

STANDARD STOCHASTIC COALESCENCE WITH SUM KERNELS

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Abstract

We build a Markovian system of particles entirely characterized by their masses, in which each pair of particles with masses x and y coalesce at rate $K(x, y) \simeq x^\lambda + y^\lambda$, for some $\lambda \in (0, 1)$, and such that the system is initially composed of infinitesimally small particles.

1 Introduction

A stochastic coalescent is a Markovian system of macroscopic particles entirely characterized by their masses, in which each pair of particles with masses x and y merge into a single particle with mass $x + y$ at some given rate $K(x, y)$. This rate K is called the coagulation kernel. We refer to the review of Aldous [3] on stochastic coalescence, on its links with the Smoluchowski coagulation equation.

When the initial state consists of a finite number of particles, the stochastic coalescent obviously exists without any assumption on K , and is known as the Marcus-Lushnikov process [9, 8]. When there are initially infinitely many particles, stochastic coalescence with constant, additive, and multiplicative kernels have been extensively studied, see Kingman [7], Aldous-Pitman [2], Aldous [1]. The case of general coagulation kernels has first been studied by Evans-Pitman [4], and their results have recently been completed in [5, 6].

Of particular importance seems to be the *standard version* of the stochastic coalescent, that is the stochastic coalescent which starts from *dust*. By *dust* we mean a state with positive total mass in which all the particles have an infinitesimally small mass. Indeed, such a version of stochastic coalescence should describe a sort of typical behaviour, since it starts with a non really specified initial datum. There are also some possible links between such a standard stochastic coalescent and the Smoluchowski coagulation equation for large times, see Aldous [3, Open Problem 14].

The well-known Kingman coalescent [7] is a stochastic coalescent with constant kernel $K(x, y) = 1$ starting from dust at time $t = 0$. The standard additive (resp. multiplicative) coalescent, see [2] (resp. [1]) is a stochastic coalescent with kernel $K(x, y) = x + y$ (resp. $K(x, y) = xy$) starting from dust at time $t = -\infty$. It seems that no result is available for general kernels. Our aim in this paper is to show the existence of a standard coalescent for kernels $K(x, y) \simeq x^\lambda + y^\lambda$, for $\lambda \in (0, 1)$, by using a refinement of the methods introduced in [5].

We have no uniqueness result, but the process we build is however a Markov process. The method we use is very restrictive: we are not able to study, for example, the case $K(x, y) = (xy)^{\lambda/2}$, for $\lambda \in (0, 1)$.

In Section 2, we introduce our notations, recall the main result of [5], and state our result. The proofs are handled in Section 3.

2 Main result

We denote by \mathcal{S}^\downarrow the set of non-increasing sequences $m = (m_k)_{k \geq 1}$ with values in $[0, \infty)$. A state $m \in \mathcal{S}^\downarrow$ represents the ordered masses in a particle system. For $\alpha > 0$ and $m \in \mathcal{S}^\downarrow$, we denote $\|m\|_\alpha := \sum_{k=1}^\infty m_k^\alpha$. Remark that the total mass of a state $m \in \mathcal{S}^\downarrow$ is simply given by $\|m\|_1$.

We will use, for $\lambda \in (0, 1]$, the set of states with total mass 1 and with a finite moment of order λ :

$$\ell_\lambda = \{m = (m_k)_{k \geq 1} \in \mathcal{S}^\downarrow, \|m\|_1 = 1, \|m\|_\lambda < \infty\}. \quad (2.1)$$

We also consider the sets of finite particle systems with total mass 1:

$$\ell_0 = \{m = (m_k)_{k \geq 1} \in \mathcal{S}^\downarrow, \|m\|_1 = 1, \inf\{k \geq 1, m_k = 0\} < \infty\}. \quad (2.2)$$

Remark that for $0 < \lambda_1 < \lambda_2$, the inclusions $\ell_0 \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$ hold.

For $i < j$, the coalescence between the i -th and j -th particles is described by the map $c_{ij} : \mathcal{S}^\downarrow \mapsto \mathcal{S}^\downarrow$, with

$$c_{ij}(m) = \text{reorder}(m_1, \dots, m_{i-1}, m_i + m_j, m_{i+1}, \dots, m_{j-1}, m_{j+1}, \dots). \quad (2.3)$$

A coagulation kernel is a function K on $[0, \infty)^2$ such that $0 \leq K(x, y) = K(y, x)$.

Remark 2.1 Consider a coagulation kernel K . For any $m \in \ell_0$, there obviously exists a unique (in law) strong Markov ℓ_0 -valued process $(M(m, t))_{t \geq 0}$ with infinitesimal generator \mathcal{L} defined, for all $\Phi : \ell_0 \mapsto \mathbb{R}$, all $\mu \in \ell_0$, by

$$\mathcal{L}\Phi(\mu) = \sum_{1 \leq i < j < \infty} K(\mu_i, \mu_j) [\Phi(c_{ij}(\mu)) - \Phi(\mu)]. \quad (2.4)$$

The process $(M(m, t))_{t \geq 0}$ is known as the Marcus-Lushnikov process.

Notice that (2.4) is well-defined for all functions Φ since the sum is actually finite. Indeed, $c_{ij}(\mu) = \mu$ as soon as $\mu_j = 0$. We refer to Aldous [3] for many details on this process.

To state our main result, we finally need to introduce some notations: for $\lambda \in (0, 1)$, and for $m, \tilde{m} \in \ell_\lambda$, we consider the distance

$$d_\lambda(m, \tilde{m}) = \sum_{k \geq 1} |m_k^\lambda - \tilde{m}_k^\lambda|. \tag{2.5}$$

Remark that for m^n, m in ℓ_λ ,

$$\lim_n d_\lambda(m^n, m) = 0 \iff \lim_n \sum_{i \geq 1} |m_i^n - m_i|^\lambda = 0. \tag{2.6}$$

The main result of [5, Corollary 2.5] (see also [6, Theorem 2.2]) is the following.

Theorem 2.2 *Let K be a coagulation kernel satisfying, for some $\lambda \in (0, 1]$ and some $a \in (0, \infty)$, for all $x, y, z \in [0, 1]$,*

$$|K(x, y) - K(x, z)| \leq a|y^\lambda - z^\lambda|. \tag{2.7}$$

Endow ℓ_λ with the distance d_λ .

(i) *For any $m \in \ell_\lambda$, there exists a unique (in law) strong Markov process $(M(m, t))_{t \geq 0} \in \mathbb{D}([0, \infty), \ell_\lambda)$ enjoying the following property. For any sequence of initial states $m^n \in \ell_0$ such that $\lim_{n \rightarrow \infty} d_\lambda(m^n, m) = 0$, the sequence of Marcus-Lushnikov processes $(M(m^n, t))_{t \geq 0}$ converges in law, in $\mathbb{D}([0, \infty), \ell_\lambda)$, to $(M(m, t))_{t \geq 0}$.*

(ii) *The obtained process is Feller in the sense that for all $t \geq 0$, the application $m \mapsto \text{Law}(M(m, t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$.*

Notation 2.3 *Under the assumptions of Theorem 2.2, we will denote by $(P_t^K)_{t \geq 0}$ the Markov semi-group of $(M(m, t))_{t \geq 0, m \in \ell_\lambda}$: for $t \geq 0$, for $\Phi : \ell_\lambda \mapsto \mathbb{R}$ measurable and bounded, and for $m \in \ell_\lambda$, $P_t^K \Phi(m) := E[\Phi(M(m, t))]$.*

The result we will prove in the present paper is the following.

Theorem 2.4 *Let K be a coagulation kernel satisfying for some $\lambda \in (0, 1)$, some $a \in (0, \infty)$ and some $\varepsilon > 0$, for all $x, y, z \in [0, \infty)$,*

$$|K(x, y) - K(x, z)| \leq a|y^\lambda - z^\lambda| \text{ and } K(x, y) \geq \varepsilon(x^\lambda + y^\lambda). \tag{2.8}$$

There exists a Markov process $(M^(t))_{t \in (0, \infty)}$ with semi-group $(P_s^K)_{s \geq 0}$, belonging a.s. to $\mathbb{D}((0, \infty), \ell_\lambda)$, such that a.s., $\lim_{t \rightarrow 0+} M_1^*(t) = 0$, where $M^*(t) = (M_1^*(t), M_2^*(t), \dots) \in \ell_\lambda$.*

This result is not obvious, because clearly, $M^*(t)$ goes out of ℓ_λ as t decreases to $0+$. Indeed, we have $\sum_i M_i^*(t) = 1$ for all $t > 0$, and $\lim_{t \rightarrow 0+} \sup_i M_i^*(t) = 0$, so that necessarily, since $\lambda \in (0, 1)$ $\limsup_{t \rightarrow 0+} \|M^*(t)\|_\lambda = \infty$.

The main ideas of the proof are the following: first, there is a *regularization* of the moment of order λ . This means that in some sense, even if the moment of order λ is infinite at time 0, it becomes finite for all positive times.

Next, we prove a refined version of the Feller property obtained in [5], which shows that the map $m \mapsto \text{Law}(M(m, t))$ is actually continuous for the distance d_1 on the level sets $\{m \in \ell_\lambda, \|m\|_\lambda \leq A\}$. We conclude using that these level sets are compact in ℓ_1 endowed with d_1 .

3 Proof

Let us first recall the following easy compactness result.

Lemma 3.1 *For any $A > 0$, any $\lambda \in (0, 1)$, the set*

$$\ell_\lambda(A) = \{m \in \ell_\lambda, \|m\|_\lambda \leq A\} \tag{3.1}$$

is compact in (ℓ_1, d_1) : for any sequence $(m^n)_{n \geq 1}$ of elements of $\ell_\lambda(A)$, we may find $m \in \ell_\lambda(A)$ and a subsequence $(m^{n_k})_{k \geq 1}$ such that $\lim_k d_1(m^{n_k}, m) = 0$.

We now check that under a suitable lowerbound assumption on the coagulation kernel, there is a *regularization* property for the moment of order λ of the stochastic coalescent.

Lemma 3.2 *Let $\lambda \in (0, 1)$ be fixed, and consider a coagulation kernel K satisfying, for some $\varepsilon \in (0, \infty)$, for all $x, y \in [0, 1]$,*

$$K(x, y) \geq \varepsilon(x^\lambda + y^\lambda). \tag{3.2}$$

For each $m \in \ell_0$, consider the Marcus-Lushnikov process $(M(m, t))_{t \geq 0}$. There exists a constant C , depending only on λ and ε , such that for all $t > 0$,

$$\sup_{m \in \ell_0} \left[\sup_{s \geq t} \|M(m, s)\|_\lambda \right] \leq C \left(1 \vee \frac{1}{t} \right) \tag{3.3}$$

Proof First of all notice that since $\lambda \in (0, 1)$, we have for all $1 \leq i < j$, for all $m \in \ell_\lambda$, $\|c_{ij}(m)\|_\lambda = \|m\|_\lambda + (m_i + m_j)^\lambda - m_i^\lambda - m_j^\lambda \leq \|m\|_\lambda$. Hence the moment of order λ of $M(m, t)$ decreases a.s. at each coalescence. Thus for any $m \in \ell_0$, the map $t \mapsto \|M(m, t)\|_\lambda$ is a.s. non-increasing. It thus suffices to check that for some constant $C > 0$, for all $t > 0$, $\sup_{m \in \ell_0} E[\Phi(M(m, t))] \leq C(1 \vee \frac{1}{t})$, where $\Phi: \ell_\lambda \mapsto \mathbb{R}_+$ is defined by $\Phi(m) = \|m\|_\lambda$. An easy computation shows that for $0 \leq y \leq x$,

$$\begin{aligned} x^\lambda + y^\lambda - (x + y)^\lambda &= x[x^{\lambda-1} - (x + y)^{\lambda-1}] + y[y^{\lambda-1} - (x + y)^{\lambda-1}] \\ &\geq y[y^{\lambda-1} - (y + y)^{\lambda-1}] \geq (1 - 2^{\lambda-1})y^\lambda. \end{aligned} \tag{3.4}$$

Using furthermore (2.4) and (3.2), we get, for any $m \in \ell_0$, since for $i < j$, $0 \leq m_j \leq m_i$,

$$\begin{aligned} \mathcal{L}\Phi(m) &= - \sum_{i < j} K(m_i, m_j)[m_i^\lambda + m_j^\lambda - (m_i + m_j)^\lambda] \\ &\leq -\varepsilon(1 - 2^{\lambda-1}) \sum_{i < j} (m_i^\lambda + m_j^\lambda)m_j^\lambda. \end{aligned} \tag{3.5}$$

Setting $c = \varepsilon(1 - 2^{\lambda-1})/2$, we obtain, using that $m_i \leq 1$ for all i ,

$$\begin{aligned} \mathcal{L}\Phi(m) &\leq -2c \sum_{i < j} m_i^\lambda m_j^\lambda = c \sum_{i \neq j} m_i^\lambda m_j^\lambda = -c \left\{ \left(\sum_{i \geq 1} m_i^\lambda \right)^2 - \sum_{i \geq 1} m_i^{2\lambda} \right\} \\ &\leq -c(\Phi^2(m) - \Phi(m)). \end{aligned} \tag{3.6}$$

Using the Cauchy-Schwarz inequality, we deduce that for all $t \geq 0$, all $m \in \ell_0$,

$$\begin{aligned} \frac{d}{dt} E[\Phi(M(m, t))] &= E[\mathcal{L}\Phi(M(m, t))] \\ &\leq -c \left(E[\Phi(M(m, t))]^2 - E[\Phi(M(m, t))] \right). \end{aligned} \tag{3.7}$$

Using finally that $E[\Phi(M(m, 0))] = \|m\|_\lambda \in [1, \infty)$ since $m \in \ell_0$, we easily deduce from this differential inequality that for all $t > 0$, $E[\Phi(M(m, t))] \leq (1 - e^{-ct})^{-1}$. For some constant $C > 0$ depending only on c , we obtain the bound $E[\Phi(M(m, t))] \leq C(1 \vee \frac{1}{t})$. \square

We next prove a sort of refined version of the Feller property obtained in [5].

Lemma 3.3 *Let $\lambda \in (0, 1)$ be fixed, and consider a coagulation kernel K satisfying, for some $a \in (0, \infty)$, for all $x, y, z \in [0, 1]$,*

$$|K(x, y) - K(x, z)| \leq a|y^\lambda - z^\lambda|. \tag{3.8}$$

It is possible to build simultaneously all the processes $(M(m, t))_{t \geq 0}$, for all $m \in \ell_\lambda$, in such a way that for all $t \geq 0$, all $m, \tilde{m} \in \ell_\lambda$,

$$\left[\sup_{[0, t]} d_1(M(m, s), M(\tilde{m}, s)) \right] \leq d_1(m, \tilde{m}) e^{8a(\|m\|_\lambda + \|\tilde{m}\|_\lambda)t}. \tag{3.9}$$

Proof We use here [5, Definition 2.1] and [5, Theorem 2.4]. We set $\bar{K} = \sup_{(x, y) \in [0, 1]^2} K(x, y)$. We consider a Poisson measure $N(dt, d(i, j), dz)$ on $[0, \infty) \times \{(i, j) \in \mathbb{N}^2, i < j\} \times [0, \bar{K}]$ with intensity measure $dt \left(\sum_{1 \leq k < l < \infty} \delta_{(k, l)} \right) dz$, and denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the associated canonical filtration.

For $m \in \ell_\lambda$, we know from [5] that there exists a unique ℓ_λ -valued càdlàg $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process $(M(m, t))_{t \geq 0}$ such that a.s., for all $t \geq 0$,

$$\begin{aligned} M(m, t) = m + \int_0^t \int_{i < j} \int_0^{\bar{K}} [c_{ij}(M(m, s-)) - M(m, s-)] \\ \mathbb{1}_{\{z \leq K(M_i(m, s-), M_j(m, s-))\}} N(ds, d(i, j), dz). \end{aligned} \tag{3.10}$$

Furthermore, this process $(M(m, t))_{t \geq 0}$ is a Markov process starting from m with semi-group $(P_s^K)_{s \geq 0}$ defined in Notation 2.3.

Remark here that the processes $(M(m, t))_{t \geq 0}$ for different initial data are coupled, in the sense that they are all built with the same Poisson measure N .

Before handling the computations, let us recall the following estimates, that can be found in [5, Corollary 3.2]: for all $1 \leq i < j$, all $m, \tilde{m} \in \ell_1$,

$$d_1(c_{ij}(m), c_{ij}(\tilde{m})) \leq d_1(m, \tilde{m}), \tag{3.11}$$

$$d_1(c_{ij}(m), \tilde{m}) \leq d_1(m, \tilde{m}) + 2m_j. \tag{3.12}$$

We may now compute. Let thus $m, \tilde{m} \in \ell_\lambda$. We have, for any $t \geq 0$,

$$d_1(M(m, t), M(\tilde{m}, t)) = d_1(m, \tilde{m}) + A_t + B_t^1 + B_t^2, \tag{3.13}$$

where, setting $\Delta_{ij}(s) := d_1(c_{ij}(M(m, s)), c_{ij}(M(\tilde{m}, s))) - d_1(M(m, s), M(\tilde{m}, s))$,

$$A_t = \int_0^t \int_{i < j} \int_0^{\bar{K}} \mathbb{1}_{\{z \leq K(M_i(m, s-), M_j(m, s-)) \wedge K(M_i(\tilde{m}, s-), M_j(\tilde{m}, s-))\}} \Delta_{ij}(s-) N(ds, d(i, j), dz), \quad (3.14)$$

while, setting $\Gamma_{ij}(s) := d_1(c_{ij}(M(m, s)), M(\tilde{m}, s)) - d_1(M(m, s), M(\tilde{m}, s))$,

$$B_t^1 = \int_0^t \int_{i < j} \int_0^{\bar{K}} \mathbb{1}_{\{K(M_i(\tilde{m}, s-), M_j(\tilde{m}, s-)) \leq z \leq K(M_i(m, s-), M_j(m, s-))\}} \Gamma_{ij}(s-) N(ds, d(i, j), dz), \quad (3.15)$$

and where B_t^2 is the same as B_t^1 permuting the roles of m and \tilde{m} .

Due to (3.11), we know that $A_t \leq 0$ for all $t \geq 0$ a.s. Next, using (3.12) and (3.8), we conclude that, setting $(x)_+ = \max(x, 0)$,

$$\begin{aligned} E \left[\sup_{[0, t]} B_s^1 \right] &\leq \int_0^t ds E \left[\sum_{i < j} 2M_j(m, s) \right. \\ &\quad \left. \left(K(M_i(m, s), M_j(m, s)) - K(M_i(\tilde{m}, s), M_j(\tilde{m}, s)) \right)_+ \right] \\ &\leq 2a \int_0^t ds E \left[\sum_{i < j} M_j(m, s) \right. \\ &\quad \left. \left(|M_i(m, s)^\lambda - M_i(\tilde{m}, s)^\lambda| + |M_j(m, s)^\lambda - M_j(\tilde{m}, s)^\lambda| \right) \right]. \end{aligned} \quad (3.16)$$

But one easily checks that $(x^{1-\lambda} + y^{1-\lambda})|x^\lambda - y^\lambda| \leq 2|x - y|$ for all $x, y \in [0, \infty)$, so that, since $M_j(m, s) \leq M_i(m, s)$ for all $i < j$,

$$\begin{aligned} &\sum_{i < j} M_j(m, s) |M_i(m, s)^\lambda - M_i(\tilde{m}, s)^\lambda| \\ &\leq \sum_{i < j} M_j(m, s)^\lambda (M_i(m, s)^{1-\lambda} + M_i(\tilde{m}, s)^{1-\lambda}) |M_i(m, s)^\lambda - M_i(\tilde{m}, s)^\lambda| \\ &\leq 2 \|M(m, s)\|_\lambda \times d_1(M(m, s), M(\tilde{m}, s)). \end{aligned} \quad (3.17)$$

By the same way,

$$\begin{aligned} &\sum_{i < j} M_j(m, s) |M_j(m, s)^\lambda - M_j(\tilde{m}, s)^\lambda| \\ &\leq \sum_{i < j} M_i(m, s)^\lambda (M_j(m, s)^{1-\lambda} + M_j(\tilde{m}, s)^{1-\lambda}) |M_j(m, s)^\lambda - M_j(\tilde{m}, s)^\lambda| \\ &\leq 2 \|M(m, s)\|_\lambda \times d_1(M(m, s), M(\tilde{m}, s)). \end{aligned} \quad (3.18)$$

We conclude that for all $t \geq 0$,

$$E \left[\sup_{[0, t]} B_s^1 \right] \leq 8a \int_0^t E \left[\|M(m, s)\|_\lambda \times d_1(M(m, s), M(\tilde{m}, s)) \right] ds. \quad (3.19)$$

Using the same computation for B_t^2 and the fact that the maps $t \mapsto \|M(m, t)\|_\lambda$ and $t \mapsto \|M(\tilde{m}, t)\|_\lambda$ are a.s. non-increasing, we finally get

$$E \left[\sup_{[0, t]} d_1(M(m, s), M(\tilde{m}, s)) \right] \leq d_1(m, \tilde{m}) \quad (3.20)$$

$$+ 8a (\|m\|_\lambda + \|\tilde{m}\|_\lambda) \int_0^t ds E [d_1(M(m, s), M(\tilde{m}, s))].$$

The Gronwall Lemma allows us to conclude. \square

We may finally handle the

Proof of Theorem 2.4 We divide the proof into several steps.

Step 1. For each $n \in \mathbb{N}$, let $m^n = (1/n, \dots, 1/n, 0, \dots) \in \ell_0$, which is an approximation of dust. We then consider, for each $n \in \mathbb{N}$, the Marcus-Lushnikov process $(M(m^n, t))_{t \geq 0}$. For each $t > 0$, using Lemma 3.2 and the notations of Lemma 3.1, we obtain

$$\lim_{A \rightarrow \infty} \inf_{n \in \mathbb{N}} P(M(m^n, t) \in \ell_\lambda(A)) = 1. \quad (3.21)$$

Due to Lemma 3.1, we deduce that for each $t > 0$, we may find a subsequence n_k such that $(M(m^{n_k}, t), \|M(m^{n_k}, t)\|_\lambda)$ converges in law in $\ell_1 \times [0, \infty)$, ℓ_1 being endowed with the distance d_1 . By (3.3) and the Fatou Lemma, the limit belongs a.s. to $\ell_\lambda \times [0, \infty)$.

Step 2. Consider now a decreasing sequence $(t_l)_{l \geq 1}$ of positive numbers such that $\lim_{l \rightarrow \infty} t_l = 0$. Using a diagonal extraction, we deduce from Step 1 that we may find a subsequence n_k such that for all $l \geq 1$, $(M(m^{n_k}, t_l), \|M(m^{n_k}, t_l)\|_\lambda)$ converges in law, in $\ell_1 \times [0, \infty)$. We denote by (M^l, X^l) the limit, which belongs a.s. to $\ell_\lambda \times [0, \infty)$. We thus may consider, using Theorem 2.2, the Markov process $(M^{l,*}(t))_{t \geq t_l}$, belonging a.s. to $\mathbb{D}([t_l, \infty), \ell_\lambda)$, with semi-group $(P_s^K)_{s \geq 0}$ starting at time t_l with the initial condition M^l .

Step 3. We now prove that for all $l \geq 1$, $(M(m^{n_k}, t))_{t \geq t_l}$ goes in law to $(M^{l,*}(t))_{t \geq t_l}$ as k tends to infinity, the convergence holding in $\mathbb{D}([t_l, \infty), \ell_1)$, ℓ_1 being equipped d_1 .

Using the Skorokhod representation Theorem, we may assume that as k tends to infinity, $(M(m^{n_k}, t_l), \|M(m^{n_k}, t_l)\|_\lambda)$ goes a.s. to (M^l, X^l) in $\ell_1 \times [0, \infty)$. This implies that a.s. for all $T > t_l$,

$$\lim_{k \rightarrow \infty} d_1(M(m^{n_k}, t_l), M^l) e^{8a(\|M(m^{n_k}, t_l)\|_\lambda + \|M^l\|_\lambda)(T - t_l)} = 0. \quad (3.22)$$

Lemma 3.3 allows us to conclude.

Step 4. We deduce from Step 3, by uniqueness of the limit, that for $p > l \geq 1$ (so that $0 < t_p < t_l$), the processes $(M^{p,*}(t))_{t \geq t_l}$ and $(M^{l,*}(t))_{t \geq t_l}$ have the same law. We may thus define a process $(M^*(t))_{t > 0}$ (using for example the Kolmogorov Theorem) in such a way that for all $l \geq 1$, the processes $(M^*(t))_{t \geq t_l}$ and $(M^{l,*}(t))_{t \geq t_l}$ have the same law. This process $(M^*(t))_{t \in (0, \infty)}$ is obviously a Markov process with semi-group $(P_s^K)_{s \geq 0}$ belonging a.s. to $\mathbb{D}([0, \infty), \ell_\lambda)$.

Step 5. It only remains to prove that a.s., $\lim_{t \rightarrow 0+} M_1^*(t) = 0$. By nature, the map $t \mapsto M_1^*(t)$ is a.s. non-decreasing, non-negative, and bounded by 1. It thus suffices to prove that $\lim_{t \rightarrow 0+} E[M_1^*(t)^2] = 0$. But for all $t > 0$, $M_1^*(t)$ is the limit in law, as $k \rightarrow \infty$, of $M_1(m^{n^k}, t)$. An easy computation, using (2.4) shows that for all $n \geq 1$,

$$\begin{aligned} E[M_1(m^n, t)^2] &= \frac{1}{n^2} + \int_0^t ds E \left[\sum_{i < j} K(M_i(m^n, s), M_j(m^n, s)) \right. \\ &\quad \left. ([M_i(m^n, s) + M_j(m^n, s)]^2 - M_1(m^n, s)^2)_+ \right] \\ &\leq \frac{1}{n^2} + 3\bar{K} \int_0^t ds E \left[\sum_{i < j} M_i(m^n, s) M_j(m^n, s) \right] \\ &\leq \frac{1}{n^2} + 3\bar{K}t, \end{aligned} \tag{3.23}$$

where $\bar{K} = \sup_{[0,1]^2} K(x, y)$. We have used that for $1 \leq i < j$, $M_j(m^n, s) \leq M_i(m^n, s) \leq M_1(m^n, s)$ and that $\sum_{i \geq 1} M_i(m^n, s) = 1$. Thus for all $t > 0$, $E[M_1^*(t)^2] \leq 3\bar{K}t$, from which the conclusion follows. \square

Remark 3.4 For the kernel $K = (xy)^{\lambda/2}$ Lemma 3.3 still holds. However, instead of Lemma 3.2, we are only able to prove a regularization of the moment of order α , for any $\alpha \in (\lambda, 1)$. Since the continuity property stated in Lemma 3.3 involves the moment of order λ , the proof breaks down.

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