsilon) + \epsilon, \quad \text{for every Borel set } A \subset \mathbb{R}^k\} \tag{4}$$

where $A^\epsilon = \{x : \|x - y\| < \varepsilon \ \text{ for some } y \in A\}$, and $\|\cdot\|$ is the Euclidean norm. Also,

$$\beta(P, Q) = \sup\left\{\left|\int f d(P - Q)\right| : \|f\|_{BL} \leq 1\right\}, \tag{5}$$

where $\|\cdot\|_{BL} = \|\cdot\|_L + \|\cdot\|_\infty$, and $\|\cdot\|_L$ is the usual Lipschitz seminorm (see [4, p. 394] for further details). The fact that we focus on the Prokhorov and the Fortet-Mounier metric is due to the following fact, proved in [4, Th. 11.7.1]. *For any two sequences* $\{P_l\}, \{Q_l\} \subset \mathbf{P}(\mathbb{R}^k)$, *the following three conditions* (**A**)–(**C**) *are equivalent:* **(A)** $\lim_{l \to +\infty} \beta(P_l, Q_l) = 0$; **(B)** $\lim_{l \to +\infty} \rho(P_l, Q_l) = 0$; **(C)** *on some auxiliary probability space* $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, *there exist sequences of random vectors* $\{\mathbf{N}^*(l) : l \geq 1\}$ *and* $\{\mathbf{I}^*(l) : l \geq 1\}$ *such that*

$$\mathcal{L}(\mathbf{I}^*(l)) = P_l \text{ and } \mathcal{L}(\mathbf{N}^*(l)) = Q_l \ \text{ for every } l, \text{ and } \ \|\mathbf{I}^*(l) - \mathbf{N}^*(l)\| \to 0, \quad \text{a.s.-}\mathbb{P}^*, \tag{6}$$

where $\mathcal{L}(\cdot)$ *indicates the law of a given random vector, and* $\|\cdot\|$ *is the Euclidean norm.*

3 Main results

Fix integers $k \geq 1$ and $d_1, ..., d_k \geq 1$, and consider a sequence of k-dimensional random vectors of the type

$$\mathbf{I}(l) = \left(I_{d_1}\left(f_l^{(1)}\right), ..., I_{d_k}\left(f_l^{(k)}\right)\right) , \quad l \geq 1 , \tag{7}$$

where, for each $l \geq 1$ and every $j = 1, ..., k$, $f_l^{(j)}$ is an element of $\mathfrak{H}^{\odot d_j}$. We will suppose the following:

- There exists $\eta > 0$ such that $\left\| f_l^{(j)} \right\|_{\mathfrak{H}^{\odot d_j}} \geq \eta$, for every $j = 1, ..., k$ and every $l \geq 1$.
- For every $j = 1, ..., k$, the sequence

$$\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^2\right] = d_j! \left\| f_l^{(j)} \right\|^2_{\mathfrak{H}^{\otimes d_j}} , \quad l \geq 1, \tag{8}$$

  is bounded.

Note that the integers $d_1, ..., d_k$ do not depend on l. For every $l \geq 1$, we denote by $\mathbf{N}(l) = (N_l^{(1)}, ..., N_l^{(k)})$ a centered k-dimensional Gaussian vector with the same covariance matrix as $\mathbf{I}(l)$, that is,

$$\mathbb{E}\left[N_l^{(i)} N_l^{(j)}\right] = \mathbb{E}\left[I_{d_i}\left(f_l^{(i)}\right) I_{d_j}\left(f_l^{(j)}\right)\right], \tag{9}$$

for every $1 \leq i, j \leq k$. For every $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$, we also use the compact notation: $\langle \lambda, \mathbf{I}(l) \rangle_k = \sum_{j=1}^k \lambda_j I_{d_j}(f_l^{(j)})$ and $\langle \lambda, \mathbf{N}(l) \rangle_k = \sum_{j=1}^k \lambda_j N_l^{(j)}$. The next result is one of the main contributions of this paper. Its proof is deferred to Section 4.

Theorem 1. *Let the above notation and assumptions prevail, and suppose that, for every $j = 1, ..., k$, the following asymptotic condition holds: for every $p = 1, ..., d_j - 1$,*

$$\left\| f_l^{(j)} \otimes_p f_l^{(j)} \right\|_{\mathfrak{H}^{\otimes 2(d_j - p)}} \to 0, \quad \textit{as } l \to +\infty. \tag{10}$$

Then, as $l \to +\infty$ and for every compact set $M \subset \mathbb{R}^k$,

$$\sup_{\lambda \in M} \left| \mathbb{E}\left[\exp\left(\mathrm{i} \langle \lambda, \mathbf{I}(l) \rangle_k\right)\right] - \mathbb{E}\left[\exp\left(\mathrm{i} \langle \lambda, \mathbf{N}(l) \rangle_k\right)\right] \right| \to 0. \tag{11}$$

We now state two crucial consequences of Theorem 1. The first one (Proposition 2) provides a formal meaning to the intuitive fact that, since (11) holds and since the variances of $\mathbf{I}(l)$ do not explode, the laws of $\mathbf{I}(l)$ and $\mathbf{N}(l)$ are "asymptotically close". The second one (Theorem 3) combines Theorem 1 and Proposition 2 to obtain an exhaustive generalization "without covariance conditions" of Theorem 0 (see the Introduction). Note that in the statement of Theorem 3 also appear Malliavin operators, so that our results are a genuine extension of the main findings by Nualart and Ortiz-Latorre in [13]. We stress that multiple stochastic integrals of the type $I_d(f)$, $d \geq 1$ and $f \in \mathfrak{H}^{\odot d}$, are always such that $I_d(f) \in \mathbb{D}^{1,2}$.

Proposition 2. *Let the assumptions of Theorem 1 prevail (in particular, (10) holds), and denote by $\mathcal{L}(\mathbf{I}(l))$ and $\mathcal{L}(\mathbf{N}(l))$, respectively, the law of $\mathbf{I}(l)$ and $\mathbf{N}(l)$, $l \geq 1$. Then, the two collections $\{\mathcal{L}(\mathbf{N}(l)) : l \geq 1\}$ and $\{\mathcal{L}(\mathbf{I}(l)) : l \geq 1\}$ are tight. Moreover, if $\gamma(\cdot, \cdot)$ metrizes the weak convergence on $\mathbf{P}\left(\mathbb{R}^k\right)$, then*

$$\lim_{l \to +\infty} \gamma\left(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))\right) = 0. \tag{12}$$

Proof. The fact that $\{\mathcal{L}(\mathbf{N}(l)) : l \geq 1\}$ and $\{\mathcal{L}(\mathbf{I}(l)) : l \geq 1\}$ are tight is a consequence of the boundedness of the sequence (8) and of the relation $\mathbb{E}[I_{d_j}(f_l^{(j)})^2] = \mathbb{E}[(N_l^{(j)})^2]$. The rest of the proof is standard, and is provided for the sake of completeness. We shall prove (12) by contradiction. Suppose there exist $\varepsilon > 0$ and a subsequence $\{l_n\}$ such that $\gamma(\mathcal{L}(\mathbf{I}(l_n)), \mathcal{L}(\mathbf{N}(l_n))) > \varepsilon$ for every n. Tightness implies that $\{l_n\}$ must contain a subsequence $\{l_{n'}\}$ such that $\mathcal{L}(\mathbf{I}(l_{n'}))$ and $\mathcal{L}(\mathbf{N}(l_{n'}))$ are both weakly convergent. Since (11) holds, we deduce that $\mathcal{L}(\mathbf{I}(l_{n'}))$ and $\mathcal{L}(\mathbf{N}(l_{n'}))$ must necessarily converge to the same weak limit, say Q. The fact that γ metrizes the weak convergence implies finally that

$$\gamma(\mathcal{L}(\mathbf{I}(l_{n'})), \mathcal{L}(\mathbf{N}(l_{n'}))) \leq \gamma(\mathcal{L}(\mathbf{I}(l_{n'})), Q) + \gamma(\mathcal{L}(\mathbf{N}(l_{n'})), Q) \underset{n' \to +\infty}{\to} 0, \tag{13}$$

thus contradicting the former assumptions on $\{l_n\}$ (note that the inequality in (13) is just the triangle inequality). This shows that (12) must necessarily take place. □

Remarks. (i) A result analogous to the arguments used in the proof of Proposition 2 is stated in [4, Exercise 3, p. 419]. Note also that, without tightness, a condition such as (11) *does not allow* to deduce the asymptotic relation (12). See for instance [4, Proposition 11.7.6] for a counterexample involving the Prokhorov metric on $\mathbf{P}(\mathbb{R})$.
(ii) Since (12) holds in particular when γ is equal to the Prokhorov metric or the Fortet-Mounier metric (as defined in (4) and (5)), Proposition 2 implies that, on some auxiliary probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, there exist sequences of random vectors $\{\mathbf{N}^*(l) : l \geq 1\}$ and $\{\mathbf{I}^*(l) : l \geq 1\}$ such that

$$\mathbf{I}^*(l) \overset{law}{=} \mathbf{I}(l) \text{ and } \mathbf{N}^*(l) \overset{law}{=} \mathbf{N}(l) \text{ for every } l, \text{ and } \|\mathbf{I}^*(l) - \mathbf{N}^*(l)\| \to 0, \text{ a.s.-}\mathbb{P}^*, \tag{14}$$

where $\|\cdot\|$ stands for the Euclidean norm (see (6), as well as [4, Theorem 11.7.1]).

Theorem 3. *Suppose that the sequence $\mathbf{I}(l)$, $l \geq 1$, verifies the assumptions of this section (in particular, for every $j = 1, ..., k$, the sequence of variances appearing in (8) is bounded). Then, the following conditions are equivalent.*

1. *As $l \to +\infty$, relation (10) is satisfied for every $j = 1, ..., k$ and every $p = 1, ..., d_j - 1$;*
2. $$\lim_{l \to +\infty} \rho(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) = \lim_{l \to +\infty} \beta(\mathcal{L}(\mathbf{I}(l)), \mathcal{L}(\mathbf{N}(l))) = 0 \tag{15}$$
   where ρ and β are, respectively, the Prokhorov metric and the Fortet-Mounier metric, as defined in (4) and (5);
3. *as $l \to +\infty$, for every $j = 1, ..., k$,*
   $$\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^4\right] - 3\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^2\right]^2 = \mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^4\right] - 3(d_j!)^2 \left\|f_l^{(j)}\right\|^4_{\mathfrak{H}^{\otimes d_j}} \to 0;$$
4. *for every $j = 1, ..., k$,*
   $$\lim_{l \to +\infty} \rho\left(\mathcal{L}\left(I_{d_j}\left(f_l^{(j)}\right)\right), \mathcal{L}\left(N_l^{(j)}\right)\right) = \lim_{l \to +\infty} \beta\left(\mathcal{L}\left(I_{d_j}\left(f_l^{(j)}\right)\right), \mathcal{L}\left(N_l^{(j)}\right)\right) = 0, \tag{16}$$
   where ρ and β are the Prokhorov and Fortet-Mounier metric on $\mathbb{R}$;

5. *for every* $j=1,...,k$,

$$\left\| D\left[I_{d_j}\left(f_l^{(j)}\right) \right] \right\|_{\mathfrak{H}}^2 - d_j\left(d_j!\right) \left\| f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^2 \to 0, \quad \textit{in } L^2\left(X\right), \tag{17}$$

as $l\to+\infty$*, where* D *is the Malliavin derivative operator defined in Section 2.*

Proof. The implication 1. $\Longrightarrow$ 2., is a consequence of Theorem 1 and Proposition 2. Now suppose (15) is in order. Then, according to [4, Theorem 11.7.1], on a probability space $(\Omega^*,\mathcal{F}^*,\mathbb{P}^*)$, there exist sequences of random vectors $\mathbf{N}^*\left(l\right)=(N_l^{*,(1)},...,N_l^{*,(j)})$, $l\geq 1$, and $\mathbf{I}^*\left(l\right)=(I_l^{*,(1)},...,I_l^{*,(k)})$, $l\geq 1$, such that (14) takes place. Now

$$3\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^2\right]^2 = 3\mathbb{E}\left[\left(N_l^{(j)}\right)^2\right]^2 = \mathbb{E}\left[\left(N_l^{(j)}\right)^4\right] = \mathbb{E}^*\left[\left(N_l^{*,(j)}\right)^4\right],$$

for every $j=1,...,k$, so that

$$\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^4\right] - 3\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^2\right]^2 = \mathbb{E}^*\left[\left(I_l^{*,(j)}\right)^4 - \left(N_l^{*,(j)}\right)^4\right] \underset{l\to+\infty}{\to} 0. \tag{18}$$

The convergence to zero in (18) is a consequence of the boundedness of the sequence (8), implying that the family $A_l^*=(I_l^{*,(j)})^4-(N_l^{*,(j)})^4$, $l\geq 1$, is uniformly integrable. To see why $\{A_l^*\}$ is uniformly integrable, one can use the fact that, since each $I_l^{*,(j)}$ has the same law as an element of the d_jth chaos of X and each $N_l^{*,(j)}$ is Gaussian, then (see e.g. [6, Ch. VI]) for every $p\geq 2$ there exists a universal positive constant $C_{p,j}$ (independent of l) such that

$$\begin{aligned}
\mathbb{E}\left[|A_l^*|^p\right]^{1/p} &= \mathbb{E}^*\left[\left| \left(I_l^{*,(j)}\right)^4 - \left(N_l^{*,(j)}\right)^4\right|^p\right]^{1/p} \\
&\leq \mathbb{E}^*\left[\left(I_l^{*,(j)}\right)^{4p}\right]^{4/4p} + \mathbb{E}\left[\left(N_l^{*,(j)}\right)^{4p}\right]^{4/4p} \\
&\leq C_{p,j}\mathbb{E}^*\left[\left(I_l^{*,(j)}\right)^2\right]^2 + C_{p,j}\mathbb{E}^*\left[\left(N_l^{*,(j)}\right)^2\right]^2 \\
&= 2C_{p,j}\times\left(d_j!\right)^2 \left\| f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^4 \leq 2C_{p,j}M_j,
\end{aligned}$$

where $M_j=\sup_l\left(d_j!\right)^2\|f_l^{(j)}\|_{\mathfrak{H}^{\otimes d_j}}^4<+\infty$, due to (8). This proves that 2. $\Longrightarrow$ 3.. The implication 3. $\Longrightarrow$ 1. can be deduced from the formula (proved in [14, p. 183])

$$\begin{aligned}
&\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^4\right] - 3\mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^2\right]^2 = \mathbb{E}\left[I_{d_j}\left(f_l^{(j)}\right)^4\right] - 3\left(d_j!\right)^2 \left\| f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^4 \\
&= \sum_{p=1}^{d_j-1} \frac{\left(d_j!\right)^4}{\left(p!\left(d_j-p\right)!\right)^2} \left\{ \left\| f_l^{(j)}\otimes_p f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes 2\left(d_j-p\right)}}^2 + \binom{2\left(d_j-p\right)}{d_j-p} \left\| f_l^{(j)}\widetilde{\otimes}_p f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes 2\left(d_j-p\right)}}^2 \right\},
\end{aligned}$$

The equivalence 1. $\Longleftrightarrow$ 4. is an immediate consequence of the previous discussion.
To conclude the proof, we shall now show the double implication 1. $\Longleftrightarrow$ 5.. To do this, we first observe that, by performing the same calculations as in [13, Proof of Lemma 2] (which

are based on an application of the multiplication formulae for multiple integrals, see [12, Proposition 1.1.3]), one obtains that

$$\begin{aligned}\left\|D\left[I_{d_j}\left(f_l^{(j)}\right)\right]\right\|_{\mathfrak{H}}^2 &= d_j\,(d_j!)\left\|f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^2 \\ &\quad + d_j^2\sum_{p=1}^{d_j-1}(p-1)!\binom{n-1}{p-1}^2 I_{2(d_j-p)}\left(f_l^{(j)}\widetilde{\otimes}_p f_l^{(j)}\right).\end{aligned}$$

Since $\left\|f_l^{(j)}\otimes_p f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \geq \left\|f_l^{(j)}\widetilde{\otimes}_p f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2$, the last relation implies immediately that 1. $\Rightarrow$ 5.. To prove the opposite implication, first observe that, due to the boundedness of (8) and the Cauchy-Schwarz inequality, there exists a finite constant M (independent of j and l) such that

$$\left\|f_l^{(j)}\otimes_p f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \leq \left\|f_l^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^4 \leq M.$$

This implies that, for every sequence $\{l_n\}$, there exists a subsequence $\{l_{n'}\}$ such that the sequences $\left\|f_{l_{n'}}^{(j)}\otimes_p f_{l_{n'}}^{(j)}\right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2$ and $d_j!\left\|f_{l_{n'}}^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^2$ are convergent for every $j=1,...,k$ and every $p=1,...,d_j-1$ (recall that, by assumption, there exists a constant $\eta>0$, such that $\left\|f_{l_{n'}}^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}} \geq \eta$, for every j and l). Now we apply Theorem 4 in [13], which implies that, if (17) takes place and $d_j!\left\|f_{l_{n'}}^{(j)}\right\|_{\mathfrak{H}^{\otimes d_j}}^2 \to c>0$, then necessarily $\left\|f_{l_{n'}}^{(j)}\otimes_p f_{l_{n'}}^{(j)}\right\|_{\mathfrak{H}^{\otimes 2(d_j-p)}}^2 \to 0$, thus proving our claim. This shows that 5. $\Rightarrow$ 1.. □

The next result says that, under the additional assumption that the variances of the elements of $\mathbf{I}(l)$ converge to one, the asymptotic approximation (15) is equivalent to the fact that each component of $\mathbf{I}(l)$ verifies a CLT. The proof is elementary, and therefore omitted.

Corollary 4. *Fix $k\geq 2$, and suppose that the sequence $\mathbf{I}(l)$, $l\geq 1$, is such that, for every $j=1,...,k$, the sequence of variances appearing in (8) converges to 1, as $l\to+\infty$. Then, each one of Conditions 1.-5. in the statement of Theorem 3 is equivalent to the following: for every $j=1,...,k$,*

$$I_{d_j}\left(f_l^{(j)}\right) \underset{l\to+\infty}{\overset{Law}{\rightarrow}} N(0,1), \tag{19}$$

where $N(0,1)$ is a centered Gaussian random variable with unitary variance.

Remark. The results of this section can be suitably extended to deal with the Gaussian approximations of random vectors of the type $(F_l^{(1)}(X),...,F_l^{(k)}(X))$, where $F_l^{(j)}(X)$, $j=1,...,k$, is a general square integrable functional of the isonormal process X, not necessarily having the form of a multiple integral. See [10, Th. 6] for a statement containing an extension of this type.

4 Proof of Theorem 1

We provide the proof in the case where

$$\mathfrak{H}=L^2([0,1],\mathcal{B}([0,1]),dx)=L^2([0,1]), \tag{20}$$

where dx stands for Lebesgue measure. The extension to a general $\mathfrak{H}$ is obtained by using the same arguments outlined in [14, Section 2.2]. If $\mathfrak{H}$ is as in (20), then for every $d \geq 2$ one has that $\mathfrak{H}^{\odot d} = L_s^2([0,1]^d)$, where the symbol $L_s^2([0,1]^d)$ indicates the class of symmetric, real-valued and square-integrable functions (with respect to the Lebesgue measure) on $[0,1]^d$. Also, the isonormal process X coincides with the Gaussian space generated by the standard Brownian motion

$$t \mapsto W_t \triangleq X\left(1_{[0,t]}\right), \quad t \in [0,1].$$

This implies in particular that, for every $d \geq 2$, the Wiener-Itô integral $I_d(f)$, $f \in L_s^2\left([0,1]^d\right)$, can be rewritten in terms of an iterated stochastic integral with respect to W, that is:

$$I_d(f) = d! \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{d-1}} f(t_1, ..., t_d)\, dW_{t_d} \cdots dW_{t_2} dW_{t_1}. \tag{21}$$

We also have that $I_1(f) = \int_0^1 f(s)\, dW_s$ for every $f \in L_s^2([0,1]^1) \equiv L^2([0,1])$. Note that the RHS of (21) is just an iterated adapted stochastic integral of the Itô type. Finally, for every $f \in L_s^2([0,1]^d)$, every $g \in L_s^2([0,1]^{d'})$ and every $p = 0, ..., d \wedge d'$, we observe that the contraction $f \otimes_p g$ is the (not necessarily symmetric) element of $L^2([0,1]^{d+d'-2p})$ given by:

$$\begin{aligned} f \otimes_p g\,(y_1, ..., y_{d+d'-2p}) &= \int_{[0,1]^p} f(y_1, ..., y_{d-p}, a_1, ..., a_p) \times \\ &\qquad \times g(y_{d-p+1}, ..., y_{d+d'-2p}, a_1, ..., a_p)\, da_1 ... da_p. \end{aligned} \tag{22}$$

In the framework of (20), the proof of Theorem 1 relies on some computations contained in [15], as well as on an appropriate use of the *Burkholder-Davis-Gundy inequalities* (see for instance [16, Ch. IV §4]). Fix $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$, and consider the random variable

$$\begin{aligned} \langle \lambda, \mathbf{I}(l) \rangle_k &= \sum_{j=1}^k \lambda_j d_j! \int_0^1 \cdots \int_0^{u_{d_j - 1}} f_l^{(j)}\left(u_1, ..., u_{d_j}\right) dW_{u_{d\,j}} \cdots dW_{u_1} \\ &\triangleq \sum_{j=1}^k \lambda_j d_j! J_{d_j}^1\left(f_l^{(j)}\right) = \int_0^1 \left(\sum_{j=1}^k \lambda_j d_j! J_{d_j - 1}^u \left(f_l^{(j)}(u, \cdot) \right) \right) dW_u \\ &= \int_0^1 \left(\sum_{j=1}^k \lambda_j d_j I_{d_j - 1}\left(f_l^{(j)}(u, \cdot)\, \mathbf{1}_{[0,u]^{d_j - 1}} \right) \right) dW_u, \end{aligned}$$

where, for every $d \geq 1$, every $t \in [0,1]$ and every $f \in L_s^2\left([0,1]^d\right)$, we define $J_d^t(f) = I_d\left(f \mathbf{1}_{[0,t]^d}\right)/d!$ (for every $c \in \mathbb{R}$, we also use the conventional notation $J_0^t(c) = c$). We start by recalling some preliminary results involving Brownian martingales. Start by setting, for every $u \in [0,1]$, $\phi_{\lambda,l}(u) = \sum_{j=1}^k \lambda_j d_j I_{d_j-1}\left(f_l^{(j)}(u, \cdot)\, \mathbf{1}_{[0,u]^{d_j-1}}\right)$, and observe that the random application

$$t \mapsto \sum_{j=1}^k \lambda_j d_j! J_{d_j}^t \left(f_l^{(j)}\right) = \int_0^t \phi_{\lambda,l}(u)\, dW_u, \quad t \in [0,1],$$

defines a (continuous) square-integrable martingale started from zero, with respect to the canonical filtration of W, noted $\left\{\mathcal{F}_t^W : t \in [0,1]\right\}$. The quadratic variation of this martingale is classically given by $t \mapsto \int_0^t \phi_{\lambda,l}(u)^2\, du$, and a standard application of the Dambis, Dubins and Schwarz Theorem (see [16, Ch. V §1]) yields that, for every $l \geq 1$, there exists a standard Brownian motion (initialized at zero) $W^{(\lambda,l)} = \left\{W_t^{(\lambda,l)} : t \geq 0\right\}$ such that

$$\langle \lambda, \mathbf{I}(l) \rangle_k = \int_0^1 \phi_{\lambda,l}(u)\, dW_u = W^{(\lambda,l)}_{\int_0^1 \phi_{\lambda,l}(u)^2 du}.$$

Note that, in general, the definition of $W^{(\lambda,l)}$ strongly depends on λ and l, and that $W^{(\lambda,l)}$ *is not* a $\mathcal{F}_t^W$-Brownian motion. However, the following relation links the two Brownian motions $W^{(\lambda,l)}$ and W: there exists a (continuous) filtration $\left\{\mathcal{G}_t^{(\lambda,l)} : t \geq 0\right\}$ such that (i) $W_t^{(\lambda,l)}$ is a $\mathcal{G}_t^{(\lambda,l)}$-Brownian motion, and (ii) for every fixed $s \in [0,1]$ the positive random variable $\int_0^s \phi_{\lambda,l}(u)^2\, du$ is a $\mathcal{G}_t^{(\lambda,l)}$-stopping time. Now define the positive constant (which is trivially a $\mathcal{G}_t^{(\lambda,l)}$-stopping time)

$$q(\lambda,l) = \int_0^1 \mathbb{E}(\phi_{\lambda,l}(u)^2) du,$$

and observe that the usual properties of complex exponentials and a standard application of the Burkholder-Davis-Gundy inequality (in the version stated in [16, Corollary 4.2, Ch. IV]) yield the following estimates:

$$\begin{aligned}
\left|\mathbb{E}\left[\exp\left(\mathrm{i}\langle \lambda, \mathbf{I}(l)\rangle_k\right)\right] - \mathbb{E}\left[\exp\left(\mathrm{i}W^{(\lambda,l)}_{q(\lambda,l)}\right)\right]\right| &= \left|\mathbb{E}\left[\exp\left(\mathrm{i}W^{(\lambda,l)}_{\int_0^1 \phi_{\lambda,l}(u)^2 du}\right)\right] - \mathbb{E}\left[\exp\left(\mathrm{i}W^{(\lambda,l)}_{q(\lambda,l)}\right)\right]\right| \\
&\leq \mathbb{E}\left[\left|W^{(\lambda,l)}_{\int_0^1 \phi_{\lambda,l}(u)^2 du} - W^{(\lambda,l)}_{q(\lambda,l)}\right|\right] \\
&\leq \mathbb{E}\left[\left|W^{(\lambda,l)}_{\int_0^1 \phi_{\lambda,l}(u)^2 du} - W^{(\lambda,l)}_{q(\lambda,l)}\right|^4\right]^{\frac{1}{4}} \\
&\leq C\mathbb{E}\left[\left|\int_0^1 \phi_{\lambda,l}(u)^2\, du - q(\lambda,l)\right|^2\right]^{\frac{1}{4}}, \qquad (23)
\end{aligned}$$

where C is some universal constant independent of λ and l. To see how to obtain the inequality (23), introduce first the shorthand notation $T(\lambda,l) \triangleq \int_0^1 \phi_{\lambda,l}(u)^2\, du$ (recall that $T(\lambda,l)$ is a $\mathcal{G}_t^{(\lambda,l)}$-stopping time), and then write

$$\left|W^{(\lambda,l)}_{\int_0^1 \phi_{\lambda,l}(u)^2 du} - W^{(\lambda,l)}_{q(\lambda,l)}\right| = \left|\int_{T(\lambda,l)\wedge q(\lambda,l)}^{T(\lambda,l)\vee q(\lambda,l)} dW_u^{(\lambda,l)}\right| = \left|\int_0^{T(\lambda,l)\vee q(\lambda,l)} H(u)\, dW_u^{(\lambda,l)}\right|,$$

where $H(u)$ is the $\mathcal{G}_u^{(\lambda,l)}$-predictable process given by $H(u) = \mathbf{1}\{u \geq T(\lambda,l) \wedge q(\lambda,l)\}$, so that

$$\begin{aligned}
\left|\int_0^{T(\lambda,l)\vee q(\lambda,l)} H(u)^2\, du\right| &= |T(\lambda,l) \vee q(\lambda,l) - T(\lambda,l) \wedge q(\lambda,l)| \\
&= |T(\lambda,l) - q(\lambda,l)| = \left|\int_0^1 \phi_{\lambda,l}(u)^2\, du - q(\lambda,l)\right|.
\end{aligned}$$

In particular, relation (23) yields that the proof of Theorem 1 is concluded, once the following two facts are proved: (A) for every $\lambda \in \mathbb{R}^k$ and every $l \geq 1$, the random variables $W^{(\lambda,l)}_{q(\lambda,l)}$ and $\langle \lambda, \mathbf{N}(l) \rangle_k$ have the same law; (B) the sequence

$$\mathbb{E}\left[\left| \int_0^1 \phi_{\lambda,l}(u)^2 \, du - q(\lambda,l) \right|^2 \right], \quad l \geq 1,$$

converges to zero, uniformly in λ, on every compact set of the type $M = [-T,T]^k$, where $T \in (0,+\infty)$. The proof of (A) is immediate: indeed, $W^{(\lambda,l)}$ is a standard Brownian motion and, by using the isometric properties of stochastic integrals and the fact that the covariance structures of $\mathbf{N}(l)$ and $\mathbf{I}(l)$ coincide,

$$q(\lambda,l) = \int_0^1 \mathbb{E}(\phi_{\lambda,l}(u)^2) du = \mathbb{E}\left[\left(\int_0^1 \phi_{\lambda,l}(u) \, dW_u \right)^2 \right] = \mathbb{E}\left[\langle \lambda, \mathbf{I}(l) \rangle_k^2 \right] = \mathbb{E}\left[\langle \lambda, \mathbf{N}(l) \rangle_k^2 \right].$$

To prove (B), use a standard version of the multiplication formula between multiple stochastic integrals (see for instance [12, Proposition 1.5.1])

$$\begin{aligned}
\int_0^1 \phi_{\lambda,l}(u)^2 \, du &= \int_0^1 \left(\sum_{j=1}^k \lambda_j d_j I_{d_j-1}\left(f_l^{(j)}(u,\cdot) \mathbf{1}_{[0,u]^{d_j-1}} \right) \right)^2 du \\
&= \int_0^1 \sum_{j,i=1}^k \lambda_j \lambda_i d_j d_i I_{d_i-1}\left(f_l^{(i)}(u,\cdot) \mathbf{1}_{[0,u]^{d_i-1}} \right) I_{d_j-1}\left(f_l^{(j)}(u,\cdot) \mathbf{1}_{[0,u]^{d_j-1}} \right) du \\
&= q(\lambda,l) + \sum_{j,i=1}^k \lambda_j \lambda_i d_j d_i \int_0^1 \sum_{p=0}^{D(i,j)} \binom{d_i-1}{p} \binom{d_j-1}{p} \\
&\qquad \times I_{d_i+d_j-2-2p}\left(\left(f_l^{(i)}(u,\cdot) \mathbf{1}_{[0,u]^{d_i-1}} \right) \otimes_p \left(f_l^{(j)}(u,\cdot) \mathbf{1}_{[0,u]^{d_j-1}} \right) \right),
\end{aligned} \tag{24}$$

where the index $D(i,j)$ is defined as

$$D(i,j) = \begin{cases} d_i - 2 & \text{if } d_i = d_j \\ \min(d_i,d_j) - 1 & \text{if } d_i \neq d_j. \end{cases}$$

Formula (24) implies that, for every $\lambda \in [-T,T]^k$ $(T > 0)$,

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_0^1 \phi_{\lambda,l}(u)^2 \, du - q(\lambda,l) \right|^2 \right]^{\frac{1}{2}} \\
\leq\ & (T \max_i d_i)^2 \sum_{i,j=1}^k \sum_{p=0}^{D(i,j)} \binom{d_i-1}{p} \binom{d_j-1}{p} \\
&\times \mathbb{E}\left[\left(\int_0^1 I_{d_i+d_j-2-2p}\left(\left(f_l^{(i)}(u,\cdot) \mathbf{1}_{[0,u]^{d_i-1}} \right) \otimes_p \left(f_l^{(j)}(u,\cdot) \mathbf{1}_{[0,u]^{d_j-1}} \right) \right) du \right)^2 \right]^{\frac{1}{2}}
\end{aligned} \tag{25}$$

(note that the RHS of (25) does not depend on λ). Finally, a direct application of the calculations contained in [15, p. 253-255] yields that, for every $i,j = 1,...,k$ and every

$p = 0, ..., D(i,j)$,

$$\mathbb{E}\left[\left(\int_0^1 I_{d_i+d_j-2-2p}\left(\left(f_l^{(i)}(u,\cdot)\mathbf{1}_{[0,u]^{d_i-1}}\right)\otimes_p\left(f_l^{(j)}(u,\cdot)\mathbf{1}_{[0,u]^{d_j-1}}\right)\right)du\right)^2\right]^{\frac{1}{2}} \to 0, \qquad (26)$$

as $l \to +\infty$. This concludes the proof of Theorem 1. ■

Remark. By inspection of the calculations contained in [15, p. 253-255], it is easily seen that, to deduce (26) from (10), it is necessary that the sequence of variances (8) is bounded.

5 Concluding remarks on applications

Theorem 1 and Theorem 3 are used in [10] to deduce high-frequency asymptotic results for subordinated spherical random fields. This study is strongly motivated by the probabilistic modelling and statistical analysis of the Cosmic Microwave Background radiation (see [7], [8], [9] and [10] for a detailed discussion of these applications). In what follows, we provide a brief presentation of some of the results obtained in [10].
Let $\mathbb{S}^2 = \left\{x \in \mathbb{R}^3 : \|x\| = 1\right\}$ be the unit sphere, and let $T = \{T(x) : x \in \mathbb{S}^2\}$ be a real-valued (centered) Gaussian field which is also *isotropic*, in the sense that $T(x) \overset{Law}{=} T(\mathcal{R}x)$ (in the sense of stochastic processes) for every rotation $\mathcal{R} \in SO(3)$. The following facts are well known:

(1) the trajectories of T admit the harmonic expansion $T(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(x)$, where $\{Y_{lm} : l \geq 0,\ \ m = -l, ..., l\}$ is the class of *spherical harmonics* (defined e.g. in [17, Ch. 5]);

(2) the complex-valued array of harmonic coefficients $\{a_{lm} : l \geq 0,\ l \geq 0,\ m = -l, ..., l\}$ is composed of centered Gaussian random variables such that the variances $\mathbb{E}\left|a_{lm}\right|^2 \triangleq C_l$ depend exclusively on l (see for instance [1]);

(3) the law of T is completely determined by the *power spectrum* $\{C_l : l \geq 0\}$ defined at the previous point.

Now fix $q \geq 2$, and consider the subordinated field

$$T^{(q)}(x) \triangleq H_q(T(x)), \quad x \in \mathbb{S}^2,$$

where H_q is the qth Hermite polynomial. Plainly, the field $T^{(q)}$ is isotropic and admits the harmonic expansion

$$T^{(q)}(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm;q}Y_{lm}(x) \triangleq \sum_{l=0}^{\infty} T_l^{(q)}(x),$$

where $a_{lm;q} \triangleq \int_{\mathbb{S}^2} T^{(q)}(z)\overline{Y_{lm}(z)}dz$. For every $l \geq 0$, the field $T_l^{(q)} = \sum_{m=-l}^{l} a_{lm;q}Y_{lm}$ is real-valued and isotropic, and it is called the lth *frequency component* of $T^{(q)}$ (see [7] or [10] for a physical interpretation of frequency components). In [10], the following problem is studied.

Problem A. *Fix $q \geq 2$. Find conditions on the power spectrum $\{C_l : l \geq 0\}$ to have that the finite dimensional distributions (f.d.d.'s) of the normalized frequency field*

$$\overline{T}_l^{(q)}(x) \triangleq \frac{T_l^{(q)}(x)}{\mathbf{Var}\left(T_l^{(q)}(x)\right)^{1/2}}, \quad x \in \mathbb{S}^2,$$

are 'asymptotically close to Gaussian', as $l \to +\infty$.

The main difficulty when dealing with Problem A is that (due to isotropy) one has always that

$$\mathbb{E}\left[\overline{T}_l^{(q)}(x)\,\overline{T}_l^{(q)}(y)\right] = P_l\left(\cos\langle x,y\rangle\right), \tag{27}$$

where P_l is the lth Legendre polynomial, and $\langle x,y\rangle$ is the angle between x and y. Indeed, since in general the quantity $P_l(\cos\langle x,y\rangle)$ does not converge (as $l \to +\infty$), one cannot prove that the f.d.d.'s of $\overline{T}_l^{(q)}$ converge to those of a Gaussian field (even if $\overline{T}_l^{(q)}(x)$ converges in law to a Gaussian random variable for every fixed x). However, as an application of Theorem 1 and Proposition 2, one can prove the following approximation result.

Proposition 5. *Under the above notation and assumptions, suppose that, for any fixed $x \in \mathbb{S}^2$,*

$$\overline{T}_l^{(q)}(x) \underset{l\to+\infty}{\overset{Law}{\to}} N(0,1). \tag{28}$$

Then, for any $k \geq 1$, any $x_1,...,x_k \in \mathbb{S}^2$ and any γ metrizing the weak convergence on $\mathbf{P}\left(\mathbb{R}^k\right)$,

$$\gamma\left(\mathcal{L}\left(\overline{T}_l^{(q)}(x_1),...,\overline{T}_l^{(q)}(x_k)\right), \mathbf{N}(l)\right) \underset{l\to+\infty}{\to} 0, \tag{29}$$

where, for every l, $\mathbf{N}(l) = \left(N_l^{(1)},...,N_l^{(k)}\right)$ is a centered real-valued Gaussian vector such that

$$\mathbb{E}\left\{N_l^{(i)}N_l^{(j)}\right\} = P_l\left(\cos\langle x_i,x_j\rangle\right).$$

Proof. Since $\overline{T}_l^{(q)}(x)$ is a linear functional involving uniquely Hermite polynomials of order q (written on the Gaussian field T) one deduces that there exists a real Hilbert space $\mathfrak{H}$ such that (in the sense of stochastic processes)

$$\overline{T}_l^{(q)}(x) \overset{Law}{=} I_q\left(f_{(q,l,x)}\right),$$

where the class of symmetric kernels

$$\left\{f_{(q,l,x)} : l \geq 0,\ x \in \mathbb{S}^2\right\}$$

is a subset of $\mathfrak{H}^{\odot q}$, and $I_q\left(f_{(q,l,x)}\right)$ stands for the qth Wiener-Itô integral of $f_{(q,l,x)}$ with respect to an isonormal Gaussian process over $\mathfrak{H}$, as defined in Section 2. Since the variances of the components of the vector $(\overline{T}_l^{(q)}(x_1),...,\overline{T}_l^{(q)}(x_k))$ are all equal to 1 by construction, we can apply Theorem 3 and Proposition 2. Indeed, by Theorem 3 we know that (28) implies that, for every $p = 1,...,q-1$ and every $j = 1,...,k$,

$$f_{(q,l,x_j)} \otimes_p f_{(q,l,x_j)} \to 0 \text{ in } \mathfrak{H}^{\odot 2(q-p)}.$$

Finally, Proposition 2 and (27) imply immediately the desired conclusion. □

The derivation of sufficient conditions to have (28) is the main object of [10]. In particular, it is proved that sufficient (and sometimes also necessary) conditions for (28) can be neatly expressed in terms of the so-called *Clebsch-Gordan coefficients* (see again [17]), that are elements of unitary matrices connecting reducible representations of $SO(3)$.

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