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Convergence rate for geometric statistics of point processes having fast decay of dependence*

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Abstract

[BYY19] established central limit theorems for geometric statistics of point processes having fast decay of dependence. As limit theorems are of limited use unless we understand their errors involved in the approximation, in this paper, we consider the rates of a normal approximation in terms of the Wasserstein distance for statistics of point processes on $\mathbb{R}^d$ satisfying fast decay of dependence. We demonstrate the use of the theorems for statistics arising from two families of point processes: the rarified Gibbs point processes and the determinantal point processes with kernels decaying fast enough.

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1 Introduction

Random events in space and time can be represented as point processes, and statistics arising from such random events are often of the form $\sum_{x\in\Upsilon\cap A}\eta(x,\Upsilon)$, where Υ is a point process on $\mathbb{R}^d$, $A\subset\mathbb{R}^d$ is a bounded Borel set, and η is a real valued function defined on (x,Υ). The function $\eta(x,\Upsilon)$ is often called a *score function*, and it represents the interaction between the point at x and the point process Υ. The form dates back to [BHH59, S81] with Υ as a binomial point process and [AB93] with Υ being a Poisson point process. As the density of points or the size of the observation window A increases, various limit theorems of the functionals have been established since then, see, e.g.,

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[S12, S16, LSY19] and references therein. If $\eta(x,\Upsilon) = \mathbf{1}_{x\in\Upsilon}$ and Υ is an α-determinantal point process with some condition on α, [S02, ST03, NS12] proved that the counting statistics are asymptotically normal. The asymptotic normality also holds when the input process Υ is a Gibbsian point process [SY13, XY15]. The common feature leading to these limit theorems is the local dependence [CS04, p. 133] in the sense that each summand is affected by a small neighbourhood only, hence making a nearly independent contribution to the statistics of interest. More generally, when the underlying point process Υ is a stationary point process and has fast decay of dependence, and the score function $\eta(x,\Upsilon)$ is determined by points of Υ not too far away from x, it is possible to establish central limit theorems for such statistics [BYY19]. In this paper, we aim to quantify the errors associated with the limit theorems in [BYY19] because the limit theorems are of limited practical value unless we understand the magnitude of the errors involved in the approximation of these statistics. To the best of our knowledge, the only quantified error bounds for the normal approximation to geometric functionals driven by non-Poisson and non-binomial point processes are in [SY13, XY15] for a Gibbs point process input, in [CRX21] for a Ginibre point process input, and [F20] for a Gibbs or determinantal point process input. For the Gibbs point process input, the Poisson-like property plays the pivotal role, and for the Ginibre point process input, the proof relies on the special structure of its first order Palm processes. [F20] can be viewed as a sequential work of [BYY19] with conditions too onerous to satisfy in applications.

The local influence of the score function η can be controlled through the concept of stabilisation [BX01, PY01, PY03, PY05, P07a, P07b, SY13, XY15]. As the distribution of a simple point process is determined by its correlation functions [BYY19, p. 840], the fast decay of dependence of the input simple point process can be controlled by the fast decay of its correlations [BYY19, Definition 1.2]. The proofs of the limit theorems in [BYY19] hinge on the Marcinkiewicz theorem [S02, Lemma 3] and the method of cumulants. The Marcinkiewicz theorem is a handy tool to prove the central limit theorems, but when we aim for the errors of approximation, it seems impractical to use the cumulants to control the errors because these quantities are hard to obtain in applications. For error bounds under the setup in [BYY19], [F20] assumed variance and moment growth conditions which are hard to verify for unbounded score functions with a non-Poisson input. For this reason, we impose the condition of the decay of dependence through the β-mixing coefficient in *Assumption 2.0 Exponential Decay of Dependence* (EDD), which is slightly stronger than the condition based on the α-mixing coefficient used in [BYY19].

We use the Palm theory of point processes [K83, §10.1] and Stein's method for the normal approximation [S72, CGS11] to establish the bounds. Stein's method was pioneered by [S72] for normal approximation of dependent random variables, and it was subsequently adapted to deal with Poisson approximation [C75], Poisson process approximation [B88, BB92], multivariate normal approximation [G91], and many other distributional approximations, see, e.g., [S86, BHJ92, CGS11]. The logic flow of the proof is: (1) to employ Stein's method to transform the comparison of the distribution of the sum and the normal distribution into the assessment of the differences between the sum and a functional form of the sum; (2) with controllable costs, to apply the stabilising condition to truncate the summands and remove the segments having long-range dependence on the point configuration, and, (3) to use the fast decay of dependence of the point process for the remaining parts of the summands so that they make nearly independent contributions to the sum. For a large class of stationary simple point processes, it is possible to work with their first order Palm distributions. However, except in some special cases, higher order Palm distributions [K83, p. 110] are generally beyond our reach. To avoid conditions of the Palm distributions of all orders imposed in [BYY19], our bounds are in terms of the first order Palm distributions at the cost of higher moments. The edge effects of the score functions play a significant role in the error bounds. Our score functions allow a variety of edge effects, including those studied in [BYY19].

In Section 2, we introduce the definitions and concepts that are needed in the paper, and state the main theorems. We will show in Section 3 that the Gibbs point processes with nearly finite range potentials, a class of the determinantal point processes with kernels decaying fast enough, the r-dependent point processes and the Boolean models all possess the EDD property. In Section 4, we demonstrate the use of the theorems for statistics arising from the rarified Gibbs point processes and the determinantal point processes with kernels decaying fast enough. For ease of reading, we postpone the proofs of the main results to Section 5.

The asymptotic normality depends on the lower bound of the variance of the statistics, and showing the order of the variance is itself an interesting but hard topic [BYY19, (1.26) and Remark (iii) after Theorem 1.14]. In an attempt to recover the volume order of the variances, we obtained Theorem 2.12, which is an analogue of [BYY19, Theorem 1.15].

2 General results

To start with, we recall the definition of the marked point processes on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ with marks in a measurable space $(T, \mathscr{T})$, where $\mathbb{R}^d$ is equipped with the Euclidean norm $\|\cdot\|$ and the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$, and $\mathscr{T}$ is a σ-algebra on T. Let $\boldsymbol{S} := \mathbb{R}^d \times T$ be equipped with the product σ-field $\mathscr{S} := \mathscr{B}(\mathbb{R}^d) \times \mathscr{T}$. We use $\boldsymbol{C_S}$ to denote the space of all locally finite (with respect to the first coordinate in $\mathbb{R}^d$) non-negative integer-valued measures ξ, often called *configurations*, on $\boldsymbol{S}$ such that $\xi(\{x\} \times T) \le 1$ for all $x \in \mathbb{R}^d$. The space $\boldsymbol{C_S}$ is endowed with the σ-field $\mathscr{C}_{\boldsymbol{S}}$ generated by the vague topology [K83, p. 169]. A *marked point process* Ξ on $\mathbb{R}^d$ is a measurable mapping from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\boldsymbol{C_S}, \mathscr{C}_{\boldsymbol{S}})$ [K17, p. 49]. The induced point process $\overline{\Xi}(\cdot) := \Xi(\cdot \times T)$ is called the *ground process* [DV08, p. 3] or projection [K17, p. 17] of the marked point process Ξ on $\mathbb{R}^d$. The point process $\overline{\Xi}$ is *simple*, i.e., $\mathbb{P}(\overline{\Xi}(\{x\}) \in \{0,1\}$ for all $x \in \mathbb{R}^d) = 1$. We use M_x to denote the mark of Ξ at x for $x \in \overline{\Xi}$.

For a marked point process Ξ, let Ξ_A be the restriction of Ξ to $A \times T$ defined as $\Xi_A(B \times D) := \Xi((A \cap B) \times D)$, and let Ξ^x be the shifted point process of Ξ by $-x$ defined as $\Xi^x(B \times D) := \Xi((B+x) \times D)$ for all $x \in \mathbb{R}^d$, $D \in \mathscr{T}$ and $A,\ B \in \mathscr{B}(\mathbb{R}^d)$. We say that the marked point process Ξ is *stationary* if $\Xi \stackrel{d}{=} \Xi^x$ for all $x \in \mathbb{R}^d$, where $\stackrel{d}{=}$ stands for 'equal in distribution'. To avoid using the Palm distributions of *all* orders and *all* cumulants in the approximation bounds, in this paper, the fast decay of dependence of the marked point process Ξ is quantified through its *β-mixing coefficient* [VR59, R17] (also known as *strong mixing coefficient* [I82]): for $A_1, A_2 \in \mathscr{B}(\mathbb{R}^d)$ such that $A_1 \cap A_2 = \emptyset$,

$$\begin{aligned}\beta_{A_1,A_2} :=& \frac{1}{2} \int_{\zeta_1 \in \boldsymbol{C}_{A_1 \times T}} \int_{\zeta_2 \in \boldsymbol{C}_{A_2 \times T}} \left| \mathbb{P}\left(\Xi_{A_1} \in d\zeta_1, \Xi_{A_2} \in d\zeta_2\right) - \mathbb{P}\left(\Xi_{A_1} \in d\zeta_1\right) \mathbb{P}\left(\Xi_{A_2} \in d\zeta_2\right) \right| \\ =& d_{TV}\left(\mathscr{L}(\Xi_{A_1 \cup A_2}), \mathscr{L}(\Xi_{A_1} + \tilde{\Xi}_{A_2})\right),\end{aligned}$$

where here and in the following, $\tilde{\Xi}$ denotes the independent copy of Ξ, $\boldsymbol{C}_{A_i \times T}$ and $\mathscr{C}_{A_i \times T}$ are defined in the same way as $\boldsymbol{C_S}$ and $\mathscr{C}_{\boldsymbol{S}}$ with $\boldsymbol{S}$ replaced by $A_i \times T$, $i = 1, 2$.

To define the decay of dependence, we set $\mathrm{diam}(A) := \sup\{\|x - y\|;\ x, y \in A\}$, $d(A, B) := \inf\{\|x - y\|; x \in A,\ y \in B\}$ for $A, B \in \mathscr{B}(\mathbb{R}^d)$, where we used the convention $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. We use $\vee$ to stand for the maximum.

Assumption 2.0 Exponential Decay of Dependence

We say that the marked point process Ξ has the *exponential decay of dependence* (EDD) if there exist constants $\theta_0 \in \mathbb{R}_0 := [0, \infty)$, $\theta_i \in \mathbb{R}_+ := (0, \infty)$, $1 \le i \le 4$, such that for any A, $B \in \mathscr{B}(\mathbb{R}^d)$ with $d(A,B) \geq \theta_3 \ln(\mathrm{diam}(A) \vee \mathrm{diam}(B) \vee \theta_4)$,

$$\beta_{A,B} \leq \theta_1(\mathrm{diam}(A)^{\theta_0} \vee 1)(\mathrm{diam}(B)^{\theta_0} \vee 1)e^{-\theta_2 d(A,B)}. \tag{2.1}$$

The idea of the EDD is that the total variation distance between the law of (Ξ_A, Ξ_B) and the law of the independent union of Ξ_A and Ξ_B decays exponentially fast as the distance between A and B becomes large.

The following lemma says that, in applications, it is sometimes more convenient to verify the EDD via the volumes of the sets. To this end, let $\mathrm{Vol}(A)$ denote the volume of the set $A \in \mathscr{B}(\mathbb{R}^d)$.

Remark 2.1. If there exist constants $\theta_0' \in \mathbb{R}_0$, $\theta_i' \in \mathbb{R}_+$, $1 \leq i \leq 4$, such that for any A, $B \in \mathscr{B}(\mathbb{R}^d)$ with $d(A,B) \geq \theta_3' \ln(\mathrm{Vol}(A) \vee \mathrm{Vol}(B) \vee \theta_4')$,

$$\begin{aligned}\beta_{A,B} \leq& \theta_1'(\mathrm{Vol}(A)^{\theta_0'} \vee 1)(\mathrm{Vol}(B)^{\theta_0'} \vee 1)e^{-\theta_2' d(A,B)}\\ =&\theta_1' e^{\theta_0' \ln((\mathrm{Vol}(A)\vee 1)(\mathrm{Vol}(B)\vee 1)) - \theta_2' d(A,B)},\end{aligned}$$

then Ξ satisfies the EDD. This can be easily checked using the property that when d is given, $\mathrm{Vol}(A) \leq \frac{\pi^{d/2}}{2^d \Gamma(\frac{d}{2}+1)} \mathrm{diam}(A)^d$.

Remark 2.2. The constants θ_4 in the definition of the EDD and θ_4' in Remark 2.1 are not essential, and they can be replaced by any positive constants because the definition of β-mixing coefficient ensures that β is non-decreasing in the sense of inclusion, i.e., $\beta_{A,B} \leq \beta_{A',B'}$ for all $A, A', B, B' \in \mathscr{B}(\mathbb{R}^d)$ such that $A \subset A'$ and $B \subset B'$.

Let Ξ be a stationary marked point process on $\boldsymbol{S}$ with independent and identically distributed ($i.i.d.$) marks that satisfies the EDD. Write the law of Ξ as $\mathscr{L}(\Xi)$ and the law of the independent marks as $\mathscr{L}_T$. The functionals we study in the paper are defined on $\Gamma_\alpha := \left[-\frac{1}{2}\alpha^{\frac{1}{d}}, \frac{1}{2}\alpha^{\frac{1}{d}}\right]^d$, the cube with volume α on $\mathbb{R}^d$ centred at $\mathbf{0}$, having respective forms

$$W_\alpha := \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m), \Xi)$$

and

$$\bar{W}_\alpha := \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) = \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m), \Xi, \Gamma_\alpha).$$

The function η is called a *score function* (resp. *restricted score function*), i.e., a measurable function mapping $(\boldsymbol{S} \times \boldsymbol{C}_{\boldsymbol{S}}, \mathscr{S} \times \mathscr{C}_{\boldsymbol{S}})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ (resp. a function mapping $\boldsymbol{S} \times \boldsymbol{C}_{\Gamma_\alpha \times T} \times \mathscr{B}(\mathbb{R}^d)$ to $\mathbb{R}$ which is $(\boldsymbol{S} \times \boldsymbol{C}_{\Gamma_\alpha \times T}, \mathscr{S} \times \mathscr{C}_{\Gamma_\alpha \times T})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ measurable when the third coordinate is fixed), which represents the interaction between a point with its mark and the configuration of the point process. The class of score functions considered here is broader than that considered in [BYY19]. More precisely, if the score function for the restricted case does not depend on the third argument, it reduces to that in [BYY19]. Because the interest is in the values of the score function of the points in a configuration, for convenience, $\eta((x,m), \mathscr{X})$ (resp. $\eta((x,m), \mathscr{X}, \Gamma_\alpha)$) is understood as 0 for all $x \in \mathbb{R}^d$ and $\mathscr{X} \in \boldsymbol{C}_{\boldsymbol{S}}$ such that $(x,m) \notin \mathscr{X}$.

We need Palm processes and reduced Palm processes for stating the conditions and constructing proofs. For ease of reading, we briefly recall their definitions. Let H be a Polish space with Borel σ-algebra $\mathscr{B}(H)$ and configuration space $(\boldsymbol{C}_H, \mathscr{C}_H)$, let Υ be a point process on $(H, \mathscr{B}(H))$ and write the mean measure of Υ as $\mathbb{E}\Upsilon$. The point processes $\{\Upsilon_x : x \in H\}$ are said to be the *reduced Palm processes* associated with Υ if for any measurable function $f : (H \times \boldsymbol{C}_H, \mathscr{B}(H) \times \mathscr{C}_H) \to (\mathbb{R}_0, \mathscr{B}(\mathbb{R}_0))$,

$$\mathbb{E}\left[\int_H f(x, \Upsilon)\Upsilon(dx)\right] = \int_H \mathbb{E}f(x, \Upsilon_x + \delta_x)\mathbb{E}\Upsilon(dx), \tag{2.2}$$

[K83, § 10.1], where δ_x is the Dirac measure at x. The distributions of Υ_x and $\Upsilon_x + \delta_x$ are respectively called the reduced Palm distribution and the Palm distribution of Υ at x. When the point process Υ is simple, the Palm distribution of $\Upsilon_x + \delta_x$ can be interpreted as the conditional distribution of Υ given $\Upsilon(\{x\}) = 1$ [K83, § 10.1].

For the stationary marked point process Ξ, since the marks are independent of each other and the ground process, we can adapt (2.2) to the marked point process Ξ. For $f : (\boldsymbol{S} \times \boldsymbol{C_S}, \mathscr{S} \times \mathscr{C}_{\boldsymbol{S}}) \to (\mathbb{R}_0, \mathscr{B}(\mathbb{R}_0))$, recalling that M_x is the mark of Ξ at the point $x \in \overline{\Xi}$, we have

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f((x, M_x), \Xi)\overline{\Xi}(dx)\right] = \int_{\mathbb{R}^d} \mathbb{E}f((x, M), \Xi_x + \delta_{(x,M)})\lambda dx, \tag{2.3}$$

where $M \sim \mathscr{L}_T$, $\lambda := \mathbb{E}\left[\overline{\Xi}(\Gamma_1)\right]$ is the intensity of $\overline{\Xi}$, Ξ_x is the point process obtained by attaching the reduced Palm process $\overline{\Xi}_x$ of $\overline{\Xi}$ with i.i.d. marks following $\mathscr{L}_T$, and M is independent of Ξ_x. Without loss of generality, we use the convention that M_x is independent of Ξ_x throughout the paper. Hence, when we need to emphasise the location of the mark, we can replace M with M_x on the right hand side of (2.3).

The following assumptions are adapted from those in [CX24], which were initiated in [PY01] and further refined by [XY15].

Assumption 2.1 Stabilisation

For a locally finite configuration $\mathscr{X}$ and $z \in \boldsymbol{S} \cup \{\emptyset\}$, write $\mathscr{X}^{\wr z \int} = \mathscr{X}$ if $z = \emptyset$ and $\mathscr{X}^{\wr z \int} = \mathscr{X} \cup \{z\}$ otherwise. We use $B(x, r)$ to stand for the ball with centre x and radius $r \geq 0$.

Definition 2.3 (unrestricted case)**.** *A score function η on $\boldsymbol{S}$ is range-bounded (resp. exponentially stabilising) with respect to $\mathscr{L}(\Xi)$ if for all $x \in \mathbb{R}^d$, $z \in \boldsymbol{S} \cup \{\emptyset\}$, and almost all realisations $\mathscr{X}$ of the marked point process Ξ_x, there exists a radius of stabilisation*

$$R := R(x) := R((x, m_x), \mathscr{X}^{\wr z \int}) \in (0, \infty),$$

such that for all locally finite $\mathscr{Y} \subset (\mathbb{R}^d \backslash B(x, R)) \times T$, we have

$$\begin{aligned} R\left((x, m_x), \left[\mathscr{X}^{\wr z \int} \cap (B(x, R) \times T)\right] \cup \mathscr{Y}\right) &= R\left((x, m_x), \mathscr{X}^{\wr z \int} \cap (B(x, R) \times T)\right), \\ \eta\left((x, m_x), \left[\mathscr{X}^{\wr z \int} \cap (B(x, R) \times T)\right] \cup \mathscr{Y}\right) &= \eta\left((x, m_x), \mathscr{X}^{\wr z \int} \cap (B(x, R) \times T)\right), \end{aligned}$$

and the tail probability

$$\tau(t) := \sup_{(x, m_x) \in \mathbb{R}^d \times \mathrm{supp}(\mathscr{L}_\mathrm{T})} \sup_{z \in \mathbf{S} \cup \{\emptyset\}} \mathbb{P}\left(R((x, m_x), \Xi_x^{\wr z \int} + \delta_{(x, m_x)}) \geq t\right)$$

satisfies $\tau(t) = 0$ for some $t > 0$ (resp. $\tau(t) \leq C_1 e^{-C_2 t}$ for all $t > 0$, where C_1 and C_2 are positive constants independent of t).

Here and in the following, we write R or $R(x)$ (resp. $\bar{R}$ or $\bar{R}(x)$ in Definition 2.4) only if it will not cause any confusion. The definition ensures that $\{R((x, m_x), \mathscr{X}^{\wr z \int}) \leq t\}$ is determined by $\mathscr{X}_{B(x,t)}$ for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$.

For the functionals with the input of a restricted marked point process, we have the following counterpart of stabilisation. Note that the score function for the restricted input is not affected by points outside Γ_α.

Definition 2.4 (restricted case)**.** *We say that the score function η is range-bounded (resp. exponentially stabilising) with respect to $\mathscr{L}(\Xi)$ if for $\alpha \in \mathbb{R}_+$, $x \in \Gamma_\alpha$, and $z \in (\Gamma_\alpha \times T) \cup \{\emptyset\}$, almost all realisations $\mathscr{X}$ of the marked point process Ξ_x, there exists a radius of stabilisation*

$$\bar{R} := \bar{R}(x, \alpha) := \bar{R}((x, m_x), \alpha, \mathscr{X}^{\wr z \int}) \in (0, \infty)$$

such that for all locally finite $\mathscr{Y} \subset (\Gamma_\alpha \backslash B(x,R)) \times T$, we have

$$\bar{R}\left((x,m_x),\left[\mathscr{X}_{\Gamma_\alpha}^{\wr z \int} \cap \left(B(x,\bar{R}) \times T\right)\right] \cup \mathscr{Y}, \Gamma_\alpha\right) = \bar{R}\left((x,m_x), \mathscr{X}_{\Gamma_\alpha}^{\wr z \int} \cap \left(B(x,\bar{R}) \times T\right), \Gamma_\alpha\right),$$
$$\eta\left((x,m_x),\left[\mathscr{X}_{\Gamma_\alpha}^{\wr z \int} \cap \left(B(x,\bar{R}) \times T\right)\right] \cup \mathscr{Y}, \Gamma_\alpha\right) = \eta\left((x,m_x), \mathscr{X}_{\Gamma_\alpha}^{\wr z \int} \cap \left(B(x,\bar{R}) \times T\right), \Gamma_\alpha\right),$$

and the tail probability

$$\bar{\tau}(t) := \sup_{(x,m_x) \in \mathbb{R}^d \times \mathrm{supp}(\mathscr{L}_{\mathrm{T}})} \sup_{\alpha \in \mathbb{R}_+} \sup_{z \in (\Gamma_\alpha \times T) \cup \{\emptyset\}} \mathbb{P}\left(\bar{R}((x,m_x),\alpha,\Xi_x^{\wr z \int} + \delta_{(x,m_x)}) \geq t\right)$$

satisfies $\bar{\tau}(t) = 0$ for some $t > 0$ (resp. $\bar{\tau}(t) \leq C_1 e^{-C_2 t}$ for all $t > 0$, where C_1 and C_2 are some positive constants independent of t).

As in the unrestricted case, the definition ensures that $\{\bar{R}((x,m_x),\alpha,\mathscr{X}^{\wr z \int}) \leq t\}$ is a function of $\mathscr{X}_{B(x,t) \cap \Gamma_\alpha}$ for all $x \in \mathbb{R}^d$ and $\alpha,\ t \in \mathbb{R}_+$.

Assumption 2.2 Translation Invariance

We write $d(x,A) := \inf\{d(x,y);\ y \in A\}$, $A \pm B := \{x \pm y;\ x \in A,\ y \in B\}$ for $x \in \mathbb{R}^d$ and $A, B \in \mathscr{B}\left(\mathbb{R}^d\right)$. Recall that the shift operator is defined as $\Xi^x(\cdot \times D) := \Xi((\cdot + x) \times D)$ for all $x \in \mathbb{R}^d$, $D \in \mathscr{T}$.

Unrestricted Case: We define the translation-invariance for the unrestricted case as in [PY01].

Definition 2.5. *The score function η is translation invariant if for all locally finite configuration $\mathscr{X}$, $x \in \mathbb{R}^d$ and $m \in T$, $\eta((\mathbf{0},m),\mathscr{X}) = \eta((x,m),\mathscr{X}^x) =: g(\mathscr{X})\mathbf{1}_{(0,m) \in \mathscr{X}}$.*

Here, $g(\mathscr{X})$ is not affected by m since $\eta\left((x,m),\mathscr{X}\right)$ is understood as 0 if $(x,m) \notin \mathscr{X}$.

Restricted Case: As a translation may send a configuration to the outside of Γ_α, resulting in a completely different configuration inside Γ_α, it is necessary to focus on the part that affects the score function. Therefore, we expect the score function to take the same value for two configurations if the parts within their stabilising radii are completely inside Γ_α and one is a translation of the other. More precisely, we have the following definition.

Definition 2.6. *A stabilising score function η with stabilisation radius $\bar{R}$ is called translation invariant if for any $\alpha > 0$, $x \in \Gamma_\alpha$ and $\mathscr{X} \in \boldsymbol{C_S}$ such that $\bar{R}((x,m),\alpha,\mathscr{X}) \leq d(x,\partial\Gamma_\alpha)$, where ∂A stands for the boundary of A, then $\eta\left((x,m),\mathscr{X},\Gamma_\alpha\right) = \eta\left((x',m),\mathscr{X}',\Gamma_{\alpha'}\right)$ and $\bar{R}((x',m),\alpha',\mathscr{X}') = \bar{R}((x,m),\alpha,\mathscr{X})$ for all $\alpha' > 0$, $x' \in \Gamma_{\alpha'}$ and $\mathscr{X}' \in \boldsymbol{C_S}$ such that $\bar{R}((x',m),\alpha',\mathscr{X}') \leq d(x',\partial\Gamma_{\alpha'})$ and $\left(\mathscr{X}'_{B(x',\bar{R}((x,m),\alpha,\mathscr{X}))}\right)^{x'} = \left(\mathscr{X}_{B(x,\bar{R}((x,m),\alpha,\mathscr{X}))}\right)^x$.*

Noting that there is a tacit assumption of consistency in Definition 2.6, which implies that if η is translation invariant in Definition 2.6, then there exists a $\bar{g} : \boldsymbol{C_S} \to \mathbb{R}$ such that

$$\lim_{\alpha \to \infty} \eta\left((0,m),\mathscr{X},\Gamma_\alpha\right) = \bar{g}\left(\mathscr{X}\right)\mathbf{1}_{(0,m) \in \mathscr{X}}$$

for $\mathscr{L}_T$ almost all $m \in T$ and almost all realisations $\mathscr{X}$ of the marked point process Ξ. The existence of the limit $\bar{g}$ ensures a mapping between a restricted score function satisfying the translation-invariance in Definition 2.6 and its unrestricted counterpart.

Definition 2.7. *For each restricted score function η that is translation-invariant as in Definition 2.6, define $\bar{\eta}$ as*

$$\bar{\eta}((x,m),\mathscr{X}) = \bar{g}(\mathscr{X}^x)\mathbf{1}_{(x,m) \in \mathscr{X}}$$

for all $(x,m) \in \boldsymbol{S}$ and $\mathscr{X} \in \boldsymbol{C_S}$.

It is not hard to check that, for any translation invariant score function η, the function $\bar{\eta}$ in Definition 2.7 is a translation invariant score function as in Definition 2.5

and $R((x,m),\mathscr{X}) = \lim_{\alpha\to\infty} \bar{R}((x,m),\alpha,\mathscr{X})$ is its stabilising radius. Consequently, $\bar{\eta}$ is range-bounded (resp. exponentially stabilising) in the sense of Definition 2.3 if η is range-bounded (resp. exponentially stabilising) in the sense of Definition 2.4. Moreover, if $B(x,R(x)) \subset \Gamma_\alpha$, then $\bar{R}(x,\alpha) = R(x)$, and if $B(x,R(x)) \not\subset \Gamma_\alpha$, then $\bar{R}(x,\alpha) > d(x,\partial\Gamma_\alpha)$, but there is no definite relationship between $\bar{R}$ and R.

Assumption 2.3 Moment condition

We need moment conditions of both the marked point process Ξ and the score function η. We say that the marked point process Ξ satisfies the kth moment condition if there exists a nonempty open set $B \subset \mathbb{R}^d$ such that

$$\mathbb{E}\left(\bar{\Xi}(B)^k\right) < \infty. \tag{2.4}$$

For the score function η, there are two cases to consider.

Unrestricted Case: The score function η is said to satisfy the kth moment condition if

$$\mathbb{E}\left[\left|\eta\left((\mathbf{0},M_{\mathbf{0}}),\Xi_{\mathbf{0}}+\delta_{(\mathbf{0},M_{\mathbf{0}})}\right)\right|^k\right] < \infty. \tag{2.5}$$

Restricted Case: The score function η is said to satisfy the kth moment condition if there exists a positive constant C such that

$$\sup_{\alpha\in\mathbb{R}_+}\sup_{x\in\Gamma_\alpha}\mathbb{E}\left[\left|\eta\left((x,M_x),(\Xi_x)_{\Gamma_\alpha}+\delta_{(x,M_x)}\right)\right|^k\right] \le C. \tag{2.6}$$

One can verify that if η is exponentially stabilising and satisfies (2.6), then its induced $\bar{\eta}$ in Definition 2.7 satisfies the moment condition of the same order in the sense of (2.5).

Assumption 2.4 Variance Condition

The speed of convergence of the normal approximation is determined by the order of $\mathrm{Var}(W_\alpha)$ or $\mathrm{Var}(\bar{W}_\alpha)$. We formulate the bounds of approximation errors in terms of the following variance conditions. If f_1 and f_2 are two functions satisfying

$$\liminf_{x\to\infty} f_1(x)/f_2(x) > 0,$$

then we write $f_1(x) = \Omega(f_2(x))$ as $x\to\infty$.

Unrestricted Case: $\mathrm{Var}(W_\alpha) = \Omega(\alpha^\nu)$ for some $\nu \in (\frac{2}{3},1]$ as $\alpha\to\infty$.

Restricted Case: $\mathrm{Var}(\bar{W}_\alpha) = \Omega(\alpha^\nu)$ for some $\nu \in (\frac{2}{3},1]$ as $\alpha\to\infty$.

The above conditions are clear, but in many cases, we need to establish the order of $\mathrm{Var}(W_\alpha)$ or $\mathrm{Var}(\bar{W}_\alpha)$. If we have the following conditions, we can prove that the variances $\mathrm{Var}(W_\alpha)$ and $\mathrm{Var}(\bar{W}_\alpha)$ have the same order as the volume α, cf. [BYY19, (1.22)].

Unrestricted Case: The unrestricted score function is said to satisfy the *variation condition* if

$$\begin{aligned}\sigma^2 :=&\mathbb{E}\left(\eta((\mathbf{0},M_{\mathbf{0}}),\Xi_{\mathbf{0}}+\delta_{(0,M_{\mathbf{0}})})^2\right)\lambda\\ &+\mathbb{E}\left(\int_{\Gamma_1}\int_{\mathbb{R}^d\setminus\{x\}}\left(\eta((y,M_y),\Xi)\bar{\Xi}(dy)-Pdy\right)\left(\eta((x,M_x),\Xi)\bar{\Xi}(dx)-Pdx\right)\right)\\ =&\mathbb{E}\left(\eta((\mathbf{0},M_{\mathbf{0}}),\Xi_{\mathbf{0}}+\delta_{(0,M_{\mathbf{0}})})^2\right)\lambda\\ &+\frac{\mathbb{E}\left(\int_{\mathbb{R}^d\setminus\{\mathbf{0}\}}\left(\eta((x,M_x),\Xi)\bar{\Xi}(dx)-Pdx\right)\left(\eta((\mathbf{0},M_{\mathbf{0}}),\Xi)\bar{\Xi}(d\mathbf{0})-Pd\mathbf{0}\right)\right)}{d\mathbf{0}} > 0,\end{aligned} \tag{2.7}$$

where $\frac{f(d\mathbf{0})}{d\mathbf{0}} := \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}f(\Gamma_\epsilon)$ if the limit is well defined, $P := \lambda\mathbb{E}(\eta((\mathbf{0},M_{\mathbf{0}}),\Xi_{\mathbf{0}}+\delta_{(0,M_{\mathbf{0}})}))$.

Restricted Case: We establish the following variation condition when the score function is exponentially stabilising. The restricted score function η satisfies the *variation* *condition* if it is exponentially stabilising, and the corresponding $\bar{\eta}$ satisfies

$$\begin{aligned}\bar{\sigma}^2 :=& \mathbb{E}\left(\bar{\eta}((\mathbf{0}, M_{\mathbf{0}}), \Xi_{\mathbf{0}} + \delta_{(0,M_{\mathbf{0}})})^2\right)\lambda \\ &+ \mathbb{E}\left(\int_{\Gamma_1}\int_{\mathbb{R}^d\setminus\{x\}}\left(\bar{\eta}((y, M_y), \Xi)\overline{\Xi}(dy) - \bar{P}dy\right)\left(\bar{\eta}((x, M_x), \Xi)\overline{\Xi}(dx) - \bar{P}dx\right)\right) \\ =& \mathbb{E}\left(\bar{\eta}((\mathbf{0}, M_{\mathbf{0}}), \Xi_{\mathbf{0}} + \delta_{(0,M_{\mathbf{0}})})^2\right)\lambda \\ &+ \frac{\mathbb{E}\left(\int_{\mathbb{R}^d\setminus\{\mathbf{0}\}}\left(\bar{\eta}((x, M_x), \Xi)\overline{\Xi}(dx) - \bar{P}dx\right)\left(\bar{\eta}((\mathbf{0}, M_{\mathbf{0}}), \Xi)\overline{\Xi}(d\mathbf{0}) - \bar{P}d\mathbf{0}\right)\right)}{d\mathbf{0}} > 0, \quad (2.8)\end{aligned}$$

where $\bar{P} := \lambda\mathbb{E}(\bar{\eta}((\mathbf{0}, M_{\mathbf{0}}), \Xi_{\mathbf{0}} + \delta_{(0,M_{\mathbf{0}})}))$.

In the proof of Theorem 2.12 and Remark 5.12, we can see that under the conditions of Theorem 2.12, the variation condition (2.7) (resp. (2.8)) holds if and only if $\mathrm{Var}(W_\alpha) = \Omega(\alpha)$ (resp. $\mathrm{Var}(\bar{W}_\alpha) = \Omega(\alpha)$).

The above variation conditions are still generally difficult to verify, we refer the interested readers to the discussion at Remark (iii) of [BYY19, Theorem 1.14].

With these assumptions, we can establish the convergence rate in terms of the Wasserstein distance defined as

$$d_W(X, Y) := \sup_{h\in\mathscr{F}_{\mathrm{Lip}}} \mathbb{E}\left(h(X) - h(Y)\right), \quad (2.9)$$

where $\mathscr{F}_{\mathrm{Lip}}$ is the set of all Lipschitz functions h on $\mathbb{R}$ such that $|h(x) - h(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Our main result for W_α (unrestricted case) is summarised below.

Theorem 2.8. *Assume that the score function η is translation invariant in Definition 2.5 and satisfies the sixth moment condition* (2.5)*, Ξ satisfies the EDD, the fifth moment condition* (2.4) *and $\mathrm{Var}(W_\alpha) = \Omega(\alpha^\nu)$ for some $\nu \in (\frac{2}{3}, 1]$ as $\alpha \to \infty$.*

(i) If η is range-bounded as in Definition 2.3, then

$$d_W\left(\frac{W_\alpha - \mathbb{E}W_\alpha}{\sqrt{\mathrm{Var}(W_\alpha)}}, Z\right) \leq O\left(\alpha^{-\frac{3}{2}\nu+1}\right).$$

(ii) If η is exponentially stabilising as in Definition 2.3, then

$$d_W\left(\frac{W_\alpha - \mathbb{E}W_\alpha}{\sqrt{\mathrm{Var}(W_\alpha)}}, Z\right) \leq O\left(\alpha^{-\frac{3}{2}\nu+1}\ln(\alpha)^{5d}\right).$$

Remark 2.9. When the variation condition (2.7) holds instead of the variance condition $\mathrm{Var}(W_\alpha) = \Omega(\alpha^\nu)$, Theorem 2.8 holds with $\nu = 1$, i.e., $d_W\left(\frac{W_\alpha - \mathbb{E}W_\alpha}{\sqrt{\mathrm{Var}(W_\alpha)}}, Z\right) \leq O\left(\alpha^{-\frac{1}{2}}\right)$ when η is range-bounded and $d_W\left(\frac{W_\alpha - \mathbb{E}W_\alpha}{\sqrt{\mathrm{Var}(W_\alpha)}}, Z\right) \leq O\left(\alpha^{-\frac{1}{2}}\ln(\alpha)^{5d}\right)$ when η is exponentially stabilising.

The restricted case is of interest in many applications. By adapting the conditions accordingly, we can show that the main result for $\bar{W}_\alpha$ (restricted case) also holds.

Theorem 2.10. *Assume that the score function η is translation invariant in Definition 2.6 and satisfies the sixth moment condition* (2.6)*, Ξ satisfies the EDD, the fifth moment condition* (2.4) *and $\mathrm{Var}(\bar{W}_\alpha) = \Omega(\alpha^\nu)$ for some $\nu \in (\frac{2}{3}, 1]$ as $\alpha \to \infty$.*

(i) If η is range-bounded as in Definition 2.4, then

$$d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\mathrm{Var}(\bar{W}_\alpha)}}, Z\right) \leq O\left(\alpha^{-\frac{3}{2}\nu+1}\right).$$

(ii) If η is exponentially stabilising as in Definition 2.4, then

$$d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\operatorname{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{3}{2}\nu+1}\ln(\alpha)^{5d}\right).$$

Remark 2.11. If the variation condition (2.8) holds instead of the variance condition $\operatorname{Var}(\bar{W}_\alpha) = \Omega(\alpha^\nu)$, Theorem 2.10 holds with $\nu = 1$, i.e., $d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\operatorname{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{1}{2}}\right)$ when η is range-bounded and $d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\operatorname{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{1}{2}}\ln(\alpha)^{5d}\right)$ when η is exponentially stabilising.

We write $f_1 = \Theta(f_2)$ if $f_1 = \Omega(f_2)$ and $f_2 = \Omega(f_1)$. Then in terms of the order of $\operatorname{Var}(\bar{W}_\alpha)$ and $\operatorname{Var}(W_\alpha)$, we have the following results, which can be regarded as the counterparts of [BYY19, Theorem 1.15].

Theorem 2.12. *(a) (unrestricted case) Assume that Ξ satisfies the EDD and the fifth moment condition* (2.4)*, and the score function η satisfies the sixth moment condition* (2.5)*. If η is exponentially stabilising in Definition 2.3, translation invariant in Definition 2.5 and $\sigma^2 > 0$, then* $\operatorname{Var}(W_\alpha) = \Theta(\alpha)$*.*

(b) (restricted case) Assume that Ξ satisfies the EDD and the fifth moment condition (2.4)*, and the score function η satisfies the sixth moment condition* (2.6)*. If η is exponentially stabilising in Definition 2.4, translation invariant in Definition 2.6 and $\bar{\sigma}^2 > 0$, then* $\operatorname{Var}(\bar{W}_\alpha) = \Theta(\alpha)$*.*

The proofs of the main and auxiliary results are postponed to Section 5, and we turn our attention to the EDD point processes and applications of the main results first.

3 EDD point processes

The cornerstone model of point processes is the Poisson point process, where points behave independently in different regions, and it is obvious that a Poisson point process satisfies the EDD. A variety of extensions from Poisson point processes have been developed to capture dependent random structures in the literature, and many of such extensions are covered in [DV03, DV08]. In particular, significant development has been made in the determinantal point processes and the Gibbs point processes [SKM95, S00, BGMSS05, DV03, D19]. The connections between the two classes were investigated in [GY05]. Both classes have also been well assessed for statistical inferences [MW04, MW07, LMR15]. In this section, we show that the Gibbs point processes with nearly finite range potentials, the determinantal point processes with kernels decaying fast enough, the r-dependent point processes and the Boolean models all satisfy the EDD. Moreover, we also show that the finite superposition of EDD point processes again satisfies EDD.

3.1 Rarified Gibbs point process

For ease of reading, we briefly introduce the idea of perfect simulation in [SY13, Section 3] for the infinite volume Gibbs point processes with nearly finite range potentials Ψ. To this end, let Ψ be a $[0, \infty]$ valued functional on the finite configuration space $\boldsymbol{C}_{\mathbb{R}^d,b} := \{\xi \in \boldsymbol{C}_{\mathbb{R}^d} : \xi(\mathbb{R}^d) < \infty\}$ satisfying i) translation invariant: $\Psi(\mathscr{X}) = \Psi(x + \mathscr{X})$ for all $x \in \mathbb{R}^d$ and $\mathscr{X} \in \boldsymbol{C}_{\mathbb{R}^d,b}$; ii) rotation invariant: $\Psi(\mathscr{X}) = \Psi(\mathscr{X}')$ for all $\mathscr{X} \in \boldsymbol{C}_{\mathbb{R}^d,b}$ and all rotations $\mathscr{X}'$ of $\mathscr{X}$; iii) non-decreasing: $\Psi(\mathscr{X}) \le \Psi(\mathscr{X}')$ for all $\mathscr{X}, \mathscr{X}' \in \boldsymbol{C}_{\mathbb{R}^d,b}$ such that $\mathscr{X} \subset \mathscr{X}'$; iv) non-degenerate: $\Psi(\{x\}) < \infty$ for all $x \in \mathbb{R}^d$. Let $D_n := [-n, n]^d$ for $n \in \mathbb{N} := \{1, 2, \dots\}$, $\Psi_D(\mathscr{X}) := \Psi(\mathscr{X} \cap D)$ for $D \in \mathscr{B}(\mathbb{R}^d)$, $\mathscr{P}^{\beta\Psi}$ and $\mathscr{P}_D^{\beta\Psi}$ denote the

Gibbs point process with inverse temperature $\beta > 0$ and potential Ψ and Ψ_D respectively. Write $\Delta(\mathbf{0}, \mathscr{X}) := \Delta^{\Psi}(\mathbf{0}, \mathscr{X}) := \Psi(\mathscr{X} \cup \{\mathbf{0}\}) - \Psi(\mathscr{X})$, $\mathbf{0} \notin \mathscr{X}$, with $\infty - \infty := 0$, and assume that $\Delta(\mathbf{0}, \mathscr{X})$ satisfies

$$\Delta_{[r]}(\mathbf{0}, \mathscr{X} \cap B_r(\mathbf{0})) \leq \Delta(\mathbf{0}, \mathscr{X}) \leq \Delta^{[r]}(\mathbf{0}, \mathscr{X} \cap B_r(\mathbf{0}))$$

for some non-negative, translation invariant functions $\Delta_{[r]}$ and $\Delta^{[r]}$, and $\Delta_{[r]}$ and $\Delta^{[r]}$ are assumed to be respectively increasing and decreasing in r. We say $\beta\Psi$ has a nearly finite range if there exists a decreasing continuous function $\psi^{(\beta)} : \mathbb{R}^+ \to [0, 1]$ satisfying that $\psi^{(\beta)}(0) = 1$, $\psi^{(\beta)}(r)$ decays exponentially fast in r and

$$e^{-\beta\Delta_{[r]}(\mathbf{0}, \mathscr{X} \cap B_r(\mathbf{0}))} - e^{-\beta\Delta^{[r]}(\mathbf{0}, \mathscr{X} \cap B_r(\mathbf{0}))} \leq \psi^{\beta}(r)$$

for all $r > 0$ and $\mathscr{X} \in \mathbf{C}_{\mathbb{R}^d}$. [SY13] established that the class of Gibbs point processes having a nearly finite range $\beta\Psi$ includes

i) the point process with a pair potential function $\Psi(\mathscr{X}) = \sum_{x \neq y} \phi(\|x - y\|)$, where $\phi : [0, \infty) \to [0, \infty)$ has a compact support or

$$\phi(r) \begin{cases} \leq K_1 \exp(-K_2 r), & r \in [r_0, \infty), \\ = \infty, & r \in (0, r_0), \end{cases}$$

for constants $K_1,\ K_2 \in \mathbb{R}_+$;

ii) the point process defined by the continuum Widom-Rowlinson model for spheres of type A having centres $\mathscr{X}$ and spheres of type B having centres $\mathscr{Y}$:

$$\Psi(\mathscr{X} \cup \mathscr{Y}) = \begin{cases} \alpha_1 \mathrm{card}(\mathscr{X}) + \alpha_2 \mathrm{card}(\mathscr{Y}) + \alpha_3, & d(\mathscr{X}, \mathscr{Y}) > 2a, \\ \infty, & \text{otherwise}, \end{cases}$$

where $\mathrm{card}(\mathscr{X})$ is the cardinality of $\mathscr{X}$, a is the common radii of the spheres and α_i's are positive constants;

iii) the area interaction point process with

$$\Psi(\mathscr{X}) = \mathrm{Vol}\left(\cup_{x \in \mathscr{X}}(x + K)\right) + \alpha_1 \mathrm{card}(\mathscr{X}) + \alpha_2,$$

where α_i's are positive constants and K is a fixed compact convex set;

iv) the hard-core process with

$$\Psi(\mathscr{X}) = \begin{cases} \alpha_1 \mathrm{card}(\mathscr{X}) + \alpha_2, & \inf_{x, y \in \mathscr{X}, x \neq y} |x - y| \geq r_0, \\ \infty, & \inf_{x, y \in \mathscr{X}, x \neq y} |x - y| < r_0, \end{cases}$$

where r_0 and α_i's are positive constants.

Moreover, extensions of the point process with a pair potential function have been developed in [G99, RMO18, F22] to detect multivariate interactions in spatial point patterns, and most of these extensions have nearly finite range potentials. [SY13, Section 3.3] states that the infinite volume Gibbs point process $\mathscr{P}^{\beta\Psi}$ exists and is the thermodynamic limit of $\mathscr{P}_{D_n}^{\beta\Psi}$. The next lemma says that $\mathscr{P}^{\beta\Psi}$ also satisfies the EDD.

Lemma 3.1. *The Gibbs point process $\mathscr{P}^{\beta\Psi}$ with a nearly finite range potential satisfies the EDD.*

Proof. Using the idea of perfect simulation introduced in [FFG02] and [SY13, Sections 3.2 & 3.3], we can construct a stationary homogeneous free birth and death process $\{\rho(t)\}_{t \in \mathbb{R}}$

such that $\rho(t) \stackrel{d}{=} \mathscr{P}^{\beta\Psi}$ for all t. For $B \in \mathscr{B}(\mathbb{R}^d)$, define the *ancestor clan* $\boldsymbol{A}_B^{\beta\Psi}(0) =: \boldsymbol{A}_B^{\beta\Psi}$ with respect to the process $\{\rho(t)\}_{t\in\mathbb{R}}$ as the accepted births in $\rho(0) \cap B$, their ancestors, the ancestors of their ancestors and so forth. From the construction, $\rho(0) \cap A$ and $\rho(0) \cap B$ are conditionally independent given $\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} = \emptyset$. [SY13, (3.6)] states that the ancestor clan $\boldsymbol{A}_B^{\beta\Psi}$ satisfies that for all $r \in \mathbb{R}_+$,

$$\mathbb{P}\left[\mathrm{diam}(\boldsymbol{A}_B^{\beta\Psi}) \geq r + \mathrm{diam}(B)\right] \leq C(\mathrm{Vol}(B) \vee 1)\exp(-r/C)$$

for some positive constant C depending on the distribution of the process $\mathscr{P}^{\beta\Psi}$ only. Noting that the ancestor clan $\boldsymbol{A}_B^{\beta\Psi}$ starts from the accepted births in $\rho(0)\cap B$, if $\rho(0)\cap B \neq \emptyset$, then $\boldsymbol{A}_B^{\beta\Psi} \cap B \neq \emptyset$, which ensures that if B is a cube in $\mathbb{R}^d$ with centre x and diagonal length $\mathrm{diam}(B) := 2r$, then by the rotation invariance, there is a constant C such that

$$\mathbb{P}\left[\boldsymbol{A}_B^{\beta\Psi} \not\subseteq B(x, 3r + r')\right] \leq C(r^d \vee 1)\exp(-r'/C) \tag{3.1}$$

for all $r' \in \mathbb{R}_+$.

For two bounded sets $A,\ B \in \mathscr{B}(\mathbb{R}^d)$ with $d(A, B) =: r_0$, without loss of generality, we assume that $r_0 \geq 1$. Let $\Xi \stackrel{d}{=} \mathscr{P}^{\beta\Psi}$ and $\{\tilde{\rho}(t)\}_{t\in\mathbb{R}}$ be an independent copy of $\{\rho(t)\}_{t\in\mathbb{R}}$. Since $\rho(0) \cap A$ and $\rho(0) \cap B$ are conditionally independent given $\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} = \emptyset$, we have

$$\begin{aligned}
&\beta_{A,B}\\
=&d_{TV}(\rho(0) \cap (A \cup B), (\rho(0) \cap A) \cup (\tilde{\rho}(0) \cap B))\\
=&d_{TV}\left(\rho(0) \cap (A \cup B), (\rho(0) \cap A) \cup (\tilde{\rho}(0) \cap B)\middle|\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} = \emptyset\right)\mathbb{P}(\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} = \emptyset)\\
&+ d_{TV}\left(\rho(0) \cap (A \cup B), (\rho(0) \cap A) \cup (\tilde{\rho}(0) \cap B)\middle|\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} \neq \emptyset\right)\mathbb{P}(\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} \neq \emptyset)\\
\leq&\mathbb{P}(\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} \neq \emptyset).
\end{aligned} \tag{3.2}$$

Since A and B are bounded, $\mathrm{diam}(A)$ and $\mathrm{diam}(B)$ are finite. We can find a set of disjoint cubes $\{\mathbb{C}_{i,j}\}_{i\in\{1,2\},0\leq j\leq n_i}$ with diagonal length $\frac{r_0}{16}$ such that $A \subset \cup_{j\leq n_1}\mathbb{C}_{1,j}$ and $B \subset \cup_{j\leq n_2}\mathbb{C}_{2,j}$ for positive integers $n_1 \leq C_1(\mathrm{diam}(A) \vee 1)^d r_0^{-d}$ and $n_2 \leq C_1(\mathrm{diam}(B)^d \vee 1)r_0^{-d}$. Write the centre of $\mathbb{C}_{i,j}$ as $c_{i,j}$. Then for any $j_1 \leq n_1$, $j_2 \leq n_2$, $d(\mathbb{C}_{1,j_1}, \mathbb{C}_{2,j_2}) \geq d(A, B) - (\mathrm{diam}(\mathbb{C}_{1,j_1}) + \mathrm{diam}(\mathbb{C}_{2,j_2})) = \frac{7r_0}{8}$, it follows from (3.1) with $2r = r' = \frac{r_0}{16}$ that

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{A}_{\mathbb{C}_{1,j_1}}^{\beta\Psi} \cap \boldsymbol{A}_{\mathbb{C}_{2,j_2}}^{\beta\Psi} \neq \emptyset)\\
\leq&\mathbb{P}\left(\boldsymbol{A}_{\mathbb{C}_{1,j_1}}^{\beta\Psi} \not\subseteq B\left(c_{1,j_1}, \frac{5r_0}{32}\right)\right) + \mathbb{P}\left(\boldsymbol{A}_{\mathbb{C}_{2,j_2}}^{\beta\Psi} \not\subseteq B\left(c_{2,j_2}, \frac{5r_0}{32}\right)\right)\\
\leq&C_2 r_0^d \exp(-C_3 r_0)
\end{aligned} \tag{3.3}$$

for some positive constants C_2 and C_3 independent of j_1 and j_2. Also, from the definition of the ancestor clans, if a set B is covered by a class of sets $\{B_1, \ldots, B_n\}$, then $\boldsymbol{A}_B^{\beta\Psi} \subset \cup_{i\leq n}\boldsymbol{A}_{B_i}^{\beta\Psi}$. Together with (3.2) and (3.3), we have

$$\begin{aligned}
\beta_{A,B} \leq&\mathbb{P}(\boldsymbol{A}_A^{\beta\Psi} \cap \boldsymbol{A}_B^{\beta\Psi} \neq \emptyset)\\
\leq&\mathbb{P}\left(\left(\cup_{j_1\leq n_1}\boldsymbol{A}_{\mathbb{C}_{1,j_1}}^{\beta\Psi}\right) \cap \left(\cup_{j_2\leq n_2}\boldsymbol{A}_{\mathbb{C}_{2,j_2}}^{\beta\Psi}\right) \neq \emptyset\right)\\
\leq&\sum_{j_1\leq n_1}\sum_{j_2\leq n_2}\mathbb{P}\left(\boldsymbol{A}_{\mathbb{C}_{1,j_1}}^{\beta\Psi} \cap \boldsymbol{A}_{\mathbb{C}_{2,j_2}}^{\beta\Psi} \neq \emptyset\right)\\
\leq&n_1 n_2 C_2 r_0^d \exp(-C_3 r_0) \leq C_1^2 C_2(\mathrm{diam}(A)^d \vee 1)(\mathrm{diam}(B)^d \vee 1)\exp(-C_3 r_0),
\end{aligned}$$

which completes the proof. □

Lemma 3.1 ensures that Theorem 2.12 is applicable to all geometric statistics arising from a Gibbs point process $\mathscr{P}^{\beta\Psi}$ with a nearly finite range potential. In Section 4, we demonstrate its use in two examples: the total edge length of a k-nearest neighbour graph and the total timber volume in a given range of forest.

3.2 Determinantal point process

The determinantal point processes are a broad class of point processes such that the distributions can be characterised by the determinants of given functions. More precisely, we say that Ξ is a *determinantal point process* on space $\mathbb{R}^d$ with kernel K if it is a simple point process on $\mathbb{R}^d$ with the joint intensities given by

$$\rho_n(x_1,\dots,x_n) = \det\left[K(x_i,x_j)\right]_{1\le i,j\le n}$$

for $n \in \mathbb{N}$ and $x_1,\dots,x_n \in \mathbb{R}^d$ [GY05]. These processes are widely used in random matrix theory and mathematical physics.

If there is a covariance function K_0 on $\mathbb{R}$ such that $K(x,y) = K_0(x-y)$ for all $x,y \in \mathbb{R}$, then the corresponding determinantal point process is stationary [LMR15, Section 3]. By [LMR15, Theorem 1] and [BL16], there exists a class of continuous complex covariance functions K_0 such that the corresponding determinantal point processes exist, and [LMR15, §3.3] further states that when K_0 is supported on $[-r,r]$ for some positive constant r, then the corresponding determinantal point process Ξ satisfies that Ξ_A and Ξ_B are independent for $d(A,B) \ge r$, i.e., it is r-dependent as defined in Section 3.3, hence it is an EDD process. In fact, this is a special case of fast decaying kernels. The next lemma says that the determinantal point process satisfies the EDD if its kernel function decays fast enough.

Lemma 3.2. *If the kernel K of the determinantal point process Ξ satisfies $\|K\|_\infty := \sup_{x,y\in\mathbb{R}^d} |K(x,y)| < \infty$ and there exist constants $C_i \in \mathbb{R}_+$, $1 \le i \le 4$, such that $|K(x,y)| \le C_1 e^{-C_2 e^{C_3|x-y|}}$ for all $x,\ y \in \mathbb{R}^d$ such that $\|x-y\| \ge C_4$, then Ξ is an EDD point process.*

Proof. We use Remark 2.1 to prove the claim. Let $p_A = \mathrm{Vol}(A)$ and $p_B = \mathrm{Vol}(B)$, [P19, Theorem 4.1] states that

$$\beta_{A,B} \le 2p_Ap_B(1+2p_A\|K\|_\infty)(1+2p_B\|K\|_\infty)e^{2(p_A+p_B)\|K\|_\infty}\omega\left(d(A,B)\right)^2, \tag{3.4}$$

where $\omega(r) := \sup_{|x-y|\ge r}|K(x,y)|$. It is easy to see that

$$1+2p_A\|K\|_\infty \le 3(p_A\vee 1)(\|K\|_\infty \vee 1) \text{ and } 1+2p_B\|K\|_\infty \le 3(p_B\vee 1)(\|K\|_\infty\vee 1),$$

hence

$$2p_Ap_B(1+2p_A\|K\|_\infty)(1+2p_B\|K\|_\infty) \le 18(\|K\|_\infty^2\vee 1)(p_A^2\vee 1)(p_B^2\vee 1). \tag{3.5}$$

For the remaining part, we take $C_5 = \left(1\vee\frac{2}{C_3}\right)\left(1\vee\ln\left(\frac{4\|K\|_\infty}{C_2}\right)\right)$, then there exists an $r_0 \in \mathbb{R}_+$ such that $s := d(A,B) \ge C_5\ln s$ for all $s \ge r_0$. Set $\theta_3' = C_5 \vee r_0$, $\theta_4' = e$ and

$\Theta = p_A \vee p_B \vee e$, if $s \geq \theta_3' \ln \Theta$, then

$$\begin{aligned}
& e^{2(p_A+p_B)\|K\|_\infty} \omega\left(d(A,B)\right)^2 \\
& \leq C_1^2 e^{4\Theta\|K\|_\infty} \exp\left\{-2C_2 e^{C_3 s}\right\} \\
& \leq C_1^2 e^{4\Theta\|K\|_\infty} \exp\left\{-C_2 e^{C_3 C_5 \ln \Theta} - C_2 e^{C_3 C_5 \ln s}\right\} \\
& = C_1^2 e^{4\Theta\|K\|_\infty} \exp\left\{-C_2 \Theta^{C_3 C_5} - C_2 s^{C_3 C_5}\right\} \\
& \leq C_1^2 e^{4\Theta\|K\|_\infty} \exp\left\{-C_2 \Theta^{1+0.5 C_3 C_5} - C_2 s\right\} \\
& \leq C_1^2 e^{4\Theta\|K\|_\infty} \exp\left\{-C_2 \Theta e^{\ln\left(\frac{4\|K\|_\infty}{C_2}\right)} - C_2 s\right\} \\
& = C_1^2 e^{-C_2 s},
\end{aligned} \tag{3.6}$$

hence the EDD follows from combining (3.4), (3.5), (3.6) and applying Remark 2.1 with $\theta_0' = 2$, $\theta_1' = 18C_1^2(\|K\|_\infty^2 \vee 1)$ and $\theta_2' = C_2$. □

Lemma 3.2 ensures that Theorem 2.12 is also applicable to all geometric statistics and the timber volume arising from a determinantal point process with its kernel satisfying the conditions in Lemma 3.2.

Now, we consider two more models that satisfy the EDD so that Theorem 2.12 can be applied to geometric statistics driven by these point processes.

3.3 r-dependent point process

A point process Ξ on $\mathbb{R}^d$ is said to be *r-dependent* if for any Borel sets A, $B \in \mathscr{B}(\mathbb{R}^d)$ with distance $d(A,B) \geq r$, Ξ_A and Ξ_B are independent. The definition implies that $\beta_{A,B} = 0$ for $d(A,B) \geq r$, hence the following lemma is trivial.

Lemma 3.3. *An r-dependent point process satisfies the EDD.*

One example of the r-dependent point processes is the Matérn hard-core process [DV03, p. 298]. Let $\Xi' := \sum_{i\in\mathbb{N}} \delta_{X_i}$ be a homogeneous Poisson point process on $\mathbb{R}^d$. Then we can construct a hard-core Poisson process by setting $\Xi := \sum_{i\in\mathbb{N}} \delta_{X_i} \mathbf{1}_{B(X_i,r/2)\cap\Xi'=\{X_i\}}$, then for any Borel sets A, $B \in \mathscr{B}(\mathbb{R}^d)$ with distance $d(A,B) \geq r$, $\sigma(\Xi_A) \subset \sigma(\Xi'_{B(A,r/2)})$ is independent of $\sigma(\Xi_B) \subset \sigma(\Xi'_{B(B,r/2)})$, where $B(A,r') := \{x : d(x,A) < r'\}$ for all $A \in \mathscr{B}(\mathbb{R}^d)$ and $r' \in \mathbb{R}_+$, so Ξ is r-dependent.

3.4 Boolean model

The Boolean model is a special class of the *germ-grain model* (see, for example, [HM99]). We call a point process $\Xi = \cup_{n\in\mathbb{N}}(X_n + \Xi(n))$ a germ-grain model, where the grains $\{\Xi(n)\}_{n\in\mathbb{N}}$ are $i.i.d.$ point processes and germs $\{X_n\}$ are independent of $\{\Xi(n)\}_{n\in\mathbb{N}}$ and form a stationary point process $\Xi' := \sum_{n\in\mathbb{N}} \delta_{X_n}$. A germ-grain model is called the *Boolean model* [DV03, p. 206] or the *Poisson cluster process* [D13, p. 101] if Ξ' is Poisson.

For the Boolean model, assume that $\Xi(n)$'s are bounded, that is, there exists some $r \in \mathbb{R}_+$ such that $\mathbb{P}(\Xi(n) \cap B(\mathbf{0},r)^c = \emptyset) = 1$, then for any Borel sets A, $B \in \mathscr{B}(\mathbb{R}^d)$ with distance $d(A,B) \geq 4r$, $\sigma(\Xi_A) \subset \sigma\left(\left\{\Xi'_{B(A,r)}, \Xi(n) \text{ such that } X_n \in B(A,r)\right\}\right)$ is independent of $\sigma(\Xi_B)$, so Ξ is a $(4r)$-dependent process and, consequently, it satisfies the EDD. More generally, we have the following conclusion.

Lemma 3.4. *For the Boolean model defined above, if there exist positive constants r_0, C_1 and C_2 such that $\mathbb{P}(\Xi(n) \cap B(\mathbf{0},r)^c \neq \emptyset) \leq C_1 \exp(-C_2 r)$ for all $r \geq r_0$, then Ξ satisfies the EDD.*

Proof. Since β_{A_1,A_2} is a non-decreasing function in the sense of inclusion (see Remark 2.2), without loss of generality, we take A_1 and A_2 as two balls with centres O_1 and O_2, radii r_1 and r_2, set $R = d(A_1, A_2)$, and we assume $R \geq 4r_0$. For the points $\mathscr{X}_i$ in A_i contributed by $\Xi(n)$ with its germ $X_n \in B(O_i, r_i + R/4)$ and $\Xi(n) \cap B(\mathbf{0}, R/4)^c = \emptyset$, these points $\mathscr{X}_i$ and their dependents are in $B(O_i, r_i + R/2)$, so $\mathscr{X}_1$ and $\mathscr{X}_2$ are independent. In other words, the contribution of dependent points is from those violating these conditions. By abuse of notation, we write $\Xi(x)$ as the grain of the germ at $x \in \Xi'$, $V(r) = \frac{\pi^{d/2}}{\Gamma(1+d/2)} r^d$ as the volume of the ball with radius r, and add up the probabilities of all possible cases leading to dependent points in A_1 and A_2 to get

$$\begin{aligned}
\beta_{A_1,A_2} \leq & \sum_{i=1}^{2} \int_{B(O_i, r_i + R/4)} \mathbb{P}\left(\Xi(x) \cap B(\mathbf{0}, R/4)^c \neq \emptyset\right) \mathbb{E}\Xi'(dx) \\
& + \sum_{i=1}^{2} \int_{B(O_i, r_i + R/4)^c} \mathbb{P}\left(\Xi(x) \cap B(\mathbf{0}, \|x - O_i\| - r_i)^c \neq \emptyset\right) \mathbb{E}\Xi'(dx) \\
\leq & \sum_{i=1}^{2} C_1 \lambda e^{-C_2 R/4} V(r_i + R/4) + \sum_{i=1}^{2} \int_{r_i + R/4}^{\infty} C_1 \lambda e^{-C_2 (r - r_i)} dV(r) \\
\leq & C_3 e^{-C_2 R/4} (R \vee r_1 \vee r_2 \vee 1)^d,
\end{aligned}$$

for $C_3 \in \mathbb{R}_+$, where $\lambda = \mathbb{E}\Xi'(d\mathbf{0})/d\mathbf{0}$. Choose $R_0 \geq 1 \vee (4r_0)$ such that $\frac{R}{\ln R} \geq \frac{8d}{C_2}$ for all $R \geq R_0$, set $\theta_4 = e^{R_0}$, $\theta_3 = 1$, then for $R \geq \ln((2r_1) \vee (2r_2) \vee \theta_4)$, we have

$$\beta_{A_1,A_2} \leq C_3((2r_1)^d \vee 1)((2r_2)^d \vee 1) e^{-C_2 R/8},$$

which implies (2.1) with $\theta_0 = d$, $\theta_1 = C_3$, $\theta_2 = C_2/8$. □

3.5 Finite superposition of independent EDD point processes

From the definition, the superposition of a sequence of point processes on the same carrier space is again a point process, which is the counterpart of the sum of random variables for point processes. Similar to the property introduced in [BYY19, F20], the superposition of finitely many independent EDD point processes still satisfies the EDD.

Lemma 3.5. *Let Ξ_1, Ξ_2, ..., Ξ_n be n independent point processes on $\mathbb{R}^d$ satisfying EDD, then $\Xi := \cup_{1 \leq i \leq n} \Xi_i$ also satisfies EDD.*

Proof. Assume that the coefficients of Ξ_j in (2.1) are $\theta_{0,j} \in \mathbb{R}_0$, $\theta_{i,j} \in \mathbb{R}_+$, $1 \leq i \leq 4$, $1 \leq j \leq n$, then Ξ satisfies EDD with coefficients $\theta_i := \max_{1 \leq j \leq n} \theta_{i,j}$, $i \in \{0, 1, 3, 4\}$ and $\theta_2 := \min_{1 \leq j \leq n} \theta_{2,j}$. □

Note that the superposition of n *i.i.d.* determinantal point processes is called *n-determinantal point process*; if each component satisfies EDD, the n-determinantal point process also satisfies EDD according to Lemma 3.5.

4 Applications

The asymptotic behaviour of geometric functionals has been of considerable interest in the last three decades, and our main normal approximation results can be applied to a large class of geometric functionals on various random graphs, including the k-nearest neighbour graph, the Voronoi graph, the sphere of influence graph, the Delaunay triangulation, the Gabriel graph and the relative neighbourhood graph [D88, T82] with vertices driven by a point process satisfying the EDD. The idea of checking the conditions can be adapted from those introduced in [PY01, CX24].

The limit theory of geometric functionals with determinantal point process input or Gibbs point process input is investigated in [BYY19]. Error bounds of a normal approximation in terms of the Kolmogorov distance for the geometric functions with Gibbsian input were derived in [XY15]. To illustrate the use of the main results in Section 2, we bound the errors of a normal approximation to the total edge length in the k-nearest neighbour graph with vertices forming a rarified Gibbs point process or a determinantal point process with a fast decaying kernel. We also bound the error of a normal approximation to the total timber volume in a given range of forest with trees following a marked Gibbs point process.

4.1 The total edge length of k-nearest neighbour graphs

The k-nearest neighbour graph $NG(\mathscr{X})$ with respect to a configuration $\mathscr{X} \in \boldsymbol{C}_{\mathbb{R}^d}$ is a graph with vertices $\mathscr{X}$ and edges $\{x, y\}$ such that y is one of the k points nearest to x or x is one of the k points nearest to y in $\mathscr{X}$. A variant $NG'(\mathscr{X})$ of the $NG(\mathscr{X})$ can be constructed by inserting directed edges $x \to y$ if y is one of the k nearest neighbours of x instead of the undirected edges in $NG(\mathscr{X})$. As in [SY13], we take the score function $\eta(x, \mathscr{X}, \Gamma_\alpha)$ (resp. $\eta'(x, \mathscr{X}, \Gamma_\alpha)$) as one half the sum of the edge lengths of edges in $NG(\Gamma_\alpha \cap (\mathscr{X} \cup \{x\}))$ (resp. $NG'(\Gamma_\alpha \cap (\mathscr{X} \cup \{x\}))$) which are incident to x, and set

$$\bar{W}_\alpha = \sum_{x \in \Xi_{\Gamma_\alpha}} \eta(x, \Xi, \Gamma_\alpha) \text{ and } \bar{W}'_\alpha = \sum_{x \in \Xi_{\Gamma_\alpha}} \eta'(x, \Xi, \Gamma_\alpha). \tag{4.1}$$

We now state the error bounds for a normal approximation of the total edge lengths $\bar{W}_\alpha$ and $\bar{W}'_\alpha$ of $NG(\Xi_{\Gamma_\alpha})$ if Ξ follows $\mathscr{P}^{\beta\Psi}$ or a determinantal point process with fast decay of dependence.

Theorem 4.1. *(a) If Ξ is an infinite range Gibbs point process with nearly finite range potential, then*

$$d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\operatorname{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{1}{2}} \ln(\alpha)^{5d}\right).$$

The statement holds if $\bar{W}_\alpha$ is replaced by $\bar{W}'_\alpha$ in (4.1).

(b) If Ξ is a determinantal point process with continuous kernel K satisfying the conditions in Lemma 3.2, and the total edge length $\bar{W}_\alpha$ satisfies $\operatorname{Var}(\bar{W}_\alpha) = \Omega(\alpha^\nu)$ for some $\nu > \frac{2}{3}$, then

$$d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\operatorname{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{3}{2}\nu+1} \ln(\alpha)^{5d}\right).$$

The statement holds true if $\bar{W}_\alpha$ is replaced by $\bar{W}'_\alpha$ in (4.1).

Remark 4.2. [XY15] proved that a normal approximation error of $W_\alpha := \sum_{x \in \Xi_{\Gamma_\alpha}} \eta(x, \Xi, \Gamma_\infty)$ in terms of the Kolmogorov distance can be bounded above by $O\left(\alpha^{-\frac{1}{2}} \ln(\alpha)^{2d}\right)$, which is slightly better than the error bound for $\bar{W}_\alpha$ in terms of the Wasserstein distance in Theorem 4.1 (a).

Proof of Theorem 4.1. We only show the claims for the undirected case, and the directed case can be handled using the same idea. In this case, the total edge length $\bar{W}_\alpha$ can be represented as

$$\eta(x, \mathscr{X}, \Gamma_\alpha) := \frac{1}{2} \sum_{y \in \mathscr{X}_{\Gamma_\alpha}} \|y - x\| \mathbf{1}_{\{(x,y) \in NG(\mathscr{X}_{\Gamma_\alpha})\}}.$$

The score function η is translation invariant according to the construction, and the EDD is ensured by Lemma 3.1 for (a) and Lemma 3.2 for (b). To apply Theorem 2.10, we need to check the stabilising condition as in Definition 2.4, the moment conditions (2.4), (2.6) and the order of $\mathrm{Var}(\bar{W}_\alpha)$.

(a) According to Lemma 3.3 in [SY13], the Gibbs point process Ξ is *Poisson-like*, which means that Ξ is stochastically dominated by a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda' > 0$, and there exist strictly positive constants $C := C(\lambda')$ and r_1 such that for all $r \geq r_1$, $x \in \mathbb{R}^d$ and $\mathscr{X} \in \boldsymbol{C}_{\mathbb{R}^d \setminus B(x,r)}$, the conditional probability that $B(x,r)$ is not hit by Ξ given $\Xi_{B(x,r)^c} = \mathscr{X}$ satisfies that

$$\mathbb{P}\left[\Xi_{B(x,r)} = \emptyset \middle| \Xi_{B(x,r)^c} = \mathscr{X}\right] \leq e^{-Cr^d}. \tag{4.2}$$

Consequently, the fifth moment condition (2.4) of Ξ is ensured by the Poisson-like property and the moment property of the Poisson point process.

To examine the remaining conditions, for simplicity, we take $d = 2$ and follow the proof of Theorem 3.1 in [CX24] using the idea initiated in [PY01] to achieve the purpose. For completeness, we recap the main steps in [CX24]. For $x \in \Gamma_\alpha$ and $t > 0$, we carve the disk with centre x and radius t into six disjoint circular sectors $T_j(t)$, $1 \leq j \leq 6$, of the same size with x as the centre and $\frac{\pi}{3}$ as their central angle. The sectors are rotated around x such that all straight edges of the sectors have at least the minimal angle $\pi/12$ with respect to the edges of Γ_α. Let $T_j(\infty) = \cup_{t>0} T_j(t)$ for $1 \leq j \leq 6$ and define

$$t_{x,\alpha}(\Xi_x) = \inf\{t: \ \mathrm{card}(T_j(t) \cap \Gamma_\alpha \cap \Xi_x) \geq k+1 \text{ or } T_j(t) \cap \Gamma_\alpha = T_j(\infty) \cap \Gamma_\alpha, \ 1 \leq j \leq 6\}$$

and $\bar{R}(x,\alpha) = 3t_{x,\alpha}(\Xi_x + \delta_x)$. It was demonstrated in [CX24] that $\bar{R}$ is a radius of stabilisation. For the tail distribution of $\bar{R}$, let A_t be an obtuse triangle with the longest side length t and two angles $\pi/12$ and $\pi/3$, define $\tau := \inf\{t: \ \mathrm{card}(\Xi_x \cap A_t) \geq k+1\}$, then [CX24] established that $\mathbb{P}\left(\bar{R}(x,\alpha) > t\right) \leq 6\mathbb{P}(\tau > t/3)$. We can find a constant $C_1 \in \mathbb{R}_+$ such that there are $k+1$ disjoint disks $\{B_1, \dots, B_{k+1}\}$ of radius $C_1 t$ such that $\cup_{i=1}^{k+1} B_i \subset A_{t/3}$. Using the Poisson-like property (4.2), we obtain

$$\begin{aligned}
&\mathbb{P}\left(\bar{R}(x,\alpha) > t\right) \leq 6\mathbb{P}(\tau > t/3) \\
&\leq 6\mathbb{P}\left(\mathrm{card}(\Xi_x \cap A_{t/3}) \leq k\right) \leq 6\mathbb{P}\left(\cup_{i=1}^{k+1}\{\Xi_x \cap B_i = \emptyset\}\right) \\
&\leq 6(k+1)e^{-C(C_1 t)^2},
\end{aligned} \tag{4.3}$$

which ensures the exponential stabilisation in Definition 2.4. For the moment condition (2.6), we again make use of the proof of Theorem 3.1 in [CX24] that

$$\eta\left(x, (\Xi_x)_{\Gamma_\alpha} + \delta_x\right) \leq 3.5k t_{x,\alpha}(\Xi_x + \delta_x). \tag{4.4}$$

Since $\bar{R}(x,\alpha) = 3t_{x,\alpha}(\Xi_x)$, (4.3) implies

$$\sup_{\alpha \in \mathbb{R}_+} \sup_{x \in \Gamma_\alpha} \mathbb{P}\left(t_{x,\alpha}(\Xi_x + \delta_x) > t\right) \leq 6(k+1)e^{-C(3C_1 t)^2}$$

for all $t > 0$. This ensures $\sup_{\alpha \in \mathbb{R}_+} \sup_{x \in \Gamma_\alpha} \mathbb{E}\left(t_{x,\alpha}(\Xi_x + \delta_x)\right)^6 < \infty$ and the moment condition (2.6) is an immediate consequence of (4.4). Finally, we establish (2.8). To this end, define

$$\bar{\eta}(x, \mathscr{X}) := \frac{1}{2} \sum_{y \in \mathscr{X}} \|y - x\| \mathbf{1}_{\{(x,y) \in NG(\mathscr{X})\}},$$

and $\bar{W}_{\infty,\alpha} = \sum_{x \in \Xi_{\Gamma_\alpha}} \bar{\eta}(x, \Xi)$, then we can apply [XY15, Theorem 1.1] to obtain (2.8). The proof of (a) is completed by applying Theorem 2.10 (ii) and Remark 2.11.

(b) Since the kernel K is continuous and fast-decreasing [BYY19, p. 842], it follows from Section 2.2.2 and Section 2.1, Remark (i) of [BYY19] that $\mathbb{E}\left(\Xi(B)^k\right)$ is finite for all bounded $B \in \mathscr{B}(\mathbb{R}^d)$ and all $k \in \mathbb{N}$, which ensures the fifth moment condition (2.4) of Ξ.

To apply Theorem 2.10, as the order of $\mathrm{Var}(\bar{W}_\alpha)$ is assumed, it remains to check the stabilising condition as in Definition 2.4 and the moment condition (2.6). For simplicity, we again take $d = 2$. Following the same argument as that for (4.3) and applying Lemma 5.6 of [BYY19, Supplement], we obtain

$$\mathbb{P}\left(\bar{R}\left(x,\alpha\right) > t\right) \le 6\mathbb{P}\left(\cup_{i=1}^{k+1}\{\Xi_x \cap B_i = \emptyset\}\right) \le 6(k+1)e^{1/8 - K(\mathbf{0},\mathbf{0})\pi(C_1 t)^2/8}, \tag{4.5}$$

which implies the exponential stabilisation as in Definition 2.4. For the moment condition (2.6), we again use the relationship $\bar{R}\left(x,\alpha\right) = 3t_{x,\alpha}(\Xi_x)$ and (4.5) to get

$$\sup_{\alpha\in\mathbb{R}_+}\sup_{x\in\Gamma_\alpha}\mathbb{P}\left(t_{x,\alpha}(\Xi_x + \delta_x) > t\right) \le 6(k+1)e^{1/8 - K(\mathbf{0},\mathbf{0})\pi(3C_1 t)^2/8}$$

for all $t > 0$. The tail behaviour of $t_{x,\alpha}(\Xi_x + \delta_x)$ and (4.4) ensure that

$$\sup_{\alpha\in\mathbb{R}_+}\sup_{x\in\Gamma_\alpha}\mathbb{E}\left(\eta\left(x,(\Xi_x)_{\Gamma_\alpha} + \delta_x\right)^6\right) \le (3.5k)^6 \sup_{\alpha\in\mathbb{R}_+}\sup_{x\in\Gamma_\alpha}\mathbb{E}\left(t_{x,\alpha}(\Xi_x + \delta_x)^6\right) < \infty.$$

The proof of (b) is completed by applying Theorem 2.10 (ii) and Remark 2.11. □

4.2 The timber volume of a forest with Gibbs point process tree locations

Marks play an important role when it is necessary to classify the points. For example, in insurance, marks may be introduced to represent the types of claims [ZS22]; in thinning [DV03, p. 32], marks may be used to stand for the points retained and discarded. In this subsection, we consider the total timber volume in a random forest [CX24, Section 3.3], where marks are used to label the species of the trees. The estimation of the total timber volume in a given range is of great interest in forest science and forest management [C80, LBHS15]. When modelling the natural forest, it is reasonable to assume the locations of trees form a Gibbs point process $\overline{\Xi}$, such as a Poisson point process or a hard-core process. As the contribution of the timber volume from different species of trees varies, we use marks to classify the species. That is, for $x \in \overline{\Xi}$, let $M_x \in T := \{1, \ldots, n\}$ to be the species of the tree at position x. We can assume that the marks are independent of other marks and the locations $\overline{\Xi}$. Then $\Xi := \sum_{x\in\overline{\Xi}} \delta_{(x,M_x)}$ forms a marked Gibbs point process recording the locations and species of trees in a forest. We can model the timber volume of the tree at location x by a function of the location, the species of the tree and the configuration of trees in a finite range around x, adjusted by a quantity ϵ_x due to other unspecified factors. Formally speaking, the timber volume of a tree at location x can be denoted by $(\eta((x,m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) + \epsilon_x) \vee 0$, where η is a non-negative bounded score function such that

$$\eta((x,m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) = \eta((x,m), \Xi_{\Gamma_\alpha \cap B(x,r)}, \Gamma_\alpha)$$

for some positive constant r and

$$\eta((x,m), \Xi_{B(x,r)}, \Gamma_{\alpha_1}) = \eta((x,m), \Xi_{B(x,r)}, \Gamma_{\alpha_2})$$

for all α_1 and α_2 with $B(x,r) \subset \Gamma_{\alpha_1 \wedge \alpha_2}$. Then we have the following result analogous to [CX24, Theorem 3.3].

Theorem 4.3. *If Ξ is an infinite range Gibbs point process with a nearly finite range potential, ϵ_x's are $i.i.d.$ random variables with the sixth moment being finite and the* *positive part* $\epsilon_x^+ := \epsilon_x \vee 0$ *is non-degenerate (i.e.,* $\mathrm{Var}(\epsilon_x^+) > 0$*), and* ϵ_x*'s are independent of* Ξ*, then the timber volume in the range* Γ_α *is*

$$\bar{W}_\alpha := \sum_{x \in \overline{\Xi}_{\Gamma_\alpha}} (\eta((x,m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) + \epsilon_x) \vee 0$$

and it satisfies

$$d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\mathrm{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{1}{2}}\right).$$

Proof. The proof is adapted from [CX24, Theorem 3.3]. We can construct a new marked Gibbs point process $\Xi' := \sum_{x \in \overline{\Xi}} \delta_{(x,(M_x,\epsilon_x))}$ by replacing the marks $\{M_x\}_{x\in\overline{\Xi}}$ of Ξ by $i.i.d.$ marks $\{(M_x,\epsilon_x)\}_{x\in\overline{\Xi}}$ on the space $(T \times \mathbb{R}, \mathscr{T} \times \mathscr{B}(\mathbb{R}))$ independent of the ground process $\overline{\Xi}' = \overline{\Xi}$. The fifth moment condition (2.4) of Ξ' follows from the Poisson-like property, as shown in the proof of Theorem 4.1 (a), and the EDD is ensured by Lemma 3.1. As $\bar{W}_\alpha$ can be represented as the sum of the score function

$$\begin{aligned}\eta'((x,(m,\epsilon_x)), \Xi', \Gamma_\alpha) :=& \eta'((x,(m,\epsilon_x)), \Xi'_{\Gamma_\alpha}, \Gamma_\alpha) \\ :=& \left[(\eta((x,m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) + \epsilon_x) \vee 0\right] \mathbf{1}_{(x,(m,\epsilon_x)) \in \Xi'_{\Gamma_\alpha}},\end{aligned}$$

the translation invariant property and the range-boundedness are direct results of the construction, the sixth moment condition (2.6) is guaranteed by the boundedness of η, the moment condition of ϵ_x's and the Minkowski inequality. Now, we can apply [XY15, Theorem 1.1] again to get $\mathrm{Var}(\bar{W}_\alpha) = \Omega(\alpha)$, and the proof is then completed by applying Theorem 2.10 (i). ☐

Remark 4.4. As discussed in [CX24, Remark 3.2], if the timber volume is determined by its k-nearest neighbouring trees, we can adjust the above proof to show the distribution of the timber volume $\bar{W}_\alpha$ satisfies $d_W\left(\frac{\bar{W}_\alpha - \mathbb{E}\bar{W}_\alpha}{\sqrt{\mathrm{Var}(\bar{W}_\alpha)}}, Z\right) \le O\left(\alpha^{-\frac{1}{2}} \ln(\alpha)^{5d}\right)$.

5 The proofs of the auxiliary and main results

Recalling the shift operator defined in Section 2, we can write $g(\mathscr{X}^x) := \eta((\mathbf{0},m), \mathscr{X}^x) = \eta((x,m), \mathscr{X})$ (resp. $g_\alpha(x,\mathscr{X}) := \eta((x,m), \mathscr{X}, \Gamma_\alpha)$) for all configurations $\mathscr{X}$ with $(x,m) \in \mathscr{X}$ and $\alpha > 0$, so that notations can be simplified, e.g.,

$$\begin{aligned}W_\alpha &= \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m), \Xi) = \int_{\Gamma_\alpha} g(\Xi^x)\overline{\Xi}(dx) = \sum_{x\in\overline{\Xi}_{\Gamma_\alpha}} g(\Xi^x), \\ \bar{W}_\alpha &= \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m), \Xi, \Gamma_\alpha) = \int_{\Gamma_\alpha} g_\alpha(x,\Xi)\overline{\Xi}(dx) = \sum_{x\in\overline{\Xi}_{\Gamma_\alpha}} g_\alpha(x,\Xi),\end{aligned}$$

where $\overline{\Xi}$ is the projection of Ξ on $\mathbb{R}^d$.

We now proceed to establish a few lemmas needed in the proofs. The following lemma bounds the difference between a normal distribution and the standard normal distribution under the Wasserstein distance, and it can be verified directly (see also [CM10, Lemma 2.4]).

Lemma 5.1. *Let* $F_{\mu,\sigma}$ *be the distribution of* $N(\mu,\sigma^2)$*, the normal distribution with mean* μ *and variance* σ^2*, and* $\Phi = F_{0,1}$*, then*

$$d_W(F_{\mu,\sigma}, \Phi) \le |\mu| + \frac{2}{\sqrt{2\pi}}|\sigma - 1|.$$

The following lemma says that the cost of throwing away the terms with large radii of stabilisation is negligible under stabilising conditions. For convenience, we define $W_{\alpha,r} := \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m),\Xi)\mathbf{1}_{R(x)\le r}$, $\bar{W}_{\alpha,r} := \sum_{(x,m)\in\Xi_{\Gamma_\alpha}} \eta((x,m),\Xi,\Gamma_\alpha)\mathbf{1}_{\bar{R}(x,\alpha)\le r}$, which means that we through away the terms with stabilisation radii greater than r from W_α and $\bar{W}_\alpha$.

Lemma 5.2. *(a) (unrestricted case) If the score function is exponentially stabilising in Definition 2.3, then we have*

$$d_{TV}(W_\alpha, W_{\alpha,r}) \le C_1 \alpha e^{-C_2 r}$$

for some positive constants C_1, C_2.

(b) (restricted case) The restricted counterpart of (a) with Definition 2.3 and W replaced by Definition 2.4 and $\bar{W}$, resp., holds.

Proof. We first prove (b). Recall that $M_x \sim \mathscr{L}_T$ is the mark of the point $x \in \overline{\Xi}$, and it is independent of Ξ_x. From the construction of $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$, we can see that the event $\{\bar{W}_\alpha \neq \bar{W}_{\alpha,r}\} \subset \{$at least one $x \in \overline{\Xi} \cap \Gamma_\alpha$ with $\bar{R}(x,\alpha) > r\}$, so from (2.3), we have

$$\begin{aligned} d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha,r}) \le & \mathbb{P}\left(\{\bar{W}_\alpha \neq \bar{W}_{\alpha,r}\}\right) \\ \le & \mathbb{P}\left(\{\text{at least one } x \in \overline{\Xi} \cap \Gamma_\alpha \text{ such that } \bar{R}(x,\alpha) > r\}\right) \\ \le & \mathbb{E}\int_{\Gamma_\alpha} \mathbf{1}_{\bar{R}(x,\alpha)>r}\overline{\Xi}(dx) \\ = & \int_{\Gamma_\alpha} \mathbb{P}\left(\bar{R}((x,M_x),\alpha,\Xi_x+\delta_{(x,M_x)}) > r\right)\lambda dx \\ \le & \alpha\lambda\bar{\tau}(r). \end{aligned} \tag{5.1}$$

This, together with the stabilisation condition in Definition 2.4, gives the claim in (b).

The claim (a) can be proved by replacing corresponding counterparts $\bar{W}_\alpha$ by W_α; $\bar{W}_{\alpha,r}$ by $W_{\alpha,r}$; $\bar{R}(x,\alpha)$ by $R(x)$; $\bar{R}((x,M_x),\alpha,\Xi_x+\delta_{(x,M_x)})$ by $R((x,M_x),\Xi_x+\delta_{(x,M_x)})$ and $\bar{\tau}$ by τ. □

The moments of $W_{\alpha,r}$ and W_α (resp. $\bar{W}_{\alpha,r}$ and $\bar{W}_\alpha$) can be established using the moment conditions required. To begin with, we first show a statement about the moments of $\overline{\Xi}(\Gamma_\alpha)$. Let $\|X\|_p := \mathbb{E}\left(|X|^p\right)^{\frac{1}{p}}$ be the L_p norm of X provided it is finite.

Lemma 5.3. *For $k \in \mathbb{N}$, if the marked point process Ξ satisfies the kth moment condition* (2.4)*, then $\mathbb{E}\left(\overline{\Xi}(\Gamma_\alpha)^k\right) \le O(\alpha^k)$ for $\alpha > 0$.*

Proof. Since $\overline{\Xi}(B)$ is non-decreasing in B in the sense of inclusion and it is also stationary, the condition (2.4) is equivalent to that there exists an $\alpha_0 > 0$ such that $\mathbb{E}\left(\overline{\Xi}(\Gamma_{\alpha_0})^k\right) = \mathbb{E}\left(\overline{\Xi}(\Gamma_{\alpha_0}+x)^k\right) =: C < \infty$ for all $x \in \mathbb{R}^d$.

We can find a cover of Γ_α of the form $\{\Gamma_{\alpha_0}+x_i\}_{i\le n_\alpha}$ with $n_\alpha = \left\lceil \left(\frac{\alpha}{\alpha_0}\right)^{\frac{1}{d}} \right\rceil^d = O(\alpha)$, it follows from Minkowski's inequality that

$$\mathbb{E}\left(\overline{\Xi}(\Gamma_\alpha)^k\right) = \|\overline{\Xi}(\Gamma_\alpha)\|_k^k \le \left(\sum_{i=1}^{n_\alpha} \|\overline{\Xi}(\Gamma_{\alpha_0}+x_i)\|_k\right)^k = n_\alpha^k C^k = O(\alpha^k),$$

as claimed. □

Remark 5.4. From the proof of Lemma 5.3, we can see that for arbitrary $A \in \mathscr{B}\left(\mathbb{R}^d\right)$, if A can be covered by $\{\Gamma_{\alpha_0}+x_i\}_{i\le n_A}$, then $\mathbb{E}\left(\overline{\Xi}(A)^k\right) \le n_A^k C$ for some positive constant C.

Lemma 5.5. *(a) (unrestricted case) If* Ξ *satisfies the* $(2n-1)$*th moment condition* (2.4) *and the score function* η *satisfies the* $(2n)$*th moment condition* (2.5)*, then*

$$\mathbb{E}\left(|W_\alpha|^k\right) \vee \mathbb{E}\left(|W_{\alpha,r}|^k\right) \le C\alpha^{2k-1}$$

for some positive constant C *for all integers* $1 \le k \le n$.

(b) (restricted case) The restricted counterpart of (a) with (2.5) *and* W *replaced by* (2.6) *and* $\bar{W}$*, resp., holds.*

Proof. We start with the restricted case, and the unrestricted case follows in the same way. For $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}\left(|\bar{W}_\alpha|^k\right) =& \mathbb{E}\left(\left|\int_{\Gamma_\alpha} g_\alpha(x,\Xi)\overline{\Xi}(dx)\right|^k\right) \\
\le& \mathbb{E}\left(\left(\int_{\Gamma_\alpha} |g_\alpha(x,\Xi)|\,\overline{\Xi}(dx)\right)^k\right) \\
=& \mathbb{E}\underbrace{\int_{\Gamma_\alpha}\cdots\int_{\Gamma_\alpha}}_{k \text{ of them}} |g_\alpha(x_1,\Xi)\dots g_\alpha(x_k,\Xi)|\,\overline{\Xi}(dx_1)\dots\overline{\Xi}(dx_k) \\
\le& \frac{1}{k}\mathbb{E}\underbrace{\int_{\Gamma_\alpha}\cdots\int_{\Gamma_\alpha}}_{k \text{ of them}} \left(|g_\alpha(x_1,\Xi)|^k + \cdots + |g_\alpha(x_k,\Xi)|^k\right) d\overline{\Xi}(dx_1)\dots\overline{\Xi}(dx_k) \\
=& \mathbb{E}\int_{\Gamma_\alpha} \overline{\Xi}(\Gamma_\alpha)^{k-1}\,|g_\alpha(x,\Xi)|^k\,\overline{\Xi}(dx) \\
\le& \frac{1}{2}\mathbb{E}\int_{\Gamma_\alpha}\left(\overline{\Xi}(\Gamma_\alpha)^{2k-2} + |g_\alpha(x,\Xi)|^{2k}\right)\overline{\Xi}(dx) \\
=& \frac{1}{2}\left[\mathbb{E}\left(\overline{\Xi}(\Gamma_\alpha)^{2k-1}\right) + \mathbb{E}\left(\int_{\Gamma_\alpha} |g_\alpha(x,\Xi_x+\delta_{(x,M_x)})|^{2k}\lambda dx\right)\right] \\
\le& \frac{1}{2}C_1\alpha^{2k-1} + \frac{1}{2}C_2\alpha,
\end{aligned} \tag{5.2}$$

for some positive constants C_1 and C_2, the second and third inequalities follow from the fact that $\Pi_{i=1}^j y_i \le \frac{1}{j}\sum_{i=1}^j y_i^j$ for all $y_i \ge 0$, $1 \le i \le j$, the last equality follows from (2.3) and the last inequality follows from Lemma 5.3. Then we can find a common positive constant C such that $\mathbb{E}\left(|\bar{W}_\alpha|^k\right) \le C\alpha^{2k-1}$ for all integers $k \le n$. The claim $\mathbb{E}\left(|\bar{W}_{\alpha,r}|^k\right) \le C\alpha^{2k-1}$ can be proved by following exactly the same steps but replacing $g_\alpha(x,\Xi)$ with $g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x,\alpha)\le r}$.

The statement for the unrestricted case is also true, which can be proved by replacing the corresponding counterparts $\bar{W}_\alpha$ by W_α, $\bar{W}_{\alpha,r}$ by $W_{\alpha,r}$, and g_α by g. □

Remark 5.6. The proof of Lemma 5.5 does not depend on the shape of Γ_α, so the claims still hold if we replace Γ_α with a set $A \in \mathscr{B}(\mathbb{R})$ such that A satisfies the assumption in Remark 5.4 with $n_A \le O(\mathrm{Vol}(A))$.

Remark 5.7. Using the same idea as in the proof of (5.2), for each $1 \le i \le k$, by taking the range of x_i in A_i satisfying the condition in Remark 5.6 instead of Γ_α, we have, for all integers $1 \le k \le n$,

$$\mathbb{E}\int_{A_1}\cdots\int_{A_k} |g_\alpha(x_1,\Xi)\dots g_\alpha(x_k,\Xi)|\,\overline{\Xi}(dx_1)\dots\overline{\Xi}(dx_k) \le C\max_{1\le j\le k}\mathrm{Vol}(A_j)^{2k-1}$$

for some positive constant C.

With these preparations, we are ready to bound the differences $|\mathrm{Var}\,(W_\alpha) - \mathrm{Var}\,(W_{\alpha,r})|$ and $\left|\mathrm{Var}\,(\bar{W}_\alpha) - \mathrm{Var}\,(\bar{W}_{\alpha,r})\right|$.

Lemma 5.8. *(a) (unrestricted case) Assume that Ξ satisfies the fifth moment condition* (2.4) *and the score function η satisfies the sixth moment condition* (2.5)*. If η is exponentially stabilising in Definition 2.3, then there exist positive constants α_0 and C such that*

$$|\mathrm{Var}\,(W_\alpha) - \mathrm{Var}\,(W_{\alpha,r})| \le \frac{1}{\alpha}$$

for all $\alpha \ge \alpha_0$ and $r \ge C\ln(\alpha)$.

(b) (restricted case) The restricted counterpart of (a) with (2.5)*, Definition 2.3 and W replaced by* (2.6)*, Definition 2.4 and $\bar{W}$, resp., holds.*

Proof. We start with (b). From Lemma 5.5 (b), taking $n = 3$ and $k = 1$ or $k = 3$, we have

$$\max\left\{\|\bar{W}_\alpha\|_1, \|\bar{W}_{\alpha,r}\|_1\right\} \le C_0\alpha, \;\; \max\left\{\|\bar{W}_\alpha\|_3, \|\bar{W}_{\alpha,r}\|_3\right\} \le C_0\alpha^{\frac{5}{3}} \tag{5.3}$$

for some positive constant $C_0 \ge 1$. Without loss of generality, we assume $\alpha > 1$. Since

$$\left|\mathrm{Var}\,(\bar{W}_\alpha) - \mathrm{Var}\,(\bar{W}_{\alpha,r})\right| \le \left|\mathbb{E}\left(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2\right)\right| + \left|\left(\mathbb{E}\bar{W}_\alpha\right)^2 - \left(\mathbb{E}\bar{W}_{\alpha,r}\right)^2\right|, \tag{5.4}$$

using the assumption of stabilisation, we show that each of the terms at the right hand side of (5.4) is bounded by $\frac{1}{2\alpha}$ for α and r sufficiently large. Clearly, the definition of $\bar{W}_{\alpha,r}$ implies that $\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2 = 0$ if $\bar{R}(x,\alpha) \le r$ for all $x \in \overline{\Xi}_{\Gamma_\alpha}$, hence it remains to tackle $E_{r,\alpha} := \{\bar{R}(x,\alpha) \le r \text{ for all } x \in \overline{\Xi}_{\Gamma_\alpha}\}^c$. As shown in the proof of (5.1), $\mathbb{P}\left(E_{r,\alpha}\right) \le \alpha C_1 e^{-C_2 r}$, which, together with (5.3), Hölder's inequality and Minkowski's inequality, ensures

$$\begin{aligned}
\left|\mathbb{E}\left(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2\right)\right| &= \left|\mathbb{E}\left[\left(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2\right)\mathbf{1}_{E_{r,\alpha}}\right]\right| \\
&\le \|\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2\|_{\frac{3}{2}}\|\mathbf{1}_{E_{r,\alpha}}\|_3 \\
&\le \left(\|\bar{W}_\alpha^2\|_{\frac{3}{2}} + \|\bar{W}_{\alpha,r}^2\|_{\frac{3}{2}}\right)\mathbb{P}(E_{r,\alpha})^{\frac{1}{3}} \\
&= \left(\|\bar{W}_\alpha\|_3^2 + \|\bar{W}_{\alpha,r}\|_3^2\right)\mathbb{P}(E_{r,\alpha})^{\frac{1}{3}} \le 2C_0^2\alpha^{\frac{10}{3}}\left(\alpha C_1 e^{-C_2 r}\right)^{\frac{1}{3}}.
\end{aligned} \tag{5.5}$$

For the remaining term of (5.4), we have

$$\left|\left(\mathbb{E}\bar{W}_\alpha\right)^2 - \left(\mathbb{E}\bar{W}_{\alpha,r}\right)^2\right| = \left|\mathbb{E}\bar{W}_\alpha - \mathbb{E}\bar{W}_{\alpha,r}\right|\left|\mathbb{E}\bar{W}_\alpha + \mathbb{E}\bar{W}_{\alpha,r}\right|.$$

The bound (5.3) implies $\left|\mathbb{E}\bar{W}_\alpha + \mathbb{E}\bar{W}_{\alpha,r}\right| \le 2C_0\alpha$. However, using Hölder's inequality, Minkowski's inequality and (5.3) again, we have

$$\begin{aligned}
\left|\mathbb{E}\bar{W}_\alpha - \mathbb{E}\bar{W}_{\alpha,r}\right| &= \left|\mathbb{E}\left[\left(\bar{W}_\alpha - \bar{W}_{\alpha,r}\right)\mathbf{1}_{E_{r,\alpha}}\right]\right| \\
&\le \|\bar{W}_\alpha - \bar{W}_{\alpha,r}\|_3\|\mathbf{1}_{E_{r,\alpha}}\|_{\frac{3}{2}} \\
&\le \left(\|\bar{W}_\alpha\|_3 + \|\bar{W}_{\alpha,r}\|_3\right)\mathbb{P}(E_{r,\alpha})^{\frac{2}{3}} \\
&\le 2C_0\alpha^{\frac{5}{3}}\left(\alpha C_1 e^{-C_2 r}\right)^{\frac{2}{3}},
\end{aligned} \tag{5.6}$$

giving

$$\left|\left(\mathbb{E}\bar{W}_\alpha\right)^2 - \left(\mathbb{E}\bar{W}_{\alpha,r}\right)^2\right| \le 4C_0^2\alpha^{\frac{8}{3}}\left(\alpha C_1 e^{-C_2 r}\right)^{\frac{2}{3}}. \tag{5.7}$$

We set $r = C\ln(\alpha)$ in the upper bounds of (5.5) and (5.7) and find C such that both bounds are bounded by $1/(2\alpha)$, completing the proof of (b).

A line-by-line repetition of the above proof with $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$ replaced by W_α and $W_{\alpha,r}$ and $\bar{R}(x,\alpha)$ replaced by $R(x)$ gives (a). □

We can now establish the lower bounds for $\mathrm{Var}\left(W_\alpha\right)$ and $\mathrm{Var}\left(\bar{W}_\alpha\right)$ using the variation conditions (2.7) and (2.8). To this end, we start with a lemma. Recall that $P := \lambda\mathbb{E}(\eta((\mathbf{0}, M_\mathbf{0}), \Xi_\mathbf{0} + \delta_{(0M_\mathbf{0})})$ for the unrestricted case and $\bar{P} := \lambda\mathbb{E}(\bar{\eta}((\mathbf{0}, M_\mathbf{0}), \Xi_\mathbf{0} + \delta_{(0,M_\mathbf{0})})$ for the restricted case.

Lemma 5.9. *(a) (unrestricted case) Assume that Ξ satisfies the EDD and the fifth moment condition* (2.4)*, and the score function η is translation invariant and satisfies the sixth moment condition* (2.5)*. If η is exponentially stabilising in Definition 2.3, then*

$$\int_{\Gamma_\alpha}\int_{\mathbb{R}^d}\mathbb{E}\left[(g(x,\Xi)\overline{\Xi}(dx)-Pdx)(g(y,\Xi)\overline{\Xi}(dy)-Pdy)\right]=\alpha\sigma^2<\infty.$$

Furthermore, for any fixed $\alpha_1 > 0$,

$$\int_{\Gamma_{\alpha_1}}\int_{\mathbb{R}^d\setminus\Gamma_{\alpha_2}}\mathbb{E}\left[(g(y,\Xi)\overline{\Xi}(dy)-Pdy)(g(x,\Xi)\overline{\Xi}(dx)-Pdx)\right]$$

converges to 0 exponentially fast as $\alpha_2 \to \infty$.

(b) (restricted case) The restricted counterpart of (a) with (2.5)*, Definition 2.3, g and P replaced by* (2.6)*, Definition 2.4, $\bar{g}$ and $\bar{P}$, resp., holds.*

Proof. We start with the restricted case first. It is sufficient to show that, for any fixed $\alpha_1 > 0$,

$$\int_{\Gamma_{\alpha_1}}\int_{\mathbb{R}^d\setminus\Gamma_{\alpha_2}}\mathbb{E}\left[(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\right]$$

converges to 0 exponentially fast as $\alpha_2 \to \infty$. Bearing in mind Remark 5.7, at the cost of no more than $C_0(\alpha_1\vee 1)^3$,without loss of generality, we may assume that $\alpha_2^{1/d} > \left(6\alpha_1^{1/d}\right)\vee 2$ and $\frac{\alpha_2^{1/d}}{\ln\left(\alpha_2^{1/d}(2\sqrt{d}+1/3)\right)} \geq 12\theta_3$ with θ_3 in the definition of the EDD. The space $\mathbb{R}^d\setminus\Gamma_{\alpha_2}$ can be divided into sets of the form $\left\{\Gamma_{\alpha_2(1+l)^d}\setminus\Gamma_{\alpha_2 l^d}\right\}_{l\in\mathbb{N}} =: \{A_l\}_{l\in\mathbb{N}}$. Then

$$\begin{aligned}&\int_{\Gamma_{\alpha_1}}\int_{\mathbb{R}^d\setminus\Gamma_{\alpha_2}}\mathbb{E}\left[(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\right]\\
=&\sum_{l\in\mathbb{N}}\int_{\Gamma_{\alpha_1}}\int_{A_l}\mathbb{E}\left[(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\right]\\
=&\sum_{l\in\mathbb{N}}\mathbb{E}\left[\int_{\Gamma_{\alpha_1}}\int_{A_l}(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\right]\end{aligned}$$

if the sum in the last line is absolutely convergent.

For $l \in \mathbb{N}$, $\mathrm{diam}\left(B\left(A_l, l\alpha_2^{1/d}/6\right)\right) = \alpha_2^{1/d}(\sqrt{d}(1+l)+l/3) \geq \mathrm{diam}\left(B\left(\Gamma_{\alpha_1}, l\alpha_2^{1/d}/6\right)\right) = \sqrt{d}\alpha_1^{1/d}+l\alpha_2^{1/d}/3$, and $d\left(B\left(\Gamma_{\alpha_1}, l\alpha_2^{1/d}/6\right), B\left(A_l, l\alpha_2^{1/d}/6\right)\right) = \frac{l\alpha_2^{1/d}}{6} - \frac{\alpha_1^{1/d}}{2} \geq \frac{l\alpha_2^{1/d}}{12}$. Since $x/\ln(ax)$ for $a > 0$ is an increasing function of $x \geq e/a$, we have

$$\frac{l\alpha_2^{1/d}}{\ln\left(\alpha_2^{1/d}(\sqrt{d}(1+l)+l/3)\right)} \geq \frac{l\alpha_2^{1/d}}{\ln\left(l\alpha_2^{1/d}(2\sqrt{d}+1/3)\right)} \geq \frac{\alpha_2^{1/d}}{\ln\left(\alpha_2^{1/d}(2\sqrt{d}+1/3)\right)} \geq 12\theta_3,$$

which ensures

$$\begin{aligned}&d\left(B\left(\Gamma_{\alpha_1}, l\alpha_2^{1/d}/6\right), B\left(A_l, l\alpha_2^{1/d}/6\right)\right)\\
&\geq \theta_3\ln\left(\mathrm{diam}\left(B\left(\Gamma_{\alpha_1}, l\alpha_2^{1/d}/6\right)\right)\vee\mathrm{diam}\left(B\left(A_l, l\alpha_2^{1/d}/6\right)\right)\vee 1\right).\end{aligned}$$

With (2.3) we can show that $\mathbb{E}\int_A(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)=0$ for all bounded measurable set $A\subset\mathbb{R}^d$. Recalling that $\tilde{\Xi}$ denotes an independent copy of Ξ, we have $\mathbb{E}(\int_{\Gamma_{\alpha_1}}(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\int_{A_l}(\bar{g}(y,\tilde{\Xi})\overline{\tilde{\Xi}}(dy)-\bar{P}dy))=0$. For simplicity, we write $S_A:=\int_A\bar{g}(x,\Xi)\overline{\Xi}(dx)$, $S_{A,r}:=\int_A\bar{g}(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx)$ and the corresponding counterparts with $\tilde{\Xi}$ instead of Ξ as $\tilde{S}_A$ and $\tilde{S}_{A,r}$ for all bounded measurable sets $A\subset\mathbb{R}^d$. Using the stabilising condition in Definition 2.4 and the EDD, we can get an upper bound for $d_{TV}((S_{\Gamma_{\alpha_1}},\tilde{S}_{A_l}),(S_{\Gamma_{\alpha_1}},S_{A_l}))$ as follows:

$$\begin{aligned}
&d_{TV}((S_{\Gamma_{\alpha_1}}-\alpha_1\bar{P},\tilde{S}_{A_l}-\mathrm{Vol}(A_l)\bar{P}),(S_{\Gamma_{\alpha_1}}-\alpha_1\bar{P},S_{A_l}-\mathrm{Vol}(A_l)\bar{P}))\\
=&d_{TV}((S_{\Gamma_{\alpha_1}},\tilde{S}_{A_l}),(S_{\Gamma_{\alpha_1}},S_{A_l}))\\
\le&\mathbb{P}(S_{\Gamma_{\alpha_1}}\neq S_{\Gamma_{\alpha_1},l\alpha_2^{1/d}/6})+2\mathbb{P}(\tilde{S}_{A_l}\neq\tilde{S}_{A_l,l\alpha_2^{1/d}/6})\\
&+d_{TV}((S_{\Gamma_{\alpha_1},l\alpha_2^{1/d}/6},\tilde{S}_{A_l,l\alpha_2^{1/d}/6}),(S_{\Gamma_{\alpha_1},l\alpha_2^{1/d}/6},S_{A_l,l\alpha_2^{1/d}/6}))\\
\le&\lambda(\alpha_1+2\mathrm{Vol}(A_l))\bar{\tau}(l\alpha_2^{1/d}/6)+\beta_{B(\Gamma_{\alpha_1},l\alpha_2^{1/d}/6),B(A_l,l\alpha_2^{1/d}/6)}\\
\le&C_3e^{-C_4l\alpha_2^{1/d}},
\end{aligned}$$

for some positive constants C_3 and C_4 independent of l and α_2, where the second inequality follows from the same argument as that for (5.1), and the last inequality follows from the stabilising condition in Definition 2.4 and the EDD.

Using [BHJ92, p. 254], we can find a suitable coupling $((X_1,X_2),(Y_1,Y_2))$ of

$$\left(S_{\Gamma_{\alpha_1}}-\alpha_1\bar{P},\tilde{S}_{A_l}-\mathrm{Vol}(A_l)\bar{P}\right)\text{ and }\left(S_{\Gamma_{\alpha_1}}-\alpha_1\bar{P},S_{A_l}-\mathrm{Vol}(A_l)\bar{P}\right)$$

such that $(X_1,X_2)\overset{d}{=}\left(S_{\Gamma_{\alpha_1}}-\alpha_1\bar{P},\tilde{S}_{A_l}-\mathrm{Vol}(A_l)\bar{P}\right)$, $(Y_1,Y_2)\overset{d}{=}\left(S_{\Gamma_{\alpha_1}}-\alpha_1\bar{P},S_{A_l}-\mathrm{Vol}(A_l)\bar{P}\right)$, $\mathbb{P}(E):=\mathbb{P}((X_1,X_2)\neq(Y_1,Y_2))\le C_3e^{-C_4l\alpha_2^{1/d}}$. With Remark 5.6, we can see that $\|S_A-\mathrm{Vol}(A)\bar{P}\|_3\le C_5\mathrm{Vol}(A)^{\frac{5}{3}}$ for $A=\Gamma_\alpha$ or A_l for $l\ge1$, $\alpha>0$ for some positive constant C_5.

From Hölder's inequality and Remark 5.6, we can see that

$$\begin{aligned}
&\left|\mathbb{E}\left(\int_{\Gamma_{\alpha_1}}(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\int_{A_l}(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)\right)\right|\\
=&\left|\mathbb{E}\left(\int_{\Gamma_{\alpha_1}}(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\int_{A_l}(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)\right)\right.\\
&\left.-\mathbb{E}\left(\int_{\Gamma_{\alpha_1}}(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\int_{A_l}(\bar{g}(y,\tilde{\Xi})\overline{\tilde{\Xi}}(dy)-\bar{P}dy)\right)\right|\\
=&\left|\mathbb{E}\left(\int_{\Gamma_{\alpha_1}}(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\int_{A_l}(\bar{g}(y,\Xi)\overline{\Xi}(dy)-\bar{P}dy)\mathbf{1}_E\right)\right.\\
&\left.-\mathbb{E}\left(\int_{\Gamma_{\alpha_1}}(\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx)\int_{A_l}(\bar{g}(y,\tilde{\Xi})\overline{\tilde{\Xi}}(dy)-\bar{P}dy)\mathbf{1}_E\right)\right|\\
\le&C_6e^{-C_7l\alpha_2^{1/d}}
\end{aligned}\tag{5.8}$$

for some positive constants C_6 and C_7. This, together with the translation invariant property and the definition of $\bar{\sigma}^2$, completes the proof for the restricted case.

The statement for the unrestricted case can be proved by replacing corresponding counterparts g by $\bar{g}$; $\bar{R}$ by R; $\bar{P}$ by P; $\bar{\tau}$ by τ and $\bar{\sigma}^2$ by σ^2. □

To establish the order of the variance, we need a lemma saying that we can approximate $\alpha\sigma^2$ (resp. $\alpha\bar{\sigma}^2$) with the score function η restricted to $R \le r$ (resp. $\bar{R} \le r$). For convenience, let $P_r := \lambda\mathbb{E}(\eta((\mathbf{0}, M_\mathbf{0}), \Xi_\mathbf{0} + \delta_{(\mathbf{0},M_\mathbf{0})})\mathbf{1}_{R((\mathbf{0},M_\mathbf{0}),\Xi_\mathbf{0}+\delta_{(\mathbf{0},M_\mathbf{0})})\le r})$, $\bar{P}_{\alpha,x,r} := \lambda\mathbb{E}(\eta((x, M_x), \Xi_x + \delta_{(x,M_x)}, \Gamma_\alpha)\mathbf{1}_{\bar{R}((x,M_x),\alpha,\Xi_x+\delta_{(x,M_x)})\le r})$.

Lemma 5.10. *(a) (unrestricted case) If the conditions in Lemma 5.9 (a) hold and $\sigma^2 > 0$, then for a fixed sufficiently large $\alpha_0 > 0$, there exist positive constants α_1 and C such that*

$$\mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} (g(x,\Xi)\mathbf{1}_{R(x)\le r}\overline{\Xi}(dx) - P_r dx)\int_{\Gamma_\alpha} (g(y,\Xi)\mathbf{1}_{R(y)\le r}\overline{\Xi}(dy) - P_r dy)\right] \in [\frac{1}{2}\alpha_0\sigma^2, \frac{3}{2}\alpha_0\sigma^2]$$

for all $\alpha \ge \alpha_1$ and $r \ge C\ln(\alpha)$, $z \in \Gamma_\alpha$ such that $d(z, \partial\Gamma_\alpha) \ge 5r$.

(b) (restricted case) If the conditions in Lemma 5.9 (b) hold and $\bar{\sigma}^2 > 0$, then for a fixed sufficiently large $\alpha_0 > 0$, there exist positive constants α_1 and C such that

$$\mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} (g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx) - \bar{P}_{\alpha,x,r}dx)\int_{\Gamma_\alpha} (g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy) - \bar{P}_{\alpha,y,r}dy)\right] \in [\frac{1}{2}\alpha_0\bar{\sigma}^2, \frac{3}{2}\alpha_0\bar{\sigma}^2]$$

for all $\alpha \ge \alpha_1$ and $r \ge C\ln(\alpha)$, $z \in \Gamma_\alpha$ such that $d(z, \partial\Gamma_\alpha) \ge 5r$.

Proof. We prove the restricted case only, and the unrestricted case can be proved similarly.

Let $\bar{P}_r := \lambda\mathbb{E}(\bar{g}(\mathbf{0}, \Xi_\mathbf{0} + \delta_{(\mathbf{0},M_\mathbf{0})})\mathbf{1}_{\bar{R}((\mathbf{0},M_\mathbf{0}),\alpha,\Xi_\mathbf{0}+\delta_{(\mathbf{0},M_\mathbf{0})})\le r})$, then from the moment condition (2.6), we can see that $\max_{\alpha\in\mathbb{R}_+,r\in\mathbb{R}_+,x\in\Gamma_\alpha}\{|\bar{P}_r|, |\bar{P}_{\alpha,x,r}|\} \le C_1$ for some positive constant C_1. By the translation-invariance, if x and r satisfy $B(x, r) \subset \Gamma_\alpha$, then $g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r} = \bar{g}(x,\Xi)\mathbf{1}_{R(x)\le r}$ and $\bar{P}_r = \bar{P}_{\alpha,x,r}$, where $R(x) = \lim_{\alpha\to\infty}\bar{R}(x)$, see the discussion after Definition 2.6. The stabilising condition ensures that $\mathbb{P}(\bar{R}((\mathbf{0}, M_\mathbf{0}), \alpha, \Xi_\mathbf{0} + \delta_{(\mathbf{0},M_\mathbf{0})}) \ge r) \le \bar{\tau}(r)$ decreases exponentially fast. Arguing in the same way as that for (5.5), it follows from the moment condition (2.6) and Hölder's inequality that $|\bar{P} - \bar{P}_r| \le \alpha^{-2}$ for all $r \ge C_2\ln(\alpha)$ for some positive constant C_2. Writing for brevity $T_{\alpha,r}(x,\Xi,dx) := g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx) - \bar{P}_{\alpha,x,r}dx$ and $T_r(x,\Xi,dx) := \bar{g}(x,\Xi)\mathbf{1}_{R(x)\le r}\overline{\Xi}(dx) - \bar{P}dx$. By pairing the two terms of $T_{\alpha,r}$ and the two terms of T_r, we can use Remark 5.7 to show that there exists a positive constant $\alpha_1' \ge e$ such that

$$\begin{aligned}&\Bigg|\mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} T_{\alpha,r}(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_{\alpha,r}(y,\Xi,dy)\right]\\ &\quad - \mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} T_r(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_r(y,\Xi,dy)\right]\Bigg|\\ &\le \frac{1}{8}\alpha_0\bar{\sigma}^2\end{aligned} \tag{5.9}$$

for all $\alpha \ge \alpha_1', r \ge C_2\ln(\alpha)$.

Following the proof of (5.1), we have $\mathbb{P}(E^c) := \mathbb{P}\left(\{\bar{R}(x) \le r \text{ for all } x \in \overline{\Xi}_{\Gamma_\alpha}\}^c\right) \le \alpha\bar{\tau}(r)$. Also, we can see that in the event E,

$$\int_{z+\Gamma_{\alpha_0}} T_r(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_r(y,\Xi,dy)$$

is the same as

$$\int_{z+\Gamma_{\alpha_0}} T_\infty(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_\infty(y,\Xi,dy),$$

where $T_\infty(x,\Xi,dx):=\bar{g}(x,\Xi)\overline{\Xi}(dx)-\bar{P}dx$. Using Hölder's inequality as for (5.5), we can find a $C_3\in\mathbb{R}_+$ such that

$$\begin{aligned}&\left|\int_{z+\Gamma_{\alpha_0}} T_r(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_r(y,\Xi,dy)\right.\\&\qquad\left.-\int_{z+\Gamma_{\alpha_0}} T_\infty(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_\infty(y,\Xi,dy)\right|\\&\le\frac{1}{8}\alpha_0\bar{\sigma}^2\end{aligned}\tag{5.10}$$

for all $r\ge C_3\ln(\alpha)$.

Using the translation invariant property, and replacing $\mathbb{R}^d\backslash\Gamma_{\alpha_2}$ by $\mathbb{R}^d\backslash B(\Gamma_{\alpha_0},3r)$ in the proof of Lemma 5.9, with necessary minor adjustments, we obtain

$$\begin{aligned}&\left|\mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} T_\infty(x,\Xi,dx)\int_{B(z+\Gamma_{\alpha_0},3r)} T_\infty(y,\Xi,dy)\right]-\alpha_0\bar{\sigma}^2\right|\\&=\left|\mathbb{E}\left[\int_{\Gamma_{\alpha_0}} T_\infty(x,\Xi,dx)\int_{B(\Gamma_{\alpha_0},3r)} T_\infty(y,\Xi,dy)\right]-\alpha_0\bar{\sigma}^2\right|\\&\le\frac{1}{8}\alpha_0\bar{\sigma}^2\end{aligned}\tag{5.11}$$

for $r\ge C_4\ln(\alpha)$. On the other hand, the EDD ensures that we can find an independent copy $\tilde{\Xi}$ of Ξ such that

$$E_1:=\{\Xi_{B(z+\Gamma_{\alpha_0},r)\cup(\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},2r))}\neq\Xi_{B(z+\Gamma_{\alpha_0},r)}\cup\tilde{\Xi}_{\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},2r)}\}$$

satisfies

$$\mathbb{P}(E_1)=\beta_{B(z+\Gamma_{\alpha_0},r),\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},2r)}\le\theta_1\left(\left(\sqrt{d}\alpha^{2\theta_0/d}\right)\vee 1\right)e^{-\theta_2 r}.\tag{5.12}$$

Since $\mathbb{E}T_{\alpha,r}(x,\Xi,dx)=0$, following the same argument as that for (5.8) and applying Hölder's inequality in the first inequality, the moment condition (2.6) and (5.12) in the last inequality below, we get

$$\begin{aligned}&\left|\mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} T_{\alpha,r}(x,\Xi,dx)\int_{\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},3r)} T_{\alpha,r}(y,\Xi,dy)\right]\right|\\&=\left|\mathbb{E}\left[\mathbf{1}_{E_1}\int_{z+\Gamma_{\alpha_0}} T_{\alpha,r}(x,\Xi,dx)\int_{\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},3r)} T_{\alpha,r}(y,\Xi,dy)\right]\right.\\&\qquad\left.-\mathbf{1}_{E_1}\ \mathbb{E}\left[\int_{z+\Gamma_{\alpha_0}} T_{\alpha,r}(x,\Xi,dx)\int_{\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},3r)} T_{\alpha,r}(y,\tilde{\Xi},dy)\right]\right|\\&\le\left\|\int_{z+\Gamma_{\alpha_0}} T_{\alpha,r}(x,\Xi,dx)\int_{\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},3r)} T_{\alpha,r}(y,\Xi,dy)\right\|_{3/2}\mathbb{P}(E_1)^{1/3}\\&\qquad+\left\|\int_{z+\Gamma_{\alpha_0}} T_{\alpha,r}(x,\Xi,dx)\int_{\Gamma_\alpha\backslash B(z+\Gamma_{\alpha_0},3r)} T_{\alpha,r}(y,\tilde{\Xi},dy)\right\|_{3/2}\mathbb{P}(E_1)^{1/3}\\&\le\frac{1}{8}\alpha_0\bar{\sigma}^2\end{aligned}\tag{5.13}$$

for all z satisfying $d(z,\partial\Gamma_\alpha) \ge 5r$, $r \ge C_5 \ln(\alpha)$ and $\alpha \ge \alpha_2'$. Collecting (5.9), (5.10), (5.11) and (5.13), we obtain claim (b) with $C = \max\{C_2, C_3, C_4, C_5\}$ and $\alpha_1 = \max\{\alpha_1', \alpha_2'\}$.

The statement for the unrestricted case can be proved by replacing $\bar{P}_{\alpha,x,r}$ and $\bar{P}_r$ by $P_{\alpha,x,r}$ and P_r; $\bar{g}$ by g; $\bar{R}$ by R; $\bar{\tau}$ by τ; $\bar{P}$ by P. □

Remark 5.11. Following the idea of the proof of Lemma 5.10, without loss of generality, we can take α_1 and C as non-decreasing functions of α_0 in Lemma 5.10.

Together with the variation conditions (2.7) and (2.8), we can show that the variances of W_α and $\bar{W}_\alpha$ are of the order α as claimed in Theorem 2.12.

Proof of Theorem 2.12. We show the statement for the restricted case first, and the statement for the unrestricted case can be shown in the same way.

To begin with, we choose α_0' and C such that Lemma 5.8 holds, i.e., $|\mathrm{Var}(\bar{W}_\alpha) - \mathrm{Var}(\bar{W}_{\alpha,r})| \le \frac{1}{\alpha}$ for $r \ge C\ln(\alpha)$ and $\alpha \ge \alpha_0'$. According to Remark 5.11, we can choose $C_1 \ge C$ and $\alpha_1 \ge (2\alpha_0') \vee e$ such that Lemma 5.10 (b) holds for all $r \ge C_1 \ln(\alpha)$, $\alpha \ge \alpha_1$ and α_0 in Lemma 5.10 (b) taking any value in $[\alpha_0', 2\alpha_0']$. In the rest of the proof, we fix $r = C_1 \ln(\alpha)$. Replacing θ_3 in the definition of the EDD with $2C_1$ if necessary, we assume $\theta_3 \ge 2C_1$. Cover $\Gamma_\alpha \backslash B(\partial\Gamma_\alpha, 5\theta_3\ln(\alpha))$ with disjoint cubes $\mathbb{C}_1, \dots, \mathbb{C}_{n_\alpha}$ each with a volume between α_0' and $2\alpha_0'$ and intersects $\Gamma_\alpha \backslash B(\partial\Gamma_\alpha, 5\theta_3\ln(\alpha))$, then the number of cubes n_α has the same order as α. For convenience, let $\mathbb{C}_\alpha := \cup_{1\le i\le n_\alpha}\mathbb{C}_i$, then from Lemma 5.10 (b) and Remark 5.11, for $\alpha \ge \alpha_1$,

$$\mathbb{E}\left[\int_{\mathbb{C}_\alpha} (g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx) - \bar{P}_{\alpha,x,r}dx) \int_{\Gamma_\alpha} (g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy) - \bar{P}_{\alpha,y,r}dy)\right]$$
$$\in [\frac{1}{2}n_\alpha\alpha_0'\bar{\sigma}^2, 3n_\alpha\alpha_0'\bar{\sigma}^2]. \tag{5.14}$$

Next, we show that

$$\left|\mathbb{E}\left[\int_{\Gamma_\alpha\backslash\mathbb{C}_\alpha} (g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx) - \bar{P}_{\alpha,x,r}dx) \int_{\Gamma_\alpha} (g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy) - \bar{P}_{\alpha,y,r}dy)\right]\right|$$
$$\le O\left((\ln\alpha)^{2d+1}\alpha^{\frac{d-1}{d}}\right). \tag{5.15}$$

To this end, we choose $C_2 > 1$ such that $\theta_2\theta_3C_2 \ge 13.5 + \theta_0/d$ and $C_2\theta_3 > 2C_1$, where θ_0, θ_2 are as in the EDD, divide $\Gamma_\alpha\backslash\mathbb{C}_\alpha$ into at most $n_{\mathbb{D}} = O\left(\alpha^{\frac{d-1}{d}}\ln(\alpha)\right)$ cubes $\{\mathbb{D}_i\}_{1\le i\le n_{\mathbb{D}}}$ having diameters between 1 and $\theta_3\ln(\alpha)$. For each $x \in \mathbb{D}_i$, $T(x,\Xi,dx) := g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx) - \bar{P}_{\alpha,x,r}dx$ is completely determined by $\Xi_{B(\mathbb{D}_i,r)}$ and $T(y,\Xi,dy)$ is completely determined by $\Xi_{\Gamma_\alpha\cap B(\mathbb{D}_i,2\theta_3C_2\ln(\alpha))^c}$ if $x \in B(\mathbb{D}_i,r)$ and $\|y-x\| > 4\theta_3C_2\ln(\alpha)$. By the EDD, we can find an independent copy $\tilde{\Xi}$ of Ξ such that

$$\begin{aligned}\mathbb{P}(E_i) :=& \mathbb{P}\left(\Xi_{B(\mathbb{D}_i,r)} \cup \Xi_{\Gamma_\alpha\cap B(\mathbb{D}_i,2\theta_3C_2\ln(\alpha))^c} \ne \Xi_{B(\mathbb{D}_i,r)} \cup \tilde{\Xi}_{\Gamma_\alpha\cap B(\mathbb{D}_i,2\theta_3C_2\ln(\alpha))^c}\right)\\ =&\beta_{B(\mathbb{D}_i,r),\Gamma_\alpha\cap B(\mathbb{D}_i,2\theta_3C_2\ln(\alpha))^c} \le C_3\ln(\alpha)^{\theta_0}\alpha^{\theta_0/d}e^{-\theta_2\theta_3C_2\ln(\alpha)}.\end{aligned} \tag{5.16}$$

Hence, following the same steps as those for (5.8) and using Hölder's inequality in the first inequality, (5.16) and Remark 5.7 with $k = 3$ in the second inequality below, we have

$$\left|\mathbb{E}\left[\int_{x\in\mathbb{D}_i} T(x,\Xi,dx) \int_{y\in\Gamma_\alpha,\|y-x\|>4\theta_3C_2\ln(\alpha)} T(y,\Xi,dy)\right]\right|$$
$$= \left|\mathbb{E}\left[\mathbf{1}_{E_i}\int_{x\in\mathbb{D}_i} T(x,\Xi,dx) \int_{y\in\Gamma_\alpha,\|y-x\|>4\theta_3C_2\ln(\alpha)} T(y,\Xi,dy)\right]\right.$$

$$
\begin{aligned}
&- \mathbb{E}\left[\mathbf{1}_{E_i}\int_{x\in\mathbb{D}_i} T(x,\Xi,dx)\int_{y\in\Gamma_\alpha,\|y-x\|>4\theta_3C_2\ln(\alpha)} T(y,\tilde{\Xi},dy)\right]\Bigg|\\
\le{}& \left\|\int_{x\in\mathbb{D}_i} T(x,\Xi,dx)\int_{y\in\Gamma_\alpha,\|y-x\|>4\theta_3C_2\ln(\alpha)} T(y,\Xi,dy)\right\|_{3/2}\mathbb{P}(E_i)^{1/3}\\
&+\left\|\int_{x\in\mathbb{D}_i} T(x,\Xi,dx)\int_{y\in\Gamma_\alpha,\|y-x\|>4\theta_3C_2\ln(\alpha)} T(y,\tilde{\Xi},dy)\right\|_{3/2}\mathbb{P}(E_i)^{1/3}\\
\le{}& O\left(\alpha^{\frac{5}{2}+\frac{\theta_0}{3d}-\frac{1}{3}\theta_2\theta_3C_2}\ln(\alpha)^{\theta_0/3}\right)\le O\left(\ln(\alpha)^{\theta_0/3}\alpha^{-2}\right). \qquad (5.17)
\end{aligned}
$$

Adding the estimates of (5.17) for $1\le i\le n_{\mathbb{D}}$ and using the fact $n_{\mathbb{D}} = O\left(\alpha^{\frac{d-1}{d}}\ln\alpha\right)$, we obtain

$$
\left|\mathbb{E}\left[\int_{x\in\Gamma_\alpha\setminus\mathbb{C}_\alpha} T(x,\Xi,dx)\int_{y\in\Gamma_\alpha,\|y-x\|>4\theta_3C_2\ln(\alpha)} T(y,\Xi,dy)\right]\right| \le O\left(\alpha^{-1}\right). \qquad (5.18)
$$

For the remaining part, we have

$$
\begin{aligned}
&\left|\mathbb{E}\left[\iint_{x\in\Gamma_\alpha\setminus\mathbb{C}_\alpha,y\in\Gamma_\alpha,\|y-x\|\le4\theta_3C_2\ln(\alpha)} g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx)g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)\right]\right|\\
&\le \mathbb{E}\left[\iint_{x\in\Gamma_\alpha\setminus\mathbb{C}_\alpha,y\in\Gamma_\alpha,\|y-x\|\le4\theta_3C_2\ln(\alpha)} \frac{1}{2}\left(g_\alpha(x,\Xi)^2\mathbf{1}_{\bar{R}(x)\le r}+g_\alpha(y,\Xi)^2\mathbf{1}_{\bar{R}(y)\le r}\right)\overline{\Xi}(dx)\overline{\Xi}(dy)\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\int_{x\in\Gamma_\alpha\setminus\mathbb{C}_\alpha}\overline{\Xi}(B(x,4\theta_3C_2\ln(\alpha)))g_\alpha(x,\Xi)^2\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx)\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\int_{y\in\Gamma_\alpha\cap B(\partial\Gamma_\alpha,9\theta_3C_2\ln(\alpha))}\overline{\Xi}(B(y,4\theta_3C_2\ln(\alpha)))g_\alpha(y,\Xi)^2\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)\right]\\
&\le \mathbb{E}\left[\int_{y\in\Gamma_\alpha\cap B(\partial\Gamma_\alpha,9\theta_3C_2\ln(\alpha))}\overline{\Xi}(B(y,4\theta_3C_2\ln(\alpha)))g_\alpha(y,\Xi)^2\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)\right]\\
&\le \mathbb{E}\left[\int_{y\in\Gamma_\alpha\cap B(\partial\Gamma_\alpha,9\theta_3C_2\ln(\alpha))}\frac{1}{2}\left(\overline{\Xi}(B(y,4\theta_3C_2\ln(\alpha)))^2+g_\alpha(y,\Xi)^4\mathbf{1}_{\bar{R}(y)\le r}\right)\overline{\Xi}(dy)\right]\\
&\le \int_{y\in\Gamma_\alpha\cap B(\partial\Gamma_\alpha,9\theta_3C_2\ln(\alpha))}\left(\left(\overline{\Xi}_{\mathbf{0}}(B(\mathbf{0},4\theta_3C_2\ln(\alpha)))+1\right)^2\lambda dy+g_\alpha(y,\Xi)^4\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)\right). \qquad (5.19)
\end{aligned}
$$

However, by Remark 5.4,

$$
\begin{aligned}
&O\left(\ln(\alpha)^{3d}\right)\\
=&\mathbb{E}\overline{\Xi}(B(\mathbf{0},8\theta_3C_2\ln(\alpha)))^3\\
\ge&\mathbb{E}\int_{B(\mathbf{0},4\theta_3C_2\ln(\alpha))}\overline{\Xi}(B(x,4\theta_3C_2\ln(\alpha)))^2\overline{\Xi}(dx)\\
=&\lambda\int_{B(\mathbf{0},4\theta_3C_2\ln(\alpha))}\mathbb{E}\left(\overline{\Xi}_x(B(x,4\theta_3C_2\ln(\alpha)))+1\right)^2dx\\
=&\frac{\lambda(4\theta_3C_2\ln(\alpha))^d\pi^{d/2}}{\Gamma(1+d/2)}\mathbb{E}\left(\overline{\Xi}_{\mathbf{0}}(B(\mathbf{0},4\theta_3C_2\ln(\alpha)))+1\right)^2,
\end{aligned}
$$

which implies

$$
\mathbb{E}\left(\overline{\Xi}_{\mathbf{0}}(B(\mathbf{0},4\theta_3C_2\ln(\alpha)))+1\right)^2\le O\left(\ln(\alpha)^{2d}\right). \qquad (5.20)
$$

Combining (2.6), (5.19) and (5.20) gives

$$\left|\mathbb{E}\left[\iint_{x\in\Gamma_\alpha\setminus\mathbb{C}_\alpha,y\in\Gamma_\alpha,\|y-x\|\le 4\theta_3C_2\ln(\alpha)} g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx)g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)\right]\right|$$
$$\le O\left((\ln\alpha)^{2d+1}\alpha^{\frac{d-1}{d}}\right). \tag{5.21}$$

Direct verification using (2.6) again gives

$$\left|\mathbb{E}\left[\iint_{x\in\Gamma_\alpha\setminus\mathbb{C}_\alpha,y\in\Gamma_\alpha,\|y-x\|\le 4\theta_3C_2\ln(\alpha)}\bar{P}_{\alpha,x,r}dx\bar{P}_{\alpha,y,r}dy\right]\right|\le O\left(\alpha^{\frac{d-1}{d}}\ln(\alpha)^{d+1}\right). \tag{5.22}$$

Collecting (5.18), (5.21) and (5.22), we have (5.15).

Now, since the variance of $\bar{W}_{\alpha,r}$ can be decomposed as

$$\begin{aligned}&\mathrm{Var}(\bar{W}_{\alpha,r})\\ =&\mathbb{E}\left[\int_{\mathbb{C}_\alpha}(g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx)-\bar{P}_{\alpha,x,r}dx)\int_{\Gamma_\alpha}(g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)-\bar{P}_{\alpha,y,r}dy)\right]\\ &+\mathbb{E}\left[\int_{\Gamma_\alpha\setminus\mathbb{C}_\alpha}(g_\alpha(x,\Xi)\mathbf{1}_{\bar{R}(x)\le r}\overline{\Xi}(dx)-\bar{P}_{\alpha,x,r}dx)\int_{\Gamma_\alpha}(g_\alpha(y,\Xi)\mathbf{1}_{\bar{R}(y)\le r}\overline{\Xi}(dy)-\bar{P}_{\alpha,y,r}dy)\right],\end{aligned}$$

it follows from (5.14) and (5.15) that, for $\alpha\ge\alpha_1$, $\mathrm{Var}(\bar{W}_{\alpha,r})\in[C_4\alpha,C_5\alpha]$ for some positive constants C_4, C_5. This, together with $|\mathrm{Var}(\bar{W}_\alpha)-\mathrm{Var}(\bar{W}_{\alpha,r})|\le\frac{1}{\alpha}$, ensures that $\mathrm{Var}(\bar{W}_\alpha)=\Theta(\mathrm{Var}(\bar{W}_{\alpha,r}))=\Theta(\alpha)$.

The statement for the unrestricted case can be proved by replacing g_α by g; $\bar{P}_{\alpha,x,r}$ by P_r; $\bar{W}_\alpha$ by W_α and $\bar{W}_{\alpha,r}$ by $W_{\alpha,r}$. □

Remark 5.12. Since the variance is always non-negative, the proof of this also shows that σ^2 (resp. $\bar{\sigma}^2$) defined in (2.7) (resp. (2.8)) is non-negative.

Remark 5.13. Using the same idea as in the proof of Theorem 2.12, we can see that $\mathrm{Var}(W_\alpha)$ and $\mathrm{Var}(\bar{W}_\alpha)$ cannot have an order greater than α.

Proof of Theorem 2.10. Let $\mu_\alpha:=\mathbb{E}\left(\bar{W}_\alpha\right)$, $\mu_{\alpha,r}:=\mathbb{E}\left(\bar{W}_{\alpha,r}\right)$, $\sigma_\alpha^2:=\mathrm{Var}\left(\bar{W}_\alpha\right)$, $\sigma_{\alpha,r}^2:=\mathrm{Var}\left(\bar{W}_{\alpha,r}\right)$ and $\bar{Z}_{\alpha,r}\sim N\left(\frac{\mu_{\alpha,r}-\mu_\alpha}{\sigma_\alpha},\frac{\sigma_{\alpha,r}^2}{\sigma_\alpha^2}\right)$, then it follows from the triangle inequality that

$$d_W\left(\frac{\bar{W}_\alpha-\mu_\alpha}{\sigma_\alpha},Z\right)\le d_W\left(\frac{\bar{W}_\alpha-\mu_\alpha}{\sigma_\alpha},\frac{\bar{W}_{\alpha,r}-\mu_\alpha}{\sigma_\alpha}\right)+d_W\left(Z,\bar{Z}_{\alpha,r}\right)+d_W\left(\frac{\bar{W}_{\alpha,r}-\mu_\alpha}{\sigma_\alpha},\bar{Z}_{\alpha,r}\right). \tag{5.23}$$

Next, we bound the terms on the right hand side of (5.23) separately. We start with the exponentially stabilising case (ii).

The first term of (5.23) can be bounded using Lemma 5.2 (b), the variance condition and the property of the Wasserstein distance. Let $U_\alpha:=\left(\bar{W}_\alpha-\mu_\alpha\right)/\sigma_\alpha$ and $U_{\alpha,r}:=\left(\bar{W}_{\alpha,r}-\mu_{\alpha,r}\right)/\sigma_\alpha$. According to the property of the total variation distance and [BHJ92, p. 254], we can find a coupling $(\bar{U}_\alpha,\bar{U}_{\alpha,r})$ of U_α and $U_{\alpha,r}$ such that $\bar{U}_\alpha\stackrel{d}{=}U_\alpha$, $\bar{U}_{\alpha,r}\stackrel{d}{=}U_{\alpha,r}$ and

$$\mathbb{P}(\bar{U}_\alpha\ne\bar{U}_{\alpha,r})=:\mathbb{P}(E_{\alpha,r})=d_{TV}(U_\alpha,U_{\alpha,r})=d_{TV}(\bar{W}_\alpha,\bar{W}_{\alpha,r})\le C_1\alpha e^{-C_2r}.$$

Then from Hölder's inequality, the variance condition and Lemma 5.5,

$$\begin{aligned}&d_W\left(\frac{\bar{W}_\alpha-\mu_\alpha}{\sigma_\alpha},\frac{\bar{W}_{\alpha,r}-\mu_\alpha}{\sigma_\alpha}\right)\\ =&\inf_{X\stackrel{d}{=}U_\alpha,Y\stackrel{d}{=}U_{\alpha,r}}\mathbb{E}(|X-Y|)\\ \le&\mathbb{E}(|\bar{U}_\alpha-\bar{U}_{\alpha,r}|)\le\mathbb{E}\left(|\bar{U}_\alpha|\mathbf{1}_{E_{\alpha,r}}\right)+\mathbb{E}\left(|\bar{U}_{\alpha,r}|\mathbf{1}_{E_{\alpha,r}}\right)\le\frac{1}{\alpha},\end{aligned} \tag{5.24}$$

for $r > C_3 \ln(\alpha)$.

For the second term of (5.23), we can establish an upper bound using Lemma 5.1. To this end, Lemma 5.8 (b) gives

$$\left|\sigma_\alpha^2 - \sigma_{\alpha,r}^2\right| \le \frac{1}{\alpha}, \tag{5.25}$$

which, together with the condition given in the theorem, implies

$$\sigma_{\alpha,r}^2 = \Omega(\alpha^\nu), \qquad \sigma_\alpha^2 = \Omega(\alpha^\nu), \tag{5.26}$$

for $r > C_4 \ln(\alpha)$. We combine (5.6) and (5.26) to obtain

$$\frac{|\mu_\alpha - \mu_{\alpha,r}|}{\sigma_\alpha} \le O\left(\alpha^{-1}\right), \tag{5.27}$$

for $r > C_5 \ln(\alpha)$. Therefore, it follows from (5.25), (5.26), (5.27) and Lemma 5.1 that

$$d_W(Z, \bar{Z}_{\alpha,r}) \le \frac{|\mu_\alpha - \mu_{\alpha,r}|}{\sigma_\alpha} + \frac{|\sigma_{\alpha,r} - \sigma_\alpha|}{\sigma_\alpha} \le O(\alpha^{-1}) \tag{5.28}$$

for $r > C_6 \ln(\alpha)$.

It remains to tackle the last term of (5.23). From the definition of the Wasserstein distance, we have

$$\begin{aligned} d_W\left(\frac{\bar{W}_{\alpha,r} - \mu_\alpha}{\sigma_\alpha}, \bar{Z}_{\alpha,r}\right) &= d_W\left(\frac{\bar{W}_{\alpha,r} - \mu_{\alpha,r}}{\sigma_\alpha}, \bar{Z}_{\alpha,r} + \frac{\mu_\alpha - \mu_{\alpha,r}}{\sigma_\alpha}\right) \\ &\le \frac{\sigma_{\alpha,r}}{\sigma_\alpha} d_W\left(\frac{\bar{W}_{\alpha,r} - \mu_{\alpha,r}}{\sigma_{\alpha,r}}, Z\right) \le 2 d_W\left(V_{\alpha,r}, Z\right) \end{aligned} \tag{5.29}$$

for $r > C_4 \ln(\alpha)$ when α large, where $V_{\alpha,r} := \left(\bar{W}_{\alpha,r} - \mu_{\alpha,r}\right)/\sigma_{\alpha,r}$. We now use Stein's method to bound the Wasserstein distance between $V_{\alpha,r}$ and Z. Stein's method for the normal approximation hinges on a Stein equation (see [CGS11, pp. 15–16])

$$f'(w) - w f(w) = h(w) - Nh, \tag{5.30}$$

where $Nh := \mathbb{E}h(Z)$. The solution of (5.30) is given by

$$f_h(w) = e^{w^2/2} \int_{-\infty}^{w} e^{-t^2/2}(h(t) - Nh)dt = -e^{w^2/2} \int_{w}^{\infty} e^{-t^2/2}(h(t) - Nh)dt.$$

Recall the definition of the Wasserstein distance (2.9), we have, for any random variable X,

$$d_W(X, Z) = \sup_{h \in \mathscr{F}_{\mathrm{Lip}}} |\mathbb{E}(h(X) - h(Z))| \le \sup_{f \in \mathscr{F}} \left|\mathbb{E}\left(f'(X) - X f(X)\right)\right|, \tag{5.31}$$

where $\mathscr{F} := \left\{f;\ \mathbb{R} \to \mathbb{R}, \|f\| \le 2,\ \|f'\| \le \sqrt{\frac{2}{\pi}},\ \|f''\| \le 2\right\}$. From the definition of $V_{\alpha,r}$, we can represent it as $V_{\alpha,r} = \frac{1}{\sigma_{\alpha,r}} \int_{\Gamma_\alpha} \left(g_\alpha(x, \Xi) \mathbf{1}_{\bar{R}(x) \le r} \overline{\Xi}(dx) - P_{\alpha,x,r} dx\right) =: \int_{\Gamma_\alpha} V(dx)$ if this does not cause confusion. Then, from the definition of $V_{\alpha,r}$, we have

$$1 = \mathrm{Var}(V_{\alpha,r}) = \mathbb{E}\left(\int_{\Gamma_\alpha} V(dx)\right)^2.$$

To bound $d_W(V_{\alpha,r}, Z)$, by (5.31), it is sufficient to bound

$$\left|\mathbb{E}\left(f'(V_{\alpha,r}) - V_{\alpha,r} f(V_{\alpha,r})\right)\right| \tag{5.32}$$

for all $f \in \mathscr{F}$. To do this, let's consider the two terms separately.

Before analysing (5.32), for large α, we can divide Γ_α into disjoint cubes with volumes at least $\alpha_0 \le \alpha$ for some positive constant α_0. To this end, we can find a partition of Γ_α, $\mathscr{C} := \{\mathbb{C}_1, \dots, \mathbb{C}_{n_\alpha}\}$, where $\mathbb{C}_i$ are cubes with edge length $\frac{\alpha^{1/d}}{\lfloor (\alpha/\alpha_0)^{1/d} \rfloor}$ for all $1 \le i \le n_\alpha$. Then each cube in $\mathscr{C}$ has a volume no more than $2^d\alpha_0$, and n_α has the same order as α. Let $N'_{i,\alpha,r} = B(\mathbb{C}_i, 3r) \cap \Gamma_\alpha$ and $N''_{i,\alpha,r} = B(\mathbb{C}_i, 6r) \cap \Gamma_\alpha$, we have $N'_{i,\alpha,r} \subset B(\mathbb{C}_i, 3r)$ and $N''_{i,\alpha,r} \subset B(\mathbb{C}_i, 6r)$, so the volumes of $N'_{i,\alpha,r}$ and $N''_{i,\alpha,r}$ are bounded by $O(r^d)$ for all $1 \le i \le n_\alpha$. Define $S_{i,\alpha,r} = \int_{\mathbb{C}_i} V(dy)$, $S'_{i,\alpha,r} = \int_{N'_{i,\alpha,r}} V(dy)$ and $S''_{i,\alpha,r} = \int_{N''_{i,\alpha,r}} V(dy)$. Clearly, $V(dx)$ is a function of $\Xi_{B(x,r)} \cap \Gamma_\alpha$, $S'_{i,\alpha,r}$, $S''_{i,\alpha,r}$ $V_{\alpha,r} - S'_{i,\alpha,r}$ and $V_{\alpha,r} - S''_{i,\alpha,r}$ are functions of $\Xi_{B(\mathbb{C}_i,4r)}$, $\Xi_{B(\mathbb{C}_i,7r)}$, $\Xi_{\Gamma_\alpha \setminus B(\mathbb{C}_i,2r)}$ and $\Xi_{\Gamma_\alpha \setminus B(\mathbb{C}_i,5r)}$ respectively. For convenience, we write $\tilde{\Xi}$ as an independent copy of Ξ and $\tilde{V}_{\alpha,r}$, $\tilde{V}(dx)$, $\tilde{S}'_{i,\alpha,r}$, $\tilde{S}''_{i,\alpha,r}$ as the corresponding counterparts of $V_{\alpha,r}$, $V(dx)$, $S'_{i,\alpha,r}$, $S''_{i,\alpha,r}$.

For the first term in (5.32),

$$
\begin{aligned}
&\mathbb{E}f'(V_{\alpha,r}) \\
=&\mathbb{E}\left(\int_{\Gamma_\alpha} V(dx)\right)^2 \mathbb{E}f'(V_{\alpha,r}) \\
=&\sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\right)\mathbb{E}f'(V_{\alpha,r}) + \mathbb{E}\left(\left(\int_{\Gamma_\alpha} V(dx)\right)^2 - \sum_{i=1}^{n_\alpha} S_{i,\alpha,r}S'_{i,\alpha,r}\right)\mathbb{E}f'(V_{\alpha,r}) \\
=&\sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\right)\left(\mathbb{E}f'(V_{\alpha,r}) - \mathbb{E}f'(V_{\alpha,r} - S''_{i,\alpha,r}) + \mathbb{E}f'(V_{\alpha,r} - S''_{i,\alpha,r})\right) + \epsilon_1 \\
=&\sum_{i=1}^{n_\alpha}\left(\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\right)\mathbb{E}\left(\int_0^{S''_{i,\alpha,r}} f''(V_{\alpha,r} - x)dx\right)\right) + \sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}f'(\tilde{V}_{\alpha,r} - \tilde{S}''_{i,\alpha,r})\right) \\
&+ \epsilon_1 \\
=&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\left(f'(\tilde{V}_{\alpha,r} - \tilde{S}''_{i,\alpha,r}) - f'(V_{\alpha,r} - S''_{i,\alpha,r}) + f'(V_{\alpha,r} - S''_{i,\alpha,r})\right)\right) + \epsilon_1 + \epsilon_2 \\
=&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}f'(V_{\alpha,r} - S''_{i,\alpha,r})\right) + \epsilon_1 + \epsilon_2 + \epsilon_3,
\end{aligned}
\tag{5.33}
$$

where

$$
\begin{aligned}
\epsilon_1 =&\mathbb{E}\left(\left(\int_{\Gamma_\alpha} V(dx)\right)^2 - \sum_{i=1}^{n_\alpha} S_{i,\alpha,r}S'_{i,\alpha,r}\right)\mathbb{E}f'(V_{\alpha,r}); \\
\epsilon_2 =&\sum_{i=1}^{n_\alpha}\left(\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\right)\mathbb{E}\left(\int_0^{S''_{i,\alpha,r}} f''(V_{\alpha,r} - x)dx\right)\right); \\
\epsilon_3 =&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\left(f'(\tilde{V}_{\alpha,r} - \tilde{S}''_{i,\alpha,r}) - f'(V_{\alpha,r} - S''_{i,\alpha,r})\right)\right).
\end{aligned}
$$

Using the same idea as in the proof of Lemma 5.10 and the fact that $\|f'\| \le \sqrt{\frac{2}{\pi}}$, we have

$$
|\epsilon_1| \le \sqrt{\frac{2}{\pi}}\left|\mathbb{E}\sum_{i=1}^{n_\alpha} S_{i,\alpha,r}\left(\int_{\Gamma_\alpha} V(dx) - S'_{i,\alpha,r}\right)\right| \le \alpha^{-1} \tag{5.34}
$$

for $r > C_7 \ln(\alpha)$ for some positive constant C_7.

For ϵ_2, using the fact that $\|f''\| \leq 2$ and Remark 5.7, we have

$$\begin{aligned}|\epsilon_2| \leq & 2\sum_{i=1}^{n_\alpha}\left(\mathbb{E}\left(\left|S_{i,\alpha,r}S'_{i,\alpha,r}S''_{i,\alpha,r}\right|\right)\right)\\ \leq & O(\sigma_{\alpha,r}^{-3}n_\alpha \max_i\{\mathrm{Vol}(N'_{i,\alpha,r}),\mathrm{Vol}(N''_{i,\alpha,r})\}^5) = O(\alpha^{-\frac{3}{2}\nu+1}r^{5d}).\end{aligned} \tag{5.35}$$

To bound ϵ_3, we can use the coupling method. Since $(S_{i,\alpha,r}, S'_{i,\alpha,r})$ is a function of $\Xi_{B(\mathbb{C}_i,4r)}$ and $V_{\alpha,r} - S''_{i,\alpha,r}$ is a function of $\Xi_{\Gamma_\alpha\setminus B(\mathbb{C}_i,5r)}$, from the EDD and [BHJ92, p. 254], there is a coupling $(X_{i,1}, X_{i,2}, X_{i,3})$ and $(Y_{i,1}, Y_{i,2}, Y_{i,3})$ of $(S_{i,\alpha,r}, S'_{i,\alpha,r}, f'(\tilde V_{\alpha,r} - \tilde S''_{i,\alpha,r}))$ and $(S_{i,\alpha,r}, S'_{i,\alpha,r}, f'(V_{\alpha,r} - S''_{i,\alpha,r}))$ such that $\mathbb{P}(E_i) := \mathbb{P}((X_{i,1}, X_{i,2}, X_{i,3}) \neq (Y_{i,1}, Y_{i,2}, Y_{i,3})) \leq \alpha^{\theta_0/d}C_8r^{\theta_0}e^{-C_9r}$, for all $1 \leq i \leq n_\alpha$. Then for $r > C_{10}\ln(\alpha)$ where C_{10} is large enough, $\mathbb{P}(E_i) \leq C_{11}\alpha^{-6}$. Now,

$$\begin{aligned}|\epsilon_3| \leq & \sum_{i=1}^{n_\alpha}\left|\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\left(f'(\tilde V_{\alpha,r} - \tilde S''_{i,\alpha,r}) - f'(V_{\alpha,r} - S''_{i,\alpha,r})\right)\right)\right|\\ = & \sum_{i=1}^{n_\alpha}\left|\mathbb{E}\left(X_{i,1}X_{i,2}X_{i,3} - Y_{i,1}Y_{i,2}Y_{i,3}\right)\right|\\ \leq & \sum_{i=1}^{n_\alpha}\left(\left|\mathbb{E}\left(X_{i,1}X_{i,2}X_{i,3}\mathbf{1}_{E_i}\right)\right| + \left|\mathbb{E}\left(Y_{i,1}Y_{i,2}Y_{i,3}\mathbf{1}_{E_i}\right)\right|\right).\end{aligned} \tag{5.36}$$

However, applying Hölder's inequality, the fact that $\|f'\| \leq \sqrt{\frac{2}{\pi}} \leq 1$ and Remark 5.7, we have

$$\begin{aligned}\left|\mathbb{E}\left(X_{i,1}X_{i,2}X_{i,3}\mathbf{1}_{E_i}\right)\right| \leq \|X_{i,1}X_{i,2}\|_{3/2}\mathbb{P}(E_i)^{1/3} \leq O\left(r^{10d/3}\right)\sigma_{\alpha,r}^{-2}\mathbb{P}(E_i)^{1/3},\\ \left|\mathbb{E}\left(Y_{i,1}Y_{i,2}Y_{i,3}\mathbf{1}_{E_i}\right)\right| \leq \|Y_{i,1}Y_{i,2}\|_{3/2}\mathbb{P}(E_i)^{1/3} \leq O\left(r^{10d/3}\right)\sigma_{\alpha,r}^{-2}\mathbb{P}(E_i)^{1/3}.\end{aligned}$$

Combining with (5.36), we get

$$|\epsilon_3| \leq n_\alpha\sigma_{\alpha,r}^{-2}O(r^{10d/3})\max_{1\leq i\leq n_\alpha}\mathbb{P}(E_i)^{1/3} \leq O(\alpha^{-1-\nu}r^{10d/3}). \tag{5.37}$$

For the second term in (5.32), we have

$$\begin{aligned}&\mathbb{E}V_{\alpha,r}f(V_{\alpha,r})\\ =&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}f(V_{\alpha,r})\right)\\ =&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}\left(f(V_{\alpha,r} - S'_{i,\alpha,r}) + f(V_{\alpha,r}) - f(V_{\alpha,r} - S'_{i,\alpha,r})\right)\right)\\ =&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\int_0^1 f'(V_{\alpha,r} - uS'_{i,\alpha,r})du\right) + \sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}f(\tilde V_{\alpha,r} - \tilde S'_{i,\alpha,r})\right)\\ &+\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}\left(f(V_{\alpha,r} - S'_{i,\alpha,r}) - f(\tilde V_{\alpha,r} - \tilde S'_{i,\alpha,r})\right)\right)\\ =&\sum_{i=1}^{n_\alpha}\mathbb{E}\left(S_{i,\alpha,r}S'_{i,\alpha,r}\int_0^1 f'(V_{\alpha,r} - uS'_{i,\alpha,r})du\right) + \epsilon_4 + \epsilon_5,\end{aligned} \tag{5.38}$$

where

$$\epsilon_4 = \sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r} f(\tilde{V}_{\alpha,r} - \tilde{S}'_{i,\alpha,r})\right);$$

$$\epsilon_5 = \sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r}\left(f(V_{\alpha,r} - S'_{i,\alpha,r}) - f(\tilde{V}_{\alpha,r} - \tilde{S}'_{i,\alpha,r})\right)\right).$$

Since $f(\tilde{V}_{\alpha,r} - \tilde{S}'_{i,\alpha,r})$ is independent of $S_{i,\alpha,r}$ for all $1 \le i \le n_\alpha$,

$$\epsilon_4 = \sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r}\right) \mathbb{E}\left(f(\tilde{V}_{\alpha,r} - \tilde{S}'_{i,\alpha,r})\right) = 0 \tag{5.39}$$

from the definition of $S_{i,\alpha,r}$ and $\|f\| \le 2$.

To bound ϵ_5, we can use the same idea as that for bounding ϵ_3, i.e., we can construct a coupling of $(S_{i,\alpha,r}, f(V_{\alpha,r} - S'_{i,\alpha,r}))$ and $(S_{i,\alpha,r}, f(\tilde{V}_{\alpha,r} - \tilde{S}'_{i,\alpha,r}))$. Since $S_{i,\alpha,r}$ is a function of $\Xi_{B(\mathbb{C}_i,r)}$ and $V_{\alpha,r} - S'_{i,\alpha,r}$ is a function of $\Xi_{\Gamma_\alpha \setminus B(\mathbb{C}_i,2r)}$, from the EDD and [BHJ92, p. 254], there is a coupling $((X_{i,1}, X_{i,2}), (Y_{i,1}, Y_{i,2}))$ of $(S_{i,\alpha,r}, f(V_{\alpha,r} - S'_{i,\alpha,r}))$ and $(S_{i,\alpha,r}, f(\tilde{V}_{\alpha,r} - \tilde{S}'_{i,\alpha,r}))$ such that $\mathbb{P}\left((X_{i,1}, X_{i,2}) \ne (Y_{i,1}, Y_{i,2})\right) =: \mathbb{P}(E') \le \alpha^{\theta_0/d} C_{12} r^{\theta_0} e^{-C_{13} r}$ for all $1 \le i \le n_\alpha$, where $C_{12},\ C_{13} \in \mathbb{R}_+$ are independent of i's. Then we follow the steps as for (5.37) to get

$$|\epsilon_5| \le O(\alpha^{-1-\nu} r^{10d/3}), \tag{5.40}$$

for $r > C_{14} \ln(\alpha)$ with sufficiently large C_{14}.

Also, from the fact that $\|f''\| \le 2$ and Remark 5.7,

$$\begin{aligned}
|\epsilon_6| :=& \left| \sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r} S'_{i,\alpha,r} f'(V_{\alpha,r} - S''_{i,\alpha,r})\right) - \sum_{i=1}^{n_\alpha} \mathbb{E}\left(S_{i,\alpha,r} S'_{i,\alpha,r} \int_0^1 f'(V_{\alpha,r} - u S'_{i,\alpha,r}) du\right) \right| \\
\le& \sum_{i=1}^{n_\alpha} \left| \mathbb{E}\left(S_{i,\alpha,r} S'_{i,\alpha,r} \int_0^1 \left(f'(V_{\alpha,r} - u S'_{i,\alpha,r}) - f'(V_{\alpha,r} - S''_{i,\alpha,r})\right) du\right) \right| \\
\le& \sum_{i=1}^{n_\alpha} \left| \mathbb{E}\left(S_{i,\alpha,r} S'_{i,\alpha,r} \int_0^1 \int_0^{S''_{i,\alpha,r} - u S'_{i,\alpha,r}} f''(V_{\alpha,r} - S''_{i,\alpha,r} + v) dv du\right) \right| \\
\le& 2 \sum_{i=1}^{n_\alpha} \mathbb{E}\left(\left|S_{i,\alpha,r} S'_{i,\alpha,r}\right| \left(\left|S''_{i,\alpha,r}\right| + \left|S'_{i,\alpha,r}\right|\right)\right) \le O(\alpha^{-\frac{3}{2}\nu+1} r^{5d}).
\end{aligned} \tag{5.41}$$

Combining (5.29), (5.32), (5.33), (5.34), (5.35), (5.37), (5.38), (5.39), (5.40) and (5.41), we obtain the bound

$$d_W\left(\frac{\bar{W}_{\alpha,r} - \mu_\alpha}{\sigma_\alpha}, \bar{Z}_{\alpha,r}\right) \le O(\alpha^{-\frac{3}{2}\nu+1} r^{5d}) \tag{5.42}$$

for $r > C_{15} \ln(\alpha)$, where $C_{15} = \max\{C_7, C_{10}, C_{14}\}$.

From (5.23), taking $r = \max\{C_3, C_4, C_6, C_{15}\} \ln(\alpha)$ for large α, together with the bounds in (5.24), (5.28) and (5.42), we have

$$d_W\left(\frac{\bar{W}_\alpha - \mu_\alpha}{\sigma_\alpha}, Z\right) \le \alpha^{-\frac{3}{2}\nu+1} \ln(\alpha)^{5d}.$$

(i) If η is range-bounded, then there exists an $r_1 > 0$ such that $\bar{W}_{\alpha,r_1} = \bar{W}_\alpha\ a.s.$ for all α, so $d_W\left(\frac{\bar{W}_\alpha - \mu_\alpha}{\sigma_\alpha}, Z\right) = d_W\left(\frac{\bar{W}_{\alpha,r_1} - \mu_\alpha}{\sigma_\alpha}, \bar{Z}_{\alpha,r_1}\right)$. Since η is range-bounded, it is also exponentially stabilising, (5.26) and (5.42) still hold, then $d_W\left(\frac{\bar{W}_\alpha - \mu_\alpha}{\sigma_\alpha}, Z\right) = O(\alpha^{-\frac{3}{2}\nu+1} r_1^{5d}) = O(\alpha^{-\frac{3}{2}\nu+1})$, which completes the proof. □

Proof of Theorem 2.8. The statement can be shown by replacing $\bar{W}_\alpha$, $\bar{W}_{\alpha,r}$, $\bar{Z}_\alpha$, $\bar{Z}_{\alpha,r}$, $g_\alpha(x,\Xi)$ and $\bar{R}(x,\alpha)$ by their counterparts W_α, $W_{\alpha,r}$, Z_α, $Z_{\alpha,r}$, $g(\Xi^x)$ and $R(x)$ in the proof of Theorem 2.10. □

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