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Limit theorems for discounted convergent perpetuities II*

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Abstract

Let (ξ_1, η_1), $(\xi_2, \eta_2), \dots$ be independent identically distributed $\mathbb{R}^2$-valued random vectors. Assuming that ξ_1 has zero mean and finite variance and imposing three distinct groups of assumptions on the distribution of η_1 we prove three functional limit theorems for the logarithm of convergent discounted perpetuities $\sum_{k\geq 0} \mathrm{e}^{\xi_1+\dots+\xi_k-ak}\eta_{k+1}$ as $a \to 0+$. Also, we prove a law of the iterated logarithm which corresponds to one of the aforementioned functional limit theorems. The present paper continues a line of research initiated in the paper Iksanov, Nikitin and Samoillenko (2022), which focused on limit theorems for a different type of convergent discounted perpetuities.

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1 Introduction

Let (ξ_1, η_1), $(\xi_2, \eta_2), \dots$ be independent copies of an $\mathbb{R}^2$-valued random vector (ξ, η) with arbitrarily dependent components. Denote by $(S_k)_{k\in\mathbb{N}_0}$ (as usual, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) the standard random walk with jumps ξ_k defined by $S_0 := 0$ and $S_k := \xi_1 + \dots + \xi_k$ for $k \in \mathbb{N}$. Whenever a random series $\sum_{k\geq 0} \mathrm{e}^{S_k}\eta_{k+1}$ converges almost surely (a.s.), its sum is called *perpetuity* because of the following financial application. Assuming temporarily that η is a.s. positive, the variables η_{k+1} and e^{ξ_k} may be interpreted as the planned payoff of a private pension fund to a client and the discount factor for year $k \in \mathbb{N}_0$, respectively.

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The pension payoffs to a client are made at the beginning of each year. The variable $\sum_{k\geq 0} \mathrm{e}^{S_k}\eta_{k+1}$ can be thought of as a perpetuity, that is, the present value of a permanent commitment to make a payoff annually into the future forever. In other words, this is the initial payment to a fund and, for $k \in \mathbb{N}_0$, $\mathrm{e}^{S_k}\eta_{k+1}$ is an amount ensuring that a client gets the planned payoff η_{k+1} at the beginning of year $k \in \mathbb{N}_0$.

It is known (see Lemma 1.7 in [20] or Theorem 2.1 in [11]) that the conditions $\mathbb{E}\xi \in [-\infty, 0)$ and $\mathbb{E}\log^+ |\eta| < \infty$ ensure that the series $\sum_{k\geq 0} \mathrm{e}^{S_k}\eta_{k+1}$ (absolutely) converges a.s. Recall that $\log^+ x = \log x$ if $x \geq 1$ and $= 0$ if $x \in (0,1)$. Further detailed information on perpetuities, accumulated up to 2016, can be found in the books [6] and [12].

1.1 Previously investigated discounted perpetuities

There are several ways to define a *discounted convergent perpetuity*. One option is

$$X(b) := \sum_{k\geq 0} b^{S_k}\eta_{k+1}, \quad b \in (0,1)$$

or equivalently $\sum_{k\geq 0} \mathrm{e}^{-S_k/t}\eta_{k+1}$ for $t > 0$. In the recent article [14] basic limit theorems for $X(b)$, properly normalized, as $b \to 1-$, were proved, namely, a strong law of large numbers, a functional central limit theorem and a law of the iterated logarithm. To be more specific, we state a combination of Theorem 1.2 and one part of Theorem 1.5 in [14] as Proposition 1.1. Denote by $C = C(0,\infty)$ the space of continuous functions defined on $(0,\infty)$ equipped with the locally uniform topology. Throughout the paper we write $\Longrightarrow$ to denote weak convergence of probability measures in a function space.

Proposition 1.1. *Assume that $\mu = \mathbb{E}\xi \in (0,\infty)$, $\mathbb{E}\eta = 0$ and $\mathrm{s}^2 := \mathrm{Var}\,\eta \in (0,\infty)$. Then, as $b \to 1-$,*

$$\Big((1-b^2)^{1/2}X(b^u)\Big)_{u>0} \implies (2\mathrm{s}^2\mu^{-1})^{1/2}\bigg(\int_{[0,\,\infty)} e^{-uy}\mathrm{d}B(y)\bigg)_{u>0} \tag{1.1}$$

on C, where $(B(t))_{t\geq 0}$ is a standard Brownian motion, and

$$\limsup\,(\liminf)_{b\to 1-}\bigg(\frac{1-b^2}{\log\log\frac{1}{1-b^2}}\bigg)^{1/2}X(b) = +(-)(2\mathrm{s}^2\mu^{-1})^{1/2} \quad \text{a.s.} \tag{1.2}$$

Note that in the cited Theorem 1.2 weak convergence was stated on the Skorokhod space $D(0,\infty)$ of càdlàg functions on $(0,\infty)$ equipped with the J_1-topology. Since the process on the left-hand side of (1.1) is a.s. continuous in u, the weak convergence also takes place on C. Corollary 1.5 in [13] is an ultimate version of the functional central limit theorem for $X(b)$ in the case $\mathrm{Var}\,\xi < \infty$ and $\mathrm{s}^2 < \infty$, which particularly strengthens (1.1). In [13], the condition $\mathbb{E}\eta \in \mathbb{R}$ is allowed, which is a new aspect in comparison to (1.1).

In both (1.1) and (1.2), the random walk (S_k) only provides a first-order contribution to the limit which is represented by the strong law of large numbers $\lim_{k\to\infty} k^{-1}S_k = \mu$ a.s. In other words, the limits remain unchanged when replacing S_k on the left-hand sides with μk; see Theorem 1.1 in [5] for the corresponding counterpart of (1.2). The limit relations (1.1) and (1.2) are mainly driven by fluctuations of the random walk $(\eta_1 + \ldots + \eta_n)$ as n becomes large. More precisely, the main driving forces behind (1.1) and (1.2) are the Donsker functional limit theorem and the law of the iterated logarithm for $(\eta_1 + \ldots + \eta_n)$, respectively.

1.2 New type of discounted perpetuities and main results

Our standing assumptions throughout the paper are: η is a.s. positive, $\mathbb{E}\log^+ \eta < \infty$ and

$$\mathbb{E}\xi = 0 \quad \text{and} \quad \sigma^2 := \mathrm{Var}\,\xi \in (0,\infty). \tag{1.3}$$

We shall investigate

$$Y(a) := \sum_{k\ge 0} \mathrm{e}^{S_k - ak}\eta_{k+1}, \quad a>0,$$

which is yet another type of discounted convergent perpetuity, and an accompanying process

$$Z(a) := \sup_{k\ge 0}\,(S_k - ak + \log \eta_{k+1}), \quad a>0.$$

By Theorem 2.1 in [11], the latter series converges a.s., that is, the perpetuity is indeed convergent. This implies that $\lim_{k\to\infty}(S_k - ak + \log\eta_{k+1}) = -\infty$ a.s., whence $|Z(a)| < \infty$ a.s. for each $a>0$. Specifically, we shall prove functional limit theorems for $(\log Y(au))_{u>0}$, properly normalized, as $a \to 0+$ and a law of the iterated logarithm for $\log Y(a)$, again properly normalized. Note that $\lim_{a\to 0+}\sum_{k\ge 0}\mathrm{e}^{S_k - ak}\eta_{k+1} = \sum_{k\ge 0}\mathrm{e}^{S_k}\eta_{k+1} = +\infty$ a.s. Here, noting that our assumptions entail $\mathbb{P}\{\eta + \mathrm{e}^{\xi}c = c\} < 1$ for all $c\in\mathbb{R}$, the a.s. divergence is justified by Theorem 2.1 in [11]. Thus, some normalizations are indeed needed in our limit theorems.

The presence of the logarithm already shows that the limit theorems for $X(b)$ and $Y(a)$ are of different nature. It will follow from our proofs that the functional limit theorems for $Y(a)$ are driven by heavy-traffic limit theorems for $Z(a)$ as $a\to 0+$. The sequence $(S_k - ak + \log\eta_{k+1})_{k\in\mathbb{N}_0}$ is a globally perturbed random walk; see [12] for a survey. The asymptotics of its supremum depend heavily upon the interplay between the asymptotic growth of $(S_k - ak)_{k\in\mathbb{N}_0}$ and that of $(\log\eta_j)_{j\in\mathbb{N}}$. This fact leads to three different functional limit theorems stated in Theorems 1.2, 1.4 and 1.7. We note in passing that the one-dimensional distributional convergence of $\sup_{k\ge 0}(S_k - ak)$ as $a\to 0+$, properly normalized, is well-understood for the random walks $(S_k)_{k\in\mathbb{N}_0}$ attracted to a centered stable Lévy process; see [18] and references therein. Also, we mention that the one-dimensional distributional convergence of $\sup_{k\ge 0}(\log\eta_{k+1} - ak)$, properly normalized, as $a\to 0$ was investigated in [8]; see, in particular, Theorem 7 therein.

Similarly, the law of the iterated logarithm for $\log Y(a)$ stated in Theorem 1.8 is a consequence of the law of the iterated logarithm for $\max_{0\le k\le n}(S_k + \log\eta_{k+1})$, properly normalized, as $n\to\infty$ and a previously known deterministic continuity result recalled in Proposition 4.1.

With the aforementioned financial interpretation, the variable $Y(a)$ is a perpetuity with a discount factor for year k being equal to $\mathrm{e}^{\xi_k - a}$. It is natural to call $-(\mathbb{E}\xi - a) = a$ the average rate of exponential wealth growth. Thus, our limit theorems describe the fluctuations of the perpetuity, when the average rate of exponential wealth growth approaches 0 while staying positive.

We are ready to formulate our main results.

1.2.1 Weak convergence

According to Lemma 2.2, the processes

$$\big(Z(au)\big)_{u>0} \quad \text{and} \quad \Big(\log Y(au)\Big)_{u>0}$$

are a.s. continuous. This enables us to formulate functional limit theorems in C.

We start with simpler situations in which the asymptotic behavior of the discounted convergent perpetuity is driven by either fluctuations of $(S_k - ak)_{k\in\mathbb{N}_0}$ (Theorem 1.2) or $(\log\eta_j)_{j\in\mathbb{N}}$ (Theorem 1.4).

Theorem 1.2. *Suppose that* (1.3) *holds and*

$$\lim_{t\to\infty} t^2\mathbb{P}\{\log\eta > t\} = 0. \tag{1.4}$$

Then

$$\big(a\,Z(au)\big)_{u>0} \quad\Longrightarrow\quad \big(\sup_{s\geq 0}\,(\sigma B(s)-us)\big)_{u>0}, \quad a\to 0+ \tag{1.5}$$

and

$$\Big(a\log Y(au)\Big)_{u>0} \quad\Longrightarrow\quad \big(\sup_{s\geq 0}\,(\sigma B(s)-us)\big)_{u>0}, \quad a\to 0+ \tag{1.6}$$

on C, where B is a standard Brownian motion.

Remark 1.3. The limit process in Theorem 1.2 is the Legendre-Fenchel transform of $s\mapsto -\sigma B(s)\,\mathbb{1}_{[0,\infty)}(s)$, $s\in\mathbb{R}$ evaluated at $-u<0$. In particular, it is a.s. convex (as a function of u), hence a.s. continuous. Similarly, the converging process in (1.5) can be thought of as a discrete version of the Legendre-Fenchel transform. These observations are implicitly used in the proof of Lemma 2.2 below when showing the a.s. convexity of the processes involved.

For positive γ and ρ, let $N^{(\gamma,\rho)}:=\sum_k \varepsilon_{(t_k^{(\gamma,\rho)},\,j_k^{(\gamma,\rho)})}$ be a Poisson random measure on $[0,\infty)\times(0,\infty]$ with intensity measure $\mathbb{LEB}\times\mu_{\gamma,\rho}$, where $\varepsilon_{(t,x)}$ is Dirac measure at $(t,x)\in[0,\infty)\times(0,\infty]$, $\mathbb{LEB}$ is Lebesgue measure on $[0,\infty)$, and $\mu_{\gamma,\rho}$ is the measure on $(0,\infty]$ defined by

$$\mu_{\gamma,\rho}\big((x,\infty]\big)=\gamma x^{-\rho},\quad x>0.$$

Theorem 1.4. *Suppose that* (1.3) *holds and that the function* $t\mapsto\mathbb{P}\{\log\eta>t\}$ *is regularly varying at* ∞ *of index* $-\beta$, $\beta\in(1,2]$. *If* $\beta=2$, *assume additionally that* $\lim_{t\to\infty}t^2\mathbb{P}\{\log\eta>t\}=\infty$. *Let* b *and* c *be arbitrary positive functions which satisfy* $\lim_{t\to\infty}t\mathbb{P}\{\log\eta>b(t)\}=1$ *and* $b(c(a))\sim ac(a)$ *as* $a\to 0+$. *Then*

$$\Big(\frac{1}{ac(a)}\,Z(au)\Big)_{u>0} \quad\Longrightarrow\quad \big(\sup_k\,\big(-ut_k^{(1,\beta)}+j_k^{(1,\beta)}\big)\big)_{u>0}, \quad a\to 0+ \tag{1.7}$$

and

$$\Big(\frac{1}{ac(a)}\log Y(au)\Big)_{u>0} \quad\Longrightarrow\quad \big(\sup_k\,\big(-ut_k^{(1,\beta)}+j_k^{(1,\beta)}\big)\big)_{u>0}, \quad a\to 0+ \tag{1.8}$$

on C.

Remark 1.5. One can choose b as an asymptotic inverse of $t\mapsto 1/\mathbb{P}\{\log\eta>t\}$. By Theorem 1.5.12 in [3], such functions exist and are regularly varying at ∞ of index $1/\beta$. In the role of $t\mapsto c(1/t)$ one can take an asymptotically inverse function of $t\mapsto t/b(t)$. Another appeal to Theorem 1.5.12 in [3] enables us to conclude that $t\mapsto c(1/t)$ is regularly varying at ∞ of index $\beta/(\beta-1)$. Hence, $a\mapsto c(a)$ is regularly varying at $0+$ of index $-\beta/(\beta-1)$. In particular, if $\mathbb{P}\{\log\eta>t\}\sim\kappa t^{-\beta}$ as $t\to\infty$ for some $\kappa>0$, then $c(a)\sim\kappa^{1/(\beta-1)}a^{-\beta/(\beta-1)}$ as $a\to 0+$. For later use, we note that

$$\lim_{a\to 0+}a^2c(a)=\infty. \tag{1.9}$$

This is obvious when $\beta\in(1,2)$ and follows from

$$a^2c(a)\sim a^2c^2(a)\mathbb{P}\{\log\eta>b(c(a))\}\sim(ac(a))^2\mathbb{P}\{\log\eta>ac(a)\}\to\infty,\quad a\to 0+$$

when $\beta=2$.

As we shall see in Proposition 2.1; see formula (2.3) below, for each $u>0$, $\sup_k\big(-ut_k^{(1,1)}+j_k^{(1,1)}\big)=+\infty$ a.s. This explains the fact that Theorem 1.4 is not applicable in the situations in which $\mathbb{E}\log^+\eta<\infty$ and $t\mapsto\mathbb{P}\{\log\eta>t\}$ is regularly varying at ∞ of index -1.

Remark 1.6. Observe that, under the assumptions of Theorem 1.2, both (1.5) and (1.6) remain true when replacing η with 1 and that, under the assumptions of Theorem 1.4, both (1.7) and (1.8) remain true when replacing ξ with 0. We think this (obvious) observation facilitates the understanding of Theorems 1.2 and 1.4.

If in addition to (1.3) the condition

$$\mathbb{P}\{\log\eta > t\} \sim \lambda t^{-2}, \quad t\to\infty, \tag{1.10}$$

holds for some $\lambda > 0$, then the contributions of $\max_{0\le k\le n} S_k$ and $\max_{1\le k\le n+1}\log\eta_k$ to the asymptotic behavior of $\max_{0\le k\le n}(S_k+\log\eta_{k+1})$ are comparable. This situation which is more interesting than the other two is treated in the following result.

Theorem 1.7. *Suppose that* (1.3) *and* (1.10) *hold. Then*

$$\big(aZ(au)\big)_{u>0} \quad\Longrightarrow\quad \big(\sup_k\big(\sigma B(t_k^{(\lambda,\,2)}) - ut_k^{(\lambda,\,2)} + j_k^{(\lambda,\,2)}\big)\big)_{u>0}, \quad a\to 0+ \tag{1.11}$$

and

$$\Big(a\log Y(au)\Big)_{u>0} \quad\Longrightarrow\quad \big(\sup_k\big(\sigma B(t_k^{(\lambda,\,2)}) - ut_k^{(\lambda,\,2)} + j_k^{(\lambda,\,2)}\big)\big)_{u>0}, \quad a\to 0+ \tag{1.12}$$

on C, where B is a standard Brownian motion independent of $N^{(\lambda,\,2)}$.

1.2.2 A law of the iterated logarithm

We shall now turn to the a.s. asymptotic behavior of $\log Y(a)$. The first part of the following theorem is a law of the iterated logarithm for $\log Y(a)$ which corresponds to the distributional convergence of Theorem 1.2. The second part is a law of the iterated logarithm for the closely related random variables $\log\int_0^\infty \mathrm{e}^{B(s)-as}\mathrm{d}s$, where $a>0$ and B is a standard Brownian motion. These variables and their appearance will be discussed in details in Section 1.3 below.

Theorem 1.8. *(a) Suppose that* (1.3) *holds and*

$$\mathbb{E}\left(\frac{(\log^+\eta)^2}{\log\log(\log^+\eta)}\,\mathbb{1}_{\{\log^+\eta>\mathrm{e}\}}\right) < \infty. \tag{1.13}$$

Then

$$\limsup_{a\to0+}\frac{2a\log Y(a)}{\log\log(1/a)} = \sigma \quad \text{a.s.} \tag{1.14}$$

(b) Let B be a standard Brownian motion. Then

$$\limsup_{a\to0+}\frac{2a\log\int_0^\infty \mathrm{e}^{B(s)-as}\mathrm{d}s}{\log\log(1/a)} = 1 \quad \text{a.s.} \tag{1.15}$$

For a family of functions or a sequence (x_t) denote by $C((x_t))$ the set of its limit points.

Corollary 1.9. *Under the assumptions of Theorem 1.8,*

$$C\Big(\Big(\frac{2a\log Y(a)}{\log\log(1/a)} : a\in(0,1/\mathrm{e})\Big)\Big) = [0,\sigma] \quad \text{a.s.}$$

and

$$C\Big(\Big(\frac{2a\log\int_0^\infty \mathrm{e}^{B(s)-as}\mathrm{d}s}{\log\log(1/a)} : a\in(0,1/\mathrm{e})\Big)\Big) = [0,1] \quad \text{a.s.}$$

Remark 1.10. There exist distributions of η which satisfy (1.4) (the assumption of Theorem 1.2) and do not satisfy (1.13) (the assumption of Theorem 1.8). To exemplify, let $\mathbb{P}\{\log\eta > t\} \sim t^{-2}(\log t)^{-1}$ as $t \to \infty$. It will be explained in Remark 4.3 that, under (1.3), relation (1.14) fails to hold for the aforementioned distributions of η.

The log-exponential normalization in the laws of iterated logarithms (1.14) and (1.15) is not new. It has already appeared in the literature for random walks with infinite second moments; see [7, 15]. A collection of related results can be found in Section 7.5 of the book [16].

1.3 A connection to the exponential functional of Brownian motion

The variable $\int_0^\infty e^{B(s)-as}ds$ appearing in (1.15) serves as a continuous-time counterpart of the discounted convergent perpetuities $Y(a)$. It is known in the literature as an *exponential functional of Brownian motion* and has been the object of intensive research in the recent past; see [22] for a collection of results in a book format.

The advantage of working with $\int_0^\infty e^{B(s)-as}ds$, $a > 0$ is availability of explicit formulae for their marginal distributions. This makes their analysis easier in comparison to that of $Y(a)$. The appearance below of the distributions of the suprema of certain Lévy processes with a drift in the role of limit distributions provides a hint towards what can be expected in the discrete setting in Theorems 1.2, 1.4 and 1.7.

In what follows, $\overset{d}{=}$ and $\overset{d}{\longrightarrow}$ denote equality of distributions and convergence in distribution, respectively. Let $\theta_{b,\,c}$ be a random variable having a gamma distribution with positive parameters b and c, that is,

$$\mathbb{P}\{\theta_{b,\,c} \in dx\} = \frac{c^b x^{b-1}}{\Gamma(b)} e^{-cx}\, \mathbb{1}_{(0,\infty)}(x)dx,$$

where Γ is the Euler gamma function. Note that $\theta_{1,\,c}$ is an exponentially distributed random variable of mean $1/c$. A known result (Proposition 3 in [17], Proposition 4.4.4 (b) in [9], Example 3.3 on p. 309 in [19]) states that, for each $a > 0$,

$$\int_0^\infty e^{B(s)-as}ds \;\overset{d}{=}\; 2/\theta_{2a,\,1}. \tag{1.16}$$

From this we infer

$$a\log\int_0^\infty e^{B(s)-as}ds \overset{d}{\longrightarrow} \theta_{1,\,2}, \quad a \to 0+, \tag{1.17}$$

The appearance of an exponential distribution may look mysterious, unless it is interpreted via the distributional equality

$$\theta_{1,\,2} \;\overset{d}{=}\; \sup_{s\ge 0}\,(B(s) - s),$$

which follows from Corollary 2 (ii) on p. 190 in [1]. Thus, (1.17) is a continuous-time counterpart of Theorem 1.2. It also serves an informal explanation of the fact that one factor of the normalization in Theorem 1.8 is $\log\log 1/a$ rather than $(\log\log 1/a)^{1/2}$ which typically arises in the cases when the limit distribution is normal.

More generally, let $L := (L(s))_{s\ge 0}$ be a centered spectrally negative Lévy process. Then, by the same Corollary 2 (ii) in [1], $\sup_{s\ge 0}(L(s) - s) \;\overset{d}{=}\; \theta_{1,\,\tau}$, where $\tau > 0$ is the largest solution to the equation $e^{-s}\mathbb{E}e^{sL(1)} = 1$. If $L = B$ a Brownian motion, then the latter equation is equivalent to $s^2/2 - s = 0$, whence $\tau = 2$. By the same reasoning, for each $u > 0$ and each $w \in \mathbb{R}$,

$$\sup_{s\ge 0}\,(wB(s) - us) \;\overset{d}{=}\; \theta_{1,\,2u/w^2}. \tag{1.18}$$

Assume additionally that L is an α-stable Lévy process, $\alpha \in (1,2]$. Then arguing along the lines of the proof of Theorem 1.2 one can show that

$$a^{\alpha-1}\log\int_0^\infty \mathrm{e}^{L(s)-as}\mathrm{d}s \overset{\mathrm{d}}{\longrightarrow} \sup_{s\geq 0}(L(s)-s) \overset{\mathrm{d}}{=} \theta_{1,\tau}, \quad a\to 0+.$$

2 Auxiliary results

2.1 Marginal limit distributions and continuity of the paths

According to (1.18), the marginal limit distributions in Theorem 1.2 are exponential with means $\sigma^2/(2u)$. In Proposition 2.1 we identify the marginal limit distributions in Theorems 1.4 and 1.7 and justify the claim made in Remark 1.5.

Proposition 2.1. *Let $x, u, T > 0$.*
(a) For $\lambda > 0$ and $\beta \in (1,2]$,

$$\mathbb{P}\Big\{\sup_{k:\, t_k^{(\lambda,\beta)}\leq T}\big(-ut_k^{(\lambda,\beta)}+j_k^{(\lambda,\beta)}\big)\leq x\Big\} = \exp(-u^{-1}(\beta-1)^{-1}\lambda(x^{1-\beta}-(x+uT)^{1-\beta}), \tag{2.1}$$

$$\mathbb{P}\Big\{\sup_k\big(-ut_k^{(\lambda,\beta)}+j_k^{(\lambda,\beta)}\big)\leq x\Big\} = \exp(-u^{-1}(\beta-1)^{-1}\lambda x^{1-\beta}) \tag{2.2}$$

and

$$\mathbb{P}\Big\{\sup_k\big(-ut_k^{(\lambda,1)}+j_k^{(\lambda,1)}\big)\leq x\Big\} = 0. \tag{2.3}$$

In particular, the random variables $\sup_{k:\, t_k^{(1,\beta)}\leq T}\big(-ut_k^{(\lambda,\beta)}+j_k^{(\lambda,\beta)}\big)$ *and* $\sup_k\big(-ut_k^{(\lambda,\beta)}+j_k^{(\lambda,\beta)}\big)$ *are a.s. finite and positive, and the latter has a rescaled Fréchet distribution with the shape parameter $\beta-1$.*
(b) For $\lambda,\sigma>0$,

$$\mathbb{P}\Big\{\sup_{k:\, t_k^{(\lambda,2)}\leq T}\big(\sigma B(t_k^{(\lambda,2)})-ut_k^{(\lambda,2)}+j_k^{(\lambda,2)}\big)\leq x\Big\}$$
$$= \mathbb{E}\exp\Big(-\lambda\int_0^T\frac{\mathrm{d}t}{(x-\sigma B(t)+ut)^2}\Big)\mathbb{1}_{\{\sup_{s\in[0,\,T]}(\sigma B(s)-us)<x\}} \tag{2.4}$$

and

$$\mathbb{P}\Big\{\sup_k\big(\sigma B(t_k^{(\lambda,2)})-ut_k^{(\lambda,2)}+j_k^{(\lambda,2)}\big)\leq x\Big\} = \mathbb{E}\exp\Big(-\lambda\int_0^\infty\frac{\mathrm{d}t}{((x-\sigma B(t)+ut)^+)^2}\Big)$$
$$= \mathbb{E}\exp\Big(-\lambda\int_0^\infty\frac{\mathrm{d}t}{(x-\sigma B(t)+ut)^2}\Big)\mathbb{1}_{\{\sup_{s\geq 0}(\sigma B(s)-us)<x\}}. \tag{2.5}$$

In particular, the random variables $\sup_{k:\, t_k^{(\lambda,2)}\leq T}\big(\sigma B(t_k^{(\lambda,2)})-ut_k^{(\lambda,2)}+j_k^{(\lambda,2)}\big)$ *and* $\sup_k\big(\sigma B(t_k^{(\lambda,2)})-ut_k^{(\lambda,2)}+j_k^{(\lambda,2)}\big)$ *are a.s. finite and positive.*

Proof. (a) We shall prove (2.1) and

$$\mathbb{P}\Big\{\sup_{k:\, t_k^{(\lambda,1)}\leq T}\big(-ut_k^{(\lambda,1)}+j_k^{(\lambda,1)}\big)\leq x\Big\} = \Big(\frac{x}{uT+x}\Big)^{\lambda/u}, \quad x>0. \tag{2.6}$$

Sending $T\to\infty$ yields (2.2) and (2.3).

The probabilities on the left-hand sides of (2.1) and (2.6) are equal to

$$\mathbb{P}\big\{N^{(\lambda,\beta)}\big((t,y): t\leq T, -ut+y>x\big)=0\big\}$$
$$= \exp\big(-\mathbb{E}N^{(\lambda,\beta)}\big((t,y): t\leq T, -ut+y>x\big)\big)$$

for $\beta\in(1,2]$ and $\beta=1$, respectively, because $N^{(\lambda,\,\beta)}\big((t,y):t\leq T,-ut+y>x\big)$ is a Poisson distributed random variable. Since

$$\mathbb{E}N^{(\lambda,\,\beta)}\big((t,y):t\leq T,-ut+y>x\big)\\ =\int_0^T\int_{[0,\,\infty)}\mathbb{1}_{\{y>ut+x\}}\,\mu_{\lambda,\,\beta}({\rm d}y){\rm d}t=\lambda\int_0^T(ut+x)^{-\beta}{\rm d}t\\ =\begin{cases}u^{-1}(\beta-1)^{-1}\lambda(x^{1-\beta}-(x+uT)^{1-\beta}), & \text{if } \beta\in(1,2],\\ u^{-1}\lambda(\log(uT+x)-\log x), & \text{if } \beta=1,\end{cases}$$

(2.1) and (2.6) follow. Letting in (2.1) and (2.2) $x\to0+$ justifies the claims about the a.s. positivity.
(b) Conditioning on B and arguing as in the proof of part (a) we arrive at

$$\mathbb{P}\Big\{\sup_{k:\,t_k^{(\lambda,\,2)}\leq T}\big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big)\leq x\Big\}=\mathbb{E}\exp\Big(-\lambda\int_0^T\frac{{\rm d}t}{((x-\sigma B(t)+ut)^+)^2}\Big).$$

In the case $u=0$ this formula can also be found in Proposition 1 of [21], along with an equivalent representation of the right-hand side. Formula (2.4) is its analogue in the case $u>0$.

By the strong law of large numbers for a Brownian motion, the integrand in (2.4) behaves as $(ut)^{-2}$ as $t\to\infty$. Hence, it is integrable on $[0,\infty)$. Sending in (2.4) $T\to\infty$ and invoking the Lebesgue dominated convergence theorem prove (2.5). In view of (1.18), the random variable $\sup_{s\geq0}(\sigma B(s)-us)$ is a.s. positive. Hence, letting in the second part of (2.5) $x\to0+$ and appealing once again to the Lebesgue dominated convergence theorem we conclude that the variable $\sup_k\big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big)$ is a.s. positive. The a.s. positivity of $\sup_{k:\,t_k^{(\lambda,\,2)}\leq T}\big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big)$ follows analogously. □

The next lemma justifies the usage of space C in our distributional limit theorems.

Lemma 2.2. *The processes $\big(Z(au)\big)_{u>0}$, $\Big(\log Y(au)\Big)_{u>0}$ and the limit processes in Theorems 1.2, 1.4 and 1.7 are a.s. convex, hence a.s. continuous.*

Proof. Recall that, according to the discussion at the beginning of Section 1.2, the first two processes (the converging processes in our distributional limit theorems) are a.s. finite for each $u>0$. The a.s. finiteness of the limit processes, for each $u>0$, follows from (1.18), (2.2) and (2.5), respectively.

Further, write, for any $\lambda_1,\lambda_2\geq0$ satisfying $\lambda_1+\lambda_2=1$ and any $u_1,u_2>0$,

$$\sup_{k\geq0}(S_k-a(\lambda_1u_1+\lambda_2u_2)k+\log\eta_{k+1})\\ =\sup_{k\geq0}(\lambda_1(S_k-au_1k+\log\eta_{k+1})+\lambda_2(S_k-au_2k+\log\eta_{k+1}))\\ \leq\sup_{k\geq0}(\lambda_1(S_k-au_1k+\log\eta_{k+1}))+\sup_{k\geq0}(\lambda_2(S_k-au_2k+\log\eta_{k+1}))\\ =\lambda_1\sup_{k\geq0}(S_k-au_1k+\log\eta_{k+1})+\lambda_2\sup_{k\geq0}(S_k-au_2k+\log\eta_{k+1})$$

having utilized subadditivity of the supremum for the inequality. This proves the claim for the first process. The proofs for the limit processes are analogous.

For each $a>0$, the function $u\mapsto Y(au)=\sum_{k\geq0}{\rm e}^{S_k-auk}\eta_{k+1}$ is the Laplace-Stieltjes transform of an infinite random measure μ_a defined by $\mu_a(\{ak\}):={\rm e}^{S_k}\eta_{k+1}$ for $k\in\mathbb{N}_0$. It is a standard fact, which is secured by Hölder's inequality, that any Laplace-Stieltjes transform f, say, is log-convex, that is, $\log f$ is convex. □

We close this section with another auxiliary result.

Lemma 2.3. *Let $\beta \in (1,2]$ and $\lambda > 0$. With probability one, for each $u > 0$,*

$$\lim_{T\to\infty} \sup_{s\geq T} (\sigma B(s) - us) = -\infty, \tag{2.7}$$

$$\lim_{T\to\infty} \sup_{k:\, t_k^{(1,\,\beta)}\geq T} \big(- ut_k^{(1,\,\beta)} + j_k^{(1,\,\beta)}\big) = -\infty$$

and

$$\lim_{T\to\infty} \sup_{k:\, t_k^{(\lambda,\,2)}\geq T} \big(\sigma B(t_k^{(\lambda,\,2)}) - ut_k^{(\lambda,\,2)} + j_k^{(\lambda,\,2)}\big) = -\infty. \tag{2.8}$$

Proof. Relation (2.7) follows from

$$\lim_{T\to\infty} \sup_{s\geq T} (\sigma B(s) - us) = \limsup_{T\to\infty} (\sigma B(T) - uT) = -\infty \quad \text{a.s.},$$

where the last equality is ensured by the strong law of large numbers for a Brownian motion.

Arguing as in the proof of Proposition 2.1 we conclude that

$$\mathbb{P}\Big\{ \sup_{k:\, t_k^{(1,\,\beta)}\geq T} \big(-ut_k^{(1,\,\beta)}+j_k^{(1,\,\beta)}\big)\leq x\Big\}=\begin{cases} 0, & \text{for } x \leq -uT, \\ \exp\big(- (u(\beta-1))^{-1}(uT+x)^{1-\beta}\big), & \text{for } x > -uT. \end{cases}$$

Letting $T\to\infty$ we infer $\lim_{T\to\infty} \sup_{k:\, t_k^{(1,\,\beta)}\geq T} \big(- ut_k^{(1,\,\beta)} + j_k^{(1,\,\beta)}\big) = -\infty$ in probability and, by monotonicity, a.s.

Using subadditivity of the supremum yields

$$\sup_{k:\, t_k^{(\lambda,\,2)}\geq T} \big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big) \leq \sup_{s\geq T} (\sigma B(s)-us/2)+ \sup_{k:\, t_k^{(\lambda,\,2)}\geq T} \big(-ut_k^{(\lambda,\,2)}/2+j_k^{(\lambda,\,2)}\big).$$

According to formula (2.2) with $\beta = 2$, $\sup_k \big(- ut_k^{(\lambda,\,2)}/2 + j_k^{(\lambda,\,2)}\big)$ is a.s. finite, whence

$$\lim_{T\to\infty} \sup_{k:\, t_k^{(\lambda,\,2)}\geq T} \big(- ut_k^{(\lambda,\,2)}/2 + j_k^{(\lambda,\,2)}\big) < \infty \quad \text{a.s.}$$

This in combination with (2.7) proves (2.8). □

2.2 Technical results

Denote by D the Skorokhod space of càdlàg functions defined on $[0,\infty)$. We assume that the space D is endowed with the J_1-topology

Lemma 2.4. *For $n\in\mathbb{N}_0$, let $f_n \in D$ and $\lim_{n\to\infty} f_n = f_0$ on D. Assume that*

$$M_0 := \sup_{t\geq 0} f_0(t) < \infty$$

and

$$\limsup_{t\to\infty} \limsup_{n\to\infty} f_n(t) < M_0. \tag{2.9}$$

Then

$$\lim_{n\to\infty} \sup_{t\geq 0} f_n(t) = M_0.$$

Proof. By (2.9), given sufficiently small $\varepsilon > 0$, there exist $T(\varepsilon) \geq 0$ and $n_0(\varepsilon) \in \mathbb{N}$ such that

$$f_n(t) \leq M_0 - \varepsilon, \quad t \geq T(\varepsilon), \quad n \geq n_0(\varepsilon).$$

By the definition of supremum, there exists $t_0(\varepsilon) \geq 0$ such that

$$M_0 - \varepsilon/2 \leq f_0(t_0(\varepsilon)) \leq M_0.$$

In view of the assumption $\lim_{n\to\infty} f_n = f_0$, there exists a sequence $(t_n(\varepsilon))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} t_n(\varepsilon) = t_0(\varepsilon) \quad \text{and} \quad \lim_{n\to\infty} f_n(t_n(\varepsilon)) = f_0(t_0(\varepsilon)).$$

Thus, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that, for $n \geq n_1(\varepsilon)$,

$$t_n(\varepsilon) \leq t_0(\varepsilon) + \varepsilon \quad \text{and} \quad f_n(t_n(\varepsilon)) \geq f_0(t_0(\varepsilon)) - \varepsilon/2 \geq M_0 - \varepsilon.$$

Put $a(\varepsilon) := \max(T(\varepsilon), t_0(\varepsilon) + \varepsilon)$. Combining the fragments together we conclude that, for $n \geq \max(n_0(\varepsilon), n_1(\varepsilon))$,

$$\sup_{t\geq 0} f_n(t) = \sup_{t\in[0,\,a(\varepsilon)]} f_n(t)$$

and thereupon

$$\lim_{n\to\infty} \sup_{t\geq 0} f_n(t) = \lim_{n\to\infty} \sup_{t\in[0,\,a(\varepsilon)]} f_n(t) = \sup_{t\in[0,\,a(\varepsilon)]} f_0(t) \in [M_0 - \varepsilon, M_0].$$

Sending $\varepsilon \to 0+$ completes the proof. □

Remark 2.5. If f_0 is continuous, then (2.9) boils down to

$$\limsup_{t\to\infty} f_0(t) < M_0. \tag{2.10}$$

Corollary 2.6. *Under the assumption of Lemma 2.4, for each $T > 0$,*

$$\lim_{n\to\infty} \sup_{t\geq T} f_n(t) = \sup_{t\geq T} f_0(t).$$

Proof. Apply Lemma 2.4 to the sequence $(f_n(T + \cdot))_{n\in\mathbb{N}_0}$. □

We shall need Theorem 1.3.17 in [12] which we state as Proposition 2.7 and a slight extension of Lemma 1.3.18 in [12] which we state as Lemma 2.8. Let $C[0,\infty)$ be the set of continuous functions defined on $[0,\infty)$ equipped with the locally uniform topology. Denote by M_p the set of point measures ν on $[0,\infty) \times (0,\infty]$ which satisfy

$$\nu([0,r] \times [\delta,\infty]) < \infty$$

for all $r > 0$ and all $\delta > 0$. The set M_p is endowed with the topology of vague convergence. Define the mapping $\mathcal{F}$ from $D \times M_p$ to D by

$$\mathcal{F}(f,\nu)(t) := \begin{cases} \sup_{k:\ \theta_k \leq t} (f(\theta_k) + y_k), & \text{if } \theta_k \leq t \text{ for some } k, \\ f(0), & \text{otherwise,} \end{cases}$$

where $\nu = \sum_k \varepsilon_{(\theta_k,\, y_k)}$.

Proposition 2.7. *For $j \in \mathbb{N}$, let $f_j \in D$ and $\nu_j \in M_p$. Assume that $f_0 \in C[0,\infty)$ and*

- $\nu_0(\{0\} \times (0,+\infty]) = 0$,
- $\nu_0((r_1,r_2) \times (0,\infty]) \geq 1$ *for all positive r_1 and r_2 such that $r_1 < r_2$,*

- $\nu_0=\sum_k \varepsilon_{\left(\theta_k^{(0)},\,y_k^{(0)}\right)}$ *does not have clustered jumps, that is,* $\theta_k^{(0)}\neq\theta_j^{(0)}$ *for* $k\neq j$.

If $\lim_{j\to\infty} f_j=f_0$ *in the* J_1*-topology on* D *and* $\lim_{j\to\infty}\nu_j=\nu_0$ *on* M_p*, then*

$$\lim_{j\to\infty}\mathcal{F}(f_j,\nu_j)=\mathcal{F}(f_0,\nu_0)$$

in the J_1*-topology on* D.

Lemma 2.8. *Let* $T\geq 0$, $\gamma,\rho>0$. *With probability one the random measure* $N_T^{(\gamma,\,\rho)}:=\sum_k \mathbb{1}_{\{t_k^{(\gamma,\,\rho)}\geq T\}}\varepsilon_{(t_k^{(\gamma,\,\rho)}-T,\,j_k^{(\gamma,\,\rho)})}$ *satisfies all the assumptions imposed on* ν_0 *in Proposition 2.7. Here,* $(t_k^{(\gamma,\,\rho)},\,j_k^{(\gamma,\,\rho)})$ *are the atoms of a Poisson random measure* $N^{(\gamma,\,\rho)}$ *defined in the paragraph preceding Theorem 1.4.*

Proof. The case $T=0$ is covered by Lemma 1.3.18 in [12]. If $T>0$, then $N_T^{(\gamma,\,\rho)}$ is just a deterministic shift of $N^{(\gamma,\,\rho)}$. Since the latter does not have atoms on any fixed deterministic vertical line with probability one, the claim follows. □

Hereafter, we write $\overset{\rm f.d.}{\longrightarrow}$ and $\overset{\mathbb{P}}{\longrightarrow}$ to denote weak convergence of finite-dimensional distributions and convergence in probability, respectively.

Proposition 2.9. *Under the assumptions of Theorem 1.2, for any* $T>0$,

$$\Big(a\sup_{0\leq k\leq \lfloor Ta^{-2}\rfloor}(S_k-auk+\log\eta_{k+1})\Big)_{u\in\mathbb{R}}\overset{\rm f.d.}{\longrightarrow}\Big(\sup_{s\in[0,\,T]}(\sigma B(s)-us)\Big)_{u\in\mathbb{R}},\quad a\to 0+,\tag{2.11}$$

where $(B(s))_{s\geq 0}$ *is a standard Brownian motion, and, for any* $T\geq 0$,

$$(a\sup_{k\geq\lfloor Ta^{-2}\rfloor}(S_k-auk+\log\eta_{k+1}))_{u>0}\overset{\rm f.d.}{\longrightarrow}(\sup_{s\geq T}(\sigma B(s)-us))_{u>0},\quad a\to 0+.\tag{2.12}$$

Proof. We shall write ζ for $\log\eta$ and ζ_k for $\log\eta_k$, $k\in\mathbb{N}$.

By Donsker's theorem,

$$(aS_{\lfloor Ta^{-2}\rfloor})_{T\geq 0}\implies(\sigma B(T))_{T\geq 0},\quad a\to 0+,$$

on D. Fix any $T>0$. Since, for all $\varepsilon>0$,

$$\begin{aligned}\mathbb{P}\{a\max_{1\leq k\leq\lfloor Ta^{-2}\rfloor+1}\zeta_k>\varepsilon\}&=1-\big(\mathbb{P}\{\zeta\leq\varepsilon a^{-1}\}\big)^{\lfloor Ta^{-2}\rfloor+1}\\&\leq(\lfloor Ta^{-2}\rfloor+1)\mathbb{P}\{\zeta>\varepsilon a^{-1}\}\to 0\end{aligned}$$

as $a\to 0+$ in view of (1.4), we infer

$$a\max_{1\leq k\leq\lfloor Ta^{-2}\rfloor+1}\zeta_k\overset{\mathbb{P}}{\longrightarrow}0,\quad a\to 0+,$$

which implies

$$(a\zeta_{\lfloor Ta^{-2}\rfloor+1})_{T\geq 0}\implies(\Xi(T))_{T\geq 0},\quad a\to 0+$$

on D where $\Xi(t)=0$ for $t\geq 0$. Hence,

$$(a(S_{\lfloor Ta^{-2}\rfloor}+\zeta_{\lfloor Ta^{-2}\rfloor+1}))_{T\geq 0}\implies(\sigma B(T))_{T\geq 0},\quad a\to 0+$$

by Slutsky's lemma and thereupon, for any $n\in\mathbb{N}$ and any $-\infty<u_1<\ldots<u_n<\infty$,

$$\begin{aligned}&(a(S_{\lfloor Ta^{-2}\rfloor}-au_1\lfloor Ta^{-2}\rfloor+\zeta_{\lfloor Ta^{-2}\rfloor+1}),\ldots,a(S_{\lfloor Ta^{-2}\rfloor}-au_n\lfloor Ta^{-2}\rfloor+\zeta_{\lfloor Ta^{-2}\rfloor+1}))_{T\geq 0}\\&\qquad\implies(\sigma B(T)-u_1T,\ldots,\sigma B(T)-u_nT)_{T\geq 0},\quad a\to 0+\end{aligned}\tag{2.13}$$

in the J_1-topology on D^n. The supremum functional is continuous in the J_1-topology. This in combination with the continuous mapping theorem proves (2.11).

By the Skorokhod representation theorem, there exist versions of the processes in (2.13), for which (2.13) holds a.s., with a replaced by $(a_k)_{k\in\mathbb{N}}$ an arbitrary convergent to 0 sequence of positive numbers. Each coordinate of the version of the limit is a.s. continuous and, for each $u>0$, $\lim_{s\to\infty}(\sigma B(s)-us)=-\infty$ a.s. by the strong law of large numbers for a Brownian motion. The latter ensures that the a.s. counterpart of (2.10) holds for each coordinate of the version of the limit. Applying Corollary 2.6 and Remark 2.5 separately to each coordinate of the versions and passing from the versions to the original processes we arrive at (2.12). □

Proposition 2.10. *Under the assumptions of Theorem 1.4, for any $T>0$,*

$$\Big(\frac{1}{ac(a)}\sup_{k\leq\lfloor Tc(a)\rfloor}(S_k-auk+\log\eta_{k+1})\Big)_{u>0}\overset{\rm f.d.}{\longrightarrow}\Big(\sup_{k:\,t_k^{(1,\,\beta)}\leq T}(-ut_k^{(1,\,\beta)}+j_k^{(1,\,\beta)})\Big)_{u>0},\quad a\to 0+ \tag{2.14}$$

and, for any $T\geq 0$,

$$\Big(\frac{1}{ac(a)}\sup_{k\geq\lfloor Tc(a)\rfloor}(S_k-auk+\log\eta_{k+1})\Big)_{u>0}\overset{\rm f.d.}{\longrightarrow}\Big(\sup_{k:\,t_k^{(1,\,\beta)}\geq T}(-ut_k^{(1,\,\beta)}+j_k^{(1,\,\beta)})\Big)_{u>0},\quad a\to 0+. \tag{2.15}$$

Under the assumptions of Theorem 1.7, as $a\to 0+$, for any $T>0$,

$$\big(a\sup_{k\leq\lfloor Ta^{-2}\rfloor}(S_k-auk+\log\eta_{k+1})\big)_{u>0}\overset{\rm f.d.}{\longrightarrow}\Big(\sup_{k,\,t_k^{(\lambda,\,2)}\geq T}(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)})\Big)_{u>0} \tag{2.16}$$

and, for any $T\geq 0$,

$$\big(a\sup_{k\geq\lfloor Ta^{-2}\rfloor}(S_k-auk+\log\eta_{k+1})\big)_{u>0}\overset{\rm f.d.}{\longrightarrow}\Big(\sup_{k:\,t_k^{(\lambda,\,2)}\geq T}(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)})\Big)_{u>0}. \tag{2.17}$$

Proof. In the setting of Theorem 1.4 put $c(a):=a^{-2}$ for $a>0$.

Fix $u>0$ and $T\geq 0$. Under the assumptions of Theorem 1.7, there is a joint convergence

$$\begin{aligned}&\Big(\Big(\frac{S_{\lfloor(t+T)c(a)\rfloor}}{\sqrt{c(a)}}-u(t+T)\Big)_{t\geq 0},\sum_{k\geq 0}\mathbb{1}_{\{k\geq\lfloor Tc(a)\rfloor\}}\,\varepsilon_{(k/c(a)-T,\,\zeta_{k+1}/\sqrt{c(a)})}\Big)\\&\quad\Longrightarrow\Big((\sigma B(t+T)-u(t+T))_{t\geq 0},\sum_k\mathbb{1}_{\{t_k^{(\lambda,\,2)}\geq T\}}\,\varepsilon_{(t_k^{(\lambda,\,2)}-T,\,j_k^{(\lambda,\,2)})}\Big),\quad a\to 0+\end{aligned} \tag{2.18}$$

in the space $D\times M_p$ endowed with the product topology; see the bottom of p. 27 in [12]. Moreover, the components on the right-hand side are independent. Fix now any $n\in\mathbb{N}$ and any $0<u_1<u_2<\ldots<u_n<\infty$. Then (2.18) immediately extends to

$$\begin{aligned}&\Big(\Big(\Big(\frac{S_{\lfloor(t+T)c(a)\rfloor}}{\sqrt{c(a)}}-u_i(t+T)\Big)_{t\geq 0}\Big)_{1\leq i\leq n},\sum_{k\geq 0}\mathbb{1}_{\{k\geq\lfloor Tc(a)\rfloor\}}\,\varepsilon_{(k/c(a)-T,\,\zeta_{k+1}/\sqrt{c(a)})}\Big)\\&\quad\Longrightarrow\Big(((\sigma B(t+T)-u_i(t+T))_{t\geq 0})_{1\leq i\leq n},\sum_k\mathbb{1}_{\{t_k^{(\lambda,\,2)}\geq T\}}\,\varepsilon_{(t_k^{(\lambda,\,2)}-T,\,j_k^{(\lambda,\,2)})}\Big),\quad a\to 0+\end{aligned} \tag{2.19}$$

in the space $D^n \times M_p$ endowed with the product topology, because the components indexed by i only differ by a deterministic term. Similarly, under the assumptions of Theorem 1.4,

$$\Big(\Big(\Big(\frac{S_{\lfloor (t+T)c(a)\rfloor}}{ac(a)} - u_i(t+T)\Big)_{t\geq 0}\Big)_{1\leq i\leq n}, \sum_{k\geq 0} \mathbb{1}_{\{k\geq \lfloor Tc(a)\rfloor\}}\, \varepsilon_{(k/c(a)-T,\, \zeta_{k+1}/ac(a))}\Big)$$
$$\Longrightarrow \Big(((-u_i(t+T))_{t\geq 0})_{1\leq i\leq n}, \sum_k \mathbb{1}_{\{t_k^{(1,\,\beta)}\geq T\}}\, \varepsilon_{(t_k^{(1,\,\beta)}-T,\, j_k^{(1,\,\beta)})}\Big), \quad a\to 0+ \qquad (2.20)$$

in the space $D^n \times M_p$ endowed with the product topology, where the convergence of the normalized random walk to the zero process Ξ follows from (1.9) which ensures that $ac(a)/\sqrt{c(a)} = a\sqrt{c(a)} \to \infty$ as $a \to 0+$.

Fix any $T_1 > T$ and let $(a_j)_{j\in\mathbb{N}}$ be any sequence of positive numbers satisfying $\lim_{j\to\infty} a_j = 0$. By the Skorokhod representation theorem there are versions of the processes, for which (2.19) and (2.20) hold a.s. Retaining the original notation for these versions we intend to apply Proposition 2.7 n times with

$$f_j(t) := \frac{S_{\lfloor (t+T)c(a_j)\rfloor}}{\sqrt{c(a_j)}} - u_i(t+T), \quad \nu_j := \sum_{k\geq 0} \mathbb{1}_{\{k\geq \lfloor Tc(a_j)\rfloor\}}\, \varepsilon_{(k/c(a_j)-T,\, \zeta_{k+1}/\sqrt{c(a_j)})}$$
$$f_0(t) := \sigma B(t+T) - u_i(t+T),$$
$$\nu_0 := \sum_k \mathbb{1}_{\{t_k^{(\lambda,\,2)}\geq T\}}\, \varepsilon_{(t_k^{(\lambda,\,2)}-T,\, j_k^{(\lambda,\,2)})}, \;\; j\in\mathbb{N},\; t\geq 0,\; i=1,\dots,n$$

and n times with

$$f_j(t) := \frac{S_{\lfloor (t+T)c(a_j)\rfloor}}{a_j c(a_j)} - u_i(t+T), \quad \nu_j := \sum_{k\geq 0} \mathbb{1}_{\{k\geq \lfloor Tc(a_j)\rfloor\}}\, \varepsilon_{(k/c(a_j)-T,\, \zeta_{k+1}/a_j c(a_j))}$$
$$f_0(t) := -u_i(t+T), \quad \nu_0 := \sum_k \mathbb{1}_{\{t_k^{(1,\,\beta)}\geq T\}}\, \varepsilon_{(t_k^{(1,\,\beta)}-T,\, j_k^{(1,\,\beta)})}, \;\; j\in\mathbb{N},\; t\geq 0,\; i=1,\dots,n.$$

The so defined converging and limit processes satisfy the assumptions of Proposition 2.7 with probability one. In particular, a.s. continuity of the limit functions f_0 is obvious, whereas Lemma 2.8 justifies the claim for the random measures ν_0. A specialization of Proposition 2.7 to the one-dimensional (rather than functional) convergence in conjunction with (2.19) and (2.20) yields

$$\Big(a \sup_{\lfloor Tc(a)\rfloor\leq k\leq \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big)\Big)_{u>0}$$
$$\overset{\text{f.d.}}{\longrightarrow} \Big(\sup_{k:\, T\leq t_k^{(\lambda,\,2)}\leq T_1} (\sigma B(t_k^{(\lambda,\,2)}) - ut_k^{(\lambda,\,2)} + j_k^{(\lambda,\,2)})\Big)_{u>0}, \quad a\to 0+ \qquad (2.21)$$

and

$$\Big(\frac{1}{ac(a)} \sup_{\lfloor Tc(a)\rfloor\leq k\leq \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big)\Big)_{u>0}$$
$$\overset{\text{f.d.}}{\longrightarrow} \Big(\sup_{k:\, T\leq t_k^{(1,\,\beta)}\leq T_1} (-ut_k^{(1,\,\beta)} + j_k^{(1,\,\beta)})\Big)_{u>0}, \quad a\to 0+, \qquad (2.22)$$

respectively. Putting in (2.21) and (2.22) $T = 0$ and then replacing T_1 with T we obtain (2.16) and (2.14).

The right-hand sides of (2.21) and (2.22) converge a.s. as $T_1 \to \infty$ to the right-hand sides of (2.17) and (2.15), respectively. According to Theorem 4.2 on p. 25 in [2], both (2.15) and (2.17) follow if we can show that, for all $\varepsilon > 0$,

$$\lim_{T_1\to\infty} \limsup_{a\to 0+} \mathbb{P}\Big\{ \sum_{i=1}^{n} \Big(\sup_{\lfloor Tc(a)\rfloor \le k} \big(S_k - auk + \zeta_{k+1}\big) - \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big)\Big)^2 > \varepsilon \Big\} = 0.$$

Plainly, it is sufficient to check that, for each fixed $u > 0$,

$$\lim_{T_1\to\infty} \limsup_{a\to 0+} \mathbb{P}\big\{ \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) \neq \sup_{\lfloor Tc(a)\rfloor \le k} \big(S_k - auk + \zeta_{k+1}\big)\big\} = 0.$$

Note that, for $T_1 \ge 2T$ and $z \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}\big\{ \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) \neq \sup_{\lfloor Tc(a)\rfloor \le k} \big(S_k - auk + \zeta_{k+1}\big)\big\} \\
&= \mathbb{P}\Big\{ \frac{1}{ac(a)} \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) < \frac{1}{ac(a)} \sup_{k > \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big)\Big\} \\
&\le \mathbb{P}\Big\{ \frac{1}{ac(a)} \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor 2Tc(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) < \frac{1}{ac(a)} \sup_{k > \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big)\Big\} \\
&\le \mathbb{P}\Big\{ \frac{1}{ac(a)} \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor 2Tc(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) \le z\Big\} \\
&+ \mathbb{P}\Big\{ \frac{1}{ac(a)} \sup_{k > \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) > z\Big\}.
\end{aligned}$$

According to (2.21) and (2.22), the random variables $(ac(a))^{-1} \sup_{\lfloor Tc(a)\rfloor \le k \le \lfloor 2Tc(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big)$ converge in distribution as $a \to 0+$ to an a.s. finite random variable ρ_T, say (the a.s. finiteness follows from Proposition 2.1). As a consequence, the first probability on the right-hand side tends to 0 as $a \to 0+$ and then $z \to -\infty$ along the sequence of continuity points of the distribution function of ρ_T. Thus, it is enough to prove that, for each fixed $z \in \mathbb{R}$,

$$\lim_{T_1\to\infty} \limsup_{a\to 0+} \mathbb{P}\Big\{ \frac{1}{ac(a)} \sup_{k > \lfloor T_1 c(a)\rfloor} \big(S_k - auk + \zeta_{k+1}\big) > z\Big\} = 0.$$

The latter probability does not exceed

$$\mathbb{P}\big\{ \sup_{k > \lfloor T_1 c(a)\rfloor} (S_k - auk/2) > ac(a)\big\} + \mathbb{P}\big\{ \sup_{k > \lfloor T_1 c(a)\rfloor} (\zeta_{k+1} - auk/2) > (z-1)ac(a)\big\}. \quad (2.23)$$

In the setting of Theorem 1.7,

$$\lim_{T_1\to\infty} \limsup_{a\to 0+} \mathbb{P}\big\{ \sup_{k > \lfloor T_1 c(a)\rfloor} (S_k - auk/2) > ac(a)\big\} = 0.$$

by (2.12) and (2.7), whereas in the setting of Theorem 1.4 this follows from (2.12) and (1.9). Indeed, for small enough $a > 0$, (1.9) entails $c(a) \ge a^{-2}$, whence

$$\mathbb{P}\big\{ \sup_{k > \lfloor T_1 c(a)\rfloor} (S_k - auk/2) > ac(a)\big\} \le \mathbb{P}\big\{ a \sup_{k > \lfloor T_1 a^{-2}\rfloor} (S_k - auk/2) > a^2 c(a)\big\}.$$

In view of (2.12), the right-hand side converges to 0 as $a \to 0+$.

As far as the second summand in (2.23) is concerned we argue as follows. For large $T_1 > \max(2(1-z),0)$ and small $a>0$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{k>\lfloor T_1 c(a)\rfloor}(\zeta_{k+1}-auk/2)>(z-1)ac(a)\Big\}\\
&\qquad\le \sum_{k>\lfloor T_1 c(a)\rfloor}\mathbb{P}\left\{\zeta_{k+1}>auk/2+(z-1)ac(a)-a/2\right\}\\
&\qquad\le \int_{\lfloor T_1 c(a)\rfloor}^{\infty}\mathbb{P}\{\zeta>aux/2+(z-1)ac(a)-a/2\}\mathrm{d}x\\
&\qquad= \frac{2}{ua}\int_{(a/2)(\lfloor T_1 c(a)\rfloor u+2(z-1)c(a)-1)}^{\infty}\mathbb{P}\{\zeta>x\}\mathrm{d}x\\
&\qquad\sim \frac{1}{ua}\frac{1}{\beta-1}(T_1u+2(z-1))ac(a)\mathbb{P}\{\zeta>(T_1u+2(z-1))ac(a)/2\}\\
&\qquad\sim \frac{1}{au}\frac{1}{\beta-1}(T_1u+2(z-1))ac(a)\mathbb{P}\{\zeta>(T_1u+2(z-1))b(c(a))/2\}\\
&\qquad\sim \frac{1}{au}\frac{2^\beta}{\beta-1}(T_1u+2(z-1))^{1-\beta}ac(a)\frac{1}{c(a)},\quad a\to 0+.
\end{aligned}$$

We have used Proposition 1.5.10 in [3] for the first asymptotic equivalence. Since $\beta>1$, the right-hand side converges to zero as $T_1\to\infty$. This completes the proof of Proposition 2.10. ☐

3 Proofs of Theorems 1.2, 1.4 and 1.7

We shall prove all the results simultaneously. To this end, for $a,T,u>0$, put

$$m(a):=a,\ c(a):=a^{-2},\ X_1(u,T):=\sup_{s\in[0,\,T]}(\sigma B(s)-us),\ X_1(u,\infty):=\sup_{s\ge 0}(\sigma B(s)-us)$$

and

$$X_1^*(u,T):=\sup_{s\ge T}(\sigma B(s)-us)$$

under the assumptions of Theorem 1.2;

$$m(a):=(ac(a))^{-1},\ c(a)\text{ is as defined in Theorem 1.4,}$$

$$X_2(u,T):=\sup_{k:\ t_k^{(1,\,\beta)}\le T}\big(-ut_k^{(1,\,\beta)}+j_k^{(1,\,\beta)}\big),\quad X_2(u,\infty):=\sup_k\big(-ut_k^{(1,\,\beta)}+j_k^{(1,\,\beta)}\big)$$
$$\text{and}\quad X_2^*(u,T):=\sup_{k:\ t_k^{(1,\,\beta)}\ge T}\big(-ut_k^{(1,\,\beta)}+j_k^{(1,\,\beta)}\big)$$

under the assumptions of Theorem 1.4; and

$$m(a):=a,\ c(a):=a^{-2},\ X_3(u,T):=\sup_{k:\ t_k^{(\lambda,\,2)}\le T}\big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big),$$

$$X_3(u,\infty):=\sup_k\big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big)$$
$$\text{and}\quad X_3^*(u,T):=\sup_{k:\ t_k^{(\lambda,\,2)}\ge T}\big(\sigma B(t_k^{(\lambda,\,2)})-ut_k^{(\lambda,\,2)}+j_k^{(\lambda,\,2)}\big)$$

under the assumptions of Theorem 1.7.

Since the converging processes

$$\big(m(a)Z(au)\big)_{u>0} \quad \text{and} \quad (m(a)\log Y(au))_{u>0}$$

are a.s. nonincreasing and, by Lemma 2.2, the limit processes $(X_l(u,\infty))_{u>0}$, $l=1,2,3$ are a.s. continuous, weak convergence of probability measures in C is equivalent to weak convergence of the corresponding finite-dimensional distributions. This follows from Skorokhod's representation theorem in combination with Dini's theorem.

Thus, limit relations (1.5), (1.7) and (1.11), dealing with the convergence of suprema, are ensured by (2.12), (2.15) and (2.17) all with $T=0$, respectively, and the last remark. As far as the perpetuities are concerned, we have to show that

$$\Big(m(a)\log Y(au)\Big)_{u>0} \overset{\text{f.d.}}{\longrightarrow} (X_l(u,\infty))_{u>0}, \quad a\to 0+, \quad l=1,2,3. \tag{3.1}$$

As a preparation, we prove that, for any $T>0$,

$$\Big(m(a)\log \sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k-auk}\eta_{k+1}\Big)_{u>0} \overset{\text{f.d.}}{\longrightarrow} (X_l(u,T))_{u>0}, \quad a\to 0+, \quad l=1,2,3. \tag{3.2}$$

Fix any $n\in\mathbb{N}$, any $\gamma_1,\dots,\gamma_n\in\mathbb{R}$ and any $0<u_1,u_2,\dots,u_n<\infty$. Assume, without loss of generality, that $\gamma_1,\dots,\gamma_{n_0}\geq 0$ and $\gamma_{n_0+1},\dots,\gamma_n<0$ for some $n_0\in\mathbb{N}_0$, $n_0\leq n$. In particular, the situation is allowed in which all γ_j are of the same sign (in which case $n_0=0$ or $n_0=n$). In view of the Cramér-Wold device, relation (3.2) is equivalent to the following: for any $T>0$, as $a\to 0+$,

$$m(a)\sum_{j=1}^n \gamma_j \log \sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k-au_jk}\eta_{k+1} \overset{\text{d}}{\longrightarrow} \sum_{j=1}^n \gamma_j X_l(u_j,T), \quad l=1,2,3. \tag{3.3}$$

To prove (3.3), write, for any $T>0$,

$$\begin{aligned}\sum_{j=1}^n \gamma_j \log \sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k-au_jk}\eta_{k+1} &= \sum_{j=1}^{n_0} \gamma_j \log \sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k-au_jk}\eta_{k+1}\\ +\sum_{j=n_0+1}^n \gamma_j \log \sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k-au_jk}\eta_{k+1} &\leq \sum_{j=1}^{n_0}\gamma_j\big(\log(\lfloor Tc(a)\rfloor+1)+\max_{0\leq k\leq\lfloor Tc(a)\rfloor}(S_k-au_jk+\zeta_{k+1})\big)\\ &\quad+\sum_{j=n_0+1}^n \gamma_j \max_{0\leq k\leq\lfloor Tc(a)\rfloor}(S_k-au_jk+\zeta_{k+1}),\end{aligned}$$

where $\zeta_j=\log\eta_j$ for $j\in\mathbb{N}$, and analogously

$$\begin{aligned}\sum_{j=1}^n \gamma_j \log \sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k-au_jk}\eta_{k+1} &\geq \sum_{j=1}^{n_0}\gamma_j \max_{0\leq k\leq\lfloor Tc(a)\rfloor}(S_k-au_jk+\zeta_{k+1})\\ &\quad+\sum_{j=n_0+1}^n \gamma_j\big(\log(\lfloor Tc(a)\rfloor+1)+\max_{0\leq k\leq\lfloor Tc(a)\rfloor}(S_k-au_jk+\zeta_{k+1})\big).\end{aligned}$$

With these at hand, (3.3) follows from

$$(m(a)\sup_{0\leq k\leq\lfloor Tc(a)\rfloor}(S_k-auk+\zeta_{k+1}))_{u>0} \overset{\text{f.d.}}{\longrightarrow} (X_l(u,T))_{u>0}, \quad a\to 0+, \quad l=1,2,3, \tag{3.4}$$

(see (2.11), (2.14) and (2.16)) and the fact that $\lim_{a\to 0+} m(a)\log c(a)=0$. In the setting of Theorem 1.2 the latter is justified by the regular variation of c at $0+$ of index $-\beta/(\beta-1)$;

see Remark 1.5. This particularly implies that the function $a \mapsto m(a) = (ac(a))^{-1}$ is regularly varying at $0+$ of positive index $(\beta-1)^{-1}$.

Plainly, $\lim_{T\to\infty}\sum_{j=1}^n \gamma_j X_l(u_j,T) = \sum_{j=1}^n \gamma_j X_l(u_j,\infty)$ a.s. Hence, according to Theorem 4.2 on p. 25 in [2] the proof of (3.1) is complete if we can show that, for all $\varepsilon>0$,

$$\lim_{T\to\infty}\limsup_{a\to 0+}\mathbb{P}\Big\{m(a)\Big|\sum_{j=1}^n \gamma_j\Big(\log Y(au_j) - \log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - au_j k}\eta_{k+1}\Big)\Big| > \varepsilon\Big\} = 0.$$

By the triangle inequality, it is enough to prove that, with $u>0$ fixed,

$$\lim_{T\to\infty}\limsup_{a\to 0+}\mathbb{P}\Big\{m(a)\Big(\log Y(au) - \log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1}\Big) > \varepsilon\Big\} = 0.$$

The latter probability is upper bounded as follows:

$$\begin{aligned}
&\le \mathbb{P}\Big\{m(a)\Big(\log^+ Y(au) - \log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1}\Big) > \varepsilon,\ \log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1} \le 0\Big\}\\
&\quad + \mathbb{P}\Big\{m(a)\Big(\log^+ Y(au) - \log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1}\Big) > \varepsilon,\ \log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1} > 0\Big\}\\
&\le \mathbb{P}\Big\{m(a)\log\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1} \le 0\Big\}\\
&\quad + \mathbb{P}\Big\{m(a)\Big(\log^+ Y(au) - \log^+\sum_{k=0}^{\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1}\Big) > \varepsilon\Big\}.
\end{aligned}$$

The first probability on the right-hand side converges to 0 as $a\to 0+$. This is secured by (3.3) with $n=1$ and $\gamma_1=1$, and the fact that the right-hand sides in (3.3) are a.s. positive. The latter follows from (1.18) and Proposition 2.1. To proceed, we need two inequalities:

$$\log^+(x+y) \le \log^+(x) + \log^+(y) + 2\log 2,\quad x,y\ge 0 \tag{3.5}$$

and

$$\log^+(xy) \le \log^+ x + \log^+ y,\quad x,y\ge 0. \tag{3.6}$$

Inequality (3.5) follows from

$$\log^+(x) \le \log(1+x) \le \log^+(x) + \log 2,\quad x\ge 0$$

and the subadditivity of $x\mapsto \log(1+x)$, namely,

$$\begin{aligned}
\log^+(x+y) &\le \log(1+x+y) \le \log(1+x) + \log(1+y)\\
&\le \log^+(x) + \log^+(y) + 2\log 2,\quad x,y\ge 0.
\end{aligned}$$

Inequality (3.6) is a consequence of the subadditivity of $x\to x^+$.

In view of (3.5), it remains to prove that

$$\lim_{T\to\infty}\limsup_{a\to 0+}\mathbb{P}\Big\{m(a)\log^+\sum_{k>\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1} > \varepsilon\Big\} = 0. \tag{3.7}$$

To this end, write, with the help of (3.6),

$$\log^+\sum_{k>\lfloor Tc(a)\rfloor} \mathrm{e}^{S_k - auk}\eta_{k+1} \le (\sup_{k>\lfloor Tc(a)\rfloor}(S_k - auk/2 + \zeta_{k+1}))^+ + \log^+\sum_{k>\lfloor Tc(a)\rfloor}\mathrm{e}^{-auk/2}.$$

While $\lim_{a\to 0+}\log^+\sum_{k>\lfloor Tc(a)\rfloor}\mathrm{e}^{-auk/2}=0$, formulae (2.12), (2.15) and (2.17) entail

$$m(a)\big(\sup_{k>\lfloor Tc(a)\rfloor}(S_k-auk/2+\zeta_{k+1})\big)^+ \ \overset{\mathrm{d}}{\longrightarrow}\ (X_l^*(u/2,T))^+,\quad a\to 0+,\ l=1,2,3.$$

Finally, by Lemma 2.3, $\lim_{T\to\infty}(X_l^*(u/2,T))^+=0$ a.s., and (3.7) follows.

The proof of Theorems 1.2, 1.4 and 1.7 is complete.

4 Proof of Theorem 1.8 and Corollary 1.9

The following deterministic result is a consequence of Corollary 4.12.5 in [3].

Proposition 4.1. *Let $A\in(0,\infty)$ and μ be a locally finite measure on $[0,\infty)$. Assume that the function φ is regularly varying at ∞ of index $\alpha>1$ and put $\psi(t):=\varphi(t)/t$ for large t. Then*

$$\limsup_{x\to\infty}\frac{\log\mu([0,\ \varphi(x)])}{x}=A \tag{4.1}$$

if, and only if,

$$\limsup_{\lambda\to\infty}\frac{\log\int_{[0,\,\infty)}\mathrm{e}^{-x/\psi(\lambda)}\mu(\mathrm{d}x)}{\lambda}=(\alpha-1)\Big(\frac{A}{\alpha}\Big)^{\alpha/(\alpha-1)}. \tag{4.2}$$

Proof. Assume that (4.1) holds. Then, according to the first implication in Corollary 4.12.5 in [3],

$$\limsup_{\lambda\to\infty}\frac{\log\int_{[0,\,\infty)}\mathrm{e}^{-x/\psi(\lambda)}\mu(\mathrm{d}x)}{\lambda}\le(\alpha-1)\Big(\frac{A}{\alpha}\Big)^{\alpha/(\alpha-1)}. \tag{4.3}$$

Suppose that the above inequality is strict, that is, for some $\varepsilon>0$ and $A(\varepsilon)<A$,

$$\limsup_{\lambda\to\infty}\frac{\log\int_{[0,\,\infty)}\mathrm{e}^{-x/\psi(\lambda)}\mu(\mathrm{d}x)}{\lambda}=(\alpha-1)\Big(\frac{A}{\alpha}\Big)^{\alpha/(\alpha-1)}-\varepsilon=(\alpha-1)\Big(\frac{A(\varepsilon)}{\alpha}\Big)^{\alpha/(\alpha-1)}.$$

Then the second implication in the aforementioned Corollary 4.12.5 yields

$$\limsup_{x\to\infty}\frac{\log\mu([0,\ \varphi(x)])}{x}\le A_1(\varepsilon)<A,$$

which is a contradiction. Thus, the inequality in (4.3) can be replaced by the equality. The inverse implication follows analogously. □

The following result is needed for the proof of Theorem 1.8(a) and also of independent interest.

Theorem 4.2. *Suppose that* (1.3) *and* (1.13) *hold. Then*

$$\limsup_{t\to\infty}\frac{\max_{0\le k\le\lfloor t\rfloor}(S_k+\log\eta_{k+1})}{(t\log\log t)^{1/2}}=2^{1/2}\sigma\quad\text{a.s.} \tag{4.4}$$

and

$$\limsup_{t\to\infty}\frac{\log\sum_{k=0}^{\lfloor t\rfloor}\mathrm{e}^{S_k}\eta_{k+1}}{(t\log\log t)^{1/2}}=2^{1/2}\sigma\quad\text{a.s.} \tag{4.5}$$

Proof. In view of

$$\max_{0\le k\le\lfloor t\rfloor}(S_k+\log\eta_{k+1})\le\log\sum_{k=0}^{\lfloor t\rfloor}\mathrm{e}^{S_k}\eta_{k+1}\le\log(\lfloor t\rfloor+1)+\max_{0\le k\le\lfloor t\rfloor}(S_k+\log\eta_{k+1})\quad\text{a.s.,}$$

it suffices to prove (4.4). Furthermore, when doing so we can and do replace $\lfloor t \rfloor$ with integer n.

Note that

$$\limsup_{n\to\infty} \frac{\max_{0\le k\le n} S_k}{(n\log\log n)^{1/2}} = 2^{1/2}\sigma \quad \text{a.s.}; \tag{4.6}$$

see, for instance, p. 439 in [4]. Recall the notation $\zeta = \log\eta$ and $\zeta_k = \log\eta_k$ for $k\in\mathbb{N}$. The assumption (1.13) in combination with the Borel-Cantelli lemma entails

$$\lim_{n\to\infty} (n\log\log n)^{-1/2}\zeta_n^+ = 0 \quad \text{a.s.}$$

and thereupon $\lim_{n\to\infty}(n\log\log n)^{-1/2}\max_{1\le k\le n+1}\zeta_k^+ = 0$ a.s. Using the latter, (4.6) and

$$\max_{0\le k\le n}(S_k+\zeta_{k+1}) \le \max_{0\le k\le n} S_k + \max_{1\le k\le n+1}\zeta_k^+ \quad \text{a.s.}$$

we infer

$$\limsup_{n\to\infty} \frac{\max_{0\le k\le n}(S_k+\zeta_{k+1})}{(n\log\log n)^{1/2}} \le 2^{1/2}\sigma \quad \text{a.s.}$$

Fix any $\delta\in(0,1)$, put $x_n(\delta) := (1-\delta)2^{1/2}\sigma(n\log\log n)^{1/2}$ for $n\ge 3$ and define the event

$$A_n = A_n(\delta) := \{\max_{0\le k\le n}(S_k+\zeta_{k+1}) > x_n(\delta)\}, \quad n\ge 3.$$

Our purpose is to show that

$$\mathbb{P}\{A_n \ \text{i.o.}\} = 1.$$

Here, as usual, 'i.o.' is a shorthand for 'infinitely often' and $\{A_n \ \text{i.o.}\} = \cap_{n\ge 3}\cup_{k\ge n} A_k$. Pick any $\gamma\in\mathbb{R}$ satisfying $\mathbb{P}\{\zeta>\gamma\}>0$. For $n\ge 3$, put $\tau_n := \inf\{k\le n : S_k > x_n(\delta)-\gamma\}$ on the event $\{\max_{0\le k\le n} S_k > x_n(\delta)-\gamma\}$ and $\tau_n := +\infty$ on the complementary event. Now define the events

$$B_n = B_n(\delta) := \{\max_{0\le k\le n} S_k > x_n(\delta)-\gamma\} \quad \text{and} \quad C_n = C_n(\delta) := \{\zeta_{\tau_n+1}>\gamma\}, \quad n\ge 3.$$

Observe that, for each $n\ge 3$, $B_n\cap C_n\subseteq A_n$, whence

$$\Big\{\sum_{n\ge 3}\mathbb{1}_{B_n\cap C_n}=\infty\Big\} = \{B_n\cap C_n \ \text{i.o.}\} \subseteq \{A_n \ \text{i.o.}\}.$$

For $n\ge 3$, denote by $\mathcal{F}_n$ the σ-algebra generated by $((\xi_k,\zeta_k))_{1\le k\le n}$. In view of

$$\zeta_{\tau_n+1}\,\mathbb{1}_{\{\tau_n\le n\}} = \zeta_1\,\mathbb{1}_{\{x_n(\delta)<\gamma\}} + \sum_{k=1}^n \zeta_{k+1}\,\mathbb{1}_{\{\max_{0\le j\le k-1} S_j\le x_n(\delta)-\gamma,\, S_k>x_n(\delta)-\gamma\}},$$

we conclude that $B_n\cap C_n\in\mathcal{F}_{n+1}$. Hence, by the conditional Borel-Cantelli lemma; see, for instance, Theorem 5.3.2 on p. 240 in [10],

$$\mathbb{P}\Big\{\sum_{n\ge 3}\mathbb{1}_{B_n\cap C_n}=\infty\Big\} = \mathbb{P}\Big\{\sum_{n\ge 3}\mathbb{P}\{B_n\cap C_n|\mathcal{F}_n\}=\infty\Big\}.$$

Since

$$\sum_{n\ge 3}\mathbb{P}\{B_n\cap C_n|\mathcal{F}_n\} = \mathbb{P}\{\zeta>\gamma\}\sum_{n\ge 3}\mathbb{1}_{B_n}$$

and (4.6) secures $\mathbb{P}\{\sum_{n\ge 3}\mathbb{1}_{B_n}=\infty\}=1$, we infer $\mathbb{P}\{B_n\cap C_n \ \text{i.o.}\} = 1 = \mathbb{P}\{A_n \ \text{i.o.}\}$. The proof of Theorem 4.2 is complete. □

Remark 4.3. If (1.3) holds and (1.13) does not hold, then both (4.4) and (4.5) fail to hold. As a consequence, so does (1.14) as follows from Proposition 4.1. By the Borel-Cantelli lemma, if the expectation on the left-hand side of (1.13) is infinite, then

$$\limsup_{n\to\infty}\frac{\max_{1\leq k\leq n+1}\,\zeta_k}{(n\log\log n)^2}=\limsup_{n\to\infty}\frac{\max_{1\leq k\leq n+1}\,\zeta_k^+}{(n\log\log n)^2}=+\infty\quad\text{a.s.}$$

Using this, relation (4.6), applied to $(-S_k)$ instead of S_k, and

$$\max_{0\leq k\leq n}\,(S_k+\zeta_{k+1})\geq\max_{1\leq k\leq n+1}\,\zeta_k-\max_{0\leq k\leq n}\,(-S_k)\quad\text{a.s.}$$

we infer

$$\limsup_{n\to\infty}\frac{\max_{0\leq k\leq n}\,(S_k+\zeta_{k+1})}{(n\log\log n)^2}=+\infty\quad\text{a.s.}$$

Proof of Theorem 1.8. Both parts of the theorem will be proved by an application of Proposition 4.1. In addition, we find it instructive to give an alternative, more probabilistic proof of the relation $\limsup_{a\to 0+}\leq 1$ a.s. in part (b) which takes advantage of formula (1.16).

(b) In the setting of Proposition 4.1, let μ be a *random* measure defined by

$$\mu([0,\,t]):=\int_0^t \mathrm{e}^{B(s)}\mathrm{d}s,\quad t\geq 0,$$

and put $\psi(x):=x/\log\log x$ for $x\geq x_0$, where $x_0>\mathrm{e}$ is chosen to ensure that ψ, hence $x\mapsto x^2/\log\log x$, are strictly increasing and continuous on (x_0,∞). We intend to show that

$$\limsup_{t\to\infty}\frac{\log\int_0^t \mathrm{e}^{B(s)}\mathrm{d}s}{(t\log\log t)^{1/2}}=2^{1/2}\quad\text{a.s.}\tag{4.7}$$

On the one hand, $\log\int_0^t \mathrm{e}^{B(s)}\mathrm{d}s\leq\log t+\max_{s\in[0,\,t]}\,B(s)$ a.s. On the other hand, let $\tau_t\in[0,\,t]$ denote any (random) point satisfying $B(\tau_t)=\max_{s\in[0,\,t]}\,B(s)$ a.s. Then, given $\varepsilon>0$ there exists a random $\delta\in(0,1)$ such that $B(u)\geq\max_{s\in[0,\,t]}\,B(s)-\varepsilon$ whenever $u\in(\tau_t-\delta,\tau_t+\delta)\cap(0,\infty)$. This yields $\log\int_0^t \mathrm{e}^{B(u)}\mathrm{d}u\geq\log\int_{\max(\tau_t-\delta,\,0)}^{\tau_t+\delta}\mathrm{e}^{B(u)}\mathrm{d}u\geq\max_{s\in[0,\,t]}\,B(s)-\varepsilon+\log\delta$ a.s. Now relation (4.7) follows from the two inequalities and

$$\limsup_{t\to\infty}\frac{\max_{s\in[0,\,t]}\,B(s)}{(t\log\log t)^{1/2}}=2^{1/2}\quad\text{a.s.}$$

For the latter see, for instance, p. 439 in [4].

Formula (4.7) entails an a.s. version of (4.1), with the present choice of μ, $\varphi=f$, $A=2^{1/2}$ and $\alpha=2$. By Proposition 4.1, (4.2) holds with the right-hand side being equal to $1/2$. Hence,

$$\limsup_{a\to 0+}\frac{\log\int_{[0,\,\infty)}\mathrm{e}^{-ax}\mu(\mathrm{d}x)}{\psi^{\leftarrow}(1/a)}=2^{-1}\quad\text{a.s.},$$

where $\psi^{\leftarrow}$ the generalized inverse function of ψ satisfies $\psi^{\leftarrow}(t)\sim t\log\log t$ as $t\to\infty$. This proves (1.15).

Here is the promised probabilistic proof. Fix any $r>1$. In view of (1.16), a distribution density of $r^{-n}\big(\log\int_0^\infty \mathrm{e}^{B(s)-r^{-(n+1)}s}\mathrm{d}s-\log 2\big)$ is

$$x\mapsto\frac{r^n\mathrm{e}^{-2r^{-1}x}\exp(-\mathrm{e}^{-r^n x})}{\Gamma(2r^{-(n+1)})},\quad x\to\infty.$$

Hence, for all $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\Big\{r^{-n}\Big(\log\int_0^\infty \mathrm{e}^{B(s)-r^{-(n+1)}s}\mathrm{d}s - \log 2\Big) > 2^{-1}r(1+\varepsilon)\log\log r^n\Big\}\\
&\quad= \frac{r^n}{\Gamma(2r^{-(n+1)})}\int_{2^{-1}r(1+\varepsilon)\log(n\log r)}^\infty \mathrm{e}^{-2r^{-1}x}\exp(-\mathrm{e}^{-r^n x})\mathrm{d}x\\
&\quad\leq \frac{r^n}{\Gamma(2r^{-(n+1)})}\int_{2^{-1}r(1+\varepsilon)\log(n\log r)}^\infty \mathrm{e}^{-2r^{-1}x}\mathrm{d}x\\
&\quad= \frac{1}{2r^{-(n+1)}\Gamma(2r^{-(n+1)})\mathrm{e}^{(1+\varepsilon)\log(n\log r)}} \sim \frac{1}{n^{1+\varepsilon}}
\end{aligned}$$

as $n \to \infty$ having utilized $\lim_{x\to 0+} x\Gamma(x) = 1$. Thus, by the Borel-Cantelli lemma,

$$\limsup_{n\to\infty}\frac{2\log\int_0^\infty \mathrm{e}^{B(s)-r^{-(n+1)}s}\mathrm{d}s}{r^n\log\log r^n} \leq r \quad \text{a.s.}$$

For each $a \in (0,1]$ there exists $n \in \mathbb{N}_0$ such that $a \in [r^{-(n+1)}, r^{-n}]$ and, by monotonicity, for such a,

$$\frac{2a\log\int_0^\infty \mathrm{e}^{B(s)-as}\mathrm{d}s}{\log\log(1/a)} \leq \frac{2\log\int_0^\infty \mathrm{e}^{B(s)-r^{-(n+1)}s}\mathrm{d}s}{r^n\log(n\log r)} \quad \text{a.s.},$$

whence $\limsup_{a\to 0+}\frac{2a\log\int_0^\infty \mathrm{e}^{B(s)-as}\mathrm{d}s}{\log\log(1/a)} \leq 1$ a.s. because $r > 1$ is arbitrary.

(a) Let μ be a *random* measure defined by

$$\mu([0,\, t]) := \sum_{k=0}^{\lfloor t\rfloor}\mathrm{e}^{S_k}\eta_{k+1}, \quad t \geq 0$$

and take the same φ and ψ as in the proof of part (a), so that $\alpha = 2$. By Theorem 4.2, relation (4.5) holds. With this at hand, the rest of the proof mimics that of part (b). Note that there is the additional factor σ which was absent in the proof of part (a). In particular, $A = 2^{1/2}\sigma$ rather than $2^{1/2}$. □

Proof of Corollary 1.9. Under the assumptions of Theorem 1.8(a), invoking (1.6) yields

$$\frac{a\log Y(a)}{\log\log 1/a} \xrightarrow{\mathbb{P}} 0, \quad a \to 0+.$$

This in combination with the inequality $\log Y(a) \geq 0$ a.s. for small $a > 0$, which is a consequence of $\lim_{a\to 0+} Y(a) = +\infty$ a.s., yields

$$\liminf_{a\to 0+}\frac{a\log Y(a)}{\log\log(1/a)} = 0 \quad \text{a.s.}$$

In the setting of Theorem 1.8(b), relation (1.17) entails

$$\liminf_{a\to 0+}\frac{a\log\int_0^\infty \mathrm{e}^{B(s)-as}\mathrm{d}s}{\log\log 1/a} = 0 \quad \text{a.s.}$$

Both claims follow from the last two limit relations and Theorem 1.8 with the help of the intermediate value theorem for continuous functions. □

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