

Lévy flows and associated stochastic PDEs^{*†}

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Abstract

In this paper, we first explore certain structural properties of Lévy flows and use this information to obtain the existence of strong solutions to a class of Stochastic PDEs in the space of tempered distributions, driven by Lévy noise. The uniqueness of the solutions follows from Monotonicity inequality. These results extend an earlier work of the second author on the diffusion case.

Keywords: S' valued process; Lévy processes; Hermite-Sobolev space; strong solution.

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1 Introduction and main results

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. Let $(B_t)_{t \geq 0}$ be a standard d -dimensional $(\mathcal{F}_t)_{t \geq 0}$ Brownian motion. Let N be a Poisson random measure with on \mathbb{R}^d with associated Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ and let \tilde{N} denote the corresponding compensated Poisson random measure. We assume that B and N are independent. For any $z \in \mathbb{R}^d$, consider the following stochastic differential equation (SDE) in \mathbb{R}^d

$$\begin{aligned} dZ_t &= b(Z_{t-})dt + \sigma(Z_{t-})dB_t + \int_{0 < |x| < 1} F(Z_{t-}, x)\tilde{N}(dt, dx), \\ Z_0 &= z, \end{aligned} \tag{1.1}$$

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where the coefficients $b : \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, F : \mathbb{R}^d \times \{x \in \mathbb{R}^d : 0 < |x| < 1\} \rightarrow \mathbb{R}^d$ satisfies the Lipschitz condition and linear growth condition as described below, i.e. there exist constants $K_1 > 0$ and $K_2 > 0$ such that for all $z, z_1, z_2 \in \mathbb{R}^d$ we have

$$|b(z_1) - b(z_2)|^2 + \|a(z_1, z_1) - 2a(z_1, z_2) + a(z_2, z_2)\|^2 + \int_{0 < |x| < 1} |F(z_1, x) - F(z_2, x)|^2 \nu(dx) \leq K_1 |z_1 - z_2|^2 \tag{1.2}$$

and

$$|b(z)|^2 + \|a(z, z)\|^2 + \int_{0 < |x| < 1} |F(z, x)|^2 \nu(dx) \leq K_2(1 + |z|^2). \tag{1.3}$$

Here, $a(x, y) := \sigma(x)\sigma^t(y), \forall x, y \in \mathbb{R}^d, \sigma^t$ being the transpose of σ . The term $|x|$ denotes the Euclidean norm of a vector x and $\|A\|$ denotes the matrix seminorm $\sum_{i=1}^d |a_{ii}|$ of any real $d \times d$ matrix $A = (a_{ij})_{d \times d}$. The component functions of b, σ and F shall be denoted by $b_i, \sigma_{ij}, F_i; i, j = 1, \dots, d$, respectively.

We denote the unique strong solution [2, Theorem 6.2.3] to the SDE (1.1) above by $(Z_t^z)_{t \geq 0}$. In the first half of the present paper, we look at the flows associated to this SDE. In stochastic analysis/dynamics, stochastic flows play an important role (see [4, 8, 12, 17, 24, 30, 31, 33–35, 37, 44] and the references therein). In this article, we are interested in specific structures on Lévy flows (see [1–3, 16, 19, 32, 33, 43] and the references therein) and we extend results of [5] on the diffusion case.

In [5], for diffusions, i.e. for $F \equiv 0, G \equiv 0$, a decomposition of $(Z_t^z)_{t \geq 0}$ into a ‘global’ component $(Z_t^0)_{t \geq 0}$ and a deterministic ‘local’ component $(Z_t^z - Z_t^0)_{t \geq 0}$ was considered. This situation was characterized by [5, Theorem 2.5], which implies that the diffusion coefficient σ is a real non-singular $d \times d$ matrix and the drift coefficient b is in an affine form, viz. $b(x) = \beta + Ax, \forall x \in \mathbb{R}^d$ for some $\beta \in \mathbb{R}^d$ and a real $d \times d$ matrix A . Since the flows generated in this class of SDEs are Gaussian, this result can also be treated as a characterization result for Gaussian flows. The well-known Ornstein-Uhlenbeck diffusion [2, p. 241] falls in this class.

In this paper, we extend the above characterization to the case of Lévy flows. We start with an analogue of [5, Definition 2.2].

Definition 1.1. We say that the solution $(Z_t^z)_{t \geq 0}$ to the SDE (1.1) depends deterministically on the initial condition if there exists a function $f : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for all $z \in \mathbb{R}^d$, we have a.s.

$$Z_t^z = f(t, z) + Z_t^0 \quad \forall t \geq 0. \tag{1.4}$$

We take f to be jointly measurable in t and z .

Based on the assumptions on the coefficients, if the deterministic dependence on the initial condition holds as above, then necessarily for all $z \in \mathbb{R}^d$ and $t \geq 0$, we have

$$f(t, z) = \mathbb{E}(Z_t^z) - \mathbb{E}(Z_t^0) = z + \int_0^t (\mathbb{E}(b(Z_{s-}^z)) - \mathbb{E}(b(Z_{s-}^0))) ds \tag{1.5}$$

For every fixed $z, t \mapsto \mathbb{E}(b(Z_{t-}^z)) - \mathbb{E}(b(Z_{t-}^0))$ is left continuous with right limits and hence, the left partial derivative of f with respect to t exists on $(0, \infty)$. We shall denote this derivative by $\frac{\partial f}{\partial t}$ and take $\frac{\partial f}{\partial t}(0, z) = 0, \forall z$. Note that, $\frac{\partial f}{\partial t}(t, z) = \mathbb{E}(b(Z_{t-}^z)) - \mathbb{E}(b(Z_{t-}^0)), t > 0$.

We now state the main result of the first half of this paper.

Theorem 1.2 (Main Result 1). Let σ, b, F be as stated above. Suppose the following happen:

- (i) there exists $z \in \mathbb{R}^d$ such that the determinant of $\sigma(z) = (\sigma_{ij}(z))$ is not zero,
- (ii) $b_i \in C^2(\mathbb{R}^d, \mathbb{R}), i = 1, 2, \dots, d$,
- (iii) for every fixed $z \in \mathbb{R}^d$, the map $t \mapsto \frac{\partial f}{\partial t}(t, z)$ is of bounded variation.

Then the solution of the SDE (1.1) depends deterministically on the initial condition through (1.4) if and only if σ is a real non-singular $d \times d$ matrix, b is of the form $b(z) = Cz + K$, $F(z, x)$ depends only on x and $f(t, z) = e^{tC}z \forall t \geq 0, z \in \mathbb{R}^d$ where $K \in \mathbb{R}^d$ and C is a real $d \times d$ matrix.

The proof of this result is similar to [5, Theorem 2.5] and has been discussed in Section 2. To avoid repetitions of similar arguments from [5], a sketch of the proof is provided for the one-dimensional case only.

Next, we discuss the existence and uniqueness results of strong solutions for a class of stochastic partial differential equations (SPDEs) (see [9–11, 15, 20, 22, 27, 29, 36, 38–40] and the references therein), which are an extension of the results considered in [5, Section 3]. To do this, we first recall a countably Hilbertian topology on the space $\mathcal{S}(\mathbb{R}^d)$ of real valued rapidly decreasing smooth functions on \mathbb{R}^d and its dual $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions (see [25, Chapter 1.3]).

Let $\mathbb{Z}_+^d := \{\alpha = (\alpha_1, \dots, \alpha_d) : \alpha_i \text{ non-negative integers}\}$. If $\alpha \in \mathbb{Z}_+^d$, we write $|\alpha| := \alpha_1 + \dots + \alpha_d$. For any $p \in \mathbb{R}$, consider the norm $\|\cdot\|_p$ defined by the inner-product

$$\langle \phi, \psi \rangle_p := \sum_{n \in \mathbb{Z}_+^d} (2|n| + d)^{2p} \langle \phi, h_n \rangle \langle \psi, h_n \rangle, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}^d). \tag{1.6}$$

In the above equation, $\{h_n : n \in \mathbb{Z}_+^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ given by the Hermite functions and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\mathbb{R}^d)$. The Hermite-Sobolev spaces $\mathcal{S}_p(\mathbb{R}^d)$ are defined to be the completion of $\mathcal{S}(\mathbb{R}^d)$ in the inner product $\langle \cdot, \cdot \rangle_p$. By construction, $\mathcal{S}_p(\mathbb{R}^d), p \in \mathbb{R}$ are real separable Hilbert spaces. Note that for $p > 0$, $\mathcal{S}_p(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ and $(\mathcal{S}_{-p}(\mathbb{R}^d), \|\cdot\|_{-p})$ is isometrically isomorphic to the dual to $(\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$. This duality extends from the identification of $\mathcal{S}_0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ duality with itself. Furthermore,

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{p \in \mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{p \in \mathbb{R}} (\mathcal{S}_p(\mathbb{R}^d), \|\cdot\|_p)$$

Given $\psi \in \mathcal{S}(\mathbb{R}^d)$ (or $\mathcal{S}_p(\mathbb{R}^d)$) and $\phi \in \mathcal{S}'(\mathbb{R}^d)$ (or $\mathcal{S}_{-p}(\mathbb{R}^d)$), the action of ϕ on ψ will be denoted by $\langle \phi, \psi \rangle$.

Now, we consider a class of SPDEs (1.9) in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ associated to the SDE (1.1) given above. In the diffusion case, it is well-known that the solution to the Kolmogorov’s Forward equation (or the Fokker-Planck equation) have a probabilistic representation through the solution of an associated SPDE (see [9–11, 13–15, 18, 25, 28, 29, 31, 39, 41, 42, 45] and the references therein). The equation (1.9) can be seen to be an extension of this SPDE from the diffusion case to equations involving jumps.

Proofs of new results involving (1.9) have been discussed in Section 3. Take σ, b, F as in Theorem 1.2. Note that

$$\int_{0 < |x| < 1} |F(0, x)|^2 \nu(dx) < \infty. \tag{1.7}$$

In addition, we assume that

$$\sup_{0 < |x| < 1} |F(0, x)| < \infty. \tag{1.8}$$

Now, look at the linear SPDE

$$\begin{aligned}
 dY_t &= A^*(Y_{t-})dB_t + L^*(Y_{t-})dt + \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id)Y_{t-}\tilde{N}(dt, dx) \\
 &+ \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id + \sum_{i=1}^d F_i(0,x)\partial_i)Y_{t-}\nu(dx)dt, \\
 Y_0 &= \psi,
 \end{aligned} \tag{1.9}$$

with the terms described as below.

- (i) We take the initial condition ψ to be in $L^1(\mathbb{R}^d)$. Note that $L^1(\mathbb{R}^d) \subset \mathcal{S}_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$, see [5, Lemma 3.2].
- (ii) Given $\phi \in \mathcal{S}'(\mathbb{R}^d)$, the distributional derivatives $\partial_i\phi \in \mathcal{S}'(\mathbb{R}^d), i = 1, \dots, d$ are defined by the duality relation

$$\langle \partial_i\phi, g \rangle = -\langle \phi, \partial_i g \rangle, \forall g \in \mathcal{S}(\mathbb{R}^d).$$

Similarly, the multiplication operators $M_i, i = 1, \dots, d$ defined by $(M_i g)(x) := x_i g(x), g \in \mathcal{S}(\mathbb{R}^d), x = (x_1, \dots, x_d) \in \mathbb{R}^d$ on $\mathcal{S}(\mathbb{R}^d)$ are extended by duality to $M_i : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, where

$$\langle M_i\phi, g \rangle = \langle \phi, M_i g \rangle, \forall g \in \mathcal{S}(\mathbb{R}^d).$$

It is well-known that $\partial_i, M_i : \mathcal{S}_p(\mathbb{R}^d) \rightarrow \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ are bounded linear operators, for any $p \in \mathbb{R}$.

- (iii) Define operators A, L as follows: for $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
 A\phi &:= (A_1\phi, \dots, A_d\phi), \\
 A_i\phi(x) &:= \sum_{k=1}^d \sigma_{ki}\partial_k\phi(x), \\
 L\phi(x) &:= \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^t)_{ij}\partial_{ij}^2\phi(x) + \sum_{i=1}^d b_i(x)\partial_i\phi(x).
 \end{aligned} \tag{1.10}$$

On $\mathcal{S}'(\mathbb{R}^d)$, define the adjoint operators A^*, L^* as follows: for $\psi \in \mathcal{S}'(\mathbb{R}^d)$,

$$\begin{aligned}
 A^*\psi &:= (A_1^*\psi, \dots, A_d^*\psi), \\
 A_i^*\psi &:= -\sum_{k=1}^d \partial_k(\sigma_{ki}\psi), \\
 L^*\psi &:= \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2((\sigma\sigma^t)_{ij}\psi) - \sum_{i=1}^d \partial_i(b_i\psi).
 \end{aligned} \tag{1.11}$$

For any $p \in \mathbb{R}, A_i^* : \mathcal{S}_{-p}(\mathbb{R}^d) \rightarrow \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d), i = 1, 2, \dots, d$ and $L^* : \mathcal{S}_p(\mathbb{R}^d) \rightarrow \mathcal{S}_{p-1}(\mathbb{R}^d)$ are bounded linear operators.

- (iv) Translation operators $\tau_x : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d), x \in \mathbb{R}^d$ are defined by the duality relation

$$\langle \tau_x\phi, g \rangle = \langle \phi, \tau_{-x}g \rangle, \forall \phi \in \mathcal{S}'(\mathbb{R}^d), g \in \mathcal{S}(\mathbb{R}^d),$$

where $(\tau_x g)(y) := g(y - x), y \in \mathbb{R}^d$.

Definition 1.3. We say that an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued $(\mathcal{F}_t)_{t \geq 0}$ adapted process $(Y_t)_{t \geq 0}$ satisfies the SPDE (1.9) with equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ if the following equality holds in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$: a.s. for all $t \geq 0$,

$$\begin{aligned}
 Y_t = & \psi + \int_0^t A^*(Y_{s-}) dB_s + \int_0^t L^*(Y_{s-}) ds + \int_0^t \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id) Y_{s-} \tilde{N}(ds, dx) \\
 & + \int_0^t \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id + \sum_{i=1}^d F_i(0,x) \partial_i) Y_{s-} \nu(dx) ds.
 \end{aligned}
 \tag{1.12}$$

Note that this equality is equivalent to the fact that a.s. for all $t \geq 0$ and for all $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
 \langle Y_t, \phi \rangle = & \langle \psi, \phi \rangle + \int_0^t \langle A^*(Y_{s-}), \phi \rangle dB_s + \int_0^t \langle L^*(Y_{s-}), \phi \rangle ds \\
 & + \int_0^t \int_{0 < |x| < 1} \langle (\tau_{F(0,x)} - Id) Y_{s-}, \phi \rangle \tilde{N}(ds, dx) \\
 & + \int_0^t \int_{0 < |x| < 1} \left\langle (\tau_{F(0,x)} - Id + \sum_{i=1}^d F_i(0,x) \partial_i) Y_{s-}, \phi \right\rangle \nu(dx) ds.
 \end{aligned}
 \tag{1.13}$$

Motivated by [5, 39], we construct a solution to (1.9) in the following way. There exists a \mathbb{P} null set \mathcal{N} such that for all $\omega \in \Omega \setminus \mathcal{N}, t \geq 0$

$$Z_t^0 = \int_0^t \sigma dB_s + \int_0^t (CZ_{s-}^0 + K) ds + \int_0^t \int_{0 < |x| < 1} F(0,x) \tilde{N}(ds, dx)$$

We take $(Z_t^z)_{t \geq 0}$ with $Z_t^z := Z_t^0 + ze^{tC}, \forall t \geq 0, z \in \mathbb{R}^d$ to be the solution to the SDE (1.1). Now define,

$$Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} \psi(z) \delta_{Z_t^z(\omega)} dz, \forall t \geq 0, \omega \in \Omega \setminus \mathcal{N}
 \tag{1.14}$$

and set $Y_t(\omega)(\psi) = 0, \forall t \geq 0, \omega \in \mathcal{N}$. Here, for $z \in \mathbb{R}^d$, let δ_z denotes the Dirac distribution supported at the point z . For $p > \frac{d}{4}$, $\sup_{z \in \mathbb{R}^d} \|\delta_z\|_{-p} \leq C(p)$ for some positive constant $C(p)$ depending on p (see [39, Theorem 4.1]). This implies the well-definedness of $(Y_t(\psi))_{t \geq 0}$ and we have the following analogue of [5, Proposition 3.5].

Proposition 1.4. Let $\psi, (Z_t^z)_{t \geq 0}, (Y_t(\psi))_{t \geq 0}$ be as above. Let $p > \frac{d}{4}$. Then $(Y_t(\psi))_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ adapted $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued process with paths which are right continuous with left limits. Furthermore, $Y_t(\psi)$ is norm-bounded, where the bound can be chosen to be independent of t .

Consider the linear map $Z_t : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ defined by $(Z_t(\phi))(z) := \phi(Z_t^z), z \in \mathbb{R}^d$. Now, we have the following analogue of [5, Lemma 3.6].

Lemma 1.5. Fix any $t \geq 0, \omega \in \Omega \setminus \mathcal{N}$. The linear map $Z_t(\omega) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous.

Observe that for any $\phi \in \mathcal{S}(\mathbb{R}^d), \psi \in L^1(\mathbb{R}^d)$, we have

$$\langle Y_t(\psi), \phi \rangle = \int_{\mathbb{R}^d} \psi(z) \phi(Z_t^z) dz = \int_{\mathbb{R}^d} \psi(z) (Z_t(\phi))(z) dz = \langle \psi, Z_t(\phi) \rangle.$$

This implies

$$Y_t(\psi) = Z_t^*(\psi),
 \tag{1.15}$$

where $Z_t^* : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is the transpose of the linear map Z_t . The next result is the main result of the second half of this paper and is an analogue of [5, Theorem 3.8].

Theorem 1.6 (Main Result 2). *Let $p > \frac{d}{4}$ and $\psi \in L^1(\mathbb{R}^d)$. Then the $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued process $(Y_t(\psi))_{t \geq 0}$ satisfies the SPDE (1.9). This solution is also unique.*

The proof of the above theorem has been discussed in Section 3. The existence part follows from Itô formula [2, Theorem 4.4.7] and duality arguments. The proof of uniqueness uses the Monotonicity inequality (see [21, Theorem 2.1] and [7, Theorem 3.1 and Theorem 4.2]). Theorem 1.6 for $F \equiv 0$ is exactly [5, Theorem 3.8]. Another example of an SPDE in the class considered here is as follows

$$Y_t = \psi + \int_0^t \partial Y_{s-} dB_s + \int_0^t \frac{1}{2} \partial^2 Y_{s-} ds + \int_{0 < |x| < 1} (\tau_x - Id) Y_{t-} \tilde{N}(dt, dx) + \int_{0 < |x| < 1} (\tau_x - Id + x\partial) Y_{t-} \nu(dx) dt, t \geq 0,$$

where $d = 1, \sigma \equiv 1, b \equiv 0, F(0, x) = x, \forall x$.

An important observation is that $Y_t(\psi)$, as above, is given by a (random) $L^1(\mathbb{R}^d)$ function. This is an analogue of [5, Proposition 3.11] and has been stated in Proposition 3.1.

Using an analogue of the interlacing technique [2, Example 1.3.13 and Theorem 6.2.9], we also show the existence and uniqueness of strong solutions for SPDEs involving large jump terms.

Theorem 1.7 (Main Result 3). *Let $p > \frac{d}{4}$ and $\psi \in L^1(\mathbb{R}^d)$. Let $G : \{x \in \mathbb{R}^d : |x| \geq 1\} \rightarrow \mathbb{R}^d$ be a measurable function. Let σ, b and F be as in Theorem 1.6. Then there exists an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued process $(Y_t(\psi))_{t \geq 0}$ satisfying the following SPDE*

$$dY_t = A^*(Y_{t-}) dB_t + L^*(Y_{t-}) dt + \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id) Y_{t-} \tilde{N}(dt, dx) + \int_{|x| \geq 1} (\tau_{G(x)} - Id) Y_{t-} N(dt, dx) + \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id + F(0, x)\partial) Y_{t-} \nu(dx) dt, \tag{1.16}$$

with equality in \mathcal{S}_{-p-1} and initial condition ψ . This solution is also unique.

The proof of Theorem 1.7 has been discussed in Section 3.

2 Proof of Theorem 1.2

In this section, we give a sketch of the proof of Theorem 1.2. As mentioned in Section 1 of introduction, the proof is provided for the one-dimensional case only.

Proof. If the relation (1.4) holds, then $f(0, z) = z, \forall z$. Rewriting (1.4), we have

$$\int_0^t \left(b(Z_{s-}^z) - b(Z_{s-}^0) - \frac{\partial f}{\partial s}(s, z) \right) ds + \int_0^t (\sigma(Z_{s-}^z) - \sigma(Z_{s-}^0)) dB_s + \int_0^t \int_{0 < |x| < 1} (F(Z_{s-}^z, x) - F(Z_{s-}^0, x)) \tilde{N}(ds, dx) = 0$$

We denote the first, second and the third term on the left hand side by A_t, M_t^1 and M_t^2 respectively. Note that $(M_t^1 + M_t^2)_{t \geq 0}$ is a martingale with paths which are right continuous and have left limits. But, $(A_t)_{t \geq 0}$ is a predictable process of integrable variation. Since $M_0^1 + M_0^2 = 0$, using [26, Lemma 25.11] we conclude a.s. for all $t \geq 0, A_t = 0 = M_t^1 + M_t^2$.

Observe that a.s. $[M^1, M^1 + M^2]_t = 0, \forall t \geq 0$. Since B and N are independent, we have a.s. $[M^1, M^2]_t = 0, \forall t \geq 0$. Hence, a.s. $[M^1]_t = [M^1, M^1]_t = 0, \forall t \geq 0$. Similarly, a.s.

$[M^2]_t = 0, \forall t \geq 0$. Using Itô isometry, the identification of σ goes as in [5, Theorem 2.5]. The identification of F is also similar.

To make the identification of b , we start with a.s. $A_t = 0, \forall t \geq 0$. Since the process $(Z_t^z)_{t \geq 0}$ has only countably many jumps in any time interval $[0, t]$, we have a.s.

$$A_t = \int_0^t \left(b(Z_s^z) - b(Z_s^0) - \frac{\partial f}{\partial s}(s, z) \right) ds = 0, \forall t \geq 0.$$

Now, observe that

$$b(Z_t^z) - b(Z_t^0) - \frac{\partial f}{\partial t}(t, z) = 0, \forall z. \tag{2.1}$$

In particular for $t = 0$, we have $b(z) = b(0) - \frac{\partial f}{\partial t}(0, z), \forall z$. Since b is a C^2 function on \mathbb{R} , we expand $b(Z_t^z)$ by Itô formula [2, Theorem 4.4.7] and look at the martingale component of $b(Z_t^z) - b(Z_t^0)$. Applying [26, Lemma 25.11], we conclude that $b(z) = Cz + k, \forall z$, where $C = b'(0)$ and k is some real number. This part of the argument is similar to the proof of [5, Theorem 2.5].

Now, we have $f(0, z) = z, \forall z$ and

$$f(t, z) - z = \int_0^t (b(Z_s^z) - b(Z_s^0)) ds = \int_0^t C (Z_s^z - Z_s^0) ds = \int_0^t C f(s, z) ds$$

and hence, $f(t, z) = ze^{tC}, \forall t \geq 0, z \in \mathbb{R}$. This completes the proof of necessity.

To prove the converse, observe that there exists a \mathbb{P} null set \mathcal{N} such that for all $\omega \in \Omega \setminus \mathcal{N}, t \geq 0$

$$Z_t^0 = \int_0^t \sigma dB_s + \int_0^t (CZ_{s-}^0 + K) ds + \int_0^t \int_{0 < |x| < 1} F(0, x) \tilde{N}(ds, dx).$$

Hence, on $\Omega \setminus \mathcal{N}, (Z_t^0 + ze^{tC})_{t \geq 0}$ solves the SDE (1.1). This completes the proof. \square

Remark 2.1. In the SDE related part of this paper, we have not considered the large jumps case. Usually the existence and uniqueness of strong solutions of SDEs of the form

$$\begin{aligned} dZ_t &= b(Z_{t-})dt + \sigma(Z_{t-})dB_t + \int_{0 < |x| < 1} F(Z_{t-}, x) \tilde{N}(dt, dx) + \int_{|x| \geq 1} G(Z_{t-}, x)N(dt, dx), \\ Z_0 &= z, \end{aligned} \tag{2.2}$$

are obtained by interlacing arguments [2, Example 1.3.13 and Theorem 6.2.9]. The deterministic dependence on initial conditions in this case can be worked out in simple situations. Let us work in one dimension and in the pure jump case with $\sigma(\cdot) \equiv b(\cdot) \equiv F(\cdot, \cdot) \equiv 0$. Assume that $z \mapsto G(z, x)$ is continuous for all x with $|x| \geq 1$. Here, as a necessary condition for deterministic dependence on the initial condition, we have $f(t, z) = Z_t^z - Z_t^0 = z, \forall t < \tau_1$, where $\{\tau_n\}_n$ are the arrival times of the compound Poisson process given by $A(t) = \int_{|x| \geq 1} x N(t, dx), \forall t$. At $t = \tau_1$, by [2, Theorem 6.2.9], we have for every fixed $z \in \mathbb{R}$

$$f(\tau_1, z) = z + [G(z, \Delta P(\tau_1)) - G(0, \Delta P(\tau_1))].$$

Since, $\Delta P(\tau_1)$ is independent of the Poisson process $N(t, \{x : |x| \geq 1\})$ (see the proof of [2, Theorem 2.3.9]), we conclude that $f(\tau_1, z)$ is dependent only on z . Since τ_1 is an exponential random variable with mean $\frac{1}{\nu(\{x : |x| \geq 1\})}$, we conclude $f(t, z) = z, \forall t > 0, z \in \mathbb{R}$. This implies, $G(z, \Delta P(\tau_1)) = G(0, \Delta P(\tau_1)), \forall z$ and hence $(z, x) \mapsto G(z, x)$ depends only on x, ν -almost every z . Conversely, if $(z, x) \mapsto G(z, x)$ depends only on x , then the relation $f(t, z) = Z_t^z - Z_t^0$ is satisfied with $f(t, z) = z, \forall t, z$.

3 Proofs of Theorem 1.6 and Theorem 1.7

This section is devoted to the proofs of Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Structurally, the proof in higher dimensions include appropriate notational modifications from the proof in dimension one. Since the argument remains the same, to avoid notational complexity, the proof is discussed in dimension one only.

Fix $\phi \in \mathcal{S}(\mathbb{R})$ and $z \in \mathbb{R}$. Then, by Itô formula [2, Theorem 4.4.7], we have a.s.

$$\begin{aligned} Z_t(\phi)(z) &= \phi(z) + \int_0^t (Z_{s-}(L\phi))(z)ds + \int_0^t (Z_{s-}(A\phi))(z)dB_s \\ &\quad + \int_0^t \int_{0 < |x| < 1} \{Z_{s-}(\tau_{-F(0,x)}\phi) - Z_{s-}(\phi)\} (z) \tilde{N}(ds, dx) \\ &\quad + \int_0^t \int_{0 < |x| < 1} \{Z_{s-}(\tau_{-F(0,x)}\phi) - Z_{s-}(\phi) - Z_{s-}(F(0,x)\partial\phi)\} (z) \nu(dx)ds. \end{aligned}$$

Hence, using (1.15), we have, a.s.

$$\begin{aligned} &\langle Y_t(\psi), \phi \rangle \\ &= \langle Z_t^*(\psi), \phi \rangle \\ &= \langle \psi, \phi \rangle + \int_0^t \langle \psi, Z_{s-}(A\phi) \rangle dB_s + \int_0^t \langle \psi, Z_{s-}(L\phi) \rangle ds \\ &\quad + \int_0^t \int_{0 < |x| < 1} \langle \psi, Z_{s-}(\tau_{-F(0,x)}\phi) - Z_{s-}(\phi) \rangle \tilde{N}(ds, dx) \\ &\quad + \int_0^t \int_{0 < |x| < 1} \langle \psi, Z_{s-}(\tau_{-F(0,x)}\phi) - Z_{s-}(\phi) - Z_{s-}(F(0,x)\partial\phi) \rangle \nu(dx)ds \\ &= \langle \psi, \phi \rangle + \left\langle \int_0^t A^*(Z_{s-}^*(\psi))dB_s, \phi \right\rangle + \left\langle \int_0^t L^*(Z_{s-}^*(\psi))ds, \phi \right\rangle \\ &\quad + \left\langle \int_0^t \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id)Z_{s-}^*(\psi)\tilde{N}(ds, dx), \phi \right\rangle \\ &\quad + \left\langle \int_0^t \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id + F(0,x)\partial)Z_{s-}^*(\psi)\nu(dx)ds, \phi \right\rangle. \end{aligned}$$

Since the above result is true for all $\phi \in \mathcal{S}(\mathbb{R})$, we conclude that $(Y_t(\psi))_{t \geq 0}$ solves (1.9) with equality in $\mathcal{S}_{-p-1}(\mathbb{R})$.

Now, we prove the uniqueness of strong solutions. Let $(Y_t^1)_{t \geq 0}$ and $(Y_t^2)_{t \geq 0}$ be two $\mathcal{S}_{-p}(\mathbb{R})$ valued solutions of the SPDE (1.9) with equality in $\mathcal{S}_{-p-1}(\mathbb{R})$ and set $V_t = Y_t^1 - Y_t^2, \forall t \geq 0$. Then $(V_t)_{t \geq 0}$ is $\mathcal{S}_{-p}(\mathbb{R})$ valued and solves the SPDE (1.9) with initial condition 0, i.e., a.s.

$$\begin{aligned} V_t &= \int_0^t L^*V_{s-}ds + \int_0^t A^*V_{s-}dB_s + \int_0^t \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id) V_{s-}\tilde{N}(ds, dx) \\ &\quad + \int_0^t \int_{0 < |x| < 1} (\tau_{F(0,x)} - Id + F(0,x)\partial) V_{s-}\nu(dx)ds \end{aligned}$$

We have a.s. for $t \geq 0$ and for all $n \in \mathbb{Z}_+$,

$$\begin{aligned} \langle V_t, h_n \rangle &= \int_0^t \langle L^*V_{s-}, h_n \rangle ds + \int_0^t \langle A^*V_{s-}, h_n \rangle dB_s \\ &\quad + \int_0^t \int_{0 < |x| < 1} \langle (\tau_{F(0,x)} - Id) V_{s-}, h_n \rangle \tilde{N}(ds, dx) \end{aligned}$$

$$+ \int_0^t \int_{0 < |x| < 1} \langle (\tau_{F(0,x)} - Id + F(0,x)\partial) V_{s-}, h_n \rangle \nu(dx) ds. \tag{3.1}$$

with the functions h_n 's as in (1.6).

Applying the Itô formula [2, Theorem 4.4.7] for the processes $(\langle V_t, h_n \rangle)_{t \geq 0}$ and the function $x \mapsto x^2$, we have a.s. for $t \geq 0$

$$\begin{aligned} & \|V_t\|_{-p-1}^2 \\ &= \sum_{n=0}^{\infty} (2n+1)^{-2(p+1)} \langle V_t, h_n \rangle^2 \\ &= \int_0^t \left\{ 2 \langle V_{s-}, L^* V_{s-} \rangle_{-p-1} + \|A^* V_{s-}\|_{-p-1}^2 \right\} ds + \int_0^t 2 \langle V_{s-}, A^* V_{s-} \rangle_{-p-1} dB_s \\ &+ \int_0^t \int_{0 < |x| < 1} \left\{ \|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 \right\} \tilde{N}(ds, dx) \\ &+ \int_0^t \int_{0 < |x| < 1} \left\{ \|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 + 2 \langle F(0,x)\partial V_{s-}, V_{s-} \rangle_{-p-1} \right\} \nu(dx) ds \end{aligned}$$

Then, for any $T > 0$,

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \\ & \leq \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \left\{ 2 \langle V_{s-}, L^* V_{s-} \rangle_{-p-1} + \|A^* V_{s-}\|_{-p-1}^2 \right\} ds \\ & + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t 2 \langle V_{s-}, A^* V_{s-} \rangle_{-p-1} dB_s \right| \\ & + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_{0 < |x| < 1} \left\{ \|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 \right\} \tilde{N}(ds, dx) \right| \\ & + \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \int_{0 < |x| < 1} \left\{ \|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 \right. \\ & \quad \left. + 2 \langle F(0,x)\partial V_{s-}, V_{s-} \rangle_{-p-1} \right\} \nu(dx) ds. \end{aligned} \tag{3.2}$$

We denote the four terms on the right hand side above by I_1, I_2, I_3 and I_4 . We now observe the following inequalities. Constants appearing in the computations below depend on one or more of the parameters p, σ, b and F only and may change their values from line to line.

(i) By the Monotonicity inequality for A^*, L^* [7, Theorem 4.2], we have

$$\begin{aligned} I_1 &= \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \int_0^t \left\{ 2 \langle V_{s-}, L^* V_{s-} \rangle_{-p-1} + \|A^* V_{s-}\|_{-p-1}^2 \right\} ds \right\} \\ & \leq \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \int_0^t C_1 \|V_{s-}\|_{-p-1}^2 ds \right\} \\ & \leq C_1 \int_0^T \mathbb{E} \left\{ \sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right\} ds. \end{aligned}$$

(ii) It was observed in [7, Proof of Theorem 3.1] that there exists a bounded linear operator $\mathbb{T} : \mathcal{S}_{-p-1}(\mathbb{R}) \rightarrow \mathcal{S}_{-p-1}(\mathbb{R})$ such that

$$\langle \phi, \partial \phi \rangle_{-p-1} = \langle \phi, \mathbb{T} \phi \rangle_{-p-1}, \forall \phi \in \mathcal{S}_{-p-\frac{1}{2}}(\mathbb{R}). \tag{3.3}$$

By the Burkholder-Davis-Gundy inequality [26, Theorem 26.12] and (3.3), we have

$$\begin{aligned}
 I_2 &= 2\mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \langle V_{s-}, A^* V_{s-} \rangle_{-p-1} dB_s \right| \right\} \\
 &\leq C_2 \mathbb{E} \left(\int_0^T \langle V_{s-}, A^* V_{s-} \rangle_{-p-1}^2 ds \right)^{\frac{1}{2}} \\
 &= C_2 \mathbb{E} \left(\int_0^T \sigma^2 \langle V_{s-}, \partial V_{s-} \rangle_{-p-1}^2 ds \right)^{\frac{1}{2}} \\
 &\leq C_2 \mathbb{E} \left(\int_0^T \|V_{s-}\|_{-p-1}^4 ds \right)^{\frac{1}{2}} \\
 &\leq C_2 \mathbb{E} \left(\sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \int_0^T \|V_{s-}\|_{-p-1}^2 ds \right)^{\frac{1}{2}} \\
 &\leq C_2 \left(\epsilon_1 \mathbb{E} \left(\sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right) + \frac{1}{\epsilon_1} \mathbb{E} \left(\int_0^T \|V_{s-}\|_{-p-1}^2 ds \right) \right) \\
 &\leq C_2 \left(\epsilon_1 \mathbb{E} \left(\sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right) + \frac{1}{\epsilon_1} \int_0^T \mathbb{E} \left(\sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right) ds \right),
 \end{aligned}$$

for any $\epsilon_1 > 0$.

- (iii) Recall the orthonormal basis $\{h_n : n \geq 0\}$ of $L^2(\mathbb{R})$ as in (1.6). Note that $\{(2n + 1)^{-p} h_n : n \geq 0\}$ is an orthonormal basis for $\mathcal{S}_p(\mathbb{R})$.

For fixed $z \in \mathbb{R}, \phi \in \mathcal{S}_{-p}(\mathbb{R}), m \geq 0$, the following C^2 function $g : [0, 1] \rightarrow \mathbb{R}$ defined by $g(v; z, \phi, m) := \langle \tau_{vz} \phi, h_m \rangle^2, \forall v \in [0, 1]$ was considered in [6, Lemma 2.5]. Using (3.3), we have

$$\begin{aligned}
 \sum_{m=1}^{\infty} (2m + 1)^{-2(p+1)} g'(v; z, \phi, m) &= -2 \sum_{m=1}^{\infty} (2m + 1)^{-2(p+1)} \langle \tau_{vz} \phi, h_m \rangle \langle z \partial \tau_{vz} \phi, h_m \rangle \\
 &= -2 \langle \tau_{vz}, z \partial \tau_{vz} \rangle_{-p-1} \\
 &= -2z \langle \tau_{vz}, \mathbb{T} \tau_{vz} \rangle_{-p-1}, \forall v \in (0, 1).
 \end{aligned}$$

Using this observation and the boundedness of the translation operators [38, Theorem 2.1], we have

$$\begin{aligned}
 &\left(\|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 \right)^2 \\
 &= \left(\sum_{m=0}^{\infty} (2m + 1)^{-2(p+1)} [g(1; F(0, x), V_{s-}, m) - g(0; F(0, x), V_{s-}, m)] \right)^2 \\
 &= \left(\sum_{m=0}^{\infty} (2m + 1)^{-2(p+1)} \int_0^1 g'(v; F(0, x), V_{s-}, m) dv \right)^2 \\
 &\leq 4(F(0, x))^2 \int_0^1 \langle \tau_{vF(0,x)} V_{s-}, \mathbb{T} \tau_{vF(0,x)} V_{s-} \rangle_{-p-1}^2 dv \\
 &\leq 4(F(0, x))^2 \int_0^1 \left(P_k(|vF(0, x)|) \|V_{s-}\|_{-p-1} \right)^4 dv \\
 &\leq 4 \left(\sup_{0 \leq |x| \leq 1} (P_k(|F(0, x)|))^4 \right) (F(0, x))^2 \|V_{s-}\|_{-p-1}^4, \tag{3.4}
 \end{aligned}$$

for some real polynomial P_k of degree $k = 2(\lfloor p + 1 \rfloor + 1)$ with non-negative coefficients.

Applying the Burkholder-Davis-Gundy inequality [26, Theorem 26.12], (3.4), (1.7) and (1.8) we have

$$\begin{aligned}
 & I_3 \\
 &= \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \int_{0 < |x| < 1} \left\{ \|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 \right\} \tilde{N}(ds, dx) \right| \right\} \\
 &\leq C_3 \mathbb{E} \left(\int_0^T \int_{0 < |x| < 1} \left(\|\tau_{F(0,x)} V_{s-}\|_{-p-1}^2 - \|V_{s-}\|_{-p-1}^2 \right)^2 \nu(dx) ds \right)^{\frac{1}{2}} \\
 &\leq C_3 \mathbb{E} \left(\int_0^T \int_{0 < |x| < 1} F(0, x)^2 \|V_{s-}\|_{-p-1}^4 \nu(dx) ds \right)^{\frac{1}{2}} \\
 &\leq C_3 \left(\int_{0 < |x| < 1} F(0, x)^2 \nu(dx) \right)^{\frac{1}{2}} \\
 &\quad \times \left(\epsilon_2 \mathbb{E} \left(\sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right) + \frac{1}{\epsilon_2} \int_0^T \mathbb{E} \left(\sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right) ds \right) \\
 &\leq C_3 \left[\epsilon_2 \mathbb{E} \left(\sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right) + \frac{1}{\epsilon_2} \int_0^T \mathbb{E} \left(\sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right) ds \right],
 \end{aligned}$$

for any $\epsilon_2 > 0$.

- (iv) The relevant estimate for I_4 is obtained similar to the estimate for Term 2 in the proof of [6, Lemma 3.4]. The difference with the present setup with that of [6] is that the function F there depended on both its arguments. In the present case, we use (1.7) and (1.8) and obtain

$$I_4 \leq C_4 \int_0^T \mathbb{E} \left(\sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right) ds.$$

Using these derived inequalities in (3.2), we end up with

$$\begin{aligned}
 & \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right\} \\
 &\leq \left(C_1 + C_4 + \frac{C_2}{\epsilon_1} + \frac{C_3}{\epsilon_2} \right) \int_0^T \mathbb{E} \left\{ \sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right\} ds \\
 &\quad + (C_2 \epsilon_1 + C_3 \epsilon_2) \mathbb{E} \left(\sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right)
 \end{aligned}$$

We choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $(1 - C_2 \epsilon_1 - C_3 \epsilon_2) > 0$ and we have

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} \|V_t\|_{-p-1}^2 \right\} \leq C_5 \int_0^T \mathbb{E} \left\{ \sup_{0 \leq t \leq s} \|V_t\|_{-p-1}^2 \right\} ds,$$

for some constant $C_5 > 0$ depending only on p, σ, b and F .

Using Gronwall's inequality we get a.s. for all $t \in [0, T]$, $V_t = Y_t^1 - Y_t^2 = 0$. The result follows. \square

An important observation useful for the proof of Theorem 1.7 is the following result that $Y_t(\psi)$, as in Theorem 1.6, is given by a (random) integrable function. This is an analogue of [5, Proposition 3.11].

Proposition 3.1. Under the assumptions of Theorem 1.6, on $\omega \in \Omega \setminus \mathcal{N}$, the distribution $Y_t(\psi)$ is given by the $L^1(\mathbb{R}^d)$ function

$$e^{-t \operatorname{tr}(C)} \tau_{Z_t^0} \psi_t(\cdot),$$

where $\psi_t(z) := \psi(e^{-tC}z)$ for $t \geq 0, z \in \mathbb{R}^d$ and $\operatorname{tr}(C)$ is the trace of the matrix C .

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} \langle Y_t(\psi), \phi \rangle &= \int_{\mathbb{R}^d} \psi(z) \phi(e^{tC}z + Z_t^0) dz \\ &= |\det(e^{-tC})| \int_{\mathbb{R}^d} \psi(e^{-tC}(z - Z_t^0)) \phi(z) dz, \\ &= e^{-t \operatorname{tr}(C)} \int_{\mathbb{R}^d} (\tau_{Z_t^0} \psi_t)(z) \phi(z) dz. \end{aligned}$$

In the last step above, we have used the equality $|\det(e^{-tC})| = e^{-t \operatorname{tr}(C)}$ (see [23, Problem 5.6.P43]). This completes the proof. \square

As a consequence of the above proposition, we obtain the next result.

Theorem 3.2. Let ψ be an $L^1(\mathbb{R}^d)$ valued \mathcal{F}_0 measurable random variable. Let σ, b and F be as in Theorem 1.6. Fix $p > \frac{d}{4}$. Then, the SPDE (1.9) with initial condition ψ has a unique $\mathcal{S}_{-p}(\mathbb{R}^d)$ strong solution.

Proof. First, assume that there exists $\alpha > 0$ such that $\|\psi(\omega)\|_{L^1 \mathbb{R}^d} \leq \alpha, \forall \omega$. Define

$$Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} (\psi(\omega))(z) \delta_{Z_t^z(\omega)} dz, \forall t \geq 0, \omega \in \Omega \setminus \mathcal{N} \tag{3.5}$$

and set $Y_t(\omega)(\psi) = 0, \forall t \geq 0, \omega \in \mathcal{N}$, where \mathcal{N} is as in (1.14). As proved in Theorem 1.6, it can be shown that $(Y_t(\psi))_{t \geq 0}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued adapted process satisfying the SPDE (1.9) with equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$.

Now, let ψ be any $L^1(\mathbb{R}^d)$ valued \mathcal{F}_0 measurable random variable. For positive integers k , consider $\psi^{(k)} : \Omega \rightarrow L^1(\mathbb{R}^d)$ defined by $\psi^{(k)} = \psi$ on the set $\{\|\psi\|_{L^1(\mathbb{R}^d)} \leq k\}$ and 0 otherwise. Consider the processes $(Y_t(\psi^{(k)}))_{t \geq 0}, k = 1, 2, \dots$ which solve the SPDE (1.9) with initial condition $\psi^{(k)}$. Observe that the consistency property $Y_t(\psi^{(k)}) = Y_t(\psi^{(n)}), \forall n \geq k, \forall t \geq 0$ on the set $\{\|\psi\|_{L^1(\mathbb{R}^d)} \leq k\}$ holds and $\{\|\psi\|_{L^1(\mathbb{R}^d)} \leq k\} \uparrow \Omega$ as $k \rightarrow \infty$. Hence, $(Y_t(\psi))_{t \geq 0}$, defined by (3.5) for general ψ solves the SPDE (1.9) with initial condition ψ . Uniqueness of the solution follows from the consistency criteria discussed above. \square

Remark 3.3. As discussed in Proposition 3.1 for deterministic ψ , we can show that $Y_t(\psi)$ obtained in Theorem 3.2 for random ψ is a (random) $L^1(\mathbb{R}^d)$ function.

Proof of Theorem 1.7. We use an analogue of interlacing argument in [2, Theorem 6.2.9] to construct a solution.

Let $(P_t)_{t \geq 0}$ be the compound Poisson process defined by $P(t) = \int_{|x| \geq 1} x N(t, dx), \forall t$. Let $\tau_n, n = 1, 2, \dots$ denote the arrival times of the jumps of $(P_t)_{t \geq 0}$. Let $(Y_t(\psi))_{t \geq 0}$ denote the $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued strong solution of SPDE (1.9) with initial condition ψ , where ψ is an $L^1(\mathbb{R}^d)$ valued random variable.

Consider the process $(\tilde{Y}_t(\psi))_{t \geq 0}$ defined as follows.

$$\begin{aligned} \tilde{Y}_t(\psi) &:= Y_t(\psi), \text{ if } 0 \leq t < \tau_1 \\ \tilde{Y}_{\tau_1}(\psi) &:= \tilde{Y}_{\tau_1-}(\psi) + \left(\tau_{G(\Delta P_{\tau_1})} - Id \right) \tilde{Y}_{\tau_1-}(\psi) \\ \tilde{Y}_t(\psi) &= \tilde{Y}_{\tau_1}(\psi) + Y_t(\tilde{Y}_{\tau_1}(\psi)) - Y_{\tau_1}(\tilde{Y}_{\tau_1}(\psi)), \text{ if } \tau_1 < t < \tau_2, \end{aligned} \tag{3.6}$$

where ΔP_{τ_1} gives the first jump of $(P_t)_{t \geq 0}$. Note that $\tilde{Y}_{\tau_1}(\psi)$ is given by a random $L^1(\mathbb{R}^d)$ function (see Remark 3.3) and $(Y_t(\tilde{Y}_{\tau_1}))_{t \geq 0}$ denotes the unique solution to (1.9) with initial condition \tilde{Y}_{τ_1} (see Theorem 3.2).

Continuing as in (3.6), recursively we define $\tilde{Y}_t(\psi)$ for all $t \geq 0$. Note that the process $(\tilde{Y}_t(\psi))_{t \geq 0}$ is adapted, $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued and has paths which are right continuous with left limits. Then, $(\tilde{Y}_t(\psi))_{t \geq 0}$ solves (1.16). Uniqueness of the solution follows from standard arguments. \square

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