

SPHERICAL AND HYPERBOLIC FRACTIONAL BROWNIAN MOTION

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Abstract

We define a Fractional Brownian Motion indexed by a sphere, or more generally by a compact rank one symmetric space, and prove that it exists if, and only if, $0 < H \leq 1/2$. We then prove that Fractional Brownian Motion indexed by an hyperbolic space exists if, and only if, $0 < H \leq 1/2$. At last, we prove that Fractional Brownian Motion indexed by a real tree exists when $0 < H \leq 1/2$.

1 Introduction

Since its introduction [10, 12], Fractional Brownian Motion has been used in various areas of applications (e.g. [14]) as a modelling tool. Its success is mainly due to the self-similar nature of Fractional Brownian Motion and to the stationarity of its increments. Fractional Brownian Motion is a field indexed by \mathbb{R}^d . Many applications, as texture simulation or geology, require a Fractional Brownian Motion indexed by a manifold. Many authors (e.g. [13, 8, 1, 7, 2]) use deformations of a field indexed by \mathbb{R}^d . Self-similarity and stationarity of the increments are lost by such deformations: they become only local self-similarity and local stationarity. We propose here to build Fractional Brownian Motion indexed by a manifold. For this purpose, the first condition is a stationarity condition with respect to the manifold. The second condition is with respect to the self-similar nature of the increments. Basically, the idea is that the variance of the Fractional Brownian Motion indexed by the manifold should be a fractional power of the distance. Let us be more precise.

The complex Brownian motion B indexed by \mathbb{R}^d , $d \geq 1$, can be defined [11] as a centered Gaussian field such that:

$$\begin{aligned} B(0) &= 0 \text{ (a.s.)}, \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= \|M - M'\|^H, \quad M, M' \in \mathbb{R}^d, \end{aligned}$$

where $\|M - M'\|$ is the usual Euclidean distance in \mathbb{R}^d . The complex Fractional Brownian Motion B_H of index H , $0 < H < 1$, indexed by \mathbb{R}^d , $d \geq 1$, can be defined [10, 12] as a centered Gaussian field such that:

$$\begin{aligned} B_H(0) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= \|M - M'\|^{2H} \quad M, M' \in \mathbb{R}^d . \end{aligned}$$

The complex Brownian motion B indexed by a sphere \mathbb{S}_d , $d \geq 1$, can be defined [11] as a centered Gaussian field such that:

$$\begin{aligned} B(O) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B(M) - B(M')|^2 &= d(M, M') \quad M, M' \in \mathbb{S}_d , \end{aligned}$$

where O is a given point of \mathbb{S}_d and $d(M, M')$ the distance between M and M' on the sphere (that is, the length of the geodesic between M and M'). Our first aim is to investigate the fractional case on \mathbb{S}_d . We start with the circle \mathbb{S}_1 . We first prove that there exists a centered Gaussian process (called Periodical Fractional Brownian Motion, in short PFBM) such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{S}_1 , \end{aligned}$$

where O is a given point of \mathbb{S}_1 and $d(M, M')$ the distance between M and M' on the circle, if and only if, $0 < H \leq 1/2$. We then give a random Fourier series representation of the PFBM. We then study the general case on \mathbb{S}_d . We prove that there exists a centered Gaussian field (called Spherical Fractional Brownian Motion, in short SFBM) such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.) ,} & (1) \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{S}_d , & (2) \end{aligned}$$

where O is a given point of \mathbb{S}_d and $d(M, M')$ the distance between M and M' on \mathbb{S}_d , if and only, if $0 < H \leq 1/2$. We then extend this result to compact rank one symmetric spaces (in short CROSS).

Let us now consider the case of a real hyperbolic space \mathbb{H}_d . We prove that there exists a centered Gaussian field (called Hyperbolic Fractional Brownian Motion, in short HFBM) such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.) ,} & (3) \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{H}_d , & (4) \end{aligned}$$

where O is a given point of \mathbb{H}_d and $d(M, M')$ the distance between M and M' on \mathbb{H}_d , if, and only if, $0 < H \leq 1/2$.

At last, we consider the case of a real tree (X, d) . We prove that there exists a centered Gaussian field such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in X , \end{aligned}$$

where O is a given point of X , for $0 < H \leq 1/2$.

2 Periodical Fractional Brownian Motion

Theorem 2.1

1. The PFBM exists, if and only if, $0 < H \leq 1/2$.
2. Assume $0 < H \leq 1/2$. Let us parametrize the points M of the circle \mathbb{S}_1 of radius r by their angles x . B_H can be represented as:

$$B_H(x) = \sqrt{r} \sum_{n \in \mathbb{Z}^*} d_n \varepsilon_n \cdot (e^{inx} - 1), \quad (5)$$

where

$$d_n = \frac{\sqrt{-\int_0^{|\pi|} u^{2H} \cos(u) du}}{\sqrt{2\pi} |n|^{1/2+H}}, \quad (6)$$

and $(\varepsilon_n)_{n \in \mathbb{Z}^*}$ is a sequence of i.i.d. complex standard normal variables.

Proof of Theorem 2.1

Without loss of generality, we work on the unit circle \mathbb{S}_1 : $r = 1$. Let M and M' be parametrized by $x, x' \in [0, 2\pi[$. We then have:

$$\begin{aligned} d(M, M') &= d(x, x') \\ &= \inf(|x - x'|, 2\pi - |x - x'|). \\ d_H(x, x') &\stackrel{\text{def}}{=} d^{2H}(x, x') \\ &= \inf(|x - x'|^{2H}, (2\pi - |x - x'|)^{2H}). \end{aligned}$$

The covariance function of B_H , if there exists, is:

$$R_H(x, x') = \frac{1}{2}(d_H(x, 0) + d_H(x', 0) - d_H(x, x')).$$

Let us expand the function $x \rightarrow d_H(x, 0)$ in Fourier series:

$$d_H(x, 0) = \sum_{n \in \mathbb{Z}} f_n e^{inx}. \quad (7)$$

We will see that the series $\sum_{n \in \mathbb{Z}} |f_n|$ converges. It follows that equality (7) holds pointwise.

Since $d_H(0, 0) = 0$, $\sum_{n \in \mathbb{Z}} f_n = 0$. The function d_H is odd: $f_{-n} = f_n$. We can therefore write, no matter if $x - x'$ is positive or negative:

$$\begin{aligned} d_H(x, x') &= \sum_{n \in \mathbb{Z}} f_n e^{in(x-x')} \\ &= \sum_{n \in \mathbb{Z}^*} f_n (e^{in(x-x')} - 1). \end{aligned}$$

We now prove that R_H is a covariance function if and only if $0 < H \leq 1/2$.

$$\begin{aligned} \sum_{i,j=1}^p \lambda_i \bar{\lambda}_j R_H(x_i, x_j) &= \frac{1}{2} \sum_{n \in \mathbb{Z}} f_n \left[\sum_{i,j=1}^p \lambda_i \bar{\lambda}_j \left(e^{inx_i} + e^{-inx_j} - e^{-in(x_i-x_j)} \right) \right] \\ &= -\frac{1}{2} \sum_{n \in \mathbb{Z}^*} f_n \left| \sum_{i=1}^p \lambda_i (1 - e^{inx_i}) \right|^2. \end{aligned} \tag{8}$$

Let us study the sign of $f_n, n \in \mathbb{Z}^*$. Since $f_{-n} = f_n$, let us only consider $n > 0$.

$$\begin{aligned} f_n &= \frac{1}{\pi} \int_0^\pi x^{2H} \cos(nx) dx \\ &= \frac{1}{\pi n^{1+2H}} \int_0^{n\pi} u^{2H} \cos(u) du. \end{aligned}$$

1. n odd.

$$\int_{2k\pi}^{2k\pi+\pi} u^{2H} \cos(u) du = \int_0^{\pi/2} \cos(v) [(v + 2k\pi)^{2H} - (2k\pi + \pi - v)^{2H}] dv \leq 0.$$

2. n even.

$$\begin{aligned} \int_{2k\pi}^{2(k+1)\pi} u^{2H} \cos(u) du &= \int_0^{\pi/2} \cos(v) [(v + 2k\pi)^{2H} - (2k\pi + \pi - v)^{2H} \\ &\quad - (2k\pi + \pi + v)^{2H} + (2k\pi + 2\pi - v)^{2H}] dv. \end{aligned}$$

Using the concavity/convexity of the functions $x \rightarrow x^{2H}$, one sees that

$[(v + 2k\pi)^{2H} - (2k\pi + \pi - v)^{2H} - (2k\pi + \pi + v)^{2H} + (2k\pi + 2\pi - v)^{2H}]$ is negative when $H \leq 1/2$ and positive when $H > 1/2$.

1. $H \leq 1/2$ All the f_n are negative and (8) is positive.

2. $H > 1/2$. We check that, if B_H exists, then we should have:

$$\begin{aligned} \mathbb{E} \left| \int_0^{2\pi} B_H(t) e^{int} dt \right|^2 &= \int_0^{2\pi} \int_0^{2\pi} (d_H(t, 0) + d_H(s, 0) - d_H(s, t)) e^{in(t-s)} dt ds \\ &= -4\pi^2 f_n. \end{aligned}$$

All the f_n , with n even, are positive, which constitutes a contradiction.

In order to prove the representation (5), we only need to compute the covariance:

$$\begin{aligned} \mathbb{E} B_H(x) \overline{B_H(x')} &= \frac{1}{2} \left(f_0 - \sum_{n \in \mathbb{Z}^*} d_n^2 e^{inx} - \sum_{n \in \mathbb{Z}^*} d_n^2 e^{inx'} + \sum_{n \in \mathbb{Z}^*} d_n^2 e^{in(x-x')} \right) \\ &= R_H(x, x'). \end{aligned}$$

3 Spherical Fractional Brownian Motion

Theorem 3.1

1. The SFBM, defined by (1), (2), exists if, and only if, $0 < H \leq 1/2$.
2. The same holds for CROSS: the Fractional Brownian Motion, indexed by a CROSS, exists if, and only if, $0 < H \leq 1/2$.

Corollary 3.1

Let (\mathcal{M}, d) be a complete Riemannian manifold such that \mathcal{M} and a CROSS are isometric. Then the Fractional Brownian Motion indexed by \mathcal{M} and defined by:

$$\begin{aligned} B_H(\tilde{O}) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= \tilde{d}^{2H}(M, M') \quad M, M' \in \mathcal{M} , \end{aligned}$$

exists if and only if $0 < H \leq 1/2$.

Proof of Theorem 3.1

Let us first recall the classification of the CROSS, also known as two points homogeneous spaces [9, 17]: spheres \mathbb{S}_d , $d \geq 1$, real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d \geq 2$, complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 2k$, $k \geq 2$, quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 4k$, $k \geq 2$ and Cayley projective plane P^{16} . [6] has proved that Brownian Motion indexed by CROSS can be defined. The proof of Theorem 3.1 begins with the following Lemma, which implies, using [6], the existence of the Fractional Brownian Motion indexed by a CROSS for $0 < H \leq 1/2$.

Lemma 3.1

Let (X, d) be a metric space. If the Brownian Motion B indexed by X and defined by:

$$\begin{aligned} B(O) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B(M) - B(M')|^2 &= d(M, M') \quad M, M' \in X , \end{aligned}$$

exists, then the Fractional Brownian Motion B_H indexed by X and defined by:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.) ,} \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in X , \end{aligned}$$

exists for $0 < H \leq 1/2$.

Proof of Lemma 3.1

For $\lambda \geq 0$, $0 < \alpha < 1$, one has:

$$\lambda^\alpha = -\frac{1}{C_\alpha} \int_0^{+\infty} \frac{e^{-\lambda x} - 1}{x^{1+\alpha}} dx ,$$

with

$$C_\alpha = \int_0^{+\infty} \frac{1 - e^{-u}}{u^{1+\alpha}} du .$$

We then have, for $0 < H < 1/2$:

$$d^{2H}(M, M') = -\frac{1}{C_{2H}} \int_0^{+\infty} \frac{e^{-xd(M, M')} - 1}{x^{1+2H}} dx .$$

Let us remark that:

$$e^{-xd(M,M')} = \mathbb{E} \left(e^{i\sqrt{2x}(B(M)-B(M'))} \right) ,$$

so that:

$$d^{2H}(M, M') = -\frac{1}{C_{2H}} \int_0^{+\infty} \frac{\mathbb{E} \left(e^{i\sqrt{2x}(B(M)-B(M'))} \right) - 1}{x^{1+2H}} dx .$$

Denote by $R_H(M, M')$ the covariance function of B_H , if exists:

$$R_H(M, M') = \frac{1}{2} (d_H(O, M) + d_H(M', O) - d_H(M, M')) .$$

Let us check that R_H is positive definite:

$$\begin{aligned} & \sum_{i,j=1}^p \lambda_i \bar{\lambda}_j R_H(M_i, M_j) = \\ & -\frac{1}{2C_{2H}} \int_0^{+\infty} \frac{\sum_{i,j=1}^p \lambda_i \bar{\lambda}_j \left[\mathbb{E} \left(e^{i\sqrt{2x}B(M_i)} \right) + \mathbb{E} \left(e^{-i\sqrt{2x}B(M_j)} \right) - \mathbb{E} \left(e^{i\sqrt{2x}(B(M_i)-B(M_j))} \right) - 1 \right]}{x^{1+2H}} dx \\ & = \frac{1}{2C_{2H}} \int_0^{+\infty} \frac{\mathbb{E} \left| \sum_{i=1}^p \lambda_i \left(1 - e^{i\sqrt{2x}B(M_i)} \right) \right|^2}{x^{1+2H}} dx . \end{aligned} \tag{9}$$

(9) is clearly positive and Lemma 3.1 is proved.

We now prove by contradiction that the Fractional Brownian Motion indexed by a CROSS does not exist for $H > 1/2$. The geodesic of a CROSS are periodic. Let \mathbf{G} be such a geodesic containing 0. Therefore, the process $B_H(M)$, $M \in \mathbf{G}$ is a PFBM. We know from Theorem 2.1 that PFBM exists if, and only if, $0 < H \leq 1/2$.

Proof of Corollary 3.1

Let ϕ be the isometric mapping between \mathcal{M} and the CROSS and let \tilde{d} (resp. d) be the metric of \mathcal{M} (resp. the CROSS). Then, for all $M, M' \in \mathcal{M}$, one has:

$$\tilde{d}(M, M') = d(\phi(M), \phi(M')) .$$

Let \tilde{O} be a given point of \mathcal{M} and $O = \phi(\tilde{O})$. Denote by \tilde{R}_H (resp. R_H) the covariance function of the Fractional Brownian Motion indexed by \mathcal{M} (resp. the CROSS).

$$\begin{aligned} \tilde{R}_H(M, M') & \stackrel{def}{=} \frac{1}{2} (\tilde{d}^{2H}(\tilde{O}, M) + \tilde{d}^{2H}(\tilde{O}, M') - \tilde{d}^{2H}(M, M')) \\ & = \frac{1}{2} (d^{2H}(O, \phi(M)) + d^{2H}(O, \phi(M')) - d^{2H}(\phi(M), \phi(M'))) \\ & = R_H(\phi(M), \phi(M')) . \end{aligned}$$

It follows that \tilde{R}_H is positive definite if and only if, R_H is positive definite. Corollary 3.1 is proved.

4 Hyperbolic Fractional Brownian Motion

Let us consider real hyperbolic spaces \mathbb{H}_d :

$$\mathbb{H}_d = \{(x_1, \dots, x_d), x_1 > 0, x_1^2 - \sum_2^d x_i^2 = 1\},$$

with geodesic distance:

$$d(M, M') = \operatorname{Arccosh}[M, M'],$$

where

$$[M, M'] = x_1 x'_1 - \sum_2^d x_i x'_i.$$

The HFBM is the Gaussian centered field such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.)}, \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{H}_d, \end{aligned}$$

where O is a given point of \mathbb{H}_d .

Theorem 4.1

The HFBM exists if, and only if, $0 < H \leq 1/2$.

Since the Brownian Motion indexed by a real hyperbolic space can be defined [4, 5], Lemma 3.1 implies the existence of HFBM when $0 < H \leq 1/2$.

Let H be a real function. [5, Prop. 7.6] prove that if $H(d(M, M'))$ is negative definite, then $H(x) = O(x)$ as $x \rightarrow +\infty$. It follows that $d^{2H}(M, M')$, when $H > 1/2$, is not negative definite: the HFBM, when $H > 1/2$ does not exist.

Corollary 4.1

Let (\mathcal{M}, d) be a complete Riemannian manifold such that \mathcal{M} and \mathbb{H}_d are isometric. Then the Fractional Brownian Motion indexed by \mathcal{M} and defined by:

$$\begin{aligned} B_H(\tilde{O}) &= 0 \text{ (a.s.)}, \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= \tilde{d}^{2H}(M, M') \quad M, M' \in \mathcal{M}, \end{aligned}$$

exists if, and only, if $0 < H \leq 1/2$.

The proof of Corollary 4.1 is identical to the proof of Corollary 3.1.

5 Real trees

A metric space (X, d) is a real tree (e.g. [3]) if the following two properties hold for every $x, y \in X$.

- There is a unique isometric map $f_{x,y}$ from $[0, d(x, y)]$ into X such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x, y)) = y$.

- If ϕ is a continuous injective map from $[0, 1]$ into X , such that $\phi(0) = x$ and $\phi(1) = y$, we have

$$\phi([0, 1]) = f_{x,y}([0, d(x, y)]).$$

Theorem 5.1

The Fractional Brownian Motion indexed by a real tree (X, d) exists for $0 < H \leq 1/2$.

[16] proves that the distance d is negative definite. It follows from [15] that function $R(x, y) = \frac{1}{2}(d(0, x) + d(0, y) - d(x, y))$ is positive definite. Lemma 3.1 then implies theorem 5.1.

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