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Density functions for QuickQuant and QuickVal*

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Abstract

We prove that, for every $0 \leq t \leq 1$, the limiting distribution of the scale-normalized number of key comparisons used by the celebrated algorithm QuickQuant to find the tth quantile in a randomly ordered list has a Lipschitz continuous density function f_t that is bounded above by 10. Furthermore, this density $f_t(x)$ is positive for every $x > \min\{t, 1-t\}$ and, uniformly in t, enjoys superexponential decay in the right tail. We also prove that the survival function $1 - F_t(x) = \int_x^\infty f_t(y)\,\mathrm{d}y$ and the density function $f_t(x)$ both have the right tail asymptotics $\exp[-x \ln x - x \ln \ln x + O(x)]$. We use the right-tail asymptotics to bound large deviations for the scale-normalized number of key comparisons used by QuickQuant.

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1 Prologue, introduction, and summary

1.1 Prologue

We use this subsection to provide an easy-to-read (we hope) references-free and somewhat nontechnical motivation for the remainder of the paper. More careful discussion, complete with historical and other references, is included in Section 1.2, and a full guide to the detailed results of the paper is provided in Section 1.3.

The sorting algorithm QuickSort and its variants is ubiquitous and has been widely studied. Its cousin QuickSelect (also called Find) is an algorithm for finding a number of a specified rank in an unsorted list of n distinct numbers. We (crudely) measure the efficiency of QuickSelect by counting the number of comparisons (of numbers in the list) needed. When scaled by $1/n$, the number of comparisons has a limiting distribution as $n \to \infty$ when the specified rank is approximately nt for fixed $t \in [0, 1]$. In fact, as t varies,

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there is *process* convergence to a process $Z = (Z(t))_{t\in[0,1]}$ satisfying a distributional identity (for which, while it aids in understanding, we have no further explicit need in this paper):

$$Z \stackrel{\mathcal{L}}{=} 1 + \mathbb{1}(U \geq \cdot)\, U Z^{(1)}\Big(\frac{\cdot}{U}\Big) + \mathbb{1}(U \leq \cdot)\,(1-U) Z^{(2)}\Big(\frac{\cdot - U}{1-U}\Big), \tag{1.1}$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law; U, $Z^{(1)}$, and $Z^{(2)}$ are independent; U is uniformly distributed on $(0,1)$; and $Z^{(1)}$ and $Z^{(2)}$ are each distributed as Z.

As an anonymous Associate Editor who reviewed the original version of this manuscript pointed out, the distributional identity (1.1) can be viewed as something of a process version of a perpetuity $Y = AY + B$ for real-valued Y, where (A,B) is independent of Y. Indeed, an immediate consequence of (1.1) is the perpetuity

$$Z(0) \stackrel{\mathcal{L}}{=} 1 + U Z^{(1)}(0),$$

whose unique solution is the famed Dickman distribution. In this paper we study the other univariate marginal distributions of Z, showing, for general t, that $Z(t)$ enjoys many of the same properties as a Dickman variate, including the existence of a bounded and Lipschitz continuous density. To our knowledge, the present paper is the first detailed analysis of the univariate marginals of such a “functional perpetuity”.

Since `QuickSelect` is a basic, fundamentally important algorithm, we believe that establishment of the existence and properties of its limiting densities is interesting in its own right. Further, as we show in Section 12, the explicit bounds produced on the density, the Lipschitz constant, and the Kolmogorov–Smirnov distance between the scaled number of comparisons used by `QuickSelect` and $Z(t)$ enables perfect simulation from the distribution of $Z(t)$.

1.2 Introduction and literature review

`QuickQuant` is closely related to an algorithm called `QuickSelect`, which in turn can be viewed as a one-sided analogue of `QuickSort`. In brief, `QuickSelect`(n, m) is an algorithm designed to find a number of rank m in an unsorted list of size n. It works by recursively applying the same partitioning step as `QuickSort` to the sublist that contains the item of rank m until the pivot we pick *has* the desired rank or the size of the sublist to be explored has size one. Let $C_{n,m}$ denote the number of comparisons needed by `QuickSelect`(n, m), and note that $C_{n,m}$ and $C_{n,n+1-m}$ have the same distribution, by symmetry. Knuth [24] finds the formula

$$\mathbb{E}\, C_{n,m} = 2\,[(n+1)H_n - (n+3-m)H_{n+1-m} - (m+2)H_m + (n+3)] \tag{1.2}$$

for the expectation, where H_k denotes the kth harmonic number.

The algorithm `QuickQuant`(n, t) refers to `QuickSelect`(n, m_n) such that the ratio m_n/n converges to a specified value $t \in [0,1]$ as $n \to \infty$. It is easy to see that (1.2) tells us about the limiting behavior of the expected number of comparisons after standardizing:

$$\lim_{n\to\infty} \mathbb{E}[n^{-1} C_{n,m_n}] = 2 + 2H(t), \tag{1.3}$$

where $H(x) := -x \ln x - (1-x)\ln(1-x)$ with $0 \ln 0 := 0$.

We follow the set-up and notation of Fill and Nakama [16], who use an infinite sequence $(U_i)_{i\geq 1}$ of independent Uniform$(0,1)$-distributed random variables to couple the number of key comparisons C_{n,m_n} for all n. Let $L_0(n) := 0$ and $R_0(n) := 1$. For $k \geq 1$, inductively define

$$\tau_k(n) := \inf\{i \leq n : L_{k-1}(n) < U_i < R_{k-1}(n)\},$$

and let $r_k(n)$ be the rank of the pivot $U_{\tau_k(n)}$ in the set $\{U_1,\dots,U_n\}$ if $\tau_k(n)<\infty$ and be m_n otherwise. [Recall that the infimum of the empty set is ∞; hence $\tau_k(n)=\infty$ if and only if $L_{k-1}(n)=R_{k-1}(n)$.] Also, inductively define

$$L_k(n) := \mathbb{1}(r_k(n)\le m_n)U_{\tau_k(n)} + \mathbb{1}(r_k(n)>m_n)L_{k-1}(n), \tag{1.4}$$
$$R_k(n) := \mathbb{1}(r_k(n)\ge m_n)U_{\tau_k(n)} + \mathbb{1}(r_k(n)<m_n)R_{k-1}(n), \tag{1.5}$$

if $\tau_k(n)<\infty$, but

$$(L_k(n),R_k(n)) := (L_{k-1}(n),R_{k-1}(n)).$$

if $\tau_k(n)=\infty$. The number of comparisons at the k^{th} step is then

$$S_{n,k} := \sum_{i:\tau_k<i\le n} \mathbb{1}(L_{k-1}(n)<U_i<R_{k-1}(n)),$$

and the normalized total number of comparisons equals

$$n^{-1}C_{n,m_n} := n^{-1}\sum_{k\ge 1} S_{n,k}. \tag{1.6}$$

Mahmoud, Modarres and Smythe [27] studied `QuickSelect` in the case that the rank m is taken to be a random variable M_n uniformly distributed on $\{1,\dots,n\}$ and assumed to be independent of the numbers in the list. They used the Wasserstein metric to prove that $Z_n := n^{-1}C_{n,M_n} \xrightarrow{\mathcal{L}} Y$ as $n\to\infty$ and identified the distribution of Y. In particular, they proved that Y has an absolutely continuous distribution function. Grübel and Rösler [19] treated all the quantiles t simultaneously by letting $m_n\equiv m_n(t)$. Specifically, they considered the normalized process X_n defined by

$$X_n(t) := n^{-1}C_{n,\lfloor nt\rfloor+1} \text{ for } 0\le t<1, \quad X_n(t) := n^{-1}C_{n,n} \text{ for } t=1. \tag{1.7}$$

Working in the Skorohod topology (see Billingsley [1, Chapter 3]), they proved that this process has a limiting distribution as $n\to\infty$, and the value of the limiting process at argument t is the sum of the lengths of all the intervals encountered in all the steps of searching for population quantile t. We can use the *same* sequence $(U_i)_{i\ge1}$ of Uniform$(0,1)$ random variables to express the limiting stochastic process. For $t\in[0,1]$, let $L_0(t):=0$ and $R_0(t):=1$, and let $\tau_0(t):=0$. For $t\in[0,1]$ and $k\ge1$, inductively define

$$\tau_k(t) := \inf\{i>\tau_{k-1}(t) : L_{k-1}(t)\le U_i\le R_{k-1}(t)\}, \tag{1.8}$$
$$L_k(t) := \mathbb{1}(U_{\tau_k(t)}\le t)\,U_{\tau_k(t)} + \mathbb{1}(U_{\tau_k(t)}>t)\,L_{k-1}(t), \tag{1.9}$$
$$R_k(t) := \mathbb{1}(U_{\tau_k(t)}\le t)\,R_{k-1}(t) + \mathbb{1}(U_{\tau_k(t)}>t)\,U_{\tau_k(t)}. \tag{1.10}$$

It is not difficult to see that

$$\mathbb{P}(\tau_k(t)<\infty \text{ and } 0\le L_k(t)\le t\le R_k(t)\le 1 \text{ for all } 0\le t\le 1 \text{ and } k\ge 0)=1$$

and that for each fixed $t\in(0,1)$ we have

$$\mathbb{P}(L_k(t)<t<R_k(t) \text{ for all } k\ge 0)=1.$$

The limiting process Z can be expressed as

$$Z(t) := \sum_{k=0}^{\infty}[R_k(t)-L_k(t)] = 1+\sum_{k=1}^{\infty}[R_k(t)-L_k(t)]; \tag{1.11}$$

it is not hard to see that

$$\mathbb{P}(1<Z(t)<\infty \text{ for all } 0\le t\le 1)=1.$$

Note also that the processes Z and $(Z(1-t))_{t\in[0,1]}$ have the same finite-dimensional distributions. Grübel and Rösler [19, Theorem 8] proved that we can replace the subscript $\lfloor nt\rfloor + 1$ in (1.7) by any $m_n(t)$ with $0 \le m_n(t) \le n$ such that $m_n(t)/n \to t$ as $n \to \infty$, and then the normalized random variables $n^{-1}C_{n,m_n(t)}$ converge (univariately) to the limiting random variable $Z(t)$ for each $t \in [0,1]$. They also established the distributional identity (1.1).

Among the univariate distributions of $Z(t)$ for $t \in [0,1]$, only the common distribution of $Z(0)$ and $Z(1)$ is known at all explicitly. As established by Hwang and Tsai [21], this distribution is the Dickman distribution; see their paper for a wealth of information about the distribution. The overarching goal of this paper is to establish basic properties of the distributions of $Z(t)$ for other values of t.

Kodaj and Móri [25] proved the (univariate) convergence of (1.6) to $Z(t)$ in the Wasserstein metric. Using the coupling technique and induction, they proved that (1.6) is stochastically smaller than its continuous counterpart (1.11). Combining this fact with knowledge of their expectations (see (1.2) and [25, Lemma 2.2]), they proved that (1.6) converges to (1.11) in the Wasserstein metric and thus in distribution.

Grübel [18] connected QuickSelect(n, m_n) to a Markov chain to identify the limiting process. For each fixed $n \ge 1$, he considered the Markov chain $(Y_m^{(n)})_{m\ge 0}$ on the state space $I_n := \{(i,j) : 1 \le j \le i \le n\}$ with $Y_0^{(n)} := (n, m_n)$. Transition probabilities of $Y^{(n)}$ from the state (i,j) are determined by the partition step of QuickSelect(i,j) as follows. If $Y_m^{(n)} = (i,j)$, then $Y_{m+1}^{(n)}$ is selected uniformly at random from the set

$$\{(i-k, j-k) : k = 1, \dots, j-1\} \cup \{(1,1)\} \cup \{(i-k, j) : k = 1, \dots, i-j\};$$

in particular, $(1,1)$ is an absorbing state for $Y^{(n)}$. If we write $Y_m^{(n)} = (S_m^{(n)}, Q_m^{(n)})$, then we know

$$n^{-1}C_{n,m_n} \overset{\mathcal{L}}{=} n^{-1}\sum_{m\ge 0}\left(S_m^{(n)} - 1\right). \tag{1.12}$$

Grübel [18] constructed another Markov chain $Y = (Y_m) = ((S_m, Q_m))$, which is a continuous-value counterpart of the process $Y^{(n)}$, and he proved that for all $m \ge 0$, the random vector $Y_m^{(n)}$ converges to Y_m almost surely. Using the dominated convergence theorem, he proved that the random variables $n^{-1}\sum_{m=0}^{\infty}\left(S_m^{(n)} - 1\right)$ converge almost surely to $\sum_{m=0}^{\infty} S_m$; the limiting random variable here is exactly $Z(t)$ of (1.11). Combining with (1.12), he concluded that $n^{-1}C_{n,m_n}$ converges in distribution to (1.11). As previously mentioned, Hwang and Tsai [21] identified the limiting distribution of (1.6) when $m_n = o(n)$ as the Dickman distribution.

Fill and Nakama [16] studied the limiting distribution of the cost of using QuickSelect for a variety of cost functions. In particular, when there is simply *unit* cost of comparing any two keys, then their work reduces to study of the number of key comparisons, to which we limit our focus here. They proved L^p-convergence of (1.6) for QuickQuant(n,t) to (1.11) for $1 \le p < \infty$ by first studying the distribution of the number of key comparisons needed for another algorithm called QuickVal, and then comparing the two algorithms. The algorithm QuickVal(n,t) finds the rank of the *population* t-quantile in the sample, while its cousin QuickQuant(n,t) looks for the sample t-quantile. Intuitively, when the sample size is large, we expect the rank of the population t-quantile to be close to nt. Therefore, the two algorithms should behave similarly when n is large. Given a set of keys $\{U_1, \dots, U_n\}$, where U_i are i.i.d. Uniform$(0,1)$ random variables, one can regard the operation of QuickVal(n,t) as that of finding the rank of the value t in the augmented set $\{U_1, ..., U_n, t\}$. It works by first selecting a pivot uniformly at random from the set of keys $\{U_1, ..., U_n\}$ and then using the pivot to partition the *augmented* set (we don't count the comparison of the pivot with t). We then recursively do the same partitioning step on the subset that contains t until the set of the keys on which the algorithm operates reduces to the singleton $\{t\}$. For `QuickVal`(n,t) with the definitions (1.8)–(1.10) and with

$$S_{n,k}(t) := \sum_{\tau_k(t)<i\le n} \mathbb{1}(L_{k-1}(t) < U_i < R_{k-1}(t)),$$

Fill and Nakama [16] showed that $n^{-1}\sum_{k\ge1} S_{n,k}(t)$ converges (for fixed t) almost surely and also in L^p for $1 \le p < \infty$ to (1.11). They then used these facts to prove the L^p convergence (for fixed t) of (1.6) to (1.11) for `QuickQuant`(n,t) for $1 \le p < \infty$.

Fill and Matterer [15] treated distributional convergence for the worst-case cost of `Find` for a variety of cost functions. Suppose, for example, that we continue, as at the start of this section, to assign unit cost to the comparison of any two keys, so that $C_{n,m}$ is the total cost for `QuickSelect`(n,m). Then (for a list of length n) the cost of worst-case `Find` is $\max_{1\le m\le n} C_{n,m}$, and its distribution depends on the joint distribution of $C_{n,m}$ for varying m. We shall not be concerned here with worst-case `Find`, but we wish to review the approach and some of the results of [15], since there is relevance of their work to `QuickQuant`(n,t) for fixed t.

Fill and Matterer [15] considered tree-indexed processes closely related to the operation of the `QuickSelect` algorithm, as we now describe. For each node in a given rooted ordered binary tree, let θ denote the binary sequence (or *string*) representing the path from the root to this node, where 0 corresponds to taking the left child and 1 to taking the right. The value of θ for the root is thus the empty string, denoted ε. Define $L_\varepsilon := 0$, $R_\varepsilon := 1$, and $\tau_\varepsilon := 1$. Given a sequence of i.i.d. Uniform$(0,1)$ random variables $U_1, U_2, \dots$, recursively define

$$\begin{aligned} \tau_\theta &:= \inf\{i : L_\theta < U_i < R_\theta\}, \\ L_{\theta 0} &:= L_\theta, \quad L_{\theta 1} := U_{\tau_\theta}, \\ R_{\theta 0} &:= U_{\tau_\theta}, \quad R_{\theta 1} := R_\theta. \end{aligned}$$

Here the concatenated string $\theta 0$ corresponds to the left child of the node with string θ, while $\theta 1$ corresponds to the right child. Observe that, when inserting a key U_i arriving at time $i > \tau_\theta$ into the binary tree, this key is compared with the “pivot” U_{τ_θ} if and only if $U_i \in (L_\theta, R_\theta)$. For n insertions, the total cost of comparing keys with pivot U_{τ_θ} is therefore

$$S_{n,\theta} := \sum_{\tau_\theta < i \le n} \mathbb{1}(L_\theta < U_i < R_\theta).$$

We define a binary-tree-indexed stochastic process $S_n = (S_{n,\theta})_{\theta\in\Theta}$, where Θ is the collection of all finite-length binary sequences.

For each $1 \le p \le \infty$, Fill and Matterer [15, Definition 3.10 and Proposition 3.11] defined a Banach space $\mathcal{B}^{(p)}$ of binary-tree-indexed stochastic processes that corresponds in a natural way to the Banach space L^p for random variables. Let $I_\theta := R_\theta - L_\theta$ and consider the process $I = (I_\theta)_{\theta\in\Theta}$. Fill and Matterer [15, Theorem 4.1 with $\beta \equiv 1$] proved the convergence of the processes $n^{-1}S_n$ to I in the Banach space $\mathcal{B}^{(p)}$ for each $2 \le p < \infty$.

For the simplest application in [15], namely, to `QuickVal`(n,t) with t fixed, let $\gamma(t)$ be the infinite path from the root to the key having value t in the (almost surely) complete binary search tree formed by successive insertions of $U_1, U_2, \dots$ into an initially empty tree. The total cost (call it V_n) of `QuickVal`(n,t) can then be computed by summing the cost of comparisons with each (pivot-)node along the path, that is,

$$V_n := \sum_{\theta\in\gamma(t)} S_{n,\theta}.$$

Using their tree-process convergence theorem described in our preceding paragraph, Fill and Matterer [15, Proposition 6.1 with $\beta \equiv 1$] established L^p-convergence, for each $0 < p < \infty$, of $n^{-1}V_n$ to $I_{\gamma(t)}$ as $n \to \infty$, where $I_{\gamma(t)} := \sum_{\theta \in \gamma(t)} I_\theta$. Moreover ([15, Theorem 6.3 with $\beta \equiv 1$]), they also proved L^p-convergence of $n^{-1}Q_n$ to the same limit, again for every $0 < p < \infty$, where Q_n denotes the cost of $\texttt{QuickQuant}(n, t)$.

Throughout this paper, we will use the standard notations $\mathbb{1}(A)$ to denote the indicator function of the event A and $\mathbb{E}[f;\, A]$ for $\mathbb{E}[f \mathbb{1}(A)]$.

1.3 Summary

In Section 2, by construction we establish the existence of densities f_t for the random variables $Z(t)$ defined in (1.11). In Section 3 we prove that these densities are uniformly bounded and in Section 4 that they are uniformly continuous. As shown in Section 5, the densities satisfy a certain integral equation for $0 < t < 1$. The right-tail behavior of the density functions is examined in Section 6, and the left-tail behavior in Section 8. In Section 7 we prove that $f_t(x)$ is positive if and only if $x > \min\{t, 1-t\}$, and we improve the result of Section 4 by showing that $f_t(x)$ is Lipschitz continuous in x for fixed t and jointly continuous in (t, x). Sections 9–10 are devoted to sharp logarithmic asymptotics for the right tail of f_t, and Section 11 uses the results of those two sections to treat right-tail large deviation behavior of $\texttt{QuickQuant}(n, t)$ for large but finite n. Section 12 establishes an algorithm for perfect simulation from f_t.

2 Existence (and construction) of density functions

In this section, we prove that $Z \equiv Z(t)$ defined in (1.11) for fixed $0 \le t \le 1$ has a density. For notational simplification, we let $L_k \equiv L_k(t)$ and $R_k \equiv R_k(t)$. Let $J \equiv J(t) := Z(t) - 1 = \sum_{k=1}^{\infty} \Delta_k$ with $\Delta_k \equiv \Delta_k(t) := R_k(t) - L_k(t)$. We use convolution notation as in Section V.4 of Feller [10]. The following lemma is well known and can be found, for example, in Feller [10, Theorem V.4.4] or Durrett [9, Theorem 2.1.11].

Lemma 2.1. *If X and Y are independent random variables with respective distribution functions F and G, then $Z = X + Y$ has the distribution function $F \star G$. If, in addition, X has a density f (with respect to Lebesgue measure), then Z has a density $f \star G$.*

Let $X = \Delta_1 + \Delta_2$ and $Y = \sum_{k=3}^{\infty} \Delta_k$. For the remainder of this paragraph, we suppose $0 < t < 1$. If we condition on $(L_3, R_3) = (l_3, r_3)$ for some $0 \le l_3 < t < r_3 \le 1$ with $(l_3, r_3) \neq (0, 1)$, we then have

$$Y = (r_3 - l_3) \sum_{k=3}^{\infty} \frac{R_k - L_k}{r_3 - l_3} = (r_3 - l_3) \sum_{k=3}^{\infty} (R'_k - L'_k), \tag{2.1}$$

where we set $L'_k = (L_k - l_3)/(r_3 - l_3)$ and $R'_k = (R_k - l_3)/(r_3 - l_3)$ for $k \ge 3$. Observe that, by definitions (1.8)–(1.10), the stochastic process $((L'_k, R'_k))_{k \ge 3}$, conditionally given $(L_3, R_3) = (l_3, r_3)$, has the same distribution as the (unconditional) stochastic process of intervals $((L_k, R_k))_{k \ge 0}$ encountered in all the steps of searching for population quantile $(t - l_3)/(r_3 - l_3)$ (rather than t) by $\texttt{QuickQuant}$. Note also that (again conditionally) the stochastic processes $((L_k, R_k))_{0 \le k \le 2}$ and $((L'_k, R'_k))_{k \ge 3}$ are independent. Thus (again conditionally) $Y/(r_3 - l_3)$ has the same distribution as the (unconditional) random variable $Z\left((t - l_3)/(r_3 - l_3)\right)$ and is independent of X. We will prove later (Lemmas 2.4–2.5) that, conditionally given $(L_3, R_3) = (l_3, r_3)$, the random variable X has a density. Let

$$f_{l_3, r_3}(x) := \mathbb{P}(X \in \mathrm{d}x \mid (L_3, R_3) = (l_3, r_3))/\mathrm{d}x$$

be such a conditional density. We can then use Lemma 2.1 to conclude that $J = X + Y$

has a conditional density

$$h_{l_3,r_3}(x) := \mathbb{P}(J \in \mathrm{d}x \mid (L_3, R_3) = (l_3, r_3))/\mathrm{d}x.$$

By mixing $h_{l_3,r_3}(x)$ for all possible values of l_3, r_3, we will obtain an unconditional density function for J, as summarized in the following theorem.

Theorem 2.2. *For each $0 \le t \le 1$, the random variable $J(t) = Z(t) - 1$ has a density*

$$f_t(x) := \int \mathbb{P}((L_3, R_3) \in \mathrm{d}(l_3, r_3)) \cdot h_{l_3,r_3}(x), \tag{2.2}$$

and hence the random variable $Z(t)$ has density $f_t(x-1)$.

Remark 2.3. One might well wonder why we do not employ the simpler decomposition $Z = \widetilde{X} + \widetilde{Y}$ with $\widetilde{X} = \Delta_1$ and $\widetilde{Y} = \sum_{k=2}^{\infty} \Delta_k$ and condition on the value of (L_1, R_1), rather than using the decomposition $Z = X + Y$ and conditioning on the value of (L_2, R_2). The reason is that, when $0 < l_2 < r_2 < 1$, the conditional distribution of $\widetilde{X} = R_1 - L_1$ given $(L_2, R_2) = (l_2, r_2)$ does not have a density with respect to Lebesgue measure. Indeed, when $(L_2, R_2) = (l_2, r_2)$ with $0 < l_2 < r_2 < 1$, the value of (L_1, R_1) must be either $(l_2, 1)$ or $(0, r_2)$, and so the conditional distribution of $\widetilde{X} = R_1 - L_1$ given $(L_2, R_2) = (l_2, r_2)$ concentrates on the two points $1 - l_2$ and r_2.

Now, as promised, we prove that, conditionally given $(L_3, R_3) = (l_3, r_3)$, the random variable X has a density f_{l_3,r_3}. We begin with the case $0 < t < 1$.

Lemma 2.4. *Let $0 \le l_3 < t < r_3 \le 1$ with $(l_3, r_3) \ne (0, 1)$. Conditionally given $(L_3, R_3) = (l_3, r_3)$, the random variable $X = \Delta_1 + \Delta_2$ has a right continuous density f_{l_3,r_3}.*

Proof. We consider three cases based on the values of (l_3, r_3).

Case 1: $l_3 = 0$ and $r_3 < 1$. Since L_k is nondecreasing in k, from $L_3 = 0$ it follows that $L_1 = L_2 = 0$. The unconditional joint distribution of $(L_1, R_1, L_2, R_2, L_3, R_3)$ satisfies

$$\begin{aligned} &\mathbb{P}(L_1 = 0, R_1 \in \mathrm{d}r_1, L_2 = 0, R_2 \in \mathrm{d}r_2, L_3 = 0, R_3 \in \mathrm{d}r_3) \\ &= \mathbb{1}(t < r_3 < r_2 < r_1 < 1)\,\mathrm{d}r_1 \frac{\mathrm{d}r_2}{r_1} \frac{\mathrm{d}r_3}{r_2} \end{aligned} \tag{2.3}$$

and hence

$$\begin{aligned} \mathbb{P}(L_3 = 0, R_3 \in \mathrm{d}r_3) &= \mathbb{1}(t < r_3 < 1)\,\mathrm{d}r_3 \int_{r_2=r_3}^{1} \frac{\mathrm{d}r_2}{r_2} \int_{r_1=r_2}^{1} \frac{\mathrm{d}r_1}{r_1} \\ &= \mathbb{1}(t < r_3 < 1)\,\mathrm{d}r_3 \int_{r_2=r_3}^{1} \frac{\mathrm{d}r_2}{r_2}(-\ln r_2) \\ &= \frac{1}{2}\,(\ln r_3)^2\,\mathbb{1}(t < r_3 < 1)\,\mathrm{d}r_3. \end{aligned} \tag{2.4}$$

Dividing (2.3) by (2.4), we find

$$\begin{aligned} &\mathbb{P}(L_1 = 0, R_1 \in \mathrm{d}r_1, L_2 = 0, R_2 \in \mathrm{d}r_2 \mid L_3 = 0, R_3 = r_3) \\ &\quad = \frac{2\, r_1^{-1} r_2^{-1}\,\mathrm{d}r_1\,\mathrm{d}r_2}{(\ln r_3)^2}\,\mathbb{1}(t < r_3 < r_2 < r_1 < 1). \end{aligned} \tag{2.5}$$

Thus for $x \in (2r_3, 2)$, we find

$$\begin{aligned}
f_{0,r_3}(x) &= \mathbb{P}(X \in \mathrm{d}x \mid L_3 = 0, R_3 = r_3)/\mathrm{d}x \\
&= \int_{r_2} \mathbb{P}(R_2 \in \mathrm{d}r_2,\, R_1 \in \mathrm{d}x - r_2 \mid L_3 = 0, R_3 = r_3)/\mathrm{d}x \\
&= \frac{2}{(\ln r_3)^2} \int_{r_2 = r_3 \vee (x-1)}^{x/2} (x - r_2)^{-1} r_2^{-1}\, \mathrm{d}r_2 \\
&= \frac{2}{\left(\ln \frac{1}{r_3}\right)^2} \frac{1}{x} \Bigg[\ln\left(\frac{x - r_3}{r_3}\right) \mathbb{1}(2r_3 \le x < 1 + r_3) \\
&\qquad\qquad + \ln\left(\frac{1}{x-1}\right) \mathbb{1}(1 + r_3 \le x < 2) \Bigg];
\end{aligned} \tag{2.6}$$

we set $f_{0,r_3}(x) = 0$ for $x \notin (2r_3, 2)$.
Case 2: $l_3 > 0$ and $r_3 = 1$. This condition implies that $R_1 = R_2 = 1$. Invoking symmetry, we can skip the derivation and immediately write

$$\begin{aligned}
f_{l_3,1}(x) = \frac{2}{\left(\ln \frac{1}{1-l_3}\right)^2} \frac{1}{x} \Bigg[&\ln\left(\frac{x - 1 + l_3}{1 - l_3}\right) \mathbb{1}(2 - 2l_3 \le x < 2 - l_3) \\
&+ \ln \frac{1}{x-1} \mathbb{1}(2 - l_3 \le x < 2) \Bigg]
\end{aligned} \tag{2.7}$$

for $x \in (2 - 2l_3, 2)$; we set $f_{l_3,1}(x) = 0$ for $x \notin (2 - 2l_3, 2)$.
Case 3: $0 < l_3 < t < r_3 < 1$. There are six possible scenarios for the random vector $(L_1, R_1, L_2, R_2, L_3, R_3)$, and to help us discuss the cases, we consider values l_2, r_2 satisfying $0 < l_2 < l_3 < t < r_3 < r_2 < 1$.

(a) $L_1 = l_2, L_2 = L_3 = l_3$ and $R_1 = R_2 = 1, R_3 = r_3$.

In this subcase, we consider the event that the first pivot we choose locates between 0 and l_3, the second pivot has value l_3, and the third pivot has value r_3. Denote this event by E_{llr} (with llr indicating that we shrink the search intervals by moving the lefthand, lefthand, and then righthand endpoints). We have

$$\begin{aligned}
&\mathbb{P}(L_1 \in \mathrm{d}l_2, R_1 = 1, L_2 \in \mathrm{d}l_3, R_2 = 1, L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3) \\
&\qquad = \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1)\, \mathrm{d}l_2 \frac{\mathrm{d}l_3}{1 - l_2} \frac{\mathrm{d}r_3}{1 - l_3}.
\end{aligned} \tag{2.8}$$

Integrating over all possible values of l_2, we get

$$\begin{aligned}
&\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3, E_{llr}) \\
&\qquad = \mathbb{1}(0 < l_3 < t < r_3 < 1)\, \frac{1}{1 - l_3} \ln\left(\frac{1}{1 - l_3}\right) \mathrm{d}l_3\, \mathrm{d}r_3.
\end{aligned}$$

(b) $L_1 = L_2 = 0, L_3 = l_3$ and $R_1 = r_2, R_2 = R_3 = r_3$.

In this and all subsequence subcases, we use notation like that in subcase (a). In this subcase, we invoke symmetry in comparison with subcase (a). The results are

$$\begin{aligned}
&\mathbb{P}(L_1 = 0, R_1 \in \mathrm{d}r_2, L_2 = 0, L_3 \in \mathrm{d}l_3, R_2 = R_3 \in \mathrm{d}r_3) \\
&\qquad = \mathbb{1}(0 < l_3 < t < r_3 < r_2 < 1)\, \mathrm{d}r_2 \frac{\mathrm{d}r_3}{r_2} \frac{\mathrm{d}l_3}{r_3}
\end{aligned} \tag{2.9}$$

and

$$\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3, E_{rrl}) = \mathbb{1}(0 < l_3 < t < r_3 < 1)\,\frac{1}{r_3}\ln\left(\frac{1}{r_3}\right)\,\mathrm{d}l_3\,\mathrm{d}r_3.$$

(c) $L_1 = L_2 = l_2, L_3 = l_3$ and $R_1 = 1, R_2 = R_3 = r_3$.

In this subcase we have

$$\begin{aligned}&\mathbb{P}(R_1 = 1, L_1 = L_2 \in \mathrm{d}l_2, L_3 \in \mathrm{d}l_3, R_2 = R_3 \in \mathrm{d}r_3)\\ &\qquad = \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1)\,\mathrm{d}l_2\frac{\mathrm{d}r_3}{1-l_2}\frac{\mathrm{d}l_3}{r_3-l_2}.\end{aligned} \tag{2.10}$$

Integrating over the possible values of l_2, we find

$$\begin{aligned}&\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3, E_{lrl})/(\mathrm{d}l_3\,\mathrm{d}r_3)\\ &= \mathbb{1}(0 < l_3 < t < r_3 < 1)\,\frac{1}{1-r_3}\left[\ln\left(\frac{1}{r_3-l_3}\right) - \ln\left(\frac{1}{r_3}\right) - \ln\left(\frac{1}{1-l_3}\right)\right].\end{aligned}$$

(d) $L_1 = 0, L_2 = L_3 = l_3$ and $R_1 = R_2 = r_2, R_3 = r_3$.

In this subcase, by symmetry with subcase (c), we have

$$\begin{aligned}&\mathbb{P}(L_1 = 0, L_2 = L_3 \in \mathrm{d}l_3, R_1 = R_2 \in \mathrm{d}r_2, R_3 \in \mathrm{d}r_3)\\ &\qquad = \mathbb{1}(0 < l_3 < t < r_3 < r_2 < 1)\,\mathrm{d}r_2\frac{\mathrm{d}l_3}{r_2}\frac{\mathrm{d}r_3}{r_2-l_3}.\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3, E_{rlr})/(\mathrm{d}l_3\,\mathrm{d}r_3)\\ &\quad = \mathbb{1}(0 < l_3 < t < r_3 < 1)\,\frac{1}{l_3}\left[\ln\left(\frac{1}{r_3-l_3}\right) - \ln\left(\frac{1}{1-l_3}\right) - \ln\left(\frac{1}{r_3}\right)\right].\end{aligned}$$

(e) $L_1 = 0, L_2 = l_2, L_3 = l_3$ and $R_1 = R_2 = R_3 = r_3$.

In this subcase we have

$$\begin{aligned}&\mathbb{P}(L_1 = 0, L_2 \in \mathrm{d}l_2, L_3 \in \mathrm{d}l_3, R_1 = R_2 = R_3 \in \mathrm{d}r_3)\\ &\qquad = \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1)\,\mathrm{d}r_3\frac{\mathrm{d}l_2}{r_3}\frac{\mathrm{d}l_3}{r_3-l_2}.\end{aligned} \tag{2.12}$$

Integrating over the possible values of l_2, we have

$$\begin{aligned}&\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3, E_{rll})\\ &\quad = \mathbb{1}(0 < l_3 < t < r_3 < 1)\,\frac{1}{r_3}\left[\ln\left(\frac{1}{r_3-l_3}\right) - \ln\left(\frac{1}{r_3}\right)\right]\,\mathrm{d}l_3\,\mathrm{d}r_3.\end{aligned}$$

(f) $L_1 = L_2 = L_3 = l_3$ and $R_1 = 1, R_2 = r_2, R_3 = r_3$.

In this final subcase, by symmetry with subcase (e), we have

$$\begin{aligned}&\mathbb{P}(L_1 = L_2 = L_3 \in \mathrm{d}l_3, R_1 = 1, R_2 \in \mathrm{d}r_2, R_3 \in \mathrm{d}r_3)\\ &\qquad = \mathbb{1}(0 < l_3 < t < r_3 < r_2 < 1)\,\mathrm{d}l_3\frac{\mathrm{d}r_2}{1-l_3}\frac{\mathrm{d}r_3}{r_2-l_3}\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3, E_{lrr})/(\mathrm{d}l_3\,\mathrm{d}r_3)\\ &\quad= \mathbb{1}(0 < l_3 < t < r_3 < 1)\,\frac{1}{1-l_3}\left[\ln\left(\frac{1}{r_3-l_3}\right) - \ln\left(\frac{1}{1-l_3}\right)\right].\end{aligned}$$

Summing results from the six subcases, we conclude in Case 3 that

$$\mathbb{P}(L_3 \in \mathrm{d}l_3, R_3 \in \mathrm{d}r_3) = \mathbb{1}(0 < l_3 < t < r_3 < 1)\, g(l_3, r_3)\,\mathrm{d}l_3\,\mathrm{d}r_3, \tag{2.14}$$

where

$$\begin{aligned}g(l_3, r_3) := \left[\frac{1}{l_3(1-l_3)} + \frac{1}{r_3(1-r_3)}\right]\ln\left(\frac{1}{r_3-l_3}\right)&\\ - \left(\frac{1}{l_3} + \frac{1}{1-r_3}\right)\left[\ln\left(\frac{1}{r_3}\right) + \ln\left(\frac{1}{1-l_3}\right)\right]&.\end{aligned} \tag{2.15}$$

The conditional joint distribution of (L_1, R_1, L_2, R_2) given $(L_3, R_3) = (l_3, r_3)$ can be derived by dividing each of (2.8)–(2.13) by (2.14), and we can then compute f_{l_3,r_3} from these conditional distributions. Let us write

$$f_{l_3,r_3}(x) = \mathbb{1}(0 < l_3 < t < r_3 < 1)\,\frac{1}{g(l_3, r_3)}\sum_{i=1}^{6} f_{l_3,r_3}^{(i)}(x), \tag{2.16}$$

where $f_{l_3,r_3}^{(i)}(x)\,\mathrm{d}l_3\,\mathrm{d}r_3\,\mathrm{d}x$ is the contribution to

$$\mathbb{P}(L_3 \in \mathrm{d}l_3,\ R_3 \in \mathrm{d}r_3,\ X \in \mathrm{d}x)$$

arising from the ith subcase of the six.

In subcase (a) we know that $X = R_1 - L_1 + R_2 - L_2 = 2 - l_2 - l_3$. Changing variables from l_2 to x, from (2.8) we find

$$f_{l_3,r_3}^{(1)}(x) = \mathbb{1}(2 - 2l_3 \le x < 2 - l_3)\,\frac{1}{1-l_3}\,\frac{1}{x-1+l_3}.$$

In subcase (b) we know that $X = r_2 + r_3$. Changing variables from r_2 to x, from (2.9) we find

$$f_{l_3,r_3}^{(2)}(x) = \mathbb{1}(2r_3 \le x < 1 + r_3)\,\frac{1}{r_3}\,\frac{1}{x-r_3}.$$

In subcase (c), we know that $X = 1 - 2l_2 + r_3$. Changing variables from l_2 to x, from (2.10) we find

$$f_{l_3,r_3}^{(3)}(x) = \mathbb{1}(1 + r_3 - 2l_3 \le x < 1 + r_3)\,\frac{1}{x+1-r_3}\,\frac{2}{x+r_3-1}.$$

In subcase (d), we know that $X = 2r_2 - l_3$. Changing variables from r_2 to x, from (2.11) we find

$$f_{l_3,r_3}^{(4)}(x) = \mathbb{1}(2r_3 - l_3 \le x < 2 - l_3)\,\frac{2}{x+l_3}\,\frac{1}{x-l_3}.$$

In subcase (e), we know that $X = 2r_3 - l_2$. Changing variables from l_2 to x, from (2.12) we find

$$f_{l_3,r_3}^{(5)}(x) = \mathbb{1}(2r_3 - l_3 \le x < 2r_3)\,\frac{1}{r_3}\,\frac{1}{x-r_3}.$$

Finally, in subcase (f), we know that $X = 1 + r_2 - l_3$. Changing variables from r_2 to x, from (2.13) we find

$$f_{l_3,r_3}^{(6)}(x) = \mathbb{1}(r_3 - 2l_3 + 1 \le x < 2 - 2l_3)\,\frac{1}{1-l_3}\,\frac{1}{x-1+l_3}.$$

The density functions f_{0,r_3} and $f_{l_3,1}$ we have found in Cases 1 and 2 are continuous. We have chosen to make the functions $f^{(i)}_{l_3,r_3}$ (for $i = 1, \ldots, 6$) right continuous in Case 3. Thus the density f_{l_3,r_3} we have determined at (2.16) in Case 3 is right continuous. $\square$

Our next lemma handles the cases $t = 0$ and $t = 1$ that were excluded from Lemma 2.4, and its proof is the same as for Cases 1 and 2 in the proof of Lemma 2.4.

Lemma 2.5. (a) *Suppose $t = 0$. Let $0 < r_3 < 1$. Conditionally given $(L_3, R_3) = (0, r_3)$, the random variable $X = \Delta_1 + \Delta_2$ has the right continuous density f_{0,r_3} specified in the sentence containing* (2.6)*.*

(b) *Suppose $t = 1$. Let $0 < l_3 < 1$. Conditionally given $(L_3, R_3) = (l_3, 1)$, the random variable $X = \Delta_1 + \Delta_2$ has the right continuous density $f_{l_3,1}$ specified in the sentence containing* (2.7)*.*

We need to check the trivariate measurability of the function $f_t(l_3, r_3, x) := f_{l_3,r_3}(x)$ before diving into the derivation of the density function of J. Given a topological space S, let $\mathcal{B}(S)$ denote its Borel σ-field, that is, the σ-field generated by the open sets of S. Also, given $0 < t < 1$, let

$$S_t := \{(l_3, r_3) \neq (0, 1) : 0 \leq l_3 < t < r_3 \leq 1\}.$$

Lemma 2.6. (a) *For $0 < t < 1$, the conditional density function $f_t(l_3, r_3, x)$, formed to be a right continuous function of x, is measurable with respect to $\mathcal{B}(S_t \times \mathbb{R})$.*

(b) *For $t = 0$, the conditional density function $f_0(r_3, x) := f_{0,r_3}(x)$, continuous in x, is measurable $\mathcal{B}((0, 1) \times \mathbb{R})$.*

(c) *For $t = 1$, the conditional density function $f_1(l_3, x) := f_{l_3,1}(x)$, continuous in x, is measurable $\mathcal{B}((0, 1) \times \mathbb{R})$.*

We introduce (the special case of real-valued f of) a lemma taken from Gowrisankaran [17, Theorem 3] giving a sufficient condition for the measurability of certain functions f defined on product spaces. The lemma will help us prove Lemma 2.6.

Lemma 2.7 (Gowrisankaran [17])**.** *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$. Suppose that the section mapping $f(\cdot, y)$ is $\mathcal{F}$-measurable for each $y \in \mathbb{R}$ and that the section mapping $f(\omega, \cdot)$ is either right continuous for each $\omega \in \Omega$ or left continuous for each $\omega \in \Omega$. Then f is measurable with respect to the product σ-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.*

Proof of Lemma 2.6. We prove (b), then (c), and finally (a).

(b) Recall the expression (2.6) for $f_0(r_3, x)$ [for $0 < r_3 < 1$ and $x \in (2r_3, 2)$]. We apply Lemma 2.7 with $(\Omega, \mathcal{F}) = ((0, 1), \mathcal{B}((0, 1)))$. The right continuity of $f_0(r_3, \cdot)$ has already been established in Lemma 2.5(a). On the other hand, when we fix x and treat $f(0, r_3, x)$ as a function of r_3, the conditional density function can be separated into the following cases:

- If $x \leq 0$ or $x \geq 2$, then $f_0(r_3, x) \equiv 0$.
- If $0 < x < 2$, then from (2.6) we see that $f_0(r_3, x)$ is piecewise continuous (with a finite number of measurable domain-intervals), and hence measurable, in r_3.

Since the product σ-field $\mathcal{B}((0, 1)) \otimes \mathcal{B}(\mathbb{R})$ equals $\mathcal{B}((0, 1) \times \mathbb{R})$, the desired result follows.

(c) Assertion (c) can be proved by a similar argument or by invoking symmetry.

(a) We apply Lemma 2.7 with $(\Omega, \mathcal{F}) = (S_t, \mathcal{B}(S_t))$. The right continuity of $f(l_3, r_3, \cdot)$ has already been established in Lemma 2.4, so it suffices to show for each $x \in \mathbb{R}$ that $f(l_3, r_3, x)$ is measurable in $(l_3, r_3) \in S_t$. For this it is clearly sufficient to show that $f(0, r_3, x)$ is measurable in $r_3 \in (t, 1)$, that $f(l_3, 1, x)$ is measurable in $l_3 \in (0, t)$, and that $f(l_3, r_3, x)$ is measurable in $(l_3, r_3) \in (0, t) \times (t, 1)$. All three of these assertions follow from the fact that piecewise continuous functions (with a finite number of measurable domain-pieces) are measurable; in particular, for the third assertion, note that the function g defined at (2.15) is continuous in $(l_3, r_3) \in (0, t) \times (t, 1)$ and that each of the six expressions $f^{(i)}_{l_3,r_3}(x)$ appearing in (2.16) is piecewise continuous (with a finite number of measurable domain-pieces) in these values of (l_3, r_3) for each fixed $x \in \mathbb{R}$.

This complete the proof. □

As explained at the outset of this section, a conditional density $h_{l_3,r_3}(\cdot)$ for $J(t)$ given $(L_3, R_3) = (l_3, r_3)$ can now be formed by convolving the conditional density function of X, namely $f_{l_3,r_3}(\cdot)$, with the conditional distribution function of Y. That is, we can write

$$h_{l_3,r_3}(x) = \int f_{l_3,r_3}(x - y)\ \mathbb{P}\left(Y \in \mathrm{d}y \mid (L_3, R_3) = (l_3, r_3)\right). \tag{2.17}$$

We now prove in the next two lemmas that the joint measurability of $f_{l_3,r_3}(x)$ with respect to (l_3, r_3, x) ensures the same for $h_{l_3,r_3}(x)$.

Lemma 2.8. *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonnegative function measurable with respect to the product σ-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$. Let V and Y be two measurable functions defined on a common probability space and taking values in Ω and $\mathbb{R}$, respectively. Then a regular conditional probability distribution $\mathbb{P}(Y \in dy \mid V)$ for Y given V exists, and the function $\mathcal{T}g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by*

$$\mathcal{T}g(v, x) := \int g(v, x - y)\, \mathbb{P}(Y \in \mathrm{d}y \mid V = v)$$

is measurable with respect to the product σ-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

Proof. This is standard. For completeness, we provide a proof making use of Kallenberg [23, Lemma 3.2(i)]. First, since Y is a real-valued random variable, by Billingsley [2, Theorem 33.3] or Durrett [9, Theorem 4.1.18] or Kallenberg [23, Theorem 8.5] there exists a regular conditional probability distribution for Y given V; this is a probability kernel from Ω to $\mathbb{R}$ and can trivially be regarded as a kernel from $\Omega \times \mathbb{R}$ to $\mathbb{R}$. Let $S := \Omega \times \mathbb{R}$, $T := \mathbb{R}$, $\mu_{v,x}(\mathrm{d}y) := \mathbb{P}(Y \in \mathrm{d}y \mid V = v)$, and $f((v, x), y) := g(v, x - y)$. The conclusion of our lemma is then an immediate consequence of the first assertion in the aforementioned Kallenberg lemma. □

We can now handle the measurability of $(l_3, r_3, x) \mapsto h_{l_3,r_3}(x)$.

Lemma 2.9.

(a) *For $0 < t < 1$, the mapping $(l_3, r_3, x) \mapsto h_{l_3,r_3}(x)$ is $\mathcal{B}(S_t \times \mathbb{R})$ measurable.*
(b) *For $t = 0$, the mapping $(r_3, x) \mapsto h_{0,r_3}(x)$ is $\mathcal{B}((0, 1) \times \mathbb{R})$ measurable.*
(c) *For $t = 1$, the mapping $(l_3, x) \mapsto h_{l_3,1}(x)$ is $\mathcal{B}((0, 1) \times \mathbb{R})$ measurable.*

Proof. We prove (a); the claims (b) and (c) are proved similarly. Choosing $\Omega = S_t$ and $g(l_3, r_3, x) = f_{l_3,r_3}(x)$ with $(l_3, r_3) \in \Omega$ in Lemma 2.8, we conclude that

$$(l_3, r_3, x) \mapsto h_{l_3,r_3}(x) = \mathcal{T}g(l_3, r_3, x)$$

is $\mathcal{B}(S_t \times \mathbb{R})$ measurable. □

Recall the definition of $f_t(x)$ at (2.2). It then follows from Lemma 2.9 that $f_t(x)$ is well defined and measurable with respect to $x \in \mathbb{R}$ for fixed $0 \le t \le 1$. This completes the proof of Theorem 2.2.

3 Uniform boundedness of the density functions

In this section, we prove that the functions f_t are uniformly bounded for $0 \le t \le 1$.

Theorem 3.1. *The densities f_t are uniformly bounded by 10 for $0 \le t \le 1$.*

The proof of Theorem 3.1 is our later Lemmas 3.2 and 3.6. In particular, the numerical value 10 comes from the bound in the last line of the proof of Lemma 3.2 plus two times the bound in the last sentence of the proof of Lemma 3.6. A bound on the function f_t is established by first finding a bound on the conditional density function f_{l_3,r_3}. Observe that the expressions in the proof of Lemma 2.4 for $f^{(i)}_{l_3,r_3}(x)$ (for $i = 1, \dots, 6$) in Case 3 all involve indicators of intervals. The six endpoints of these intervals are

$$2r_3 - l_3,\ 2r_3,\ 1 + r_3 - 2l_3,\ 1 + r_3,\ 2 - 2l_3, \text{ and } 2 - l_3,$$

with $0 < l_3 < t < r_3 < 1$. The relative order of these six endpoints is determined once we know the value of $\rho = \rho(l_3, r_3) := l_3/(1 - r_3)$. Indeed:

- When $\rho \in (0, 1/2)$, the order is

$$2r_3 - l_3 < 2r_3 < 1 + r_3 - 2l_3 < 1 + r_3 < 2 - 2l_3 < 2 - l_3.$$

- When $\rho \in (1/2, 1)$, the order is

$$2r_3 - l_3 < 1 + r_3 - 2l_3 < 2r_3 < 2 - 2l_3 < 1 + r_3 < 2 - l_3.$$

- When $\rho \in (1, 2)$, the order is

$$1 + r_3 - 2l_3 < 2r_3 - l_3 < 2 - 2l_3 < 2r_3 < 2 - l_3 < 1 + r_3.$$

- When $\rho \in (2, \infty)$, the order is

$$1 + r_3 - 2l_3 < 2r_3 - l_3 < 2 - 2l_3 < 2 - l_3 < 2r_3 < 1 + r_3.$$

When $\rho = 0$ (i.e., in Case 1 in the proof of Lemma 2.4: $l_3 = 0 < r_3 < 1$), the function f_{l_3,r_3} is given by f_{0,r_3} at (2.6). When $\rho = \infty$ (i.e., in Case 2 in the proof of Lemma 2.4: $0 < l_3 < r_3 = 1$), the function f_{l_3,r_3} is given by $f_{l_3,1}$ at (2.7). The result (2.16) for Case 3 in the proof of Lemma 2.4 can be reorganized as follows, where we define the following functions to simplify notation:

$$\begin{aligned}
m_1(x, l_3, r_3) &:= \frac{1}{r_3(x - r_3)} + \frac{2}{(x + l_3)(x - l_3)}, \\
m_2(x, l_3, r_3) &:= \frac{1}{(1 - l_3)(x - 1 + l_3)} + \frac{2}{(x + l_3)(x - l_3)}, \\
m_3(x, l_3, r_3) &:= \frac{2}{(x + 1 - r_3)(x + r_3 - 1)} + \frac{1}{(1 - l_3)(x + l_3 - 1)}, \\
m_4(x, l_3, r_3) &:= \frac{1}{r_3(x - r_3)} + \frac{2}{(x + 1 - r_3)(x + r_3 - 1)}.
\end{aligned}$$

When $\rho \in (0, 1)$, the conditional density f_{l_3,r_3} satisfies

$$\begin{aligned}
f_{l_3,r_3}(x)\, g(l_3, r_3) = {} & \mathbb{1}(2r_3 - l_3 \le x < 1 + r_3 - 2l_3)\, m_1(x, l_3, r_3) \\
& + \mathbb{1}(1 + r_3 - 2l_3 \le x < 1 + r_3)\, [m_2(x, l_3, r_3) + m_4(x, l_3, r_3)] \\
& + \mathbb{1}(1 + r_3 \le x < 2 - l_3)\, m_2(x, l_3, r_3).
\end{aligned} \tag{3.1}$$

Lastly, when $\rho \in (1, \infty)$, the conditional density f_{l_3,r_3} satisfies

$$\begin{aligned} f_{l_3,r_3}(x)\, g(l_3, r_3) &= \mathbb{1}(1 + r_3 - 2l_3 \le x < 2r_3 - l_3)\, m_3(x, l_3, r_3) \\ &\quad + \mathbb{1}(2r_3 - l_3 \le x < 2 - l_3)\, [m_2(x, l_3, r_3) + m_4(x, l_3, r_3)] \\ &\quad + \mathbb{1}(2 - l_3 \le x < 1 + r_3)\, m_4(x, l_3, r_3). \end{aligned} \tag{3.2}$$

Recall the definition of $f_t(x)$ at (2.2). For any $x \in \mathbb{R}$ we can decompose $f_t(x)$ into three contributions:

$$\begin{aligned} f_t(x) &= \mathbb{E}\; h_{L_3,R_3}(x) \\ &= \mathbb{E}[h_{L_3,R_3}(x);\, \rho(L_3, R_3) = 0] + \mathbb{E}[h_{L_3,R_3}(x);\, \rho(L_3, R_3) = \infty] \\ &\quad + \mathbb{E}[h_{L_3,R_3}(x);\, 0 < \rho(L_3, R_3) < \infty]. \end{aligned}$$

We first consider the contribution from the case $0 < \rho(L_3, R_3) < \infty$ for any $0 < t < 1$, noting that this case doesn't contribute to $f_t(x)$ when $t = 0$ or $t = 1$. Define

$$b(l_3, r_3) = \frac{1}{g(l_3, r_3)} \frac{3}{2} \left[\frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)} \right]. \tag{3.3}$$

Lemma 3.2. *The contribution to the density function f_t from the case $0 < \rho(L_3, R_3) < \infty$ is uniformly bounded for $0 < t < 1$. More precisely, given $0 < t < 1$ and $0 < l_3 < t < r_3 < 1$, we have*

$$f_{l_3,r_3}(x) \le b(l_3, r_3) \;\text{ and }\; h_{l_3,r_3}(x) \le b(l_3, r_3) \text{ for all } x \in \mathbb{R}; \tag{3.4}$$

and, moreover, $\mathbb{E}[h_{L_3,R_3}(x);\, 0 < \rho(L_3, R_3) < \infty]$ is uniformly bounded for $0 < t < 1$.

Proof. For (3.4), we need only establish the bound on f.

We start to bound (3.1) for $0 < \rho < 1$. The function m_1 is a decreasing function of x and thus reaches its maximum in (3.1) when $x = 2r_3 - l_3$:

$$m_1(x, l_3, r_3) \le m_1(2r_3 - l_3, l_3, r_3) = \frac{3}{2} \frac{1}{r_3(r_3 - l_3)}.$$

The function $m_2 + m_4$ is also a decreasing function of x, and the maximum in (3.1) occurs at $x = 1 + r_3 - 2l_3$. Plug in this x-value and use the fact that $1 - l_3 > r_3$ when $\rho < 1$ to obtain

$$(m_2 + m_4)(x, l_3, r_3) \le \frac{3}{2} \frac{1}{r_3(r_3 - l_3)} + \frac{3}{2} \frac{1}{(1 - l_3)(r_3 - l_3)}.$$

Finally, the function m_2 is again a decreasing function of x, and the maximum in (3.1) occurs at $x = 1 + r_3$. Plug in this x-value and use the facts that $1 + r_3 - l_3 > 2r_3$ and $1 + r_3 + l_3 > r_3 - l_3$ to conclude

$$m_2(x, l_3, r_3) \le \frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)}.$$

By the above three inequalities, we summarize that for $0 < \rho < 1$ we have for all x the inequality

$$f_{l_3,r_3}(x) \le \frac{1}{g(l_3, r_3)} \frac{3}{2} \left[\frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)} \right].$$

The method to upper-bound (3.2) is similar to that for (3.1), or one can again invoke symmetry, and we skip the proof here.

For the expectation of $b(L_3, R_3)$, we see immediately that

$$\begin{aligned} \mathbb{E}[b\,(L_3, R_3)\,;\, 0 < \rho(L_3, R_3) < \infty] &= \frac{3}{2} \int_0^t \int_t^1 \left[\frac{1}{r(r - l)} + \frac{1}{(1 - l)(r - l)} \right] \mathrm{d}r\, \mathrm{d}l \\ &= \frac{\pi^2}{4} + \frac{3}{2} (\ln t)[\ln(1 - t)] \le \frac{\pi^2}{4} + \frac{3}{2} (\ln 2)^2. \end{aligned}$$ □

For the cases $\rho(L_3, R_3) = 0$ and $\rho(L_3, R_3) = \infty$, we cannot find a constant bound $b(l_3, r_3)$ on the function f_{l_3,r_3} such that the corresponding contributions to $\mathbb{E}\, b(L_3, R_3)$ are bounded for $0 \le t \le 1$. Indeed, although we shall omit the proof since it would take us too far afield, there exists no such bound $b(l_3, r_3)$.

Instead, to prove the uniform boundedness of the contributions in these two cases, we take a different approach. The following easily-proved lemma comes from Grübel and Rösler [19, proof of Theorem 9].

Lemma 3.3. *Consider a sequence of independent random variables $V_1, V_2, \ldots$, each uniformly distributed on $(1/2, 1)$, and let*

$$V := 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} V_k. \tag{3.5}$$

Then the random variables $Z(t)$, $0 \le t \le 1$, defined at (1.11) are all stochastically dominated by V. Furthermore, $\mathbb{E}\, V = 4$; and V has everywhere finite moment generating function m and therefore superexponential decay in the right tail, in the sense that for any $\theta \in (0, \infty)$ we have $\mathbb{P}(V \ge x) = o(e^{-\theta x})$ as $x \to \infty$.

The following lemma pairs the stochastic upper bound V on $Z(t)$ with a stochastic lower bound. These stochastic bounds will be useful in later sections. Recall that the Dickman distribution with support $[1, \infty)$ is the distribution of $Z(0)$.

Lemma 3.4. *Let D be a random variable following the Dickman distribution with support $[1, \infty)$. Then for all $0 \le t \le 1$ we have $D \le Z(t) \le V$ stochastically.*

Proof. Recall that $\Delta_1(t) = R_1(t) - L_1(t)$. We first use a coupling argument to show that $\Delta_1(t)$ is stochastically increasing for $0 \le t \le 1/2$. Let $U = U_1 \sim$ uniform$(0, 1)$ be the first key in the construction of Z as described in (1.8)–(1.11). Let $0 \le t_1 < t_2 \le 1/2$. It is easy to see that $\Delta_1(t_1) = \Delta_1(t_2)$ unless $t_1 < U < t_2$, in which case $\Delta_1(t_1) = U < t_2 \le 1/2 \le 1 - t_2 < 1 - U = \Delta_1(t_2)$.

Let $V_1 \sim$ uniform$(1/2, 1)$, as in Lemma 3.3, and let $0 \le t \le 1/2$. Since $\Delta_1(0) \stackrel{\mathcal{L}}{=} U$ and $\Delta_1(1/2) \stackrel{\mathcal{L}}{=} V_1$, we immediately have $U \le \Delta_1(t) \le V_1$ stochastically. This implies by a simple induction argument on k involving the conditional distribution of $\Delta_k(t)$ given $\Delta_{k-1}(t)$ that $D \le Z(t) \le V$ stochastically. $\square$

Remark 3.5. (a) Note that we do *not* claim that $Z(t)$ is stochastically increasing in $t \in [0, 1/2]$. Indeed, other than the stochastic ordering $D = Z(0) \le Z(t)$, we do not know whether any stochastic ordering relations hold among the random variables $Z(t)$.

(b) The random variable V can be interpreted as a sort of "limiting greedy (or 'on-line') worst-case `QuickQuant` normalized key-comparisons cost". Indeed, if upon each random bisection of the search interval one always chooses the half of greater length and sums the lengths to get $V^{(n)}$, then the limiting distribution of $V^{(n)}/n$ is that of V.

Lemma 3.6. *The contributions to the density function f_t from the cases $\rho(L_3, R_3) = 0$ and $\rho(L_3, R_3) = \infty$ are uniformly bounded for $0 \le t \le 1$.*

Proof. Because the Dickman density is bounded above by $e^{-\gamma}$, we need only consider $0 < t < 1$. The case $\rho(L_3(t), R_3(t)) = 0$ corresponds to $L_3(t) = 0$, while the case $\rho(L_3(t), R_3(t)) = \infty$ corresponds to $R_3(t) = 1$. By symmetry, the contribution from $R_3(t) = 1$ is the same as the contribution from $L_3(1 - t) = 0$, so we need only show that the contribution from $L_3(t) = 0$ is uniformly bounded. We will do this by showing that the larger contribution from $L_2(t) = 0$ is uniformly bounded.

By conditioning on the value of $R_2(t)$, the contribution from $L_2(t) = 0$ is

$$\begin{aligned} c_t(x) &:= \mathbb{P}(L_2(t) = 0,\ J(t) \in \mathrm{d}x)/\mathrm{d}x \\ &= \int_{r\in(t,1)} \int_{z>1} \mathbf{1}(r \le x - rz < 1)\,(x - rz)^{-1}\ \mathbb{P}(Z(t/r) \in \mathrm{d}z)\,\mathrm{d}r. \end{aligned} \tag{3.6}$$

If $z > 1$ and $r \le x - rz$, then $(x/r) - 1 \ge z > 1$ and so $r < x/2$. Therefore we find

$$c_t(x) \le \int_t^{\min\{1,\,x/2\}} \int_{\max\{1,\,(x-1)/r\}<z\le(x/r)-1} (x - rz)^{-1}\ \mathbb{P}(Z(t/r) \in \mathrm{d}z)\,\mathrm{d}r.$$

The integrand (including the implicit indicator function) in the inner integral is an increasing function of z over the interval $(-\infty, (x/r) - 1]$, with value r^{-1} at the upper endpoint of the interval. We can thus extend it to a nondecreasing function $\phi \equiv \phi_{x,r}$ with domain $\mathbb{R}$ by setting $\phi(z) = r^{-1}$ for $z > (x/r) - 1$. It then follows that

$$\begin{aligned} c_t(x) &\le \int_0^{\min\{1,\,x/2\}} \left[\int_{\max\{1,\,(x-1)/r\}<z\le(x/r)-1} (x - rz)^{-1}\ \mathbb{P}(V \in \mathrm{d}z) \right. \\ &\qquad\qquad \left. + \, r^{-1}\ \mathbb{P}\left(V > \frac{x}{r} - 1\right)\right] \mathrm{d}r \\ &\le \int_0^{\min\{1,\,x/2\}} \int_{\max\{1,\,(x-1)/r\}<z\le(x/r)-1} (x - rz)^{-1}\ \mathbb{P}(V \in \mathrm{d}z)\,\mathrm{d}r \\ &\qquad\qquad + \int_0^{x/2} r^{-1}\ \mathbb{P}\left(V > \frac{x}{r} - 1\right)\,\mathrm{d}r. \end{aligned} \tag{3.7}$$

By the change of variables $v = (x/r) - 1$, the second term in (3.7) equals

$$\int_1^\infty (v+1)^{-1}\,\mathbb{P}(V > v)\,\mathrm{d}v \le \frac{1}{2}\int_0^\infty \mathbb{P}(V > v)\,\mathrm{d}v = \frac{1}{2}\,\mathbb{E}\,V = 2.$$

Comparing the integrals $c_t(x)$ at (3.6) and the first term in (3.7), we see that the only constraint that has been discarded is $r > t$. We therefore see by the same argument that produces (3.6) that the first term in (3.7) is the value of the density for $W := U_1(1 + U_2 V)$ at x, where U_1, U_2, and V are independent and U_1 and U_2 are uniformly distributed on $(0, 1)$. Thus to obtain the desired uniform boundedness of f_t we need only show that W has a bounded density. For that, it suffices to observe that the conditional density of W given U_2 and V is bounded above by 1 (for any values of U_2 and V), and so the unconditional density is bounded by 1. We conclude that $c_t(x) \le 3$, and this completes the proof. □

Remark 3.7. Based on simulation results, we conjecture that the density functions f_t are uniformly bounded by $e^{-\gamma}$ (the sup-norm of the right-continuous Dickman density f_0) for $0 \le t \le 1$.

4 Uniform continuity of the density function f_t

From the previous section, we know that for $0 < t < 1$ in the case $0 < l < t < r < 1$ (i.e., the case $0 < \rho < \infty$) the function $f_{l,r}$ is càdlàg (that is, a right continuous function with left limits) and bounded above by $b(l, r)$, where the corresponding contribution $\mathbb{E}[b(L_3, R_3);\, 0 < \rho(L_3, R_3) < \infty]$ is finite. Applying the dominated convergence theorem, we conclude that the contribution to f_t from this case is also càdlàg.

For the cases $0 = l < t < r < 1$ ($\rho = 0$) and $0 < l < t < r = 1$ ($\rho = \infty$), the functions $f_{0,r}$ and $f_{l,1}$ are both continuous on the real line. In this section, we will build bounds $b_t(l,r)$ (note that these bounds depend on t) for these two cases (Lemma 4.1 for $\rho = 0$ and Lemma 4.2 for $\rho = \infty$) in similar fashion as for Lemma 3.6 such that both $\mathbb{E}[b_t(L_3, R_3)\,; \rho(L_3, R_3) = 0]$ and $\mathbb{E}[b_t(L_3, R_3); \rho(L_3, R_3) = \infty]$ are finite for any $0 < t < 1$. Given these bounds, we can apply the dominated convergence theorem to conclude that the density f_t is càdlàg. Later, this result will be sharpened substantially in Theorem 4.4.

Let $\alpha \approx 3.59112$ be the unique real solution of $1 + x - x \ln x = 0$ and let $\beta := 1/\alpha \approx 0.27846$. Define

$$b_1(r) := \frac{2}{\ln r^{-1}} \frac{1}{1+r} \quad \text{and} \quad b_2(r) := \frac{2}{(\ln r^{-1})^2} \frac{1}{r} \beta.$$

Lemma 4.1. *Suppose $\rho = 0$, i.e., $0 = l_3 < t < r_3 < 1$. If $t \geq \beta$, then the optimal constant upper bound on f_{l_3,r_3} is*

$$b_t(l_3, r_3) = b_1(r_3),$$

with corresponding contribution

$$\mathbb{E}[b_t(L_3(t), R_3(t));\, \rho\,(L_3(t), R_3(t)) = 0] = \int_t^1 \frac{\ln r^{-1}}{1+r}\,\mathrm{d}r \leq \int_\beta^1 \frac{\ln r^{-1}}{1+r}\,\mathrm{d}r < \frac{1}{4}$$

to $\mathbb{E}\, b_t(L_3(t), R_3(t))$. If $t < \beta$, then the optimal constant upper bound on f_{l_3,r_3} is the continuous function

$$b_t(l_3, r_3) = b_1(r_3)\mathbb{1}(\beta \leq r_3 < 1) + b_2(r_3)\mathbb{1}(t < r_3 < \beta)$$

of $r_3 \in (t, 1]$, with corresponding contribution

$$\begin{aligned}\mathbb{E}[b_t\,(L_3(t), R_3(t))\,;\, \rho\,(L_3(t), R_3(t)) = 0] &= \int_\beta^1 \frac{\ln r^{-1}}{1+r}\,\mathrm{d}r + \beta(\ln\beta - \ln t) \\ &< \frac{1}{4} + \beta(\ln\beta - \ln t).\end{aligned}$$

Lemma 4.2. *Suppose $\rho = \infty$, i.e., $0 < l_3 < t < r_3 = 1$. If $t \leq 1 - \beta$, then the optimal constant upper bound on f_{l_3,r_3} is*

$$b_t(l_3, r_3) = b_1(1 - l_3),$$

with corresponding contribution

$$\mathbb{E}[b_t\,(L_3(t), R_3(t))\,;\, \rho\,(L_3(t), R_3(t)) = 0] = \int_{1-t}^1 \frac{\ln r^{-1}}{1+r}\,\mathrm{d}r \leq \int_\beta^1 \frac{\ln r^{-1}}{1+r}\,\mathrm{d}r < \frac{1}{4}.$$

If $t > 1 - \beta$, then the optimal constant upper bound on f_{l_3,r_3} is the continuous function

$$b_t(l_3, r_3) = b_1(1 - l_3)\mathbb{1}(0 < l_3 \leq 1 - \beta) + b_2(1 - l_3)\mathbb{1}(1 - \beta < l_3 < t)$$

of $l_3 \in [0, t)$, with corresponding contribution

$$\begin{aligned}\mathbb{E}[b_t\,(L_3(t), R_3(t))\,;\, \rho\,(L_3(t), R_3(t)) = 0] &= \int_\beta^1 \frac{\ln r^{-1}}{1+r}\,\mathrm{d}r + \beta[\ln\beta - \ln(1 - r)] \\ &< \frac{1}{4} + \beta[\ln\beta - \ln(1 - t)].\end{aligned}$$

We prove Lemma 4.1 here, and Lemma 4.2 follows similarly or by symmetry.

Proof of Lemma 4.1. When $\rho = 0$, we have $l_3 = 0$, and the conditional density function is f_{0,r_3} in (2.6). The expression in square brackets at (2.6) is continuous and unimodal in x, with maximum value at $x = 1 + r_3$. Because the factor $1/x$ is decreasing, it follows that the maximum value of $f_{0,r_3}(x)$ is the maximum of

$$\frac{2}{\left(\ln r_3^{-1}\right)^2} \frac{1}{x} \ln\left(\frac{x - r_3}{r_3}\right)$$

over $x \in [2r_3, 1 + r_3]$, i.e., the maximum of

$$\frac{2}{r_3\left(\ln r_3^{-1}\right)^2} \frac{\ln y}{1 + y}$$

over $y \in [1, 1/r_3]$. A simple calculation shows that the displayed expression is strictly increasing for $y \in [1, \alpha]$ and strictly decreasing for $y \in [\alpha, \infty)$. Thus the maximum for $y \in [1, 1/r_3]$ is achieved at $y = \alpha$ if $\alpha \le 1/r_3$ and at $y = 1/r_3$ if $\alpha \ge 1/r_3$. Equivalently, $f_{0,r_3}(x)$ is maximized at $x = r_3(\alpha + 1)$ if $r_3 \le \beta$ and at $x = 1 + r_3$ if $r_3 \ge \beta$. The claims about the optimal constant upper bound on f_{l_3,r_3} and the contribution to $\mathbb{E}\, b_t(L_3(t), R_3(t))$ now follow readily. $\square$

Remark 4.3. If we are not concerned about finding the best possible upper bound, then for the case $\rho = 0$ we can choose $b_t(l, r) := b_2(r)$; for the case $\rho = \infty$, we can choose $b_t(l, r) := b_2(1 - l)$. These two bounds still get us the desired finiteness of the contributions to $\mathbb{E}\, b_t(L_3, R_3)$ for any $0 < t < 1$.

Theorem 4.4. *For $0 < t < 1$, the density function $f_t : \mathbb{R} \to [0, \infty)$ is uniformly continuous.*

Proof. Fix $0 < t < 1$. By the dominated convergence theorem, the contributions to $f_t(x)$ from $0 = l < t < r < 1$ and $0 < l < t < r = 1$, namely, the functions

$$c_0(x) := \int_{r,y} f_{0,r}(x - y)\ \mathbb{P}(L_3(t) = 0,\, R_3(t) \in \mathrm{d}r,\, Y \in \mathrm{d}y)$$

and

$$c_1(x) := \int_{l,y} f_{l,1}(x - y)\ \mathbb{P}(L_3(t) \in \mathrm{d}l,\, R_3(t) = 1,\, Y \in \mathrm{d}y),$$

are continuous for $x \in \mathbb{R}$. Further, according to (2.14) and (2.16), the contribution from $0 < l < t < r < 1$ is $\sum_{i=1}^{6} c^{(i)}(x)$, where we define

$$c^{(i)}(x) := \int_{l,r,y} f_{l,r}^{(i)}(x - y)\ \mathbb{P}(Y \in \mathrm{d}y \mid (L_3(t), R_3(t)) = (l, r))\, \mathrm{d}l\, \mathrm{d}r.$$

It is easy to see that all the functions c_0, c_1, and $c^{(i)}$ for $i = 1, \dots, 6$ vanish for arguments $x \le 0$. To prove the uniform continuity of $f_t(x)$ for $x \in \mathbb{R}$, it thus suffices to show that each of the six functions $c^{(i)}$ for $i = 1, \dots, 6$ is continuous on the real line and that each of the eight functions c_0, c_1, and $c^{(i)}$ for $i = 1, \dots, 6$ vanishes in the limit as argument $x \to \infty$.

Fix $i \in \{1, \dots, 6\}$. Continuity of $c^{(i)}$ holds since $f_{l,r}^{(i)}$ is bounded by $b(l, r)$ defined at (3.3) and is continuous except at the boundary of its support. To illustrate, consider, for example, $i = 3$. For each fixed $0 < l < t < r < 1$ and $x \in \mathbb{R}$, we have $f_{l,r}^{(3)}(x + h - y) \to f_{l,r}^{(3)}(x - y)$ as $h \to 0$ for all but two exceptional values of y, namely, $y = x - (1 + r - 2l)$ and $y = x - (1 - r)$. From the discussion following (2.1) and from Theorem 2.2, we know that the conditional law of Y given $(L_3(t), R_3(t)) = (l, r)$ has a density with respect to Lebesgue measure, and hence the set of two exceptional points has zero measure under this law. We conclude from the dominated convergence theorem that

$$\int_y f_{l,r}^{(i)}(x - y)\ \mathbb{P}(Y \in \mathrm{d}y \mid (L_3(t), R_3(t)) = (l, r))$$

is continuous in $x \in \mathbb{R}$. It now follows by another application of the dominated convergence theorem that $c^{(i)}$ is continuous on the real line.

Since the eight functions $f_{0,r}$, $f_{l,1}$, and $f_{l,r}^{(i)}$ for $i = 1, \ldots 6$ all vanish for all sufficiently large arguments, another application of the dominated convergence theorem shows that $c_0(x)$, $c_1(x)$, and $c^{(i)}(x)$ for $i = 1, \ldots, 6$ all vanish in the limit as $x \to \infty$. This completes the proof. □

Remark 4.5. For any $0 < t < 1$, by the fact that

$$J(t) \geq R_1(t) - L_1(t) \geq \min(t, 1-t),$$

we have $\mathbb{P}(J(t) < \min(t, 1-t)) = 0$ and thus $f_t(\min(t, (1-t))) = 0$ by Theorem 4.4. This is a somewhat surprising result since we know that the right-continuous Dickman density f_0 satisfies $f_0(0) = e^{-\gamma} > 0$.

Remark 4.6. Since f_0 is both (uniformly) continuous on and piecewise differentiable on $(0, \infty)$, it might be natural to conjecture that the densities f_t for $0 < t < 1$ are also piecewise differentiable.

Later, in Theorem 7.4, we prove that the densities f_t are Lipschitz continuous, which implies that each of them is almost everywhere differentiable.

5 Integral equation for the density functions

In this section we prove that for $0 \leq t < 1$ and $x \in \mathbb{R}$, the density function $f_t(x)$ is jointly Borel measurable in (t, x). By symmetry, we can conclude that $f_t(x)$ is jointly Borel measurable in (t, x) for $0 \leq t \leq 1$. We then use this result to establish an integral equation for the densities.

Let F_t denote the distribution function for $J(t)$. Because F_t is right continuous, it is Borel measurable (for each t).

Lemma 5.1. *For each positive integer n, the mapping*

$$(t, x) \mapsto F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) \quad (0 \leq t < 1,\ x \in \mathbb{R})$$

is Borel measurable.

Proof. Note that

$$F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) = \sum_{j=1}^{n} \mathbb{1}\left(\frac{j-1}{n} \leq t < \frac{j}{n}\right) F_{\frac{j}{n}}(x).$$

Each term is the product of a Borel measurable function of t and a Borel measurable function of x and so is a Borel measurable of (t, x). The same is then true of the sum. □

Lemma 5.2. *For each $0 \leq t < 1$ and $x \in \mathbb{R}$, as $n \to \infty$ we have*

$$F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) \to F_t(x).$$

Proof. We reference Grübel and Rösler [19], who construct a process $J = (J(t))_{0 \leq t \leq 1}$ with $J(t)$ having distribution function F_t for each t and with right continuous sample paths. It follows (for each $t \in [0, 1)$) that F_u converges weakly to F_t as $u \downarrow t$. But we know that F_t is a continuous (and even continuously differentiable) function, so for each $x \in \mathbb{R}$ we have $F_u(x) \to F_t(x)$ as $u \downarrow t$. The result follows. □

Proposition 5.3. *The mapping*

$$(t, x) \mapsto F_t(x) \quad (0 \leq t < 1,\ x \in \mathbb{R})$$

is Borel measurable.

Proof. According to Lemmas 5.1–5.2, this mapping is the pointwise limit as $n \to \infty$ of the Borel measurable mappings in Lemma 5.2. □

Let f_t denote the continuous density for F_t, as in Theorem 4.4.

Theorem 5.4. *The mapping*

$$(t, x) \mapsto f_t(x) \quad (0 \le t < 1,\ x \in \mathbb{R})$$

is Borel measurable.

Proof. By the fundamental theorem of integral calculus, $f_t = F_t'$. The mapping in question is thus the (sequential) limit of difference quotients that are Borel measurable by Proposition 5.3 and hence is Borel measurable. □

Now we are ready to derive integral equations. We start with an integral equation for the distribution functions F_t.

Proposition 5.5. *The distribution functions (F_t) satisfy the following integral equation for $0 \le t \le 1$ and $x \in \mathbb{R}$:*

$$F_t(x) = \int_{l \in (0,t)} F_{\frac{t-l}{1-l}}\left(\frac{x}{1-l} - 1\right) \mathrm{d}l + \int_{r \in (t,1)} F_{\frac{t}{r}}\left(\frac{x}{r} - 1\right) \mathrm{d}r. \tag{5.1}$$

Proof. This follows by conditioning on the value of $(L_1(t), R_1(t))$. Observe that each of the two integrands is (by Proposition 5.3 for $t \notin \{0,1\}$ and by right continuity of F_0 and F_1 for $t \in \{0,1\}$) indeed [for fixed (t,x)] a Borel measurable function of the integrating variable. □

Remark 5.6. It follows from (i) the changes of variables from l to $v = (t-l)/(1-l)$ in the first integral in (5.1) and from r to $v = t/r$ in the second integral, (ii) the joint continuity of $f_t(x)$ in (t,x) established later in Corollary 7.12, and (iii) Leibniz's formula that $F_t(x)$ is differentiable with respect to $t \in (0,1)$ for each fixed $x \in \mathbb{R}$.

Integral equation (5.1) for the distribution functions F_t immediately leads us to an integral equation for the density functions f_t.

Proposition 5.7. *The continuous density functions (f_t) satisfy the following integral equation for $0 < t < 1$ and $x \in \mathbb{R}$:*

$$f_t(x) = \int_{l \in (0,t)} (1-l)^{-1} f_{\frac{t-l}{1-l}}\left(\frac{x}{1-l} - 1\right) \mathrm{d}l + \int_{r \in (t,1)} r^{-1} f_{\frac{t}{r}}\left(\frac{x}{r} - 1\right) \mathrm{d}r.$$

Proof. Fix $t \in (0,1)$. Differentiate (5.1) with respect to x. It is easily proved by an argument applying the dominated convergence theorem to difference quotients and the mean value theorem that we can differentiate under the integral signs in (5.1) provided that

$$\int_{l \in (0,t)} (1-l)^{-1} f^*_{\frac{t-l}{1-l}} \,\mathrm{d}l + \int_{r \in (t,1)} r^{-1} f^*_{\frac{t}{r}} \,\mathrm{d}r \tag{5.2}$$

is finite, where f_t^* denotes any upper bound on $f_t(x)$ as x varies over $\mathbb{R}$. By Theorem 3.1 we can simply choose $f_t^* = 10$. Then (5.2) equals 10 times

$$-\ln(1-t) - \ln t,$$

which is finite. □

In the next proposition, we provide an integral equation based on the formula for f_t in (2.2); this integral equation will be useful in the next section. Recall that $Y(t) = \sum_{k=3}^{\infty} \Delta_k(t)$. Using (2.1), the conditional distribution of $Y(t)/(r_3 - l_3)$ given $(L_3, R_3) = (l_3, r_3)$ is the (unconditional) distribution of $Z(\frac{t-l_3}{r_3-l_3}) = 1 + J(\frac{t-l_3}{r_3-l_3})$. Apply Theorem 2.2 on $Z(\frac{t-l_3}{r_3-l_3})$ leads us to an integral equation for the density function of $J(t)$.

Proposition 5.8. *The continuous density functions f_t for the random variables $J(t) = Z(t) - 1$ satisfy the integral equation*

$$f_t(x) = \int \mathbb{P}((L_3(t), R_3(t)) \in \mathrm{d}(l_3, r_3)) \cdot h_t(x \mid l_3, r_3)$$

for $x \geq 0$, where

$$h_t(x \mid l_3, r_3) = \int f_{l_3,r_3}(x - y)\,(r_3 - l_3)^{-1}\, f_{\frac{t-l_3}{r_3-l_3}}\left(\frac{y}{r_3 - l_3} - 1\right) \mathrm{d}y.$$

6 Right-tail behavior of the density function

In this section we will prove, *uniformly* for $0 < t < 1$, that the continuous density functions f_t enjoy the same superexponential decay bound as Grübel and Rösler [19, Theorem 9] proved for the survival functions $1 - F_t$. By a separate and easier argument, one could include the cases $t = 0, 1$. Let m_t denote the moment generating function of $Z(t)$ and recall that m denotes the moment generating function of V at (3.5). By Lemma 3.3, the random variables $Z(t)$, $0 \leq t \leq 1$, are stochastically dominated by V. As a consequence, if $\theta \geq 0$, then

$$m_t(\theta) \leq m(\theta) < \infty$$

for every $t \in (0, 1)$.

Theorem 6.1. *Uniformly for $t \in (0,1)$, the continuous `QuickQuant` density functions $f_t(x)$ enjoy superexponential decay when x is large. More precisely, for any $\theta > 0$ we have*

$$f_t(x) < 4\theta^{-1} e^{2\theta} m(\theta) e^{-\theta x}$$

for $x \geq 3$, where m is the moment generating function of the random variable V at (3.5).

Proof. Our starting point is the following equation from the discussion preceding Proposition 5.8:

$$\begin{aligned} f_t(x) &= \int_{l,r} \mathbb{P}((L_3, R_3) \in (\mathrm{d}l, \mathrm{d}r)) \int_y f_{l,r}(x - y)\ \mathbb{P}\left((r - l)Z\left(\frac{t - l}{r - l}\right) \in \mathrm{d}y\right) \qquad (6.1) \\ &= \int_{l,r} \mathbb{P}((L_3, R_3) \in (\mathrm{d}l, \mathrm{d}r)) \int_z f_{l,r}(x - (r - l)z)\ \mathbb{P}\left(Z\left(\frac{t - l}{r - l}\right) \in \mathrm{d}z\right). \end{aligned}$$

By Lemma 3.3, for any $\theta \in \mathbb{R}$ we can obtain a probability measure $\mu_{t,\theta}(\mathrm{d}z) := m_t(\theta)^{-1} e^{\theta z}\, \mathbb{P}(Z(t) \in \mathrm{d}z)$ by exponential tilting. Since $m_t(\theta) \leq m(\theta) < \infty$ for every $\theta \geq 0$ and $t \in (0, 1)$, we can rewrite and bound (6.1) as follows (for any $\theta \geq 0$):

$$\begin{aligned} f_t(x) &= \int_{l,r} \mathbb{P}((L_3, R_3) \in (\mathrm{d}l, \mathrm{d}r))\, m_{\frac{t-l}{r-l}}(\theta) \int_z e^{-\theta z}\, f_{l,r}(x - (r - l)z)\, \mu_{\frac{t-l}{r-l},\theta}(\mathrm{d}z) \\ &\leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (\mathrm{d}l, \mathrm{d}r)) \int_z e^{-\theta z}\, f_{l,r}(x - (r - l)z)\, \mu_{\frac{t-l}{r-l},\theta}(\mathrm{d}z). \end{aligned}$$

Recall that $f_{l,r}(x)$ is bounded above by $b_t(l,r)$ (Lemmas 3.2 and 4.1–4.2) and vanishes for $x \geq 2$. Therefore, if $\theta \geq 0$ then

$$\begin{aligned} f_t(x) &\leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (\mathrm{d}l, \mathrm{d}r))\, b_t(l,r) \int_{z > \frac{x-2}{r-l}} e^{-\theta z}\, \mu_{\frac{t-l}{r-l},\theta}(\mathrm{d}z) \\ &\leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (\mathrm{d}l, \mathrm{d}r))\, b_t(l,r) \exp\left(-\theta\, \frac{x-2}{r-l}\right). \end{aligned} \tag{6.2}$$

Suppose $x \geq 3$ and $\theta > 0$. We now consider in turn the contribution to (6.2) for $l = 0$, for $r = 1$, and for $0 < l < t < r < 1$. For $l = 0$, the contribution is $m(\theta)$ times the following:

$$\begin{aligned} \int_t^1 r^{-1} \beta \exp[-\theta r^{-1}(x-2)]\, \mathrm{d}r &\leq \beta \int_0^1 r^{-2} \exp[-\theta r^{-1}(x-2)]\, \mathrm{d}r \\ &= \beta\, [\theta(x-2)]^{-1} \exp[-\theta(x-2)] \leq \beta\theta^{-1} e^{2\theta} e^{-\theta x}. \end{aligned}$$

Similarly (or symmetrically), the contribution for $r = 1$ is bounded by the same $\beta\theta^{-1}e^{2\theta}m(\theta)e^{-\theta x}$. For $0 < l < t < r < 1$, by symmetry we may without loss of generality suppose that $0 < t \leq 1/2$, and then the contribution is $\frac{3}{2}m(\theta)$ times the following:

$$\begin{aligned} &\int_0^t \int_t^1 \left[\frac{1}{r(r-l)} + \frac{1}{(1-l)(r-l)}\right] \exp\left(-\theta\, \frac{x-2}{r-l}\right) \mathrm{d}r\, \mathrm{d}l \\ &= \int_0^t \int_{t-l}^{1-l} \left[\frac{1}{(s+l)s} + \frac{1}{(1-l)s}\right] \exp[-\theta s^{-1}(x-2)]\, \mathrm{d}s\, \mathrm{d}l \\ &= \int_0^t \int_{t-l}^{1-l} (1-l)^{-1}(s+l)^{-1}(1+s)s^{-1} \exp[-\theta s^{-1}(x-2)]\, \mathrm{d}s\, \mathrm{d}l \\ &\leq 4 \int_0^t \int_{t-l}^{1-l} s^{-2} \exp[-\theta s^{-1}(x-2)]\, \mathrm{d}s\, \mathrm{d}l \\ &\leq 4 \int_0^{1/2} \int_0^1 s^{-2} \exp[-\theta s^{-1}(x-2)]\, \mathrm{d}s\, \mathrm{d}l \\ &= 2[\theta(x-2)]^{-1} \exp[-\theta(x-2)] \leq 2\theta^{-1} e^{2\theta} e^{-\theta x}. \end{aligned}$$

Summing all the contributions, we find

$$f_t(x) \leq (3 + 2\beta)\theta^{-1} e^{2\theta} m(\theta) e^{-\theta x} < 4\,\theta^{-1} e^{2\theta} m(\theta) e^{-\theta x}, \tag{6.3}$$

for any $0 < t < 1$, $x \geq 3$, and $\theta > 0$, demonstrating the uniform superexponential decay. $\square$

Remark 6.2. Since f_t is uniformly bounded by 10 by Theorem 3.1, for any $\theta > 0$, by choosing the coefficient $C_\theta := \max\{10e^{3\theta}, 4\theta^{-1}e^{2\theta}m(\theta)\}$, we can extend the superexponential bound on $f_t(x)$ in Theorem 6.1 for $x \geq 3$ to $x \in \mathbb{R}$ as

$$f_t(x) \leq C_\theta e^{-\theta x} \text{ for } x \in \mathbb{R} \text{ and } 0 < t < 1. \tag{6.4}$$

Note that this bound is not informative for $x \leq \min\{t, 1-t\}$ since we know $f_t(x) = 0$ for such x by Theorem 4.4, but it will simplify our proof of Theorem 7.4.

7 Positivity and Lipschitz continuity of the continuous density functions

In this section we establish several properties of the continuous density function f_t. We prove that $f_t(x)$ is positive for every $x > \min\{t, 1-t\}$ (Theorem 7.1), Lipschitz continuous for $x \in \mathbb{R}$ (Theorem 7.4), and jointly continuous for $(t,x) \in (0,1) \times \mathbb{R}$ (Corollary 7.12).

7.1 Positivity

Theorem 7.1. *For any* $0 < t < 1$*, the continuous density* f_t *satisfies*

$$f_t(x) > 0 \text{ if and only if } x > \min\{t, 1-t\}.$$

We already know that $f_t(x) = 0$ if $x \le \min\{t, 1-t\}$, so we need only prove the "if" assertion. Our starting point for the proof is the following lemma. Recall from Chung [4, Exercise 1.6] that a point x is said to belong to the support of a distribution function F if for every $\epsilon > 0$ we have

$$F(x+\epsilon) - F(x-\epsilon) > 0. \tag{7.1}$$

Note that to prove that x is in the support of F we may choose any $\epsilon_0(x) > 0$ and establish (7.1) for all $\epsilon \in (0, \epsilon_0(x))$.

Lemma 7.2. *For any* $0 < t < 1$*, the support of* F_t *is* $[\min\{t, 1-t\}, \infty)$*.*

Proof. Clearly the support of F_t is contained in $[\min\{t, 1-t\}, \infty)$, so we need only establish the reverse containment. Since $F_t = F_{1-t}$ by symmetry, we may fix $t \le 1/2$. Also fixing $x \ge t$, write

$$x = t + K + b$$

where $K \ge 0$ is an integer and $b \in [0, 1)$. We will show that x belongs to the support of F_t. Let

$$A := \bigcap_{k=1}^{K} \{1 - k\epsilon < R_k < 1 - (k-1)\epsilon\}.$$

We break our analysis into four cases: (i) $t < b < 1$, (ii) $b = t$, (iii) $0 < b < t$, and (iv) $b = 0$.

(i) $t < b < 1$. Let

$$B := \{b < R_{K+1} < b + \epsilon\} \bigcap \{t < R_{K+2} < t + \epsilon\} \bigcap \{t - \epsilon < L_{K+3} < t\}$$

and

$$C := \left\{ 0 \le \sum_{k=K+4}^{\infty} \Delta_k < 6\epsilon \right\}. \tag{7.2}$$

Upon observing that for $\delta_1, \delta_2 \in (0, \epsilon)$ we have by use of Markov's inequality that

$$\begin{aligned}
&\mathbb{P}(C \mid (L_{K+3}, R_{K+3}) = (t - \delta_1, t + \delta_2)) \\
&= \mathbb{P}\left((\delta_1 + \delta_2) J\left(\frac{\delta_1}{\delta_1 + \delta_2} \right) < 6\epsilon \right) \ge \mathbb{P}\left(J\left(\frac{\delta_1}{\delta_1 + \delta_2} \right) < 3 \right) \\
&\ge 1 - \frac{1}{3} \mathbb{E}\, J\left(\frac{\delta_1}{\delta_1 + \delta_2} \right) \ge 1 - \frac{1}{3}\left[1 + 2H\left(\frac{1}{2} \right) \right] > 0.2 > 0.
\end{aligned} \tag{7.3}$$

We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ. But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + b + t = x - \binom{K+1}{2}\epsilon$$

and

$$J(t) < \sum_{k=1}^{K} [1 - (k-1)\epsilon] + (b + \epsilon) + (t + \epsilon) + 2\epsilon + 6\epsilon \le x + 10\epsilon.$$

We conclude that x is in the support of F_t.

(ii) $b = t$. Let

$$B := \{t < R_{K+2} < R_{K+1} < t + \epsilon\} \bigcap \{t - \epsilon < L_{K+3} < t\}$$

and define C by (7.2). We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ. But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + t + t = x - \binom{K+1}{2}\epsilon$$

and

$$J(t) < \sum_{k=1}^{K} [1 - (k-1)\epsilon] + 2(t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 10\epsilon.$$

We conclude that x is in the support of F_t.

(iii) $0 < b < t$. Let

$$B := \{t < R_{K+1} < t + \epsilon\} \bigcap \{t - b - \epsilon < L_{K+2} < t - b\} \bigcap \{t - \epsilon < L_{K+3} < t\}$$

and define C by (7.2). We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ. But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + t + b = x - \binom{K+1}{2}\epsilon$$

and

$$J(t) < \sum_{k=1}^{K} [1 - (k-1)\epsilon] + (t + \epsilon) + (b + 2\epsilon) + 2\epsilon + 6\epsilon \leq x + 11\epsilon.$$

We conclude that x is in the support of F_t.

(iv) $b = 0$. Let

$$B := \{t < R_{K+1} < t + \epsilon\} \bigcap \{t - \epsilon < L_{K+2} < t\}$$

and define C by (7.2), but with $K + 4$ there changed to $K + 3$. We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ. But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + t = x - \binom{K+1}{2}\epsilon$$

and

$$J(t) < \sum_{k=1}^{K} [1 - (k-1)\epsilon] + (t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 9\epsilon.$$

We conclude that x is in the support of F_t. □

We next use (3.6) together with Lemma 7.2 to establish Theorem 7.1 in a special case.

Lemma 7.3. *For any $0 < t < 1$, the continuous density f_t satisfies*

$$f_t(x) > 0 \text{ for all } x > 2\min\{t, 1 - t\}.$$

Proof. We may fix $t \le 1/2$ and $x > 2t$ and prove $f_t(x) > 0$. To do this, we first note from (3.6) that

$$\begin{aligned} f_t(x) &\ge c_t(x) = \mathbb{P}(L_2(t) = 0,\, J(t) \in \mathrm{d}x)/\mathrm{d}x \\ &= \int_{r \in (t,1)} \int \mathbf{1}(r \le x - rz < 1)\,(x - rz)^{-1}\; \mathbb{P}(Z(t/r) \in \mathrm{d}z)\,\mathrm{d}r \\ &\ge \int_{r \in (t,1)} \int_{(x-1)/r}^{(x/r)-1} \mathbb{P}(Z(t/r) \in \mathrm{d}z)\,\mathrm{d}r \\ &\ge \int_{r \in (t,1)} \mathbb{P}\left(\frac{x-1}{r} < Z\left(\frac{t}{r}\right) < \frac{x}{r} - 1\right)\mathrm{d}r. \end{aligned}$$

According to Lemma 7.2, for the integrand in this last integral to be positive, it is necessary and sufficient that $(x-1)/r < (x/r) - 1$ (equivalently, $r < 1$) and

$$\tfrac{x}{r} - 1 > 1 + \min\{\tfrac{t}{r}, 1 - \tfrac{t}{r}\}$$

[for which it is sufficient that $r < (x+t)/3$]. Thus

$$f_t(x) \ge \int_{r \in (t, \min\{(x+t)/3, 1\})} \mathbb{P}\left(\frac{x-1}{r} < Z\left(\frac{t}{r}\right) < \frac{x}{r} - 1\right)\mathrm{d}r > 0$$

because (recalling $x > 2t$) the integrand here is positive over the nondegenerate interval of integration. $\square$

Finally, we use a different contribution to $f_t(x)$ together with Lemma 7.3 to establish Theorem 7.1.

Proof of Theorem 7.1. We may fix $t \ge 1/2$ and $x > 1 - t$ and prove $f_t(x) > 0$. To do this, we first note that

$$\begin{aligned} f_t(x) \ge \int_{l \in (0,t)} \int_{r \in (t,1)} & \mathbb{P}(L_1(t) = L_2(t) \in \mathrm{d}l,\, R_2(t) \in \mathrm{d}r) \\ & \left[\mathbb{P}\left((r-l)J\left(\frac{t-l}{r-l}\right) \in \mathrm{d}x - [(1-l) + (r-l)]\right)/\mathrm{d}x\right] \\ = \int_{l \in (0,t)} \int_{r \in (t,1)} & (1-l)^{-1}(r-l)^{-1} f_{\frac{t-l}{r-l}}\left(\frac{x - (1 + r - 2l)}{r - l}\right)\mathrm{d}r\,\mathrm{d}l. \end{aligned}$$

According to Lemma 7.3, for the integrand in this double integral to be positive, it is sufficient that

$$\frac{x - (1 + r - 2l)}{r - l} > 2\min\left\{\frac{t-l}{r-l}, \frac{r-t}{r-l}\right\},$$

or, equivalently,

$$x > \min\{1 + 2t + r - 4l, 1 - 2t + 3r - 2l\}.$$

This strict inequality is true (because $x > 1 - t$) when $l = t$ and $r = t$ and so, for sufficiently small $\epsilon > 0$ is true for $l \in (t - \epsilon, t)$ and $r \in (t, t + \epsilon)$. Thus

$$f_t(x) \ge \int_{l \in (t-\epsilon,t)} \int_{r \in (t,t+\epsilon)} (1-l)^{-1}(r-l)^{-1} f_{\frac{t-l}{r-l}}\left(\frac{x - (1 + r - 2l)}{r - l}\right)\mathrm{d}r\,\mathrm{d}l > 0$$

because the integrand here is positive over the fully two-dimensional rectangular region of integration. $\square$

7.2 Lipschitz continuity

We now prove that, for each $0 < t < 1$, the density function f_t is Lipschitz continuous, which is a result stronger than Theorem 4.4.

Theorem 7.4. *For each $0 < t < 1$, the density function f_t is Lipschitz continuous.*

Remark 7.5. That is, there exists a constant $\Lambda_t \in (0, \infty)$ such that for any $x, z \in \mathbb{R}$, we have $|f_t(z) - f_t(x)| \le \Lambda_t |z - x|$. The proof of Theorem 7.4 will reveal that one can take $\Lambda_t = \lambda[t^{-1} \ln t^{-1} + (1-t)^{-1} \ln((1-t)^{-1})]$ for some constant $\lambda < \infty$. Thus the densities f_t are in fact uniformly Lipschitz continuous for t in any compact subinterval of $(0, 1)$.

We can numerically bound the constant λ in the preceding paragraph once we bound the moment generating function m in Lemma 3.3 used in the proof of Theorem 7.4. [See, for example, Term (i) in Subcase 1(a) in the proof of Lemma 7.7.] For $\theta = 1/2$, the choice $\epsilon = 1/4$ in [19, proof of Theorem 9] yields

$$m(1/4) < 247.55.$$

Using this bound on $m(1/4)$, we find from our proof of Theorem 7.4 that $\lambda < 64000$; the "true" value

$$\sup_{t \in (0,1)} \Lambda_t / [t^{-1} \ln t^{-1} + (1-t)^{-1} \ln((1-t)^{-1})]$$

of λ might be considerably smaller than our bound.

We break the proof of Theorem 7.4 into two lemmas. Lemma 7.6 deals with the contribution to f_t from the disjoint-union event $\{0 = L_3(t) < t < R_3(t) < 1\} \cup \{0 < L_3(t) < t < R_3(t) = 1\}$ while Lemma 7.7 deals with the contribution from the event $\{0 < L_3(t) < t < R_3(t) < 1\}$.

Lemma 7.6. *For each $0 < t < 1$, the contribution to f_t from the event $\{0 = L_3(t) < t < R_3(t) < 1\} \cup \{0 < L_3(t) < t < R_3(t) = 1\}$ is Lipschitz continuous.*

Proof. Fix $0 < t < 1$. By symmetry, we need only consider the contribution to $f_t(x)$ from the event $\{0 = L_3(t) < t < R_3(t) < 1\}$. Recall that this contribution is

$$c_0(x) := \frac{1}{2} \int_{r,y} (\ln r)^2 \, f_{0,r}(x - y) \; \mathbb{P}(Y \in dy \,|\, L_3(t) = 0,\, R_3(t) = r) \, \mathrm{d}r,$$

and that the conditional probability in the integrand can be written as

$$\mathbb{P}(Y \in \, \mathrm{d}y \,|\, L_3(t) = 0,\, R_3(t) = r) = \frac{1}{r} \, f_{\frac{t}{r}} \left(\frac{y}{r} - 1 \right) \mathrm{d}y.$$

Let $z, x \in \mathbb{R}$ with $z > x$ and fixed $r \in (t, 1)$. Writing

$$d_r(x, z, y) := \frac{1}{2} (\ln r)^2 [f_{0,r}(z - y) - f_{0,r}(x - y)],$$

we are interested in bounding the absolute difference

$$|c_0(z) - c_0(x)| \le \int_{r,y} |d_r(x, z, y)| \, \frac{1}{r} \, f_{\frac{t}{r}} \left(\frac{y}{r} - 1 \right) \mathrm{d}y.$$

Case 1. $z - x \le 1 - r$. We bound $d_r(x, z, y)$ for y in each of the seven subintervals of the real line determined by the six partition points

$$x - 2 < z - 2 \le x - (1 + r) < z - (1 + r) \le x - 2r < z - 2r,$$

and then the contribution to our bound on $|c_0(z) - c_0(x)|$ from all y in that subinterval (and all r satisfying the restriction of Case 1). For the two subcases $y \le x - 2$ and $y > z - 2r$, we have $d_r(x, z, y) = 0$. We bound the five nontrivial subcases as follows.

Subcase 1(a). $x-2<y\le z-2$. We have

$$|d_r(x,z,y)| = \left|\frac{1}{x-y}\right| \ln\left(\frac{1}{x-y-1}\right) \le \frac{1}{1+r}\ln\frac{1}{r},$$

and the contribution to $|c_0(z)-c_0(x)|$ is bounded by

$$\int_{r=t}^{1} \frac{1}{r(1+r)}\left(\ln\frac{1}{r}\right)\int_{y=x-2}^{z-2} f_{\frac{t}{r}}\left(\frac{y}{r}-1\right)\mathrm{d}y\,\mathrm{d}r \le 10(z-x)\frac{1-t}{t(1+t)}\ln\frac{1}{t}$$

since $f_{t/r}$ is bounded by 10.

Subcase 1(b). $z-2<y\le x-(1+r)$. We have

$$\begin{aligned}
d_r(x,z,y) &= \frac{1}{z-y}\ln\left(\frac{1}{z-y-1}\right) - \frac{1}{x-y}\ln\left(\frac{1}{x-y-1}\right)\\
&= \frac{1}{z-y}\left[\ln\left(\frac{1}{z-y-1}\right) - \ln\left(\frac{1}{x-y-1}\right)\right]\\
&\quad + \left(\frac{1}{z-y}-\frac{1}{x-y}\right)\ln\left(\frac{1}{x-y-1}\right).
\end{aligned}$$

Observe that $z-y>x-y>1+r$ and that the function $\ln[1/(x-1)]$ is differentiable for $x>1$. We then use the mean value theorem to obtain

$$\begin{aligned}
|d_r(x,z,y)| &\le \frac{1}{1+r}\left|\ln\left(\frac{1}{z-y-1}\right) - \ln\left(\frac{1}{x-y-1}\right)\right| + \frac{(z-x)}{(1+r)^2}\ln\frac{1}{r}\\
&\le (z-x)\left[\frac{1}{r(1+r)} + \frac{1}{(1+r)^2}\ln\frac{1}{r}\right].
\end{aligned}$$

The contribution to $|c_0(z)-c_0(x)|$ is then bounded by

$$(z-x)\int_{r=t}^{1}\left[\frac{1}{r(1+r)}+\frac{1}{(1+r)^2}\ln\frac{1}{r}\right]\mathrm{d}r \le (z-x)\frac{1-t}{1+t}\left(\frac{1}{t}+\frac{1}{1+t}\ln\frac{1}{t}\right).$$

Subcase 1(c). $x-(1+r)<y\le z-(1+r)$. We have

$$\begin{aligned}
d_r(x,z,y) &= \frac{1}{z-y}\ln\left(\frac{1}{z-y-1}\right) - \frac{1}{x-y}\ln\left(\frac{x-y-r}{r}\right)\\
&= \frac{1}{z-y}\left[\ln\left(\frac{1}{z-y-1}\right) - \ln\left(\frac{x-y-r}{r}\right)\right]\\
&\quad + \left(\frac{1}{z-y}-\frac{1}{x-y}\right)\ln\left(\frac{x-y-r}{r}\right).
\end{aligned}$$

Using the inequalities $z-y\ge 1+r$ and $x-y>2r$, we have

$$|d_r(x,z,y)| = \frac{1}{1+r}\left|\ln\left(\frac{1}{z-y-1}\right) - \ln\left(\frac{x-y-r}{r}\right)\right| + \frac{z-x}{2r(1+r)}\ln\frac{1}{r}.$$

We can bound the absolute-value term here by

$$\begin{aligned}
&\left|\ln\frac{1}{z-y-1} - \ln\frac{1}{(1+r)-1}\right| + \left|\ln\frac{(1+r)-r}{r} - \ln\frac{x-y-r}{r}\right|\\
&\le \frac{1}{r}[z-y-(1+r)] + \frac{1}{r}[(1+r)-(x-y)] = (z-x)\frac{1}{r},
\end{aligned}$$

where the above inequality comes from two applications of the mean value theorem. The contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$(z-x)\int_{r=t}^{1}\left[\frac{1}{r(1+r)}+\frac{1}{2r(1+r)}\ln\frac{1}{r}\right]\,\mathrm{d}r \le (z-x)\frac{1-t}{t(1+t)}\left(1+\frac{1}{2}\ln\frac{1}{t}\right).$$

Subcase 1(d). $z-(1+r) < y \le x-2r$. We have

$$\begin{aligned} d_r(x,z,y) &= \frac{1}{z-y}\ln\left(\frac{z-y-r}{r}\right) - \frac{1}{x-y}\ln\left(\frac{x-y-r}{r}\right) \\ &= \frac{1}{z-y}\left[\ln\left(\frac{z-y-r}{r}\right) - \ln\left(\frac{x-y-r}{r}\right)\right] \\ &\quad + \left(\frac{1}{z-y}-\frac{1}{x-y}\right)\ln\left(\frac{x-y-r}{r}\right). \end{aligned}$$

Using the inequality $z-y > x-y \ge 2r$, we obtain

$$\begin{aligned} |d_r(x,z,y)| &\le \frac{1}{2r}[\ln(z-y-r)-\ln(x-y-r)] + \frac{z-x}{(2r)^2}\ln\frac{1}{r} \\ &\le (z-x)\left[\frac{1}{2r}\frac{1}{r}+\frac{1}{(2r)^2}\ln\frac{1}{r}\right] \end{aligned}$$

by the differentiability of $\ln(x-r)$ for $x > r$ and the mean value theorem. The contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$(z-x)\int_{r=t}^{1}\left[\frac{1}{2r}\frac{1}{r}+\frac{1}{(2r)^2}\ln\frac{1}{r}\right]\,\mathrm{d}r = (z-x)\,\frac{(1-t)+\ln(1/t)}{4t}.$$

Subcase 1(e). $x-2r < y \le z-2r$. Using the inequality $2r \le z-y < 1+r$, we have

$$|d_r(x,z,y)| = \frac{1}{z-y}\ln\left(\frac{z-y-r}{r}\right) \le \frac{1}{2r}\ln\frac{1}{r},$$

and the contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$\int_{r=t}^{1}\frac{1}{2r^2}\left(\ln\frac{1}{r}\right)\int_{y=x-2r}^{z-2r} f_{\frac{t}{r}}\left(\frac{y}{r}-1\right)\,\mathrm{d}y\,\mathrm{d}r \le 10(z-x)\frac{1-t}{2t}\ln\frac{1}{t}.$$

This completes the proof for Case 1.

Case 2. $z-x > 1-r$. We directly bound

$$|d_r(x,z,y)| \le \frac{1}{2}(\ln r)^2[f_{0,r}(z-y)+f_{0,r}(x-y)].$$

If $z-x \le 1-t$, use the bound in Remark 4.3; we can then bound the contribution to $|c_0(z) - c_0(x)|$ by

$$\int_{r=1-(z-x)}^{1}\frac{2\beta}{r}\,\mathrm{d}r \le (z-x)\frac{2\beta}{t}.$$

On the other hand, if $z-x > 1-t$, then we can bound the contribution to $|c_0(z) - c_0(x)|$ by

$$\frac{z-x}{1-t}\int_{r=t}^{1}\frac{2\beta}{r}\,\mathrm{d}r \le (z-x)\frac{2\beta}{t}.$$

This completes the proof for Case 2. We conclude that c_0 is a Lipschitz continuous function; note that the Lipschitz constant we have obtained depends on t. $\square$

Lemma 7.7. *For each $0 < t < 1$, the contribution to f_t from the event $\{0 < L_3(t) < t < R_3(t) < 1\}$ is Lipschitz continuous.*

Proof. Fix $0 < t < 1$. According to (2.14) and (2.16), the contribution from the event in question to $f_t(x)$ is $\sum_{i=1}^{6} c^{(i)}(x)$, where we define

$$c^{(i)}(x) := \int_{l,r,y} f_{l,r}^{(i)}(x-y)\ \mathbb{P}(Y \in \mathrm{d}y \mid (L_3(t), R_3(t)) = (l,r))\,\mathrm{d}l\,\mathrm{d}r.$$

We show here that $c^{(3)}$ is Lipschitz continuous, and the claims that the other contributions $c^{(i)}$ are Lipschitz continuous are proved similarly.

Let $x, z \in \mathbb{R}$ with $z > x$ and consider (l,r) satisfying $0 < l < t < r < 1$. Define

$$d_{l,r}(x,z,y) := f_{l,r}^{(3)}(z-y) - f_{l,r}^{(3)}(x-y)$$

and reformulate

$$f_{l,r}^{(3)}(x) = \mathbb{1}(1+r-2l \le x < 1+r)\frac{1}{x}\left(\frac{1}{x+1-r} + \frac{1}{x+r-1}\right)$$

from the expression for $f_{l,r}^{(3)}(x)$ found in Section 2. We are interested in bounding the quantity

$$|c^{(3)}(z) - c^{(3)}(x)| \le \int_{l,r,y} |d_{l,r}(x,z,y)|\ \mathbb{P}(Y \in \mathrm{d}y \mid (L_3, R_3) = (l,r))\,\mathrm{d}l\,\mathrm{d}r, \tag{7.4}$$

where the conditional probability can also be written in density terms as

$$\mathbb{P}(Y \in \mathrm{d}y \mid (L_3, R_3) = (l,r)) = \frac{1}{r-l} f_{\frac{t-l}{r-l}}\left(\frac{y}{r-l} - 1\right)\,\mathrm{d}y.$$

Just as we did for Lemma 7.6, we break the proof into consideration of two cases.

Case 1. $z - x < 2l$. As in the proof for Case 1 of Lemma 7.6, we bound $d_{l,r}(x,z,y)$ for y in each of the five subintervals of the real line determined by the four partition points

$$x - (1+r) < z - (1+r) < x - (1+r-2l) < z - (1+r-2l).$$

For the two subcases $y \le x - (1+r)$ and $y > z - (1+r-2l)$, we have $d_r(x,z,y) = 0$. We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

Subcase 1(a). $z - (1+r) < y \le x - (1+r-2l)$. We have

$$\begin{aligned}
&d_{l,r}(x,z,y)\\
&= \frac{1}{z-y}\left(\frac{1}{z-y+1-r} - \frac{1}{z-y+r-1}\right)\\
&- \frac{1}{x-y}\left(\frac{1}{x-y+1-r} - \frac{1}{x-y+r-1}\right)\\
&= \frac{1}{z-y}\left(\frac{1}{z-y+1-r} - \frac{1}{x-y+1-r}\right) + \left(\frac{1}{z-y} - \frac{1}{x-y}\right)\frac{1}{x-y+1-r}\\
&- \frac{1}{z-y}\left(\frac{1}{z-y+r-1} - \frac{1}{x-y+r-1}\right) - \left(\frac{1}{z-y} - \frac{1}{x-y}\right)\frac{1}{x-y+r-1}.
\end{aligned}$$

Using the inequality $z - y > x - y \geq 1 + r - 2l$, we obtain

$$|d_{l,r}(x,z,y)| \leq \frac{1}{1+r-2l}\frac{z-x}{(2-2l)^2} + \frac{z-x}{(1+r-2l)^2}\frac{1}{2-2l} + \frac{1}{1+r-2l}\frac{z-x}{(z-y+r-1)(x-y+r-1)} + \frac{z-x}{(1+r-2l)^2}\frac{1}{2(r-l)}.$$

Except for the third term, it is easy to see (by direct computation) that the corresponding contribution to the bound (7.4) on $|c^{(3)}(z) - c^{(3)}(x)|$ is bounded by a constant (depending on t) times $z - x$. So we now focus on bounding the contribution from the third term. Note that since $1 + r - 2l > 1 - t > 0$, we need only bound

$$\int_{l,r,y} \frac{1}{(z-y+r-1)(x-y+r-1)}\ \mathbb{P}(Y \in \mathrm{d}y \mid (L_3, R_3) = (l,r))\,\mathrm{d}l\,\mathrm{d}r \tag{7.5}$$

by a constant (which is allowed to depend on t, but our constant will not).

We first focus on the integral in (7.5) with respect to y and write it, using a change of variables, as

$$\int_{y \in I} d^*_{l,r}(x,z,y)\, f_{\frac{t-l}{r-l}}(y)\,\mathrm{d}y, \tag{7.6}$$

with

$$d^*_{l,r}(x,z,y) = \frac{1}{[z-(r-l)(y+1)+r-1][x-(r-l)(y+1)+r-1]}$$

and $I := \left\{y : \frac{z-(1+r)}{r-l} - 1 < y \leq \frac{x-(1+r-2l)}{r-l} - 1\right\}$. Because the support of the density $f_{\frac{t-l}{r-l}}$ is contained in the nonnegative real line, the integral (7.6) vanishes unless the right endpoint of the interval I is positive, which is true if and only if

$$r < \frac{x-1+3l}{2}.$$

So we see that the integral of (7.6) over $r \in (t,1)$ vanishes unless this upper bound on r is larger than t, which is true if and only if

$$l > \frac{1-x+2t}{3}. \tag{7.7}$$

But then the integral of (7.6) over $\{(l,r) : 0 < l < t < r < 1\}$ vanishes unless this lower bound on l is smaller than t, which is true if and only if $x > 1 - t$; we conclude that for $x \leq 1 - t$, that integral vanishes.

So we may now suppose $x > 1 - t$, and we have seen that the integral of (7.6) over $\{(l,r) : 0 < l < t < r < 1\}$ is bounded above by its integral over the region

$$R := \left\{(l,r) : \frac{1-x+2t}{3} \vee 0 < l < t < r < 1 \wedge \frac{x-1+3l}{2}\right\}.$$

Observe that on R we have

$$\frac{x-(1+r-2l)}{r-l} - 1 = \frac{x-1+l}{r-l} - 2 > \frac{2}{3}\frac{x+t-1}{r-l} - 2 > \frac{1}{2}\frac{x+t-1}{r-l} - 2. \tag{7.8}$$

Define

$$B := \left\{(l,r) : \frac{x+t-1}{2(r-l)} - 2 > 0\right\}.$$

We now split our discussion of the contribution to the integral of (7.6) over $(l,r) \in R$ into two terms, corresponding to (i) $R \cap B^c$ and (ii) $R \cap B$.

Term (i). $R \cap B^c$. Using (7.8), we can bound (7.6) by extending the range of integration from I to

$$I^* := \left\{ y : \frac{x+t-1}{2(r-l)} - 2 < y \leq \frac{x-(1+r-2l)}{r-l} - 1 \right\}.$$

Making use of the inequality (6.4), the integral (7.6) is bounded, for any $\theta > 0$, by

$$\int_{y \in I^*} \frac{1}{4(r-l)^2} C_\theta e^{-\theta y} \, \mathrm{d}y \leq \frac{C_\theta}{4\theta(r-l)^2} \exp\left[-\frac{x+t-1}{2(r-l)}\theta + 2\theta \right].$$

The integral over $(l,r) \in R \cap B^c$ of (7.6) is therefore bounded by

$$\begin{aligned}
&\frac{C_\theta}{4\theta} e^{2\theta} \int_{l=(1-x+2t)/3}^{t} \int_{r=t}^{(x-1+3l)/2} \frac{1}{(r-l)^2} \exp\left[-\frac{x+t-1}{2(r-l)}\theta \right] \mathrm{d}r \, \mathrm{d}l \\
&= \frac{C_\theta}{4\theta} e^{2\theta} \int_{l=(1-x+2t)/3}^{t} \int_{s=t-l}^{(x-1+l)/2} \frac{1}{s^2} \exp\left(-\frac{x+t-1}{2}\theta s^{-1} \right) \mathrm{d}s \, \mathrm{d}l \\
&\leq \frac{C_\theta}{4\theta} e^{2\theta} \frac{2}{\theta(x+t-1)} \int_{l=(1-x+2t)/3}^{t} \exp\left(-\frac{x+t-1}{2}\theta \frac{2}{x-1+l} \right) \mathrm{d}l \\
&\leq \frac{C_\theta}{2\theta^2} e^{2\theta} \frac{1}{x+t-1} e^{-\theta} \left(t - \frac{1-x+2t}{3} \right) = \frac{C_\theta}{6\theta^2} e^{\theta} < \infty.
\end{aligned} \tag{7.9}$$

Term (ii). $R \cap B$. We can bound (7.6) by the sum of the integrals of the same integrand over the intervals I^* and

$$I' := \left\{ y : 0 < y \leq \frac{x+t-1}{2(r-l)} - 2 \right\}.$$

The bound for the integral over I^* is the same as the bound for the $R \cap B^c$ term. To bound the integral over I', we first observe that

$$d^*_{l,r}(x,z,y) \leq \frac{1}{[\frac{1}{2}(x-t-1) + 2r - l]^2} \leq \frac{4}{(x+t-1)^2},$$

where the last inequality holds because $l < t < r$. The contribution to (7.5) can be bounded by integrating $4/(x+t-1)^2$ with respect to $(l,r) \in R \cap B$. We then extend this region of integration to R, and thus bound the contribution by

$$\frac{4}{(x+t-1)^2} \int_{l=\frac{2t+1-x}{3}}^{t} \left(\frac{x-1+3l}{2} - t \right) \mathrm{d}l \leq \frac{2}{(x+t-1)} \left(t - \frac{2t+1-x}{3} \right) = 2/3.$$

This completes the proof for Subcase 1(a).

Subcase 1(b). $x - (1+r-2l) < y \leq z - (1+r-2l)$. First note that in this subcase we have $f^{(3)}(x-y) = 0$. We proceed in similar fashion as for Subcase 1(a), this time setting

$$I := \left\{ y : \frac{x-1+l}{r-l} - 2 < y \leq \frac{z-1+l}{r-l} - 2 \right\}.$$

Again using a linear change of variables, the integral (with respect to y only, in this subcase) appearing on the right in (7.4) in this subcase can be written as

$$\int_{y \in I} d^*_{l,r}(z,y) f_{\frac{t-l}{r-l}}(y) \, \mathrm{d}y \tag{7.10}$$

where now

$$d^*_{l,r}(z,y) = \frac{1}{z-(r-l)(y+1)+1-r} \times \frac{2}{z-(r-l)(y+1)+r-1}.$$

Note that, unlike its analogue in Subcase 1(a), here $d^*_{l,r}(z,y)$ does not possess an explicit factor $z-x$.

By the same discussion as in Subcase 1(a), we are interested in the integral of (7.10) with respect to $(l,r) \in R$, where this time

$$R := \left\{(l,r) : \frac{1-z+2t}{3} \vee 0 < l < t < r < 1 \wedge \frac{z-1+3l}{2}\right\}.$$

and we may suppose that $z > 1-t$.

Observe that on R we have

$$\frac{z-1+l}{r-l} - 2 > \frac{2}{3}\frac{z+t-1}{r-l} - 2 > \frac{1}{2}\frac{z+t-1}{r-l} - 2.$$

Following a line of attack similar to that for Subcase 1(a), we define

$$W := \left\{(l,r) : \frac{x-1+l}{r-l} - 2 > \frac{z+t-1}{2(r-l)} - 2\right\}$$

and split our discussion of the integral of (7.10) over $(l,r) \in R$ into two terms, corresponding to (i) $R \cap W^c$ and (ii) $R \cap W$.

Term (i). $R \cap W$. We bound (7.10) by using the inequality (6.4) (for any $\theta > 0$) and obtain

$$\begin{aligned}\int_{y\in I} \frac{1}{2-2l}\frac{1}{r-l} C_\theta \exp\left[-\theta\left(\frac{z+t-1}{2(r-l)} - 2\right)\right] \mathrm{d}y \\ \le \frac{1}{2}\frac{1}{1-t}\frac{1}{(r-l)^2} C_\theta e^{2\theta} \exp\left[-\theta\left(\frac{z+t-1}{2(r-l)}\right)\right](z-x).\end{aligned}$$

Integrating this expression with respect to $(l,r) \in R \cap W$, we get no more than

$$\frac{1}{2}\frac{(z-x)}{1-t} C_\theta e^{2\theta} \int_{l=(1-z+2t)/3}^{t} \int_{r=t}^{(z-1+3l)/2} \frac{1}{(r-l)^2} \exp\left[-\frac{z+t-1}{2(r-l)}\theta\right] \mathrm{d}r\,\mathrm{d}l,$$

which [consult (7.9)] is bounded by $(z-x)$ times a constant depending only on t and θ.

Term (ii). $R \cap W^c$. We partition the interval I of y-integration into the two subintervals

$$I^* := \left\{y : \frac{z+t-1}{2(r-l)} - 2 < y \le \frac{z-1+l}{r-l} - 2\right\}$$

and

$$I' := \left\{y : \frac{x-1+l}{r-l} - 2 < y \le \frac{z+t-1}{2(r-l)} - 2\right\}.$$

Observe that the length of each of the intervals I^* and I' is no more than the length of I, which is $(z-x)/(r-l)$. We can bound the integral over $y \in I^*$ and $(l,r) \in R \cap W^c$ just as we did for Term (i). For the integral over $y \in I'$ and $(l,r) \in R \cap W^c$, observe the following inequality:

$$d^*_{l,r}(z,y) \le \frac{1}{2-2t}\frac{2}{\frac{1}{2}(z+t-1)+2r-l-t}.$$

Using the constant bound in Theorem 3.1, the integral of $d^*_{l,r}(z,y) f_{\frac{t-l}{r-l}}(y)$ with respect to $y \in I'$ and $(l,r) \in R \cap W^c$ is bounded above by

$$10\frac{(z-x)}{1-t} \int_{l=(1-z+2t)/3}^{t} \int_{r=t}^{(z-1+3l)/2} \frac{1}{r-l}\frac{1}{\frac{1}{2}(z+t-1)+2r-l-t} \mathrm{d}r\,\mathrm{d}l. \tag{7.11}$$

Write the integrand here in the form

$$\frac{1}{r-l}\frac{1}{\frac{z+t-1}{2}+2r-l-t}=\left(\frac{1}{r-l}-\frac{2}{2r-l-t+\frac{z+t-1}{2}}\right)\frac{1}{l-t+\frac{z+t-1}{2}},$$

and observe that $l-t+\frac{z+t-1}{2}>\frac{z+t-1}{6}>0$. Hence we can bound (7.11) by

$$\begin{aligned}
&10\frac{(z-x)}{1-t}\int_{l=(1-z+2t)/3}^{t}\frac{1}{l-t+\frac{z+t-1}{2}}\left[\ln\frac{z-1+l}{2}-\ln(t-l)\right]\mathrm{d}l\\
&\le 10\frac{(z-x)}{1-t}\frac{6}{z+t-1}\left[\frac{z+t-1}{3}\ln\frac{z+t-1}{2}-\int_{l=\frac{1-z+2t}{3}}^{t}\ln(t-l)\,\mathrm{d}l\right]\\
&=20\frac{(z-x)}{1-t}\left(\ln\frac{z+t-1}{2}-\ln\left(\frac{z+t-1}{3}\right)+1\right)\\
&=20\left(1+\ln\frac{3}{2}\right)\frac{(z-x)}{1-t}.
\end{aligned}$$

This completes the proof for Subcase 1(b).

Subcase 1(c). $x-(1+r)<y\le z-(1+r)$. In this case, the contribution from $f^{(3)}(z-y)$ vanishes. Without loss of generality we may suppose $z-x<t$, otherwise we can insert a factor $(z-x)/t$ in our upper bound, and the desired upper bound follows from the fact that the densities f_τ are all bounded by 10. Observe that the integrand $|d_{l,r}(x,z,y)|$ in the bound (7.4) is

$$\frac{1}{x-y+1-r}\frac{2}{x-y+r-1}\le\frac{1}{x-z+2}\frac{2}{x-z+2r}\le\frac{1}{2-t}\frac{2}{2r-t}\le\frac{2}{t(2-t)}.$$

Integrate this constant bound directly with respect to

$$P(Y\in\mathrm{d}y|(L_3,R_3)=(l,r))\,\mathrm{d}r\,\mathrm{d}l$$

on the region $x-(1+r)<y\le z-(1+r)$ and $0<l<t<r<1$ and use the fact that the density is bounded by 10; we conclude that this contribution is bounded by $(z-x)$ times a constant that depends on t. This completes the proof for Subcase 1(c) and also for Case 1.

Case 2. $z-x\ge 2l$. In this case we simply use

$$|d_{l,r}(x,z,y)|\le f^{(3)}(z-y)+f^{(3)}(x-y),$$

and show that each of the two terms on the right contributes at most a constant (depending on t) times $(z-x)$ to the bound in (7.4). Accordingly, let w be either x or z. We are interested in bounding

$$\int_{l=0}^{\frac{z-x}{2}\wedge t}\int_{r=t}^{1}\int_{y=w-(1+r)}^{w-(1+r-2l)}\frac{1}{w-y+1-r}\frac{2}{w-y+r-1}\mu(\mathrm{d}y,\mathrm{d}r,\mathrm{d}l)\tag{7.12}$$

with $\mu(\mathrm{d}y,\mathrm{d}r,\mathrm{d}l):=P(Y\in\mathrm{d}y\mid(L_3,R_3)=(l,r))\,\mathrm{d}r\,\mathrm{d}l$. We bound the integrand as follows:

$$\frac{1}{w-y+1-r}\frac{2}{w-y+r-1}\le\frac{1}{2-2l}\frac{2}{2r-2l}\le\frac{1}{2}\frac{1}{1-t}\frac{1}{r-l}.$$

We first suppose $z-x<t$ and bound (7.12) by

$$\begin{aligned}
\frac{1}{2}\frac{1}{1-t}\int_{l=0}^{\frac{z-x}{2}}\int_{r=t}^{1}\frac{1}{r-l}\,\mathrm{d}r\,\mathrm{d}l&\le\frac{1}{2}\frac{1}{1-t}\int_{l=0}^{\frac{z-x}{2}}[-\ln(t-l)]\,\mathrm{d}l\\
&\le\frac{1}{2}\frac{1}{1-t}\left[-\ln\left(t-\frac{z-x}{2}\right)\right]\frac{z-x}{2}\\
&\le(z-x)\frac{\ln(2/t)}{4(1-t)}.
\end{aligned}$$

If instead $z - x \geq t$, we bound (7.12) by

$$\frac{1}{2}\frac{z-x}{t}\int_{l=0}^{t}\int_{r=t}^{1}\frac{1}{1-l}\frac{1}{r-l}\,\mathrm{d}r\,\mathrm{d}l \leq (z-x)\frac{\pi^2}{12\,t}.$$

This completes the proof for Case 2 and thus the proof of Lipschitz continuity of $c^{(3)}$. □

We immediately get the following corollary from the proof of Theorem 7.4.

Corollary 7.8. *For any $0 < \eta < 1/2$, the uniform continuous family $\{f_t : t \in [\eta, 1-\eta]\}$ is a uniformly equicontinous family.*

Proof. We observe from the proof of Theorem 7.4 that for any $0 < \eta < 1/2$, the Lipschitz constants L_t in Theorem 7.4 are bounded for $t \in [\eta, 1-\eta]$ by some universal constant $C < \infty$. The result follows. □

7.3 Joint continuity

As noted in the proof of Lemma 5.2, we reference Grübel and Rösler [19] to conclude that for each $t \in [0, 1)$, the distribution functions F_u converge weakly to F_t as $u \downarrow t$. It follows by symmetry that the convergence also holds for each $t \in (0, 1]$ as $u \uparrow t$. We now deduce the convergence from f_u to f_t for each $t \in (0, 1)$ as $u \to t$, according to the following lemma.

Lemma 7.9. *For each $0 < t < 1$ we have $f_u \to f_t$ uniformly as $u \to t$.*

Proof. We fix $0 < t \leq 1/2$ and choose $0 < \eta < t$. By the weak convergence of F_u to F_t as $u \to t$, the uniform boundedness of the density functions (Theorem 3.1), the fact that $f_t(x) \to 0$ as $x \to \pm\infty$, and the uniform equicontinuity of the family $\{f_u : u \in [\eta, 1-\eta]\}$ (Corollary 7.8), we conclude from Boos [3, Lemma 1] (a converse to Scheffé's theorem) that $f_u \to f_t$ uniformly as $u \to t$. □

Remark 7.10. The uniform equicontinuity in Corollary 7.8 does not hold for the family $\{f_t : t \in (0, 1)\}$. Here is a proof. For the sake of contradiction, suppose to the contrary. We symmetrize $f_t(x)$ at $x = 0$ for every $0 \leq t \leq 1$ to create another family of continuous densities g_t; that is, consider $g_t(x) := [f_t(x) + f_t(-x)]/2$. Observe that the supposed uniform equicontinuity of the functions f_t for $t \in (0, 1)$ extends to the functions g_t. Now suppose (for each $t \in [0, 1]$) that $W(t)$ is a random variable with density g_t. By a simple calculation we have $W(t) \Rightarrow W(0)$, and it follows by Boos [3, Lemma 1] that $g_t(x) \to g_0(x)$ uniformly in x. This contradicts to the fact that $g_t(0) = 0$ for all $t \in (0, 1)$ but $g_0(0) = e^{-\gamma}$.

Remark 7.11. Since $(F_t)_{t\in[0,1]}$ is weakly continuous in t and F_t is atomless for $0 \leq t \leq 1$, it follows from a theorem of Pólya ([4, Exercise 4.3.4]) that $(F_t)_{t\in[0,1]}$ is continuous in the sup-norm metric, i.e., that $(J(t))$ [or $(Z(t))$] is continuous in the Kolmogorov–Smirnov metric on distributions.

Corollary 7.12. *The density $f_t(x)$ is jointly continuous in $(t, x) \in (0, 1) \times \mathbb{R}$.*

Proof. As $(t', x') \to (t, x) \in (0, 1) \times \mathbb{R}$, we have

$$\begin{aligned}\limsup |f_{t'}(x') - f_t(x)| &\leq \limsup |f_{t'}(x') - f_t(x')| + \limsup |f_t(x') - f_t(x)| \\ &\leq \limsup \|f_{t'} - f_t\|_\infty + \delta_t(|x' - x|) \\ &= 0\end{aligned}$$

where the sup-norm $\|f_{t'} - f_t\|_\infty$ tends to 0 as $t' \to t$ by Lemma 7.9 and the modulus of uniform continuity δ_t of the function f_t tends to 0 as $x' \to x$ by Theorem 4.4. □

Remark 7.13. The positivity of $f_t(x)$ for each $0 < t < 1$ and $x > \min\{t, 1-t\}$ in Theorem 7.1 can be proved alternatively by using the integral equation Proposition 5.7 and the joint continuity result of Corollary 7.12. Here is the proof.

Fix (for now) $t_0, t_1, t_2 \in (0,1)$ with $t_1 > t_0 > t_2$. We will show that $f_{t_0}(x) > 0$ for all $x > t_0$, using t_1 and t_2 in auxiliary fashion. Since this is true for arbitrarily chosen t_0, invoking symmetry ($f_t \equiv f_{1-t}$) then completes the proof.

We certainly know that $f_{t_0}(y_0) > 0$ for some $y_0 > t_0$; choose and fix such a y_0. Use Proposition 5.7 to represent the density $f_{t_1}(x)$. We observe that the integrand of the integral with respect to l is positive at $l = l_1 = (t_1 - t_0)/(1-t_0)$ and $x = y_1 = (1-l_1)(y_0+1)$. From Corollary 7.12 we conclude that the integrand is positive in a neighborhood of l_1 and thus $f_{t_1}(y_1) > 0$.

Further, use Proposition 5.7 to represent the density $f_{t_0}(x)$. We observe that the integrand of the integral with respect to r is positive at $r = r_2 = \frac{t_0}{t_1}$ and $x = y_2 = r_2(y_1+1)$. From $f_{t_1}(y_1) > 0$ and Corollary 7.12 we conclude that $f_{t_0}(y_2) > 0$.

Now letting $y_2 = y_0 + \epsilon_1$, we have

$$\epsilon_1 = \left(\frac{t_0}{t_1} \frac{1-t_1}{1-t_0} - 1 \right) y_0 + \frac{t_0}{t_1} \left(1 + \frac{1-t_1}{1-t_0} \right).$$

Observe that as $t_1 \downarrow t_0$ we have $\epsilon_1 \to 2$, while as $t_1 \uparrow 1$ we have $\epsilon_1 \downarrow -y_0 + t_0 < 0$. Thus, given $\delta \in (0, 2 - t_0 + y_0)$ it is possible to choose $t_1 \in (t_0, 1)$ such that $\epsilon_1 = -y_0 + t_0 + \delta$, i.e., $y_2 = t_0 + \delta$. We conclude that $f_{t_0}(x)$ is positive for every $x > t_0$, as desired.

8 Left-tail behavior of the density function

We consider the densities f_t with $t \in (0,1)$; since $f_t \equiv f_{1-t}$ by symmetry, we may without loss of generality suppose $t \in (0, 1/2]$. As previously noted (recall Theorems 7.1 and 4.4), $f_t(x) = 0$ for all $x \le t$ and $f_t(x) > 0$ for all $x > t$. In this section we consider the left-tail behavior of f_t, by which we mean the behavior of $f_t(x)$ as $x \downarrow t$.

As a warm-up, we first show that f_t has a positive right-hand derivative at t that is large when t is small.

Lemma 8.1. (a) *Fix $t \in (0, 1/2)$. Then the density function f_t has right-hand derivative $f_t'(t)$ at t equal to c_1/t, where*

$$c_1 := \int_0^1 \mathbb{E}[2 - w + J(w)]^{-2} \, \mathrm{d}w \in (0.0879, 0.3750).$$

(b) *Fix $t = 1/2$. Then the density function f_t has right-hand derivative $f_t'(t)$ at t equal to $2c_1/t = 4c_1$.*

Proof. (a) We begin with two key observations. First, if $L_1(t) > 0$, then $J(t) > 1-t$. Second, if $1 > R_1(t) > R_2(t)$, then $J(t) > 2t$. It follows that if $0 < z < \min\{1-2t, t\}$, then, with $Y \equiv Y(t)$ as defined at (2.1),

$$\begin{aligned}
f_t(t+z)\,\mathrm{d}z &= \mathbb{P}(J(t) - t \in \mathrm{d}z) \\
&= \mathbb{P}(R_1(t) < 1,\, L_2(t) > 0,\, J(t) - t \in \mathrm{d}z) \\
&= \iint_{\substack{y>x>0:\, x+y<z, \\ x<1-t,\ y-x<t}} \mathbb{P}(R_1(t) - t \in \mathrm{d}x,\, t + x - L_2(t) \in \mathrm{d}y,\, Y(t) \in \mathrm{d}z - x - y) \\
&= \iint_{\substack{y>x>0:\, x+y<z, \\ x<1-t,\ y-x<t}} \mathrm{d}x \, \frac{\mathrm{d}y}{t+x} \, \mathbb{P}\left(y\, J\left(\frac{y-x}{y} \right) \in \mathrm{d}z - x - y \right) \\
&= \iint_{\substack{y>x>0:\, x+y<z, \\ x<1-t,\ y-x<t}} \mathrm{d}x \, \frac{\mathrm{d}y}{t+x} \, y^{-1} f_{1-\frac{x}{y}} \left(\frac{z-x-y}{y} \right) \mathrm{d}z.
\end{aligned}$$

Now make the changes of variables from x to $u = x/z$ and from y to $v = y/z$. We then find

$$f_t(t+z) = z \iint_{\substack{v>u>0:\, u+v<1 \\ u<(1-t)/z,\ v-u<t/z}} (t+uz)^{-1}\, v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v$$
$$= z \iint_{v>u>0:\, u+v<1} (t+uz)^{-1}\, v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v,$$

where the second equality follows because $(1-t)/z > (1-2t)/z > 1$ and $t/z > 1$ by assumption. Thus, as desired,

$$f_t(t+z) \sim \frac{c_1 z}{t}$$

as $z \downarrow 0$ by the dominated convergence theorem, if we can show that

$$\tilde{c} := \iint_{v>u>0:\, u+v<1} v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v$$

equals c_1. For that, make another change of variables from u to $w = u/v$; then we find

$$\tilde{c} = \int_0^1 \int_0^{(1+w)^{-1}} f_{1-w}(v^{-1} - (1+w))\, \mathrm{d}v\, \mathrm{d}w$$
$$= \int_0^1 \int_0^{(2-w)^{-1}} f_w(v^{-1} + w - 2)\, \mathrm{d}v\, \mathrm{d}w.$$

Make one last change of variables, from v to $s = v^{-1} + w - 2$, to conclude

$$\tilde{c} = \int_0^1 \int_0^{\infty} (2-w+s)^{-2}\, f_w(s)\, \mathrm{d}s\, \mathrm{d}w = c_1,$$

as claimed.

To obtain the claimed upper bound on c_1, we note, using the facts that $J(w)$ and $J(1-w)$ have the same distribution and that $J(w) > w$ for $w \in (0, 1/2)$, that

$$c_1 = \int_0^{1/2} \mathbb{E}[2-w+J(w)]^{-2}\, \mathrm{d}w + \int_{1/2}^1 \mathbb{E}[2-w+J(w)]^{-2}\, \mathrm{d}w$$
$$= \int_0^{1/2} \mathbb{E}[2-w+J(w)]^{-2}\, \mathrm{d}w + \int_0^{1/2} \mathbb{E}[1+w+J(w)]^{-2}\, \mathrm{d}w$$
$$< \int_0^{1/2} \tfrac{1}{4}\, \mathrm{d}w + \int_0^{1/2} (1+2w)^{-2}\, \mathrm{d}w = \tfrac{1}{8} + \tfrac{1}{4} = \tfrac{3}{8} = 0.3750.$$

To obtain the claimed lower bound on c_1, we combine Jensen's inequality with the known fact [cf. (1.3)] that $\mathbb{E}\, J(w) = 1 + 2H(w)$ with $H(w) = -w \ln w - (1-w) \ln(1-w)$:

$$c_1 = \int_0^1 \mathbb{E}[2-w+J(w)]^{-2}\, \mathrm{d}w$$
$$\geq \int_0^1 (\mathbb{E}[2-w+J(w)])^{-2}\, \mathrm{d}w = \int_0^1 (3-w+2H(w))^{-2}\, \mathrm{d}w > 0.0879.$$

(b) By an argument similar to that at the start of the proof of (a), if $0 < z < 1/2$, then, using symmetry at the third equality,

$$f_t(t+z)\, \mathrm{d}z = \mathbb{P}(J(t) - t \in \mathrm{d}z)$$
$$= \mathbb{P}(R_1(t) < 1,\ L_2(t) > 0,\ J(t) - t \in \mathrm{d}z)$$
$$+ \mathbb{P}(L_1(t) > 0,\ R_2(t) < 1,\ J(t) - t \in \mathrm{d}z)$$
$$= 2\, \mathbb{P}(R_1(t) < 1,\ L_2(t) > 0,\ J(t) - t \in \mathrm{d}z)$$
$$\sim \frac{2c_1 z}{t} = 4c_1 z.$$

Here the asymptotic equivalence is as $z \downarrow 0$ and follows by the same argument as used for (a). $\square$

We are now prepared for our main result about the left-tail behavior of f_t.

Theorem 8.2.

(a) *Fix* $t \in (0, 1/2)$. *Then* $f_t(t+tz)$ *has the uniformly absolutely convergent power series expansion*

$$f_t(t+tz) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k$$

for $z \in [0, \min\{t^{-1} - 2, 1\})$, *where for* $k \geq 1$ *the coefficients*

$$c_k := \int_0^1 (1-w)^{k-1}\, \mathbb{E}[2 - w + J(w)]^{-(k+1)}\, \mathrm{d}w,$$

not depending on t, *are strictly positive, have the property that* $2^k c_k$ *is strictly decreasing in* k, *and satisfy*

$$0 < (0.0007) 2^{-(k+1)} (k+1)^{-2} < c_k < 2^{-(k+1)} k^{-1} (1 + 2^{-k}) < 0.375 < \infty.$$

[*In particular,* $2^k c_k$ *is both* $O(k^{-1})$ *and* $\Omega(k^{-2})$.]

(b) *Fix* $t = 1/2$. *Then* $f_t(t+tz)$ *has the uniformly absolutely convergent power series expansion*

$$f_t(t+tz) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k$$

for $z \in [0, 1)$.

Proof. (a) As shown in the proof of Lemma 8.1, for $z \in [0, \min\{t^{-1} - 2, 1\})$ we have

$$f_t(t+tz) = z \iint_{v>u>0:\, u+v<1} (1+uz)^{-1}\, v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v. \tag{8.1}$$

Note that the expression on the right here doesn't depend on t. Further, since $z \leq 1$ and $0 < u < 1/2$ in the range of integration,

$$\begin{aligned}
&\frac{1}{2} \iint_{v>u>0:\, u+v<1} (1-uz)^{-1}\, v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v \\
&\qquad < \iint_{v>u>0:\, u+v<1} v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v \\
&\qquad = \tilde{c} = c_1 < 3/8 < \infty,
\end{aligned}$$

with $\tilde{c}$ and c_1 as in the proof of Lemma 8.1. It follows that $f_t(t+tz)$ has the uniformly absolutely convergent power series expansion

$$f_t(t+tz) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k$$

for $z \in [0, \min\{t^{-1} - 2, 1\})$, where for $k \geq 1$ we have

$$\begin{aligned}
c_k &= 2 \times 2^{-k} \iint_{v>u>0:\, u+v<1} (2u)^{k-1}\, v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^1 (1-w)^{k-1}\, \mathbb{E}[2 - w + J(w)]^{-(k+1)}\, \mathrm{d}w;
\end{aligned}$$

the second equality follows just as for $c = c_1$ in the proof of Lemma 8.1. From the first equality it is clear that these coefficients have the property that $2^k c_k$ is strictly decreasing in k.

To obtain the claimed upper bound on c_k, proceed just as in the proof of Lemma 8.1 to obtain

$$\begin{aligned} c_k &< 2^{-(k+1)} \int_0^{1/2} (1-w)^{k-1} \, \mathrm{d}w + \int_0^{1/2} w^{k-1} (1+2w)^{-(k+1)} \, \mathrm{d}w \\ &= 2^{-(k+1)} k^{-1} (1 - 2^{-k}) + k^{-1} 4^{-k} = 2^{-(k+1)} k^{-1} (1 + 2^{-k}). \end{aligned}$$

The claimed lower bound on c_k follows from Lemma 8.1 for $k = 1$ but for $k \geq 2$ requires more work. We begin by establishing a lower bound on $\mathbb{P}(J(w) \leq 2w)$ for $w \leq 1/3$, using what we have already proved:

$$\begin{aligned} \mathbb{P}(J(w) \leq 2w) &= \int_0^w f_w(w+x) \, \mathrm{d}x = w \int_0^1 f_w(w + wz) \, \mathrm{d}z \\ &\geq w \int_0^1 (c_1 z - c_2 z^2) \, \mathrm{d}z = w(\tfrac{1}{2} c_1 - \tfrac{1}{3} c_2) \\ &> [0.04395 - (1/3)(1/8)(1/2)(5/4)] w > 0.0179 \, w. \end{aligned}$$

Thus c_k is at least $0.0179 \; 2^{-(k+1)}$ times the following expression:

$$\begin{aligned} 2^{k+1} \int_0^{1/3} w(1-w)^{k-1} (2+w)^{-(k+1)} \, \mathrm{d}w &\geq \int_0^{1/3} w \exp[-2(k+1)w] \exp[-(k+1)w/2] \, \mathrm{d}w \\ &\geq \int_0^{1/(k+1)} w \exp[-5(k+1)w/2] \, \mathrm{d}w \\ &\geq e^{-5/2} \int_0^{1/(k+1)} w \, \mathrm{d}w = \frac{1}{2} e^{-5/2} (k+1)^{-2}. \end{aligned}$$

(b) The claim of part (b) is clear from the proof of Lemma 8.1. □

Corollary 8.3.

(a) *Fix* $t \in (0, 1/2)$. *Then, for all* $x \in (t, \min\{1-t, 2t\})$, *the density* $f_t(x)$ *is infinitely differentiable, strictly increasing, strictly concave, and strictly log-concave.*

(b) *Fix* $t = 1/2$. *Then for all* $x \in [1/2, 1)$, *the density* $f_{1/2}(x)$ *is infinitely differentiable, strictly increasing, strictly concave, and strictly log-concave.*

Proof. Once again it is clear that we need only prove (a). The result is actually a corollary to (8.1), rather than to Theorem 8.2. It is easy to justify repeated differentiation with respect to z under the double integral of (8.1). In particular, for $z \in (0, \min\{t^{-1} - 2, 1\})$ we have

$$\begin{aligned} t f_t'(t + tz) &= \iint_{v > u > 0 \colon u + v < 1} (1 + uz)^{-1} \, v^{-1} \, f_{1 - \frac{u}{v}} \left(\frac{1 - u - v}{v} \right) \mathrm{d}u \, \mathrm{d}v \\ &\quad - z \iint_{v > u > 0 \colon u + v < 1} u (1 + uz)^{-2} \, v^{-1} \, f_{1 - \frac{u}{v}} \left(\frac{1 - u - v}{v} \right) \mathrm{d}u \, \mathrm{d}v \\ &= \iint_{v > u > 0 \colon u + v < 1} (1 + uz)^{-2} \, v^{-1} \, f_{1 - \frac{u}{v}} \left(\frac{1 - u - v}{v} \right) \mathrm{d}u \, \mathrm{d}v > 0 \end{aligned}$$

and

$$\begin{aligned} t^2 f_t''(t + tz) &= -2 \iint_{v > u > 0 \colon u + v < 1} u (1 + uz)^{-3} \, v^{-1} \, f_{1 - \frac{u}{v}} \left(\frac{1 - u - v}{v} \right) \mathrm{d}u \, \mathrm{d}v \\ &< 0. \end{aligned}$$

Strict log-concavity of the positive function f_t follows immediately from strict concavity. □

Remark 8.4. (a) By extending the computations of the first and second derivatives of f_t in the proof of Corollary 8.3 to higher-order derivatives, it is easy to see that $f_t(x)$ is real-analytic for x in the intervals as specified in Corollary 8.3(a)–(b). For the definition of real analytic function, see Krantz and Parks [26, Definition 1.1.5].

(b) It may be that, like the Dickman density f_0, the densities f_t with $0 < t < 1$ are log-concave everywhere and hence strongly unimodal. Even if this is false, we conjecture that the densities f_t are all unimodal.

9 Improved right-tail asymptotic upper bound

In this section, we will prove that for $0 < t < 1$ and $x > 4$, the continuous density function f_t satisfies

$$f_t(x) \le \exp[-x \ln x - x \ln \ln x + O(x)]$$

uniformly in t. We first bound the moment generating function of the random variable V treated in Lemma 3.3.

Lemma 9.1. *Denote the moment generating function of V by m. Then for every $\epsilon > 0$ there exists a constant $a \equiv a(\epsilon) > 0$ such that for all $\theta > 0$ we have*

$$m(\theta) \le \exp[(2+\epsilon)\theta^{-1}e^{\theta} + a\theta]. \tag{9.1}$$

Proof. The idea of the proof comes from Janson [22, Lemma 6.1]. Observe that the random variable V satisfies the following distributional identity

$$V \stackrel{\mathcal{L}}{=} 1 + V_1 \cdot V$$

where $V_1 \sim \text{Uniform}(1/2, 1)$ is independent of V. It follows by conditioning on V_1 that the moment generating function m satisfies

$$m(\theta) = 2e^{\theta} \int_{v=1/2}^{1} m(\theta v)\,\mathrm{d}v = 2e^{\theta} \int_{u=0}^{1/2} m(\theta(1-u))\,\mathrm{d}u. \tag{9.2}$$

Since m is continuous and $m(0) = 1$, there exists a $\theta_1 > 0$ such that the inequality (9.1) holds (for *any* constant $a > 0$) for $\theta \in [0, \theta_1]$. Choose and fix $\theta_2 > \max\{\theta_1, 5\}$ and choose $a \in [1, \infty)$ large enough such that the inequality (9.1) holds for $\theta \in [\theta_1, \theta_2]$.

We now suppose for the sake of contradiction that (9.1) fails at some $\theta > \theta_2$. Define $T := \inf\{\theta > \theta_2 : \text{(9.1) fails}\}$; then by continuity we have $m(T) = \exp[(2+\epsilon)T^{-1}e^{T} + aT]$.

Since $m(\theta u) \ge 1$ for any $\theta > 0$ and $0 < u < 1/2$, we can conclude from (9.2) that m satisfies

$$m(\theta) \le 2e^{\theta} \int_{u=0}^{1/2} m(\theta u)\, m(\theta(1-u))\,\mathrm{d}u \tag{9.3}$$

for every $\theta > 0$, including for $\theta = T$. The proof is now completed effortlessly by applying exactly the same argument as for the limiting `QuickSort` moment generating function in Fill and Hung [13, proof of Lemma 2.1]; indeed, using only (9.3) they prove that when $\theta = T$ the right-hand side of (9.3) is strictly smaller than $m(T)$, which is the desired contradiction. □

Thus, for $\epsilon > 0$ and $\theta > 0$, the moment generating functions m_t all satisfy

$$m_t(\theta) \le m(\theta) \le \exp[(2+\epsilon)\theta^{-1}e^{\theta} + a\theta]. \tag{9.4}$$

We now deduce a uniform right-tail upper bound on the survival functions $1 - F_t$ for $0 < t < 1$.

Theorem 9.2. *Uniformly in* $0 < t < 1$*, for* $x > 1$ *the distribution function* F_t *for* $J(t)$ *satisfies*

$$1 - F_t(x) \le \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. The proof is essentially the same as for Fill and Hung [13, proof of Proposition 1.1], but for completeness we sketch the simple proof here. Fix $\epsilon > 0$. For any $\theta > 0$ we have the Chernoff bound

$$1 - F_t(x) = \mathbb{P}(J(t) > x) \le \mathbb{P}(Z(t) > x) \le e^{-\theta x} m_t(\theta) \le e^{-\theta x} \exp[(2+\epsilon)\theta^{-1} e^{\theta} + a\theta]$$

by (9.4). Letting $\theta = \ln[(2+\epsilon)^{-1} x \ln x]$, and then $\epsilon \downarrow 0$ we get the desired upper bound—in fact, with the following improvement we will not find useful in the sequel:

$$1 - F_t(x) \le \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)]. \qquad \square$$

The continuous density function $f_t(x)$ enjoys the same uniform asymptotic bound for $0 < t < 1$ and $x > 4$.

Theorem 9.3. *Uniformly in* $0 < t < 1$*, for* $x > 4$ *the continuous density function* f_t *satisfies*

$$f_t(x) \le \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. Fix $0 < t < 1$ and let $x > 4$. We first use the integral equation in Proposition 5.8, namely,

$$f_t(x) = \int \mathbb{P}((L_3(t), R_3(t)) \in \mathrm{d}(l, r)) \cdot h_t(x \mid l, r),$$

for $x \ge 0$, where, by a change of variables,

$$h_t(x \mid l, r) = \int f_{l,r}((r-l)(y-1))\, f_{\frac{t-l}{r-l}}\left(\frac{x}{r-l} - y\right) \mathrm{d}y; \tag{9.5}$$

we consider the contribution to $f_t(x)$ from values (l, r) satisfying $0 < l < t < r < 1$. Recall that the conditional density $f_{l,r}(z)$ vanishes if $z \ge 2$. Thus the only nonzero contribution to (9.5) is from values of y satisfying

$$y \le \frac{2}{r-l} + 1.$$

If this inequality holds, then the argument for the factor $f_{(t-l)/(r-l)}$ satisfies

$$\frac{x}{r-l} - y \ge \frac{x-2}{r-l} - 1 \ge x - 3.$$

Using $b(l, r)$ of Lemma 3.2 and (3.3) to bound the $f_{l,r}$ factor, we obtain

$$h_t(x \mid l, r) \le b(l, r) \left(1 - F_{\frac{t-l}{r-l}}(x-3)\right).$$

By Theorem 9.2 and the last display in the proof of Lemma 3.2, the contribution in question is thus bounded by $\exp[-x \ln x - x \ln \ln x + O(x)]$, uniformly in t, for $x > 4$.

For the contribution to $f_t(x)$ corresponding to the cases $0 = L_3(t) < t < R_3(t) < 1$ and $0 < L_3(t) < t < R_3(t) = 1$, we use the same idea as in the proof of Lemma 3.6. By symmetry, we need only consider the first of these two cases. Recall from the proof of Lemma 3.6 that the contribution in question is bounded by the sum of $f_W(x)$, which is the density of $W = U_1(1 + U_2 V)$ evaluated at x [where U_1, U_2, and V are independent, U_1 and U_2 are uniformly distributed on $(0, 1)$, and V is as in Lemma 3.3], and the integral

$$\begin{aligned}\int_{r=0}^{1} r^{-1}\,\mathbb{P}\left(V > \frac{x}{r} - 1\right)\,\mathrm{d}r &= \int_{v=x-1}^{\infty} (v+1)^{-1}\,\mathbb{P}(V > v)\,\mathrm{d}v \\ &\le x^{-1}\int_{v=x-1}^{\infty} \mathbb{P}(V > v)\,\mathrm{d}v \\ &\le \exp[-x\ln x - x\ln\ln x + O(x)].\end{aligned}$$

The last inequality here is obtained by applying a Chernoff bound and Lemma 9.1 to the integrand and integrating; we omit the straightforward details. To bound the density of W at x, observe that by conditioning on the values of U_2 and V, we have

$$\begin{aligned}f_W(x) &= \int_{u,v} (1+uv)^{-1}\,\mathbb{1}(0 \le x \le 1 + uv)\;\mathbb{P}(U_2 \in \mathrm{d}u,\, V \in \mathrm{d}v) \\ &= \int_{u=0}^{1}\int_{v=(x-1)/u}^{\infty} (1+uv)^{-1}\;\mathbb{P}(V \in \mathrm{d}v)\,\mathrm{d}u \\ &\le x^{-1}\int_{u=0}^{1} \mathbb{P}\left(V > \frac{x-1}{u}\right)\,\mathrm{d}u \\ &\le x^{-1}\,\mathbb{P}(V > x-1) \le \exp[-x\ln x - x\ln\ln x + O(x)].\end{aligned}$$

This completes the proof. □

10 Matching right-tail asymptotic lower bound

In this section we will prove for each fixed $t \in (0, 1)$ that the continuous density function f_t satisfies

$$f_t(x) \ge \exp[-x\ln x - x\ln\ln x + O(x)] \text{ as } x \to \infty,$$

matching the upper bound of Theorem 9.2 to two logarithmic asymptotic terms, with remainder of the same order of magnitude. While we are able to get a similarly matching lower bound to Theorem 9.3 for the survival function $1 - F_t$ that is uniform in t, we are unable to prove uniformity in t for the density lower bound.

We begin with consideration of the survival function.

Theorem 10.1. *Uniformly in $0 < t < 1$, the distribution function F_t for $J(t)$ satisfies*

$$1 - F_t(x) \ge \exp[-x\ln x - x\ln\ln x + O(x)].$$

Proof. With D denoting a random variable having the Dickman distribution with support $[1, \infty)$, for any $0 < t < 1$ we have from Lemma 3.4 that

$$\begin{aligned}1 - F_t(x) = \mathbb{P}(J(t) > x) = \mathbb{P}(Z(t) > x+1) &\ge \mathbb{P}(D > x+1) \\ &= \exp[-x\ln x - x\ln\ln x + O(x)] \text{ as } x \to \infty.\end{aligned}$$

The asymptotic lower bound here follows by substitution of $x + 1$ for u in equation (1.6) (for the unnormalized Dickman function) of Xuan [28], who credits earlier work of de Bruijn [5] and of Hua [20]. □

Now we turn our attention to the densities.

Theorem 10.2. *For each fixed $t \in (0, 1)$ we have*

$$f_t(x) \ge \exp[-x\ln x - x\ln\ln x + O(x)] \text{ as } x \to \infty.$$

Proof. From the calculations at the beginning of the proof of Lemma 8.1, for all $z > 0$ we have

$$f_t(t+z) \geq z \iint_{\substack{v>u>0:\, u+v<1,\\ u<(1-t)/z,\ v-u<t/z}} (t+uz)^{-1}\, v^{-1}\, f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right)\, \mathrm{d}u\, \mathrm{d}v.$$

Thus, changing variables from u to $w = 1 - (u/v)$, we have

$$f_t(t+tz) \geq z \int_0^1 \int_0^{\Upsilon(t,z,w)} [1 + v(1-w)z]^{-1}\, f_w\left(v^{-1} + w - 2\right)\, \mathrm{d}v\, \mathrm{d}w,$$

where $\Upsilon(t,z,w) := \min\{(2-w)^{-1},\ (1-t)(tz)^{-1}(1-w)^{-1},\ z^{-1}w^{-1}\}$. Now let

$$\Lambda(t,z,w) := \max\{0, t(1-t)^{-1}z(1-w) + w - 2, zw + w - 2\}$$

and change variables from v to $s = v^{-1} + w - 2$ to find

$$f_t(t+tz) \geq z \int_0^1 \int_{\Lambda(t,z,w)}^{\infty} [1 + (2-w+s)^{-1}(1-w)z]^{-1}\, (2-w+s)^{-2} f_w(s)\, \mathrm{d}s\, \mathrm{d}w.$$

Observe that if $\delta > 0$ and $t \leq w \leq (1+\delta)t \leq 1$, then

$$\Lambda(t,z,w) < (1+\delta)tz$$

and so

$$f_t(t+tz) \geq z \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{\infty} [1 + (2-w+s)^{-1}(1-w)z]^{-1}\, (2-w+s)^{-2} f_w(s)\, \mathrm{d}s\, \mathrm{d}w.$$

If $\delta \leq 1$, it follows that

$$\begin{aligned}
f_t(t+tz) &\geq z \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} [1 + (2-w+s)^{-1}(1-w)z]^{-1}\, (2-w+s)^{-2} f_w(s)\, \mathrm{d}s\, \mathrm{d}w \\
&\geq \frac{z}{(2+2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} \frac{1}{1 + (2-w+(1+\delta)tz)^{-1}(1-w)z}\, f_w(s)\, \mathrm{d}s\, \mathrm{d}w \\
&\geq \frac{z}{(2+2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} \left[1 + \frac{1-w}{(1+\delta)t}\right]^{-1} f_w(s)\, \mathrm{d}s\, \mathrm{d}w \\
&\geq \frac{z}{(2+2tz)^2} \frac{(1+\delta)t}{1+\delta t} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} f_w(s)\, \mathrm{d}s\, \mathrm{d}w \\
&\geq \frac{tz}{(2+2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} f_w(s)\, \mathrm{d}s\, \mathrm{d}w \\
&= \frac{tz}{(2+2tz)^2} \int_t^{(1+\delta)t} \big[\, \mathbb{P}(J(w) > (1+\delta)tz) - \mathbb{P}(J(w) > 2tz)\big]\, \mathrm{d}w.
\end{aligned}$$

Recall that D defined in Lemma 3.4 is a random variable having the Dickman distribution on $[1,\infty)$ and that V is defined in (3.5). By Lemma 3.4, we have $D - 1 \leq J(w) \leq V - 1$ stochastically, and thus we can further lower-bound the density function as follows:

$$\begin{aligned}
f_t(t+tz) &\geq \frac{tz}{(2+2tz)^2} \int_t^{(1+\delta)t} \big[\, \mathbb{P}(D-1 > (1+\delta)tz) - \mathbb{P}(V > 2tz)\big]\, \mathrm{d}w \\
&= \delta t \frac{tz}{(2+2tz)^2} \big[\, \mathbb{P}(D-1 > (1+\delta)tz) - \mathbb{P}(V > 2tz)\big].
\end{aligned}$$

That is, if $0 < \delta \le \min\{1, t^{-1} - 1\}$, then for every $z > 0$ we have

$$f_t(t+z) \ge \delta t \frac{z}{(2+2z)^2} \big[\mathbb{P}(D - 1 > (1+\delta)z) - \mathbb{P}(V > 2z) \big].$$

If $z \ge \max\{1, t/(1-t)\}$, then we can choose $\delta \equiv \delta_z = z^{-1}$ and conclude

$$f_t(t+z) \ge t(2+2z)^{-2} \big[\mathbb{P}(D - 1 > z+1) - \mathbb{P}(V > 2z) \big].$$

Moreover, as $z \to \infty$, we have

$$(2+2z)^{-2} \big[\mathbb{P}(D - 1 > z+1) - \mathbb{P}(V > 2z) \big] = \exp[-z \ln z - z \ln\ln z + O(z)].$$

The stated result follows readily. □

Remark 10.3. The proof of Theorem 10.1 reveals that the result in fact holds uniformly for t in any closed subinterval of $(0,1)$. In fact, the proof shows that the result follows uniformly in $t \in (0,1)$ and $x \to \infty$ satisfying $x = \Omega(\ln[1/\min\{t, 1-t\}])$.

11 Right-tail large deviation behavior of `QuickQuant`(n,t)

In this section, we investigate the right-tail large deviation behavior of `QuickQuant`(n,t), that is, of `QuickSelect`$(n, m_n(t))$. Throughout this section, for each fixed $0 \le t \le 1$ we consider any sequence $1 \le m_n(t) \le n$ such that $m_n(t)/n \to t$ as $n \to \infty$. We abbreviate the normalized number of key comparisons of `QuickSelect`$(n, m_n(t))$ discussed in Section 1 as $C_n(t) := n^{-1} C_{n, m_n(t)}$.

Kodaj and Móri [25, Corollary 3.1] bound the convergence rate of $C_n(t)$ to its limit $Z(t)$ in the Wasserstein d_1-metric, showing that the distance is $O(\delta_{n,t} \log(\delta_{n,t}^{-1}))$, where $\delta_{n,t} = |n^{-1} m_n(t) - t| + n^{-1}$. Using their result, we bound the convergence rate in Kolmogorov–Smirnov distance in the following lemma.

Lemma 11.1. *Let $d_{\mathrm{KS}}(\cdot,\cdot)$ be Kolmogorov–Smirnov* (KS) *distance. Then*

$$d_{\mathrm{KS}}(C_n(t), Z(t)) = \exp\left[-\frac{1}{2} \ln \frac{1}{\delta_{n,t}} + \frac{1}{2} \ln\ln \frac{1}{\delta_{n,t}} + O(1) \right]. \tag{11.1}$$

Proof. The lemma is an immediate consequence of Fill and Janson [14, Lemma 5.1], since the random variable $Z(t)$ has a density function bounded by 10, according to Theorem 3.1. Indeed, by that result we have

$$d_{\mathrm{KS}}(C_n(t), Z(t)) \le 2^{1/2} [10\, d_1(C_n(t), Z(t))]^{1/2} = O([\delta_{n,t} \log(\delta_{n,t}^{-1})]^{1/2}).$$ □

Using the right-tail asymptotic bounds on the limiting `QuickQuant`(t) distribution function F_t in Theorems 9.2 and 10.1 (which extend to $t \in \{0,1\}$ by known results about the Dickman distribution), we can now derive the right-tail large-deviation behavior of $C_n(t)$.

Theorem 11.2. *Fix $t \in [0,1]$ and abbreviate $\delta_{n,t}$ as δ_n. Let (ω_n) be any sequence diverging to $+\infty$ as $n \to \infty$ and let $c > 1$. For integer $n \ge 3$, consider the interval*

$$I_n := \left[c, \frac{1}{2} \frac{\ln \delta_n^{-1}}{\ln\ln \delta_n^{-1}} \left(1 - \frac{\omega_n}{\ln\ln \delta_n^{-1}} \right) \right].$$

(a) *Uniformly for $x \in I_n$ we have*

$$\mathbb{P}(C_n(t) > x) = (1 + o(1)) \mathbb{P}(Z(t) > x) \quad \text{as } n \to \infty. \tag{11.2}$$

(b) *If $x_n \in I_n$ for all large n, then*

$$\mathbb{P}(C_n(t) > x_n) = \exp[-x_n \ln x_n - x_n \ln\ln x_n + O(x_n)]. \tag{11.3}$$

Proof. The proof is similar to that of Fill and Hung [13, Theorem 3.3] or its improvement in [12, Theorem 3.3]. We prove part (a) first. By Lemma 11.1, it suffices to show that

$$\exp\left[-\frac{1}{2}\ln\frac{1}{\delta_n}+\frac{1}{2}\ln\ln\frac{1}{\delta_n}+O(1)\right]\le o(\mathbb{P}(Z(t)>x_n))$$

with $x_n \equiv \frac{1}{2}\frac{\ln\delta_n^{-1}}{\ln\ln\delta_n^{-1}}\left(1-\frac{\omega_n}{\ln\ln\delta_n^{-1}}\right)$ and $\omega_n = o(\ln\ln\delta_n^{-1})$. Since, by Theorem 10.1, we have

$$\mathbb{P}(Z(t)>x_n)\ge \exp[-x_n\ln x_n - x_n\ln\ln x_n + O(x_n)],$$

it suffice to show that for any constant $C<\infty$ we have

$$-\frac{1}{2}\ln\frac{1}{\delta_n}+\frac{1}{2}\ln\ln\frac{1}{\delta_n}+C+x_n\ln x_n+x_n\ln\ln x_n+Cx_n\to-\infty. \tag{11.4}$$

This is routine and similar to what is done in [13, proof of Theorem 3.3]. This completes the proof of part (a).

Part (b) is immediate from part (a) and Theorems 9.2 and 10.1. □

Remark 11.3. Consider the particular choice $m_n(t) = \lfloor nt\rfloor + 1$ (for $t\in[0,1)$, with $m_n(1)=n$) of the sequences $(m_n(t))$. That is, suppose that $C_n(t)=X_n(t)$ as defined in (1.7). In this case, large-deviation upper bounds based on tail estimates of the limiting F_t have broader applicability than as described in Theorem 11.2 and are easier to derive, too. The reason is that, by Kodaj and Móri [25, Lemma 2.4], the random variable $X_n(t)$ is stochastically dominated by its continuous counterpart $Z(t)$. Then, by Theorem 10.1, uniformly in $t\in[0,1]$, we have

$$\mathbb{P}(X_n(t)>x)\le\mathbb{P}(Z(t)>x)\le\exp[-x\ln x - x\ln\ln x+O(x)] \tag{11.5}$$

for $x>1$; there is *no restriction at all* on how large x can be in terms of n or t.

Here is an example of a *very* large value of x for which the tail probability is nonzero and the aforementioned bound still matches logarithmic asymptotics to lead order of magnitude, albeit not to lead-order term. The largest possible value for the number $C_{n,m}$ of comparisons needed by `QuickSelect`(n,m) is $\binom{n}{2}$, corresponding in the natural coupling to any permutation of the n keys for which the $m-1$ keys smaller than the target key appear in increasing order, the $n-m$ keys larger than the target key appear in decreasing order, and the target key appears last; thus

$$\mathbb{P}\left(C_{n,m}=\binom{n}{2}\right)=\frac{1}{n!}\binom{n-1}{m-1},$$

which lies between $1/n!$ and $\binom{n-1}{\lceil(n-1)/2\rceil}/n! \sim 2^{n-(1/2)}/(n!\sqrt{\pi n})$. We conclude that for $x_n=(n-1)/2$ we have, uniformly in $t\in[0,1]$, that

$$P\left(X_n(t)\ge x_n\right)=\mathbb{P}\left(X_n(t)=x_n\right)=\exp[-2x_n\ln x_n+O(x_n)].$$

The bound (11.5) on $\mathbb{P}(X_n(t)>x)$ is in fact also (by the same proof) a bound on the larger probability $\mathbb{P}(X_n(t)\ge x)$, and in this case implies

$$P\left(X_n(t)\ge x_n\right)=\exp[-x_n\ln x_n+O(x_n\log\log x_n)].$$

The bound (11.5) is thus loose only by an asymptotic factor of 2 in the logarithm of the tail probability.

Remark 11.4. (a) We can use another result of Kodaj and Móri, namely, [25, Lemma 3.2], in similar fashion to quantify the Kolmogorov–Smirnov continuity of the process Z discussed in Remark 7.11. Let $0 \le t < u \le 1/2$ and $\delta = u - t$. Then the lemma asserts

$$d_1(Z(t), Z(u)) < 4\delta(1 + 2\log\delta^{-1}).$$

It follows using Fill and Janson [14, Lemma 5.1] that

$$d_{\mathrm{KS}}(Z(t), Z(u)) \le O((\delta\log\delta^{-1})^{1/2}) = \exp\left[-\tfrac{1}{2}\ln\delta^{-1} + \tfrac{1}{2}\ln\ln\delta^{-1} + O(1)\right],$$

uniformly for $|u - t| \le \delta$, as $\delta \downarrow 0$. We thus have *uniform* Kolmogorov–Smirnov continuity of Z.

(b) Kodaj and Móri [25] did not consider a lower bound on either of the distances in (a), but we can rather easily obtain a lower bound on the KS distance that is of order δ^2 uniformly for t and u satisfying $0 < t < t + \delta = u \le \min\{1/2, 2t\}$.

Indeed, for such t and u we have $\mathbb{P}(J(u) \le u) = 0$ and, by Theorem 8.2 (since $t \le u \le 1/2 \le \min\{1 - t, 2t\}$, as required by the hypotheses of the theorem) and in the notation of that theorem,

$$\begin{aligned}
\mathbb{P}(J(t) \le u) = \int_t^u f_t(x)\,\mathrm{d}x &= t\int_0^{(u/t)-1}\sum_{k=1}^{\infty}(-1)^{k-1}c_k z^k\,\mathrm{d}z \\
&\ge t\int_0^{(u/t)-1}(c_1 z - c_2 z^2)\,\mathrm{d}z \\
&= t\left[\frac{1}{2}c_1\left(\frac{u}{t} - 1\right)^2 - \frac{1}{3}c_2\left(\frac{u}{t} - 1\right)^3\right] \\
&\ge \frac{1}{3}c_1 t\left(\frac{u}{t} - 1\right)^2 > \frac{1}{150}(u - t)^2 = \frac{1}{150}\delta^2,
\end{aligned}$$

where the penultimate inequality holds because $\frac{u}{t} - 1 = \frac{\delta}{t} < 1$ and $0 < c_2 \le \frac{1}{2}c_1$.

(c) The lower bound in (b) can be improved to order δ when $t = 0$. Then for every $u \in [0, 1]$ we have $\mathbb{P}(J(0) \le u) = e^{-\gamma}u$, and so for $u \in [0, 1/2]$ we have

$$d_{\mathrm{KS}}(Z(0), Z(u)) \ge e^{-\gamma}u.$$

12 Perfect simulation

In this final section, we show how results of the preceding sections can be used to produce an algorithm for perfect simulation of the limiting `QuickQuant` random variable $J(t) = Z(t) - 1$ for given $t \in (0, 1)$. Perfect simulation from the common (Dickman) distribution of $J(0)$ and $J(1)$ can be done in similar fashion, or one can use highly efficient perfect simulation algorithms based on Markov Chain Monte Carlo due to Fill and Huber [11] or Devroye and Fawzi [7]. We make no claim whatsoever of efficiency for the algorithm presented in this section; rather, this section should be regarded as “proof of concept”.

The algorithm is based on ideas from the encyclopedic treatment of nonuniform random variate generation by Devroye [6] and is quite similar to the algorithm for perfect simulation from the limiting distribution for the normalized number of comparisons used by `QuickSort` devised in [8]. We have endeavored, however, to make our treatment here reasonably (though not entirely) self-contained.

Henceforth, fix $t \in (0, 1)$. Let G_n denote the distribution of $n^{-1}C_{n,m_n} - 1$, referencing here the normalized number of comparisons $n^{-1}C_{n,m_n}$ in (1.6), for n large enough that the specific choice $m_n = \lfloor nt + \frac{1}{2}\rfloor$ we hereby make satisfies $m_n \ge 1$ [i.e., for $n \ge 1/(2t)$].

We begin by asserting four properties enjoyed by the distribution of $J \equiv J(t)$, with continuous density $f \equiv f_t$ on $(0, \infty)$ and distribution function $F \equiv F_t$, namely, that there are finite constants K_1, K_2, K_3 and positive sequences (δ_n) and (ϵ_n), all explicitly identifiable, satisfying

(P1) $\mathbb{E}\, J^4 \le K_1$;
(P2) f is bounded by K_2;
(P3) the Lipschitz constant Λ for f satisfies $\Lambda \le K_3$; and
(P4) the sequences (δ_n) and (ϵ_n) vanish in the limit as $n \to \infty$, and

$$\left| \frac{G_n(x + (\delta_n/2)) - G_n(x - (\delta_n/2))}{\delta_n} - f(x) \right| \le \epsilon_n.$$

Note that by Theorem 3.1 we can choose $K_2 = 10$, and by Remark 7.5 we can choose $K_3 = \lambda[t^{-1} \ln t^{-1} + (1-t)^{-1} \ln(1-t)^{-1}]$ with $\lambda = 64000$. In Lemma 12.1 we will show that we can choose $K_1 = 196$, and in Section 12.3 we will prove the "semi-local limit theorem" (P4) for suitably chosen (δ_n) and (ϵ_n).

12.1 The algorithm

The algorithm is based on classical von Neumann rejection sampling. Just as argued in [8, Section 2], from the nonnegativity of J, Theorem VII.3.5 in Devroye [6], Markov's inequality, and (P1) we have the bound

$$f(x) \le \sqrt{2\Lambda[1 - F(x)]} \le \sqrt{2K_3 K_1 x^{-4}}, \qquad x > 0.$$

Therefore, if we define

$$g(x) := \min\{K_2, (2K_1K_3)^{1/2} x^{-2}\}, \qquad x > 0,$$

then $f \le g$. Note that then $\tilde{g} := \xi g$ is a probability density when $\xi := 1/\|g\|_{L_1} = [2K_2^{1/2}(2K_1K_3)^{1/4}]^{-1}$. According to Devroye [6, Theorem VII.3.3], a sample (call it W) from density $\tilde{g}$ is given by $W = \left[(2K_1K_3)^{1/4}/K_2^{1/2}\right] U_1/U_2$, where U_1 and U_2 are independent uniform$(0,1)$ random variables.

Applying rejection sampling, this observation $W = w$ should be accepted as a realization of J with probability $f(w)/g(w)$. Our algorithm therefore generates a third independent uniform$(0,1)$ random variable U and accepts w if and only if $U \le f(w)/g(w)$. We don't have a formula for f, but from (P4) with n sufficiently large we can (with probability one) determine whether or not to accept by computing

$$g_n(w) := [G_n(w + (\delta_n/2)) - G_n(w - (\delta_n/2))]/\delta_n.$$

Using recursive computations like those in [8, Section 3], computing this approximation to $f(w)$ can be done in time $O(n^5\delta_n)$ [the brief explanation being that there are respectively $O(n)$, $O(n^2)$, and $O(n)$ values of m, the number of comparisons, and the initial pivot to consider in computing the probability mass functions for the random variables $C_{n,m}$, $1 \le m \le n$, and then computation of $g_n(w)$ requires summing over $O(n\delta_n)$ values].

12.2 Bounds on the moments of J

Let $\|\cdot\|$ denote L^p-norm. Choosing $p = 4$, the following lemma shows that we can take $K_1 = 196$ in (P1).

Lemma 12.1. *For any $t \in (0,1)$ and any $p \in [1, \infty)$, we have*

$$\|J(t)\|_p \le \left(\frac{2 - 2^{-p}}{p+1}\right)^{1/p} \Bigg/ \left[1 - \left(\frac{2 - 2^{-p}}{p+1}\right)^{1/p}\right].$$

Proof. Recall $J(t) = Z(t) - 1$ and (1.11). By Lemma 3.5 in Fill and Nakama [16], for any $p > 0$ and $k \geq 1$ we have

$$\mathbb{E}\left[R_k(t) - L_k(t)\right]^p \leq \left(\frac{2 - 2^{-p}}{p+1}\right)^k.$$

The lemma now follows from subadditivity of the L^p-norm for $p \in [1, \infty)$. □

12.3 Proof of the semi-local limit theorem (P4)

To establish the semi-local limit theorem (P4) for suitably chosen sequences (δ_n) and (ϵ_n), we will need the following quantitative sharpening of Corollary 3.1 in Kodaj and Móri [25]. Recall the notation at the start of Section 11; we assume $n \geq \max\{2, 1/(2t)\}$ and make the specific choice $m_n(t) = \lfloor nt + \frac{1}{2} \rfloor$.

Proposition 12.2. *For the Wassertstein d_1-metric we have*

$$d_1(C_n(t), Z(t)) \leq K_4 \frac{\ln n}{n}$$

with $K_4 < 29$.

Granted Proposition 12.2 for the moment, we now establish (P4). By the same argument as in the proof of Lemma 11.1, it follows from Proposition 12.2 that

$$d_{\mathrm{KS}}(C_n(t), Z(t)) \leq \left(2K_2K_4 \frac{\ln n}{n}\right)^{1/2}.$$

Therefore, for any positive sequence (δ_n) we have

$$\begin{aligned}\Big|\,[G_n(x + (\delta_n/2)) - G_n(x - (\delta_n/2))] - [F(x + (\delta_n/2)) - F(x - (\delta_n/2))]\,\Big|\\ \leq 2d_{\mathrm{KS}}(C_n(t), Z(t)) \leq \left(8K_2K_4 \frac{\ln n}{n}\right)^{1/2}\end{aligned}$$

while also

$$\begin{aligned}&\Big|F(x + (\delta_n/2)) - F(x - (\delta_n/2)) - \delta_n f(x)\Big|\\ &\quad\leq \int_{-\delta_n/2}^{\delta_n/2} \Big|f(x+y) - f(x)\Big| \,\mathrm{d}y \leq \int_{-\delta_n/2}^{\delta_n/2} K_3\, |y| \,\mathrm{d}y = \frac{K_3}{4}\delta_n^2.\end{aligned}$$

Combining these two inequalities we find

$$\left|\frac{G_n(x + (\delta_n/2)) - G_n(x - (\delta_n/2))}{\delta_n} - f(x)\right| \leq \left(8K_2K_4 \frac{\ln n}{n}\right)^{1/2} \delta_n^{-1} + \frac{K_3}{4}\delta_n.$$

Choosing

$$\delta_n := 2\left(8\frac{K_2K_4}{K_3^2}\frac{\ln n}{n}\right)^{1/4}$$

to minimize the right-hand side, we obtain the desired inequality

$$\left|\frac{G_n(x + (\delta_n/2)) - G_n(x - (\delta_n/2))}{\delta_n} - f(x)\right| \leq \epsilon_n$$

with

$$\epsilon_n := \left(8K_2K_3^2K_4 \frac{\ln n}{n}\right)^{1/4}.$$

We conclude this section with the proof of Proposition 12.2.

Proof of Proposition 12.2. We use results from [25], translated to our notation. It follows from [25, Lemma 2.4] (and a stochastic comparison of what the authors call Y's and Z's) that for any $i \in \{1, \dots, n\}$ we have

$$\Delta(n,i) := d_1(C_{n,i}, nZ((i-(1/2))/n) = n\,\mathbb{E}\,Z((i-(1/2))/n) - \mathbb{E}\,C_{n,i}.$$

By [25, Lemma 2.2],

$$\begin{aligned} & n\,\mathbb{E}\,Z((i-(1/2))/n) \\ & = 2n(1+\ln n) - (2i-1)\ln(i-(1/2)) - [2(n-i)+1]\ln(n-i+(1/2)). \end{aligned}$$

By (1.2),

$$\mathbb{E}\,C_{n,i} = 2[(n+1)H_n - (n+3-i)H_{n+1-i} - (i+2)H_i + (n+3)].$$

Letting γ denote the Euler–Mascheroni constant and using the inequalities

$$\ln k + \gamma \le H_k \le \ln k + \gamma + (2k)^{-1}$$

for the harmonic numbers and other simple inequalities, one can show (we suppress the details) that

$$\Delta(n,i) \le 5[\ln i + \ln(n-i+1)] - 2\ln n + 8\gamma + 2; \tag{12.1}$$

compare [25, Theorem 3.2].

Now we are prepared to sharpen [25, Corollary 3.1] and obtain our desired inequality. With

$$W_n := nZ((m_n(t) - (1/2))/n),$$

observe

$$\begin{aligned} nd_1(C_n(t), Z(t)) &= d_1(C_{n,m_n(t)}, nZ(t)) \\ &\le d_1(C_{n,m_n(t)}, W_n) + d_1(W_n, nZ(t)) \\ &= \Delta(n, m_n(t)) + d_1(W_n, nZ(t)). \end{aligned}$$

To bound this last expression we employ (12.1) for the first term and [25, Lemma 3.2] (for which we assume, without loss of generality, that $t \le 1/2$) for the second term; the result is

$$\begin{aligned} & nd_1(C_n(t), Z(t)) \\ & \le 5[\ln m_n + \ln(n-m_n+1)] - 2\ln n + 8\gamma + 2 + 4n\tilde{\delta}_n\left(1 + 2\ln\frac{1}{\tilde{\delta}_n}\right), \end{aligned}$$

where

$$0 \le \tilde{\delta}_n := t - \frac{m_n - (1/2)}{n} < n^{-1}.$$

Using this bound on $\tilde{\delta}_n$ and the inequality $m_n(n-m_n+1) \le (n+1)^2/4$, it follows from simple bounds that for $n \ge 2$ we have the asserted inequality; we suppress the routine details. □

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