Electronic Journal of Probability

Electron. J. Probab. **27** (2022), article no. 72, 1–32.
ISSN: 1083-6489 https://doi.org/10.1214/22-EJP796

Distorted Brownian motions on space with varying dimension*

Liping Li[†] Shuwen Lou[‡]

Abstract

In this paper we introduce and study distorted Brownian motion on state spaces with varying dimension (dBMV in abbreviation). Roughly speaking, the state space of dBMV is embedded in $\mathbb{R}^4$ and consists of two components: a 3-dimensional component and a 1-dimensional component. These two parts are joined together at the origin. 3-dimensional dBMV models homopolymer with attractive potential at the origin and has been studied in [9], [8], [7]. dBMV restricted on the 1-dimensional component can be viewed as a Brownian motion with drift of Kato-class type. Such a process with varying dimensional can be concisely characterized in terms of Dirichlet forms. Using the method of radial process developed in [5] combined with some calculation specifically for dBMV, we get its short-time heat kernel estimates.

Keywords: distorted Brownian motions; Dirichlet forms; varying dimension; heat kernel estimates.
MSC2020 subject classifications: 60J45; 60J46; 60J60; 60J65.
Submitted to EJP on May 11, 2021, final version accepted on May 15, 2022.

1 Introduction

To give a brief background on the studies for processes with varying dimension, in [5], Brownian motion with varying dimension (BMVD) is studied as an example of Brownian motion on state spaces with singularities. The state space of BMVD can be visualized as a 2-dimensional plane with a 1-dimensional pole installed on it. Since a 2-dimensional Brownian motion does not hit a singleton, the construction of BMVD relies on the method of “darning”: Setting the resistance on a 2-dimensional disc equal to zero. The disc is centered at the intersection of the plane and the pole. From there we are motivated

*The first named author is partially supported by NSFC (No. 11688101 and No. 11801546) and Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (No. 2008DP173182).

[†]RCSDS, HCMS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. E-mail: liliping@amss.ac.cn

[‡]Loyola University Chicago, United States of America.
E-mail: slou1@luc.edu

to consider other more natural models with varying dimension. Three-dimensional distorted Brownian motion arises in statistical physics by providing a continuum model for homopolyer with attractive potential at the origin. Unlike a 3-dimensional standard Brownian motion which does not hit any singleton, a 3-dimensional distorted Brownian motion has positive capacity at the origin, which allows us to “extend” such a process onto a state space with varying dimension. Many of the properties of 3-dimensional distorted Brownian motion have been investigated in [9], [8], and [7], including its explicit transition densities and behaviors near the origin. Later in [11], Fitzsimmons and the first named author give a description to dBMV in terms of Dirichlet forms.

The state space of dBMV is embedded in $\mathbb{R}^4$. We let $\mathbb{R}^4 \supset \mathfrak{R}^3 := \{(\mathbf{x}, 0_1) : \mathbf{x} \in \mathbb{R}^3\} \cong \mathbb{R}^3$ and $\mathbb{R}^4 \supset \mathfrak{R}_+ := \{(0_3, x) : x \in \mathbb{R}_+\} \cong \mathbb{R}_+$. Set

$$E := \mathfrak{R}^3 \cup \mathfrak{R}_+.$$

Clearly, $\mathfrak{R}^3 \cap \mathfrak{R}_+ = (0_3, 0_1) =: \mathbf{0} \in \mathbb{R}^4$. The restriction of dBMV on $\mathfrak{R}^3$ behaves like 3-dimensional distorted Brownian motion studied in [9], [8], [7]. Roughly speaking, a distorted Brownian motion subjects to a strong push towards the origin. The restriction of dBMV on the $\mathfrak{R}$ is equivalent to a 1-dimensional Brownian motion with drift function b satisfying Kato-class condition. The rigorous definition for 3-dimensional dBM and dBM with varying dimension is given in Section 3.2. Similar to [11], dBMV can be similarly characterized by means of Dirichlet forms.

The main result of this paper is the short-time heat kernel estimates for this process. We use the method of radial process developed in [5]. The major difficulty is that, unlike standard Brownian motion, the existing results on heat kernels of distorted Brownian is very limited. We derive such results by recognizing that the Dirichlet form of distorted Brownian motion can be viewed as that for standard Brownian motion via an h-transform.

Before we state the main results, we introduce underlying measure and the metric equipped on the state space. The measure m on E is given by the Lebesgue measures on $\mathfrak{R}^3$ and $\mathfrak{R}_+$, i.e.,

$$m|_{\mathfrak{R}^3} := d^3\mathbf{x}|_{\mathfrak{R}^3}, \quad m|_{\mathfrak{R}_+} := d^1x|_{\mathfrak{R}_+}. \tag{1.1}$$

Here $d^3\mathbf{x}$ means the Lebesgue measure on a three-dimensional Euclidean space and d^1x is that on a one-dimensional space. m is well-defined because 0 is of zero-Lebesgue-measure for both 1-dimensional and 3-dimensional spaces. For the sake of brevity, we ignore these superscripts if no confusions arises.

Throughout this paper, we use $|\cdot|$ to denote the Euclidean distance. To be more exact, $|x-y|$ is the Euclidean distance between x and y if either $x, y \in \mathfrak{R}^3$ or $x, y \in \mathfrak{R}_+$. By slightly abusing the notation,

$$|x-y| := |x-\mathbf{0}| + |y-\mathbf{0}|, \quad x \in \mathfrak{R}^3,\, y \in \mathfrak{R}_+. \tag{1.2}$$

The rest of this paper is organized as follows. Section 3.3 gives a Dirichlet form construction and characterization of dBM with varying dimension (denoted by M), as well as its infinitesimal generator. Some important properties of M are given in Section 3.4, including the fact that the “origin” of the state space of M has positive capacity, as well as the SDE characterization for M. In section 5 we give the proof to the short-time heat kernel estimates for M.

We follow the convention that in the statements of the theorems or propositions $C, C_1, \cdots$ denote positive constants, whereas in their proofs $c, c_1, \cdots$ denote positive constants whose exact value is unimportant and may change from line to line.

2 Distorted BMs on $\mathbb{R}^3$

In this section, we give a brief overview for a special family of 3-dimensional dBMs, which will be used as building blocks to construct dBMs on space with varying dimension in §3. The name of dBMs, to our knowledge, is due to [2]. They are introduced by so-called energy forms that are special Dirichlet forms thereof. What we are concerned with are those induced by a family of concrete density functions on $\mathbb{R}^3$. More precisely, fix a constant $\gamma \in \mathbb{R}$ and set

$$\psi_\gamma(x) := \frac{\mathrm{e}^{-\gamma|x|}}{2\pi|x|}, \quad x \in \mathbb{R}^3. \tag{2.1}$$

Further set a measure $m_\gamma(dx) := \psi_\gamma(x)^2 dx$ on $\mathbb{R}^3$. It is easy to verify that m_γ is a positive Radon measure with full support. Note that m_γ is finite, i.e. $\psi_\gamma \in L^2(\mathbb{R}^3)$, if and only if $\gamma > 0$. Define an energy form on $L^2(\mathbb{R}^3, m_\gamma)$ as follows:

$$\begin{aligned} \mathcal{F}^3 &:= \left\{ f \in L^2(\mathbb{R}^3, m_\gamma) : \nabla f \in L^2(\mathbb{R}^3, m_\gamma) \right\}, \\ \mathcal{E}^3(f,g) &:= \frac{1}{2}\int_{\mathbb{R}^3} \nabla f(x) \cdot \nabla g(x) m_\gamma(dx), \quad f, g \in \mathcal{F}^3, \end{aligned}$$

where ∇f stands for the weak derivative of f. We shall write $(\mathcal{E}^{3,\gamma}, \mathscr{F}^{3,\gamma})$ for $(\mathcal{E}^3, \mathscr{F}^3)$ when there is a risk of ambiguity.

2.1 Associated dBM

Some basic facts about $(\mathcal{E}^3, \mathcal{F}^3)$ are collected in the following theorem. Since $(\mathcal{E}^3, \mathcal{F}^3)$ is indicated to be a regular Dirichlet form, we denote its associated dBM by $X^3 = \{(X^3_t)_{t\geq 0}, \left(\mathbf{P}^3_x\right)_{x\in\mathbb{R}^3}\}$ henceforth. Note that the case $\gamma > 0$ has been considered in [11] but we present a different (and simpler) proof as below.

Theorem 2.1. *The following statements hold:*

- *(i)* $(\mathcal{E}^3, \mathcal{F}^3)$ *is a regular and irreducible Dirichlet form on* $L^2(\mathbb{R}^3, m_\gamma)$ *with* $C_c^\infty(\mathbb{R}^3)$ *being its special standard core.*
- *(ii) When* $\gamma \geq 0$, $(\mathcal{E}^3, \mathcal{F}^3)$ *is recurrent. When* $\gamma < 0$*, it is transient.*

Proof. Recall that $(\mathcal{E}^{3,\gamma}, \mathcal{F}^{3,\gamma})$ also denotes the Dirichlet form $(\mathcal{E}^3, \mathcal{F}^3)$. It is straightforward to verify that $(\mathcal{E}^{3,\gamma}, \mathcal{F}^{3,\gamma})$ is a Dirichlet form on $L^2(\mathbb{R}^3, m_\gamma)$ and $C_c^\infty(\mathbb{R}^3) \subset \mathcal{F}^{3,\gamma}$. The irreducibility of $(\mathcal{E}^{3,\gamma}, \mathcal{F}^{3,\gamma})$ for the case $\gamma > 0$ has been proved in [11, Proposition 2.4]. The case $\gamma \leq 0$ can be concluded by the comparison of irreduciblity presented in [14, Corollary 4.6.4]. To show $C_c^\infty(\mathbb{R}^3)$ is $\mathcal{E}_1^{3,\gamma}$-dense in $\mathcal{F}^{3,\gamma}$, we first note that this is true for the case $\gamma = 0$ since ψ_0 belongs to the so-called Muckenhoupt's class; see e.g. [16]. Then it suffices to consider the case $\gamma \neq 0$. Let $\mathcal{F}_\mathrm{c}^{3,\gamma}$ (resp. $\mathcal{F}_\mathrm{c}^{3,0}$) be the family of all bounded functions with compact support in $\mathcal{F}^{3,\gamma}$ (resp. $\mathcal{F}^{3,0}$). We first assert that $\mathcal{F}_\mathrm{c}^{3,\gamma}$ is $\mathcal{E}_1^{3,\gamma}$-dense in $\mathcal{F}^{3,\gamma}$. To do this, fix $f \in \mathcal{F}^{3,\gamma}$ and assume without loss of generality that f is bounded (see [14, Theorem 1.4.2]). Take $\eta \in C_c^\infty(\mathbb{R}^3)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\{x : |x| \leq 1\}$. Set $\eta_n(x) := \eta(x/n)$ and $f_n := f \cdot \eta_n \in \mathcal{F}_\mathrm{c}^{3,\gamma}$ for all $n \in \mathbb{N}$. Since $0 \leq \eta_n \leq 1$ and $\eta_n \to 1$ pointwisely, it follows from the dominated convergence theorem that

$$\int |f - f_n|^2 dm_\gamma = \int |f|^2 \cdot |1 - \eta_n|^2 dm_\gamma \to 0$$

as $n \uparrow \infty$. On the other hand,

$$\nabla f - \nabla f_n = (1 - \eta_n)\nabla f - \frac{f}{n}\nabla\eta\left(\frac{x}{n}\right).$$

Since $\nabla f, f \in L^2(\mathbb{R}^3, m_\gamma)$ and $\nabla \eta$ is bounded, one can obtain that

$$\int |\nabla f - \nabla f_n|^2 dm_\gamma \to 0.$$

Hence $\mathcal{E}_1^{3,\gamma}(f_n - f, f_n - f) \to 0$ as $n \to \infty$. Next fix $g \in \mathcal{F}_{\mathrm{c}}^{3,\gamma}$ and take $L > 0$ such that $\mathrm{supp}[g] \subset \{x : |x| < L\}$. Clearly, there exist two appropriate positive constants c_1 and c_2 (depending on L and γ) such that for all x with $|x| < L$,

$$c_1 \psi_0(x) \leq \psi_\gamma(x) \leq c_2 \psi_0(x). \tag{2.2}$$

This implies $g \in \mathcal{F}_{\mathrm{c}}^{3,0}$. Then there exists a sequence of functions $g_n \in C_c^\infty(\mathbb{R}^3)$ with $\mathrm{supp}[g_n] \subset \{x : |x| < L\}$ converging to g relative to the $\mathcal{E}_1^{3,0}$-norm. By using (2.2) again, we can conclude that g_n also converges to g relative to the $\mathcal{E}_1^{3,\gamma}$-norm. Therefore, $(\mathcal{E}^{3,\gamma}, \mathcal{F}^{3,\gamma})$ is regular and $C_c^\infty(\mathbb{R}^3)$ is a special standard core of it.

The recurrence of $(\mathcal{E}^{3,\gamma}, \mathcal{F}^{3,\gamma})$ for the case $\gamma > 0$ has been also illustrated in [11, Proposition 2.4]. For the case $\gamma = 0$, let η_n be as above. Then $\eta_n \in \mathcal{F}^{3,0}$ and $\eta_n \to 1$ pointwisely. To attain the recurrence of $(\mathcal{E}^{3,0}, \mathcal{F}^{3,0})$, it suffices to show $\mathcal{E}^{3,0}(\eta_n, \eta_n) \to 0$ as $n \to \infty$. Indeed,

$$\mathcal{E}^{3,0}(\eta_n, \eta_n) = \frac{1}{2n^2} \int |(\nabla \eta)(x/n)|^2 \frac{dx}{|x|^2} = \frac{1}{2n} \int |\nabla \eta(x)|^2 \frac{dx}{|x|^2} \to 0.$$

Finally consider the case $\gamma < 0$. Since $\psi_\gamma(x) \geq |\gamma|$ for all x, it follows that for all $f \in C_c^\infty(\mathbb{R}^3)$,

$$\mathcal{E}^{3,\gamma}(f, f) \geq \frac{\gamma^2}{2} \int |\nabla f|^2 dx =: \gamma^2 \mathbf{D}(f, f),$$

where $\mathbf{D}$ is the Dirichlet integral that induces the associated Dirichlet form of 3-dimensional Brownian motion. Clearly, 3-dimensional Brownian motion is transient. By virtue of [14, Theorem 1.6.4], we can conclude the transience of $(\mathcal{E}^{3,\gamma}, \mathcal{F}^{3,\gamma})$. That completes the proof. □

When $\gamma \geq 0$, $(\mathcal{E}^3, \mathcal{F}^3)$ is not only recurrent but also ergodic in the following sense: For $\gamma > 0$ and any $x \in \mathbb{R}^3$,

$$\frac{1}{t} \int_0^t \mathbf{P}_x^3(X_s^3 \in \cdot) ds \to \frac{m_\gamma(\cdot)}{m_\gamma(\mathbb{R}^3)} = 2\pi\gamma m_\gamma(\cdot), \quad \text{weakly as } t \uparrow \infty; \tag{2.3}$$

For $\gamma = 0$, the probability measure on the left hand side is vaguely convergent to 0 as $t \uparrow \infty$. See e.g. [14, Theorem 4.7.3].

2.2 Generator and motivated polymer model

The dBM X^3 is motivated by a singular polymer model explored in e.g. [7, 8, 9]. Let us use a few lines to explain it. Fix $T > 0$ and let $\Omega_T := C([0, T], \mathbb{R}^d)$, i.e. the family of all continuous paths of size T in $\mathbb{R}^d$, be the configuration space of the system. Then the polymer model is described by a Gibbs ensemble at each inverse temperature β (≥ 0), realized as a probability measure $\mathbf{Q}_{\beta,T}$ on Ω_T, which is also called a Gibbs measure. More precisely, the underlying probability measure $\mathbf{Q}_{0,T}$ is identified with the Wiener measure on Ω_T in this model, and we also denote it by $\mathbf{Q}_T$ in abbreviation. For $\beta > 0$, $\mathbf{Q}_{\beta,T}$ is determined by the so-called Hamiltonian H_T, which is given by a certain potential function v on $\mathbb{R}^d$ in the following manner:

$$H_T(\omega) = -\int_0^T v(\omega(t)) dt, \quad \omega \in \Omega_T. \tag{2.4}$$

In other words,

$$\mathbf{Q}_{\beta,T}(d\omega)=\frac{\exp\{-\beta H_T(\omega)\}}{Z_{\beta,T}}\mathbf{Q}_T(d\omega)=\frac{\exp\{\beta\int_0^T v(\omega(t))dt\}}{Z_{\beta,T}}\mathbf{Q}_T(d\omega), \tag{2.5}$$

where $Z_{\beta,T}:=\mathbf{E}_T\exp\{-\beta H_T\}$ is the so-called partition function. The motivated model is on dimension 3, i.e. $d=3$, and given by a singular potential function $v=\delta_0$, i.e. the delta function at the origin. In this case, the Hamiltonian H_T is understood as a limitation $-\lim_{\varepsilon\downarrow 0}\int_0^T A_\varepsilon\cdot 1_{(-\varepsilon,\varepsilon)}(\omega_t)dt$ in a certain manner, where A_ε ($\uparrow\infty$ as $\varepsilon\downarrow 0$) is a constant depending on a crucial parameter $\gamma\in\mathbb{R}$, and meanwhile at the heuristic level the inverse temperature β in (2.5) is also retaken to be a function of γ, i.e. $\beta:=\beta_\gamma>0$ (and $\beta_{-\infty}:=0$); see e.g. [8]. There are at least three ways to manifest the phase transition parametrized by γ and the critical value is $\gamma_{cr}=0$ — The first two are already mentioned in [8] and the last one is due to the Dirichlet form description of X^3:

(1) The first way is to observe the thermodynamic limit of $\mathbf{Q}_{\beta_\gamma,T}$ as $T\uparrow\infty$. It can be shown that (see e.g. [8])

- (i) When $\gamma>\gamma_{cr}=0$, the limiting measure of $\mathbf{Q}_{\beta_\gamma,T}$ under suitable scaling as $T\uparrow\infty$ exists and induces a diffusion process on $\mathbb{R}^3$. In fact, this process is nothing but X^3 obtained in Theorem 2.1, which possesses an ergodic distribution $2\pi\gamma m_\gamma(dx)=2\pi\gamma\psi_\gamma(x)^2dx$ (see (2.3)). In this case, the ensemble is called in the globular state;
- (ii) When $\gamma=\gamma_{cr}=0$, the limiting process is mixed Gaussian;
- (iii) When $\gamma<\gamma_{cr}=0$, the scaling is taken to be a different one, and the limiting process is nothing but 3-dimensional Brownian motion. In this case, the ensemble is called in the diffusive state.

(2) The second way is to analyse the spectrum of the informal Schrödinger operator

$$\frac{1}{2}\Delta+\beta_\gamma\cdot\delta_0: L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3), \tag{2.6}$$

where Δ is the Laplacian operator. Note that all self-adjoint extensions on $L^2(\mathbb{R}^3)$ of $\frac{1}{2}\Delta$ restricting to $C_c^\infty(\mathbb{R}^3\setminus\{0\})$ can be parametrized by a constant $\gamma\in\{-\infty\}\cup\mathbb{R}$; see e.g. [8, Theorem 2.1]. Denote the family of all these extensions by $\{\mathcal{L}_\gamma:\gamma=-\infty \text{ or } \gamma\in\mathbb{R}\}$ and particularly $\mathcal{L}_{-\infty}=\frac{1}{2}\Delta$ corresponds to the underlying case (Recall that $\beta_{-\infty}=0$). Then (2.6) should be understood as $\mathcal{L}_\gamma$. Denote the spectrum set of $\mathcal{L}_\gamma$ by $\sigma(\mathcal{L}_\gamma)$. It is well known that

- (i) When $\gamma>\gamma_{cr}=0$, $\sigma(\mathcal{L}_\gamma)=(-\infty,0]\cup\{\gamma^2/2\}$. Moreover, ψ_γ is the ground state of $\mathcal{L}_\gamma$, i.e.
$$\mathcal{L}_\gamma\psi_\gamma=\frac{\gamma^2}{2}\psi_\gamma,$$
and $\gamma^2/2$ is exactly the free energy of the ensemble, i.e.
$$\lim_{T\uparrow\infty}\frac{\log Z_{\beta_\gamma,T}}{T}=\frac{\gamma^2}{2};$$
- (ii) When $\gamma\le\gamma_{cr}=0$, $\sigma(\mathcal{L}_\gamma)=(-\infty,0]$ and no eigenvalues exist.

The third way is based on the relation between $\mathcal{L}_\gamma$ and the generator of $(\mathcal{E}^3,\mathcal{F}^3)$. Define an operator $\mathcal{A}_\gamma$ on $L^3(\mathbb{R}^3,m_\gamma)$ by informal h-transform as follows:

$$\begin{aligned}&\mathcal{D}(\mathcal{A}_\gamma)=\{f\in L^2(\mathbb{R}^3,m_\gamma):f\psi_\gamma\in\mathcal{D}(\mathcal{L}_\gamma)\},\\&\mathcal{A}_\gamma f=\frac{\mathcal{L}_\gamma(\psi_\gamma f)}{\psi_\gamma}-\frac{\gamma^2}{2}f,\qquad f\in\mathcal{D}(\mathcal{A}_\gamma).\end{aligned} \tag{2.7}$$

It is not hard to verify that $C_c^\infty(\mathbb{R}^3\setminus\{0\})\subset\mathcal{D}(\mathcal{A}_\gamma)$ and for $f\in C_c^\infty(\mathbb{R}^3\setminus\{0\})$ (see e.g. [11, (2.3)]),

$$\mathcal{A}_\gamma f=\frac{1}{2}\Delta f+\frac{\nabla\psi_\gamma}{\psi_\gamma}\cdot\nabla f.$$

The lemma below links $(\mathcal{E}^3,\mathcal{F}^3)$ with $\mathcal{L}_\gamma$.

Lemma 2.2. *The operator $\mathcal{A}_\gamma$ defined by* (2.7) *is the generator of $(\mathcal{E}^3,\mathcal{F}^3)$.*

Proof. The case $\gamma>0$ has been shown in [11]. For the case $\gamma\leq 0$, see [1, Appendix F]. □

Remark 2.3. Since the semigroup associated with $\mathcal{L}_\gamma$ admits a symmetric transition density with respect to the Lebesgue measure, i.e. there exists a suitable function $r^\gamma_t(x,y)$ such that $r^\gamma_t(x,y)=r^\gamma_t(y,x)$ and $R^\gamma_t f(x):=\int_{\mathbb{R}^3}r^\gamma_t(x,y)f(y)dy$ forms this semigroup (see e.g. [8]), this lemma tells us the semigroup associated with $\mathcal{A}_\gamma$ admits a symmetric transition density with respect to m_γ:

$$p^\gamma_t(x,y):=\frac{\mathrm{e}^{-\frac{\gamma^2}{2}t}\cdot r^\gamma_t(x,y)}{\psi_\gamma(x)\psi_\gamma(y)}.$$

In other words,

$$P^\gamma_t f(x):=\int_{\mathbb{R}^3}p^\gamma_t(x,y)f(y)m_\gamma(dy)=\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-\frac{\gamma^2}{2}t}\psi_\gamma(y)r^\gamma_t(x,y)}{\psi_\gamma(x)}f(y)dy,\quad f\in L^2(\mathbb{R}^3,m_\gamma),$$

is the semigroup associated with $\mathcal{A}_\gamma$.

Then Theorem 2.1 leads to the third reflection of the same phase transition:

(3) Under the h-transform (2.7), $\mathcal{L}_\gamma$ corresponds to the dBM X^3. The global property of X^3 depending on γ manifests the same phase transition as the two mentioned above: When $\gamma\geq\gamma_{cr}=0$, X^3 is recurrent; otherwise it is transient. The difference between the diffusive state $\gamma>0$ and the critical state $\gamma=0$ has already illustrated by the ergodicity of X^3 after the proof of Theorem 2.1.

Remark 2.4. A similar discussion about the critical phenomenon of certain Markovian Schrödinger forms appeared in [23], where h-transform and the global property of Dirichlet forms are employed as well. However in the current paper, the Schrödinger form induced by $\mathcal{L}_\gamma$ (or informally by (2.6)) is not Markovian. In other words, $\mathcal{L}_\gamma$ can not be the generator of a certain Markov process.

2.3 Characterization via h-transform

This subsection is devoted to illustrating some connections between X^3 and three-dimensional Brownian motion. We use the notation $R_t:=R_t^{-\infty}$ to stand for the probability transition semigroup of 3-dimensional Brownian motion $W=(W_t)_{t\geq 0}$ as well as its L^2-semigroup if no confusions caused. Note that ψ_γ is finite out of $\{0\}$. Consider the following h-transform with $h:=\psi_\gamma$:

$${}_hR^\gamma_t(x,dy):=\begin{cases}\mathrm{e}^{-\frac{\gamma^2}{2}t}\dfrac{\psi_\gamma(y)}{\psi_\gamma(x)}R_t(x,dy), & x\in\mathbb{R}^3\setminus\{0\},\\ 0, & x=0.\end{cases}\tag{2.8}$$

It is not hard to verify that ψ_γ is $\frac{\gamma^2}{2}$-excessive relative to R_t in the sense that $\mathrm{e}^{-\gamma^2t/2}R_t\psi_\gamma\leq\psi_\gamma$ and ${}_hR^\gamma_t$ is a sub-Markovian semigroup on $\mathbb{R}^3\setminus\{0\}$. Denote the induced Markov process of ${}_hR^\gamma_t$ on $\mathbb{R}^3\setminus\{0\}$ by ${}_hW^\gamma=\{({}_hW^\gamma_t)_{t\geq0},({}_h\mathbf{P}^\gamma_x)_{x\in\mathbb{R}^3},\zeta_h\}$, where ${}_h\mathbf{P}^\gamma_x$ is the law of ${}_hW^\gamma$ starting from x and ζ_h is its life time.

Remark 2.5. When $\gamma \geq 0$, ψ_γ is nothing but the $\gamma^2/2$-resolvent kernel of W. More precisely, let $r(t,x)$ be the Gaussian kernel, i.e. $r(t,x) = \frac{1}{(2\pi t)^{3/2}} \mathrm{e}^{-\frac{|x|^2}{2t}}$. Then

$$\psi_\gamma(x) = \int_0^\infty \mathrm{e}^{-\frac{\gamma^2 t}{2}} r(t,x) dt.$$

Particularly ψ_0 coincides with the 3-dimensional Newtonian potential kernel.

To phrase an alternative characterization of X^3, we prepare two notions. Let E be a locally compact separable metric space and $\mathfrak{m}$ be a positive Radon measure on it. The first one is the so-called *part process*; see [14, §4.4]. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(E,\mathfrak{m})$ associated with a Markov process X and $F \subset E$ be a closed set of positive capacity relative to $(\mathcal{E},\mathcal{F})$. Then the part process X^G of X on $G := E \setminus F$ is obtained by killing X once upon leaving G. In other words,

$$X_t^G = \begin{cases} X_t, & t < \sigma_F := \{s > 0 : X_s \in F\}, \\ \partial, & t \geq \sigma_F, \end{cases}$$

where ∂ is the trap of X^G. Note that X^G is associated with the *part Dirichlet form* $(\mathcal{E}^G,\mathcal{F}^G)$ of $(\mathcal{E},\mathcal{F})$ on G:

$$\begin{aligned} &\mathcal{F}^G = \{f \in \mathcal{F} : \tilde{f} = 0,\ \mathcal{E}\text{-q.e. on } F\}, \\ &\mathcal{E}^G(f,g) = \mathcal{E}(f,g), \quad f,g \in \mathcal{F}^G, \end{aligned} \tag{2.9}$$

where $\tilde{f}$ stands for the quasi-continuous version of f. The second is the one-point reflection of a Markov process studied in [4]; see also [3, §7.5]. Let $a \in E$ be a non-isolated point with $\mathfrak{m}(\{a\}) = 0$ and X^0 be an $\mathfrak{m}$-symmetric Borel standard process on $E_0 := E \setminus \{a\}$ with no killing inside. Then a right process X on E is called a *one-point reflection* of X^0 (at a) if X is $\mathfrak{m}$-symmetric and of no killing on $\{a\}$, and the part process of X on E_0 is X^0.

Theorem 2.6. *Fix $\gamma \in \mathbb{R}$ and let X^3 and $(\mathcal{E}^3,\mathcal{F}^3)$ be in Theorem 2.1. Then $\{0\}$ is of positive 1-capacity relative to $\mathcal{E}^3$. Furthermore, the following hold:*

(1) ${}_hW^\gamma$ is identified with the part process of X^3 on $\mathbb{R}^3 \setminus \{0\}$;

(2) X^3 is the unique (in law) one-point reflection of ${}_hW^\gamma$ at 0.

Proof. To show the 1-capacity of $\{0\}$ is positive, the case $\gamma > 0$ has been considered in [11, Proposition 3.1]. Denote the 1-capacity relative to $\mathcal{E}^{3,1}$ by Cap^1, and then $\mathrm{Cap}^1(\{0\}) > 0$. For the case $\gamma \leq 0$, denote the 1-capacity relative to $\mathcal{E}^{3,\gamma}$ by Cap^γ. It suffices to note that $\mathrm{Cap}^\gamma(A) \geq \mathrm{Cap}^1(A)$ for any Borel set $A \subset \mathbb{R}^3$ due to $\mathcal{F}^{3,\gamma} \subset \mathcal{F}^{3,1}$ and $\mathcal{E}^{3,1}(f,f) \leq \mathcal{E}^{3,\gamma}(f,f)$ for all $f \in \mathcal{F}^{3,\gamma}$. Particularly, $\mathrm{Cap}^\gamma(\{0\}) \geq \mathrm{Cap}^1(\{0\}) > 0$.

Denote the associated Dirichlet form of 3-dimensional BM by $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^3))$, i.e. $H^1(\mathbb{R}^3)$ is the 1-Sobolev space and $\mathbf{D}$ is the Dirichlet integral. To prove the first assertion, it is straightforward to verify that $({}_hR_t^\gamma)$ is symmetric with respect to $\mathfrak{m}_\gamma(dx) = \psi_\gamma(x)^2 dx$ and then associated with the Dirichlet form (see [14, (1.3.17)])

$$\begin{aligned} &{}_h\mathcal{F} = \{f \in L^2(\mathbb{R}^3,\mathfrak{m}_\gamma) : {}_h\mathcal{E}(f,f) < \infty\}, \\ &{}_h\mathcal{E}(f,g) = \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^3} \left(f(x) - {}_hR_t^\gamma f(x)\right) g(x)\mathfrak{m}_\gamma(dx), \quad f,g \in {}_h\mathcal{F}. \end{aligned}$$

One can easily deduce that for any $f \in L^2(\mathbb{R}^3, \mathfrak{m}_\gamma)$,

$$\begin{aligned}
{}_h\mathcal{E}(f,f) &= \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^3} \left(f(x)\psi_\gamma(x) - \mathrm{e}^{-\frac{\gamma^2}{2}t} R_t(f\psi_\gamma)(x) \right) (f\psi_\gamma)(x)dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla(f\psi_\gamma)|^2(x)dx + \frac{\gamma^2}{2} \int_{\mathbb{R}^3} |(f\psi_\gamma)|^2(x)dx \\
&= \frac{1}{2}\mathbf{D}_{\gamma^2}(f\psi_\gamma, f\psi_\gamma),
\end{aligned}$$

whenever the limit exists. This leads to

$$ {}_h\mathcal{F} = \{f : f\psi_\gamma \in H^1(\mathbb{R}^3)\}, \quad {}_h\mathcal{E}(f,f) = \frac{1}{2}\mathbf{D}_{\gamma^2}(f\psi_\gamma, f\psi_\gamma), \quad f \in {}_h\mathcal{F}. \tag{2.10}$$

Since $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ is a core of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^3))$ and $\psi_\gamma \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ is positive, we can conclude that $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ is also a core of $({}_h\mathcal{E}, {}_h\mathcal{F})$. On the other hand, the part process $X^{3,\mathbb{R}^3\setminus\{0\}}$ of X^3 on $\mathbb{R}^3 \setminus \{0\}$ is associated with the Dirichlet form $(\mathcal{E}^{3,\mathbb{R}^3\setminus\{0\}}, \mathcal{F}^{3,\mathbb{R}^3\setminus\{0\}})$ given by (2.9) with $(\mathcal{E},\mathcal{F}) = (\mathcal{E}^3, \mathcal{F}^3)$ and $G = \mathbb{R}^3 \setminus \{0\}$. Particularly, $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ is also a core of $(\mathcal{E}^{3,\mathbb{R}^3\setminus\{0\}}, \mathcal{F}^{3,\mathbb{R}^3\setminus\{0\}})$ by [14, Theorem 4.4.3]. It follows from Lemma 2.2 and $\mathcal{L}_\gamma = \frac{1}{2}\Delta$ on $C_c^\infty(\mathbb{R}^3)$ that for any $f \in C_c^\infty(\mathbb{R}^3 \setminus \{0\}) \subset \mathcal{D}(\mathcal{A}_\gamma)$,

$$\begin{aligned}
\mathcal{E}^{3,\mathbb{R}^3\setminus\{0\}}(f,f) = \mathcal{E}^3(f,f) &= (-\mathcal{A}_\gamma f, f)_{m_\gamma} \\
&= -\int_{\mathbb{R}^3} \mathcal{L}_\gamma(\psi_\gamma f)(x)\,(\psi_\gamma f)\,(x)dx + \frac{\gamma^2}{2} \int_{\mathbb{R}^3} |(f\psi_\gamma)|^2(x)dx \\
&= \frac{1}{2}\mathbf{D}_{\gamma^2}(f\psi_\gamma, f\psi_\gamma).
\end{aligned}$$

Applying (2.10), one can obtain that

$$\mathcal{E}^{3,\mathbb{R}^3\setminus\{0\}}(f,f) = {}_h\mathcal{E}(f,f), \quad \forall f \in C_c^\infty(\mathbb{R}^3 \setminus \{0\}).$$

As a result, $(\mathcal{E}^{3,\mathbb{R}^3\setminus\{0\}}, \mathcal{F}^{3,\mathbb{R}^3\setminus\{0\}}) = ({}_h\mathcal{E}, {}_h\mathcal{F})$. Therefore, ${}_hW^\gamma$ is identified with the part process of X^3 on $\mathbb{R}^3 \setminus \{0\}$.

Finally we prove the second assertion. Clearly, $X^{(\gamma)}$ is a one-point reflection of ${}_hW^\gamma$ by the first assertion. To show the uniqueness, we shall apply [3, Theorem 7.5.4]. It suffices to note that for every $x \neq 0$,

$$ {}_h\mathbf{P}_x^\gamma(\zeta_h < \infty, {}_hW^\gamma_{\zeta_h-} = 0) = {}_h\mathbf{P}_x^\gamma(\zeta_h < \infty) = \mathbf{P}_x^3(\sigma_0 < \infty) > 0, \tag{2.11}$$

where $\sigma_0 := \inf\{t > 0 : X_t^3 = 0\}$. The second equality is due to the conservativeness of X^3 (see Corollary 2.10), and the first one is the consequence of that ${}_hW^\gamma$ has no killing inside (on $\mathbb{R}^3 \setminus \{0\}$) and the quasi-left continuity of X^3. The last equality holds because $\{0\}$ is positive capacity (or by virtue of Lemma 2.8 (2)). That completes the proof. $\square$

Remark 2.7. The second assertion in Theorem 2.6 leads to that 0 is regular for itself with respect to X^3, i.e. $\mathbf{P}_0^3(\sigma_0 = 0) = 1$.

2.4 Characterization via skew product decomposition

Due to the fact that ψ_γ is a radial function, the part process ${}_hW^\gamma$ of X^3 on $\mathbb{R}^3 \setminus \{0\}$ is rotationally invariant in the following sense: Let T be an arbitrary orthogonal transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, then

$${}_h\hat{W}^\gamma := \left\{ {}_h\hat{W}_t^\gamma := T({}_hW_t^\gamma), {}_h\hat{\mathbf{P}}_x^\gamma := {}_h\mathbf{P}^\gamma_{T^{-1}x} \right\}$$

defines an equivalent Markov process to ${}_hW^\gamma$. Hence we can characterize ${}_hW^\gamma$ by obtaining its skew product decomposition. Unsurprisingly, X^3 is also rotation invariant (Cf. [11, pp.11]) and it is not hard to figure out its radial process. The following lemma is an extension of [11, Proposition 3.7], and the proof can be completed by the same argument (So we omit it).

Lemma 2.8. *(1) The process ${}_hW^\gamma$ admits a skew-product representation*

$$ {}_hW_t^\gamma = \varrho_t^0 \vartheta_{A_t^0}, \quad t \geq 0, \tag{2.12}$$

where $\varrho^0 := (\varrho_t^0)_{t\geq 0} = (|{}_hW_t^\gamma|)_{t\geq 0}$ is a symmetric diffusion on $(0,\infty)$, killed at $\{0\}$, whose speed measure ℓ_γ^0 and scale function $\mathtt{s}_\gamma^0$ are

$$\ell_\gamma^0(dr) = \frac{\mathrm{e}^{-2\gamma r}}{\pi} dr, \quad \mathtt{s}_\gamma^0(r) = \begin{cases} \dfrac{\pi}{2\gamma}\mathrm{e}^{2\gamma r}, & \text{when } \gamma \neq 0, \\ \pi r, & \text{when } \gamma = 0, \end{cases} \qquad r \in (0,\infty);$$

$A^0 := (A_t^0)_{t\geq 0}$ is the PCAF of ϱ^0 with the Revuz measure

$$\mu_{A^0}(dr) := \frac{\ell_\gamma^0(dr)}{r^2}$$

and ϑ is a spherical Brownian motion on $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, which is independent of ϱ^0.

(2) The radial process $\varrho = (\varrho_t)_{t\geq 0} := (|X_t^3|)_{t\geq 0}$ is a symmetric diffusion on $[0,\infty)$, reflected at $\{0\}$, whose speed measure ℓ_γ and scale function $\mathtt{s}_\gamma$ are

$$\ell_\gamma(dr) = \frac{\mathrm{e}^{-2\gamma r}}{\pi} dr, \quad \mathtt{s}_\gamma(r) = \begin{cases} \dfrac{\pi}{2\gamma}\mathrm{e}^{2\gamma r}, & \text{when } \gamma \neq 0, \\ \pi r, & \text{when } \gamma = 0, \end{cases} \qquad r \in [0,\infty). \tag{2.13}$$

Remark 2.9. When $\gamma = 0$, ϱ^0 is nothing but the absorbing Brownian motion on $(0,\infty)$ (killed at 0), and ϱ is the reflecting Brownian motion on $[0,\infty)$. It is not expected that X^3 admits an analogical representation of (2.12), because $\ell_\gamma(dr)/r^2$ is not Radon on $[0,\infty)$ and hence not smooth (by e.g. [17, Theorem A.3.(4)]) relative to ϱ; see further explanation below [11, Corollary 3.11].

It is worth noting two facts about the radial processes ϱ^0 and ϱ. The first one is to derive the global properties of ϱ, which lead to those of X^3, by employing the scale function and the speed measure.

Corollary 2.10. *Let $\varrho = (\varrho_t)_{t\geq 0}$ be the radial process of X^3. Then*

(1) ϱ is irreducible and conservative. Particularly, X^3 is conservative.
(2) ϱ is recurrent, if and only if $\gamma \geq 0$. Otherwise it is transient.

Proof. The irreducibility of ϱ is clear. Note that ϱ is conservative, if and only if (see e.g. [3, Example 3.5.7])

$$\int_1^\infty \ell_\gamma((1,r)) d\mathtt{s}_\gamma(r) = \infty. \tag{2.14}$$

This is true by a straightforward computation.

By [3, Theorem 2.2.11], ϱ is transient, if and only if $\mathtt{s}_\gamma(\infty) := \lim_{r\uparrow\infty} \mathtt{s}_\gamma(r) < \infty$ and $\ell_\gamma((1,\infty)) = \infty$ (Otherwise it is recurrent). This amounts to $\gamma < 0$. That completes the proof. □

Remark 2.11. The recurrence/transience of ϱ coincides with that of X^3, as stated in Theorem 2.1.

The second is concerned with their pathwise decompositions as below. The proof is analogical to that of [11, (3.6)] and we omit it.

Corollary 2.12. *The radial processes ϱ^0 and ϱ admit the following pathwise decompositions:*

$$\varrho^0_t - \varrho^0_0 = \beta_t - \gamma t, \quad 0 \le t < \zeta^0 (= \sigma_0),$$
$$\varrho_t - \varrho_0 = \beta_t - \gamma t + \pi\gamma \cdot l^0_t, \quad t \ge 0,$$

where ζ^0 is the lift time of ϱ^0, $(\beta_t)_{t\ge 0}$ is a certain one-dimensional standard Brownian motion, and $l^0 := (l^0_t)_{t\ge 0}$ is the local time of ϱ at 0, i.e. the PCAF associated with the smooth measure δ_0 relative to ϱ.

2.5 Pathwise representation

This short subsection is devoted to the pathwise representation of X^3 by virtue of so-called Fukushima's decomposition. Let f_i be the ith coordinate function for $i = 1, 2, 3$, i.e. $f_i(x) := x_i$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Obviously $f_i \in \mathcal{F}^3_{\mathrm{loc}}$. Then we can write the Fukushima's decomposition of X^3 relative to f_i:

$$f_i(X^3_t) - f_i(X^3_0) = M^{f_i}_t + N^{f_i}_t, \quad t \ge 0, \quad \mathbf{P}^3_x\text{-a.s.}, \quad \text{q.e. } x, \tag{2.15}$$

where $M^{f_i} := (M^{f_i}_t)_{t\ge 0}$ is an MAF locally of finite energy and $N^{f_i} := (N^{f_i}_t)_{t\ge 0}$ is a CAF locally of zero energy. Note that M^{f_i} and N^{f_i} in this decomposition are unique in law. Set $M_t := (M^{f_1}_t, M^{f_2}_t, M^{f_3}_t)$ and $N_t := (N^{f_1}_t, N^{f_2}_t, N^{f_3}_t)$. Recall that an additive functional $A = (A_t)_{t\ge 0}$ is called of bounded variation if $A_t(\omega)$ is of bounded variation in t on each compact subinterval of $[0, \zeta(\omega))$ for every fixed ω in the defining set of A, where ζ ($= \infty$ for X^3 due to its conservativeness) is the life time of the underlying Markov process. We say $N := (N_t)_{t\ge 0}$ is of bounded variation if N^{f_i} is of bounded variation for $i = 1, 2, 3$.

By repeating the arguments in [11, §4], we can conclude the following characterizations of M and N.

Theorem 2.13. *Let X^3 be in Theorem 2.1 and $M = (M_t)_{t\ge 0}, N = (N_t)_{t\ge 0}$ be in the Fukushima's decomposition* (2.15)*. Then the following hold:*

(1) For q.e. $x \in \mathbb{R}^3$, M is equivalent to a 3-dimensional Brownian motion under $\mathbf{P}^3_x$.

(2) For $t < \sigma_0$,

$$N_t = -\int_0^t \frac{\gamma |X^3_s| + 1}{|X^3_s|^2} \cdot X^3_s ds.$$

However, N is not of bounded variation.

Comparing to the final property of N, the radial process ϱ of X is a semi-martingale as presented in Corollary 2.12. At a heuristic level, this behavior of X is a consequence of the following fact: As noted by Erickson [10], the excursions of X away from 0 oscillate so violently that each neighborhood of each point of the unit sphere is visited infinitely often by the angular part of X.

3 Distorted Brownian motion on space with varying dimension

Let

$$E := \mathfrak{R}_+ \cup \mathfrak{R}^3,$$

where $\mathfrak{R}_+ := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3 = 0, x_4 \ge 0\} (\simeq \mathbb{R}_+ := [0, \infty))$ and $\mathfrak{R}^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 = 0\} (\simeq \mathbb{R}^3)$, be the state space. For convenience, set the following maps:

$$\iota_+ : \mathbb{R}_+ \to \mathfrak{R}_+, \quad r \mapsto (0, 0, 0, r),$$

and

$$\iota_3 : \mathbb{R}^3 \to \mathfrak{R}^3, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0). \tag{3.1}$$

This section is devoted to the study of so-called distorted Brownian motions with varying dimension (dBMV in abbreviation) on E.

3.1 One-dimensional part

Let ρ be a function on $\mathbb{R}_+$ such that

$$\rho > 0, \text{ a.e.}, \quad \rho \text{ and } \frac{1}{\rho} \in L^1_{\mathrm{loc}}(\mathbb{R}_+), \tag{3.2}$$

and

$$\int_0^\infty \frac{dr}{\rho(r)} \int_0^r \rho(s)ds = \infty. \tag{3.3}$$

Consider the Dirichlet form $(\mathcal{E}^+, \mathcal{F}^+)$ on $L^2(\mathbb{R}_+, m_+) := L^2(\mathbb{R}_+, \rho(r)dr)$:

$$\begin{aligned} \mathcal{F}^+ &:= \left\{ f \in L^2(\mathbb{R}_+, m_+) : f' \in L^2(\mathbb{R}_+, m_+) \right\}, \\ \mathcal{E}^+(f,g) &:= \frac{1}{2} \int_{\mathbb{R}_+} f'(r) g'(r) m_+(dr), \quad f, g \in \mathcal{F}^+. \end{aligned}$$

The following lemma summarizes the basic facts about $(\mathcal{E}^+, \mathcal{F}^+)$.

Lemma 3.1. *The following hold:*

(i) $(\mathcal{E}^+, \mathcal{F}^+)$ *is a regular strongly local Dirichlet form on* $L^2(\mathbb{R}_+, m_+)$ *with a special standard core* $C_c^\infty(\mathbb{R}_+)$*. It is also irreducible and conservative. For all* $r \in \mathbb{R}_+$*, the singleton* $\{r\}$ *is of positive capacity relative to* $\mathcal{E}^+$.

(ii) *The associated diffusion* X^+ *of* $(\mathcal{E}^+, \mathcal{F}^+)$ *is an irreducible conservative diffusion on* $\mathbb{R}_+$ *reflecting at* 0*, whose speed measure is* m_+ *and scale function is*

$$\mathtt{s}_+(r) = \int_0^r \frac{1}{\rho(s)} ds, \quad r \geq 0.$$

Furthermore, X^+ *is transient (resp. recurrent), if and only if* $1/\rho \in L^1(\mathbb{R}_+)$ *(resp.* $1/\rho \notin L^1(\mathbb{R}_+)$*).*

The diffusion X^+ is usually called a *distorted Brownian motion* on $\mathbb{R}_+$; see e.g. [17]. The proof of Lemma 3.1 is referred to [19, §3.4]. We should point out that $1/\rho \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ implies the irreducibility of $(\mathcal{E}^+, \mathcal{F}^+)$ and the conservativeness is a consequence of (3.3). The recurrence or transience of X^+ is indicated by [3, Theorem 2.2.11].

Example 3.2. An interesting example is the radial process $\varrho = (\varrho_t)_{t \geq 0}$ of X^3 appearing in Lemma 2.8. In this case, $\rho(r) = \mathrm{e}^{-2\gamma r}/\pi$ satisfies (3.2) and (3.3).

Let $\hat{\varrho}$ be the diffusion on $\mathbb{R}$ obtained by the symmetrization of ϱ. In other words, $\hat{\varrho}$ is associated with the energy form induced by the symmetric measure

$$\hat{\ell}_\gamma(dr) := \frac{\mathrm{e}^{-2\gamma|r|}}{\pi} dr, \quad r \in \mathbb{R}.$$

With $\hat{\phi}_\gamma(r) = \mathrm{e}^{-\gamma|r|}/\sqrt{\pi}$ ($r \in \mathbb{R}$) in place of ψ_γ, $\hat{\varrho}$ plays the same role as X^3 in the analogical one-dimensional model (parametrized by γ) of that explained in §2.2. Particularly, under a similar h-transform to (2.7), the generator of $\hat{\varrho}$ corresponds to a self-adjoint extension of $\frac{1}{2}\Delta$ restricting to $C_c^\infty(\mathbb{R} \setminus \{0\})$. See e.g. [1, Appendix F].

3.2 Definition

Fix $\gamma \in \mathbb{R}$ and a function ρ satisfying (3.2) and (3.3) as above. Take a positive constant $\mathtt{p} > 0$. In this subsection, we rigorously give the definition for the so-called (ρ, γ)-dBMV with the parameter $\mathtt{p}$ on E.

Recall that E consists of two components $\mathfrak{R}_+$ and $\mathfrak{R}^3$. Roughly speaking, the distribution of such a dBMV on $\mathfrak{R}_+$ (resp. $\mathfrak{R}^3$) is induced by the dBM X^+ (resp. X^3) in §3.1 (resp. Theorem 2.1). To be more precise, set $M^+ = (M^+_t)_{t\geq 0} := (\iota_+(X^+_t))_{t\geq 0}$ and $M^3 = (M^3_t)_{t\geq 0} := (\iota_3(X^3_t))_{t\geq 0}$. Then M^+ is symmetric with respect to $\mathfrak{m}_+ := m_+ \circ \iota_+^{-1}$ and associated with the Dirichlet form on $L^2(\mathfrak{R}_+, \mathfrak{m}_+)$:

$$\begin{aligned} &\mathscr{F}^+ := \{f : f \circ \iota_+ \in \mathcal{F}^+\}, \\ &\mathscr{E}^+(f,g) := \mathcal{E}^+(f\circ\iota_+, g\circ\iota_+), \quad f,g \in \mathscr{F}^+. \end{aligned}$$

Accordingly, M^3 is symmetric with respect to $\mathfrak{m}_3 := m_\gamma \circ \iota_+^{-1}$ and associated with the Dirichlet form on $L^2(\mathfrak{R}^3, \mathfrak{m}_3)$:

$$\begin{aligned} &\mathscr{F}^3 := \{f : f \circ \iota_3 \in \mathcal{F}^3\}, \\ &\mathscr{E}^3(f,g) := \mathcal{E}^3(f\circ\iota_3, g\circ\iota_3), \quad f,g \in \mathscr{F}^3. \end{aligned}$$

Define a measure $\mathfrak{m}^{(\mathtt{p})}$ on E by $\mathfrak{m}^{(\mathtt{p})}|_{\mathfrak{R}_+} := \mathtt{p}\cdot\mathfrak{m}_+$ and $\mathfrak{m}^{(\mathtt{p})}|_{\mathfrak{R}^3} := \mathfrak{m}_3$. Denote the density function by

$$h^{(\mathtt{p})}_{\rho,\gamma}(x) := \begin{cases} \sqrt{\mathtt{p}\cdot\rho(r)}, & x = (0,0,0,r) \in \mathfrak{R}_+, \\ \psi_\gamma((x_1,x_2,x_3)), & x = (x_1,x_2,x_3,0) \in \mathfrak{R}^3. \end{cases} \tag{3.4}$$

In other words, $\mathfrak{m}^{(\mathtt{p})}(dx) = h^{(\mathtt{p})}_{\rho,\gamma}(x)^2 \mathtt{l}(dx)$, where $\mathtt{l}|_{\mathfrak{R}_+}$ is the one-dimensional Lebesgue measure and $\mathtt{l}|_{\mathfrak{R}^3}$ is the three-dimensional Lebesgue measure. Then we introduce the following definition.

Definition 3.3 (Distorted Brownian motion with varying dimension)**.** *Fix $\gamma \in \mathbb{R}$ and ρ satisfying* (3.2) *and* (3.3)*, $\mathtt{p} > 0$ and denote the zero element of $\mathbb{R}^4$ by $\mathbf{0}$. A (ρ,γ)-distorted Brownian motion with varying dimension ((ρ,γ)-dBMV in abbreviation) with parameter $\mathtt{p}$ on E is an $\mathfrak{m}^{(\mathtt{p})}$-symmetric irreducible diffusion $M = \{(M_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E}\}$ of no killing on $\{\mathbf{0}\}$ such that*

(i) The part process of M on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ is equivalent to that of M^+;

(ii) The part process of M on $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$ is equivalent to that of M^3.

Hereafter $\mathfrak{m}^{(\mathtt{p})}$ and $h^{(\mathtt{p})}_{\rho,\gamma}$ will be written as $\mathfrak{m}$ and $h_{\rho,\gamma}$ respectively and this notion (ρ,γ)-dBMV with parameter $\mathtt{p}$ will be called dBMV for short if no confusions caused. The uniqueness of dBMV in law can be concluded by the following argument. Let $M^{+,\mathbf{0}}$ (resp. $M^{3,\mathbf{0}}$) be the part process of M^+ (resp. M^3) on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ (resp. $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$). Define a new Markov process $M^{\mathbf{0}}$ on $E \setminus \{\mathbf{0}\}$ by $M^{\mathbf{0}}|_{\mathfrak{R}_+\setminus\{0\}} := M^{+,\mathbf{0}}$ and $M^{\mathbf{0}}|_{\mathfrak{R}^3\setminus\{0\}} := M^{3,\mathbf{0}}$. Then the dBMV M is nothing but the one-point reflection of $M^{\mathbf{0}}$ at $\mathbf{0}$ by definition. The uniqueness of dBMV in law is a consequence of [3, Theorem 7.5.4], Theorem 2.6 and Corollary 2.10.

Remark 3.4. It is worth noting that the parameter $\mathtt{p}$ plays a role only in the symmetric measure $\mathfrak{m}$ (see the notes before Corollary 4.4). For different $\mathtt{p}$, the dBMVs are different as will be shown in Remark 4.2, but their motions out of the origin $\mathbf{0}$ are exactly the same according to the definition.

3.3 Dirichlet form characterization of dBMV

The main result of this section as below gives the associated Dirichlet form of dBMV. Recall that the Dirichlet forms $(\mathscr{E}^+, \mathscr{F}^+)$ and $(\mathscr{E}^3, \mathscr{F}^3)$ are given in §3.2. Usually every function in a Dirichlet space is taken to be its quasi-continuous version tacitly. For $f \in \mathscr{F}^+$ (resp. $f \in \mathscr{F}^3$), the $\mathscr{E}^+$-quasi-continuous (resp. $\mathscr{E}^3$-quasi-continuous) version of f will be denoted by ${}^+\tilde{f}$ (resp. ${}^3\tilde{f}$) when there is a risk of ambiguity. Since **0** is of positive capacity relative to $\mathscr{E}^+$ or $\mathscr{E}^3$ due to Lemma 3.1 or Theorem 2.6, ${}^+\tilde{f}$ or ${}^3\tilde{f}$ is well defined at **0**.

Theorem 3.5. *Let $(\mathscr{E}^+, \mathscr{F}^+)$ and $(\mathscr{E}^3, \mathscr{F}^3)$ be given in §3.2. Then the quadratic form*

$$\mathscr{F} := \left\{ f \in L^2(E, \mathfrak{m}) : f|_{\mathfrak{R}_+} \in \mathscr{F}^+, f|_{\mathfrak{R}^3} \in \mathscr{F}^3, {}^+\widetilde{f|_{\mathfrak{R}_+}}(\mathbf{0}) = {}^3\widetilde{f|_{\mathfrak{R}^3}}(\mathbf{0}) \right\},$$
$$\mathscr{E}(f,g) := \mathtt{p} \cdot \mathscr{E}^+(f|_{\mathfrak{R}_+}, g|_{\mathfrak{R}_+}) + \mathscr{E}^3(f|_{\mathfrak{R}^3}, g|_{\mathfrak{R}^3}), \quad f, g \in \mathscr{F}$$

is a regular, strongly local and irreducible Dirichlet form on $L^2(E, \mathfrak{m})$, whose associated Markov process is identified with the unique (ρ, γ)-dBMV M.

Proof. Clearly, $(\mathscr{E}, \mathscr{F})$ is a symmetric bilinear form satisfying the Markovian property. The strong locality of $(\mathscr{E}, \mathscr{F})$ is led by that of $(\mathscr{E}^+, \mathscr{F}^+)$ and $(\mathscr{E}^3, \mathscr{F}^3)$. To show its closeness, take an $\mathscr{E}_1$-Cauchy sequence $\{f_n : n \geq 1\} \subset \mathscr{F}$. Then $f_n|_{\mathfrak{R}_+}$ is $\mathscr{E}_1^+$-Cauchy and $f_n|_{\mathfrak{R}^3}$ is $\mathscr{E}_1^3$-Cauchy. It follows from [14, Theorem 2.1.4] that there exists a subsequence $\{f_{n_k} : k \geq 1\}$ of $\{f_n\}$ and ${}^+f \in \mathscr{F}^+, {}^3f \in \mathscr{F}^3$ such that

$$\begin{aligned} {}^+\widetilde{f_{n_k}|_{\mathfrak{R}_+}} &\to {}^+f, \quad \mathscr{E}^+\text{-q.e.}, \\ {}^3\widetilde{f_{n_k}|_{\mathfrak{R}^3}} &\to {}^3f, \quad \mathscr{E}^3\text{-q.e.} \end{aligned}$$

and

$$\mathtt{p} \cdot \mathscr{E}_1^+(f_n|_{\mathfrak{R}_+} - {}^+f, f_n|_{\mathfrak{R}_+} - {}^+f) + \mathscr{E}_1^3(f_n|_{\mathfrak{R}^3} - {}^3f, f_n|_{\mathfrak{R}^3} - {}^3f) \to 0, \quad n \uparrow \infty.$$

Note that $\{\mathbf{0}\}$ is of positive capacity relative to $\mathscr{E}^+$ or $\mathscr{E}^3$. This implies

$${}^+f(\mathbf{0}) = \lim_{k \uparrow \infty} {}^+\widetilde{f_{n_k}|_{\mathfrak{R}_+}}(\mathbf{0}) = \lim_{k \uparrow \infty} {}^3\widetilde{f_{n_k}|_{\mathfrak{R}^3}}(\mathbf{0}) = {}^3f(\mathbf{0}).$$

Hence the function f is well defined on E by $f|_{\mathfrak{R}_+} := {}^+f$ and $f|_{\mathfrak{R}^3} := {}^3f$. In addition, $f \in \mathscr{F}$ and

$$\mathscr{E}_1(f_n - f, f_n - f) = \mathtt{p} \cdot \mathscr{E}_1^+(f_n|_{\mathfrak{R}_+} - {}^+f, f_n|_{\mathfrak{R}_+} - {}^+f) + \mathscr{E}_1^3(f_n|_{\mathfrak{R}^3} - {}^3f, f_n|_{\mathfrak{R}^3} - {}^3f) \to 0.$$

Therefore the closeness of $(\mathscr{E}, \mathscr{F})$ is verified.

Next, let us prove the regularity of $(\mathscr{E}, \mathscr{F})$. Take a special standard core $\mathscr{C}^+$ of $(\mathscr{E}^+, \mathscr{F}^+)$ and a special standard core $\mathscr{C}^3$ of $(\mathscr{E}^3, \mathscr{F}^3)$; for example, $\mathscr{C}^+ := C_c^\infty(\mathbb{R}_+) \circ \iota_+^{-1}$ and $\mathscr{C}^3 := C_c^\infty(\mathbb{R}^3) \circ \iota_3^{-1}$. Set

$$\mathscr{C} := \{f \in \mathscr{F} : f|_{\mathfrak{R}_+} \in \mathscr{C}^+, f|_{\mathfrak{R}^3} \in \mathscr{C}^3\}. \tag{3.5}$$

It suffices to show $\mathscr{C}$ is dense in $C_c(E)$ relative to the uniform norm and dense in $\mathscr{F}$ relative to the $\mathscr{E}_1$-norm. On one hand, $\mathscr{C}$ is clearly an algebra, i.e. $f, g \in \mathscr{C}$ implies $c_1 \cdot f + c_2 \cdot g, f \cdot g \in \mathscr{C}$ for any constants c_1, c_2. In addition, $\mathscr{C}$ can separate the points in E by the following argument: Without loss of generality, consider $x \in \mathfrak{R}_+, y \in \mathfrak{R}^3 \setminus \{\mathbf{0}\}$. Since $\mathscr{C}^+$ is a special standard core of $(\mathscr{E}^+, \mathscr{F}^+)$, there exists a function ${}^+f \in \mathscr{C}^+$ such that ${}^+f(\mathbf{0}) = {}^+f(x) = 1$. Another function ${}^3f \in \mathscr{C}^3$ can be taken to separate **0** and y, i.e. ${}^3f(\mathbf{0}) \neq {}^3f(y)$. Define a function f on E by

$$f|_{\mathfrak{R}_+} := {}^3f(\mathbf{0}) \cdot {}^+f, \quad f|_{\mathfrak{R}^3} := {}^3f.$$

Then $f \in \mathscr{C}$ and $f(x) = {}^3f(\mathbf{0}) \neq {}^3f(y) = f(y)$. Thus by the Stone-Weierstrass theorem, $\mathscr{C}$ is dense in $C_c(E)$ relative to the uniform norm. On the other hand, fix $f \in \mathscr{F}$ and a small constant $\varepsilon > 0$. Take ${}^+g \in \mathscr{C}^+$ with ${}^+g(\mathbf{0}) = 1$ and ${}^3g \in \mathscr{C}^3$ with ${}^3g(\mathbf{0}) = 1$. Let $C_+ := \|{}^+g\|_{\mathscr{E}_1^+}$ and $C_3 := \|{}^3g\|_{\mathscr{E}_1^3}$. By [14, Theorem 2.1.4], there exist two functions ${}^+h_\varepsilon \in \mathscr{C}^+$ and ${}^3h_\varepsilon \in \mathscr{C}^3$ such that

$$\|{}^+h_\varepsilon - f|_{\mathfrak{R}_+}\|_{\mathscr{E}_1^+} < \frac{\varepsilon}{4\sqrt{\mathtt{p}}}, \quad |{}^+h_\varepsilon(\mathbf{0}) - f(\mathbf{0})| < \frac{\varepsilon}{4C_+\sqrt{\mathtt{p}}};$$
$$\|{}^3h_\varepsilon - f|_{\mathfrak{R}^3}\|_{\mathscr{E}_1^3} < \varepsilon/4, \quad |{}^3h_\varepsilon(\mathbf{0}) - f(\mathbf{0})| < \frac{\varepsilon}{4C_3}.$$

Define a function f_ε on E by

$$f_\varepsilon|_{\mathfrak{R}_+} := {}^+h_\varepsilon + \big(f(\mathbf{0}) - {}^+h_\varepsilon(\mathbf{0})\big) \cdot {}^+g,$$
$$f_\varepsilon|_{\mathfrak{R}^3} := {}^3h_\varepsilon + \big(f(\mathbf{0}) - {}^3h_\varepsilon(\mathbf{0})\big) \cdot {}^3g.$$

Then $f_\varepsilon \in \mathscr{C}$ and

$$\begin{aligned}\|f_\varepsilon - f\|_{\mathscr{E}_1} &\le \sqrt{\mathtt{p}} \cdot \|f_\varepsilon|_{\mathfrak{R}_+} - f|_{\mathfrak{R}_+}\|_{\mathscr{E}_1^+} + \|f_\varepsilon|_{\mathfrak{R}^3} - f|_{\mathfrak{R}^3}\|_{\mathscr{E}_1^3} \\ &\le \sqrt{\mathtt{p}} \cdot \|{}^+h_\varepsilon - f|_{\mathfrak{R}_+}\|_{\mathscr{E}_1^+} + \sqrt{\mathtt{p}} \cdot |{}^+h_\varepsilon(\mathbf{0}) - f(\mathbf{0})| \cdot \|{}^+g\|_{\mathscr{E}_1^+} \\ &\qquad + \|{}^3h_\varepsilon - f|_{\mathfrak{R}^3}\|_{\mathscr{E}_1^3} + |{}^3h_\varepsilon(\mathbf{0}) - f(\mathbf{0})| \cdot \|{}^3g\|_{\mathscr{E}_1^3} \\ &< \varepsilon.\end{aligned}$$

This tells us $\mathscr{C}$ is dense in $\mathscr{F}$ relative to the $\mathscr{E}_1$-norm.

Furthermore, we derive the irreducibility of $(\mathscr{E}, \mathscr{F})$. Take an $\mathfrak{m}$-invariant set $A \subset E$, and we need to show $\mathfrak{m}(A) = 0$ or $\mathfrak{m}(A^c) = 0$. Firstly, let $(\mathscr{A}^+, \mathscr{G}^+)$ be the part Dirichlet form of $(\mathscr{E}, \mathscr{F})$ on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ and consider $f, g \in \mathscr{G}^+ \subset \mathscr{F}$. It follows from [14, Theorem 1.6.1] that $f \cdot 1_A, g \cdot 1_A \in \mathscr{F}$ and

$$\mathscr{E}(f,g) = \mathscr{E}(f1_A, g1_A) + \mathscr{E}(f1_{A^c}, g1_{A^c}).$$

Set $A_+ := A \cap (\mathfrak{R}_+ \setminus \{\mathbf{0}\})$. Since $f|_{\mathfrak{R}^3} = g|_{\mathfrak{R}^3} \equiv 0$, the expression of $\mathscr{F}$ yields $f \cdot 1_{A_+}, g \cdot 1_{A_+} \in \mathscr{G}^+$ and

$$\mathscr{A}^+(f,g) = \mathscr{A}^+(f1_{A_+}, g1_{A_+}) + \mathscr{A}^+(f1_{A_+^c}, g1_{A_+^c}).$$

Using [14, Theorem 1.6.1] again, we have A_+ is an $\mathfrak{m}_+$-invariant set relative to $\mathscr{A}^+$. Note that $(\mathscr{A}^+, \mathscr{G}^+)$ is clearly irreducible and thus $\mathfrak{m}_+(A_+) = 0$ or $\mathfrak{m}_+(A_+^c) = 0$. Analogically set $A_3 := A \cap (\mathfrak{R}^3 \setminus \{\mathbf{0}\})$ and we can also obtain that $\mathfrak{m}_3(A_3) = 0$ or $\mathfrak{m}_3(A_3^c) = 0$. Secondly, it suffices to show that $\mathfrak{m}_+(A_+) = \mathfrak{m}_3(A_3^c) = 0$ or $\mathfrak{m}_+(A_+^c) = \mathfrak{m}_3(A_3) = 0$ is impossible. Take a function $f \in \mathscr{C}$ such that $f(x) = 1$ for $|x| \le 1$. These two cases both contradict to $f \cdot 1_A \in \mathscr{F}$. Eventually we can conclude the irreducibility of $(\mathscr{E}, \mathscr{F})$.

Finally, the part Dirichlet form of $(\mathscr{E}, \mathscr{F})$ on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ (resp. $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$) is clearly associated with the same Markov process as that of $(\mathscr{E}^+, \mathscr{F}^+)$ (resp. $(\mathscr{E}^3, \mathscr{F}^3)$) on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ (resp. $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$). Therefore, the associated Markov process of $(\mathscr{E}, \mathscr{F})$ is nothing but the dBMV by definition. That completes the proof. □

It is worth noting that M as well as $(\mathscr{E}, \mathscr{F})$ is always conservative as will be shown in Remark 4.2.

Remark 3.6. A motivated model named Brownian motion on space with varying dimension (DBV in abbreviation) appears in a recent paper [5]. Its building blocks are two-dimensional Brownian motion and one-dimensional Brownian motion. The crucial point is that two-dimensional Brownian motion cannot hit the origin. To obtain BMV, the so-called darning method was employed in [5] to collapse a small ball to an abstract "point", so that the two-dimensional Brownian motion can reach it. However in our current argument, thanks to that the origin is of positive capacity relative to $\mathscr{E}^3$ (see Theorem 2.6), the darning method is not necessary to the construction of dBMV.

In addition, a number of irreducible Markov processes on E can be analogically obtained by using other processes on $\mathfrak{R}_+$ and $\mathfrak{R}^3$, relative to which the origin is of positive capacity. Every irreducible and symmetric diffusion on $\mathfrak{R}_+$ (reflecting at **0**) is such an example on $\mathfrak{R}_+$, see e.g. [19]. An example of pure-jump process on $\mathfrak{R}^3$ appears in a recent work [18].

3.4 Basics of $(\mathscr{E},\mathscr{F})$ and M

Denote the capacities with respect to $\mathscr{E}$, $\mathscr{E}^+$ and $\mathscr{E}^3$ by Cap, ${}^+$Cap and ^{3}Cap respectively. The following proposition characterizes the sets of capacity zero with respect to $\mathscr{E}$.

Proposition 3.7. *The set $A \subset E$ is of capacity zero with respect to $\mathscr{E}$, if and only if $A \subset \mathfrak{R}^3 \setminus \{\mathbf{0}\}$ and A is of capacity zero relative to $\mathscr{E}^3$. Particularly for any $x \in \mathfrak{R}_+$, Cap$(\{x\}) > 0$ but for any $x \in \mathfrak{R}^3 \setminus \{\mathbf{0}\}$, Cap$(\{x\}) = 0$.*

Proof. Let $(\mathscr{A}^+,\mathscr{G}^+)$ (resp. $(\mathscr{A}^3,\mathscr{G}^3)$) be the part Dirichlet form of $(\mathscr{E}^+,\mathscr{F}^+)$ (resp. $(\mathscr{E}^3,\mathscr{F}^3)$) on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ (resp. $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$). Then $(\mathscr{A}^+,\mathscr{G}^+)$ and $(\mathscr{A}^3,\mathscr{G}^3)$ are also the part Dirichlet forms of $(\mathscr{E},\mathscr{F})$ on $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ and $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$ respectively. Since every singleton of $\mathfrak{R}_+$ is of positive capacity with respect to $\mathscr{E}^+$, it is of positive capacity with respect to $\mathscr{A}^+$ as well as $\mathscr{E}$ by applying [14, Theorem 4.4.3]. Hence any set of capacity zero with respect to $\mathscr{E}$ must be a subset of $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$. By using [14, Theorem 4.4.3] again, a set $B \subset \mathfrak{R}^3 \setminus \{\mathbf{0}\}$ is of capacity zero with respect to $\mathscr{E}$, if and only if B is of capacity zero with respect to $\mathscr{A}^3$ as well as $\mathscr{E}^3$. That completes the proof. □

The behaviour of M near the origin **0** is crucial to the understanding of it. As indicated in Proposition 3.7, **0** is of positive capacity with respect to $\mathscr{E}$. This leads to

$$\mathbb{P}_x(\sigma_{\{\mathbf{0}\}} < \infty) > 0 \tag{3.6}$$

for $\mathscr{E}$-q.e. $x \in E$, where $\sigma_{\{\mathbf{0}\}} := \inf\{t > 0 : M_t = \mathbf{0}\}$ is the hitting time of $\{\mathbf{0}\}$, by applying [14, Theorem 4.7.1 (i)]. Furthermore, the origin **0** is called regular for a set $B \subset E$ with respect to M if $\mathbb{P}_{\mathbf{0}}(\sigma_B = 0) = 1$ where $\sigma_B := \inf\{t > 0 : M_t \in B\}$. Then we have the following.

Corollary 3.8. ***0** is regular for $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$, $\{\mathbf{0}\}$ and $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$ with respect to M respectively.*

Proof. If $\mathbb{P}_{\mathbf{0}}(\sigma_{\{\mathbf{0}\}} = 0) = 0$, then $\{\mathbf{0}\}$ is a thin set. Thin set is always semipolar and thus $\mathfrak{m}$-polar. This contradicts Proposition 3.7. Hence $\mathbb{P}_{\mathbf{0}}(\sigma_{\{\mathbf{0}\}} = 0) = 1$ by the 0-1 law.

Let $B := \mathfrak{R}_+ \setminus \{\mathbf{0}\}$ or $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$. Since B is open, it is also finely open. Thus $B \subset B^r \subset \mathfrak{R}_+$ or $\mathfrak{R}^3$, where B^r stands for its regular set. If $\mathbf{0} \notin B^r$, then B would be finely open and finely closed simultaneously. By [14, Corollary 4.6.3], we would have B is invariant. Hence the irreducibility of $(\mathscr{E},\mathscr{F})$ would imply $\mathfrak{m}(B) = 0$. This is impossible. Eventually we can conclude that $\mathbf{0} \in B^r$, i.e. **0** is regular for $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$ or $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$. That completes the proof. □

Remark 3.9. At a heuristic level, this fact tells us that starting from **0**, M enters $\mathfrak{R}_+ \setminus \{\mathbf{0}\}$, $\{\mathbf{0}\}$, and $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$ immediately.

Denote the extended Dirichlet spaces of $(\mathscr{E}^+,\mathscr{F}^+)$ and $(\mathscr{E}^3,\mathscr{F}^3)$ by $\mathscr{F}^+_{\mathrm{e}}$ and $\mathscr{F}^3_{\mathrm{e}}$ respectively. Note that $\mathscr{F}^+_{\mathrm{e}} = \{f : f \circ \iota_+ \in \mathcal{F}^+_{\mathrm{e}}\}$ where

$$\mathcal{F}^+_{\mathrm{e}} = \mathcal{F}^+_\rho := \{g : g \text{ is absolutely continuous on } \mathbb{R}_+,\ \int_{\mathbb{R}_+} g'(r)^2 m_+(dr) < \infty\}$$

when X^+ is recurrent, i.e. $1/\rho \notin L^1(\mathbb{R}_+)$, and

$$\mathcal{F}^+_{\mathrm{e}} = \{g \in \mathcal{F}^+_\rho : \lim_{r\uparrow\infty} g(r) = 0\}$$

when X^+ is transient, i.e. $1/\rho \in L^1(\mathbb{R}_+)$; see e.g. [3, Theorem 2.2.11]. The expression of $\mathscr{F}^3_{\mathrm{e}}$ is stated in [11, Corollary 3.5]. The extended Dirichlet space $\mathscr{F}_{\mathrm{e}}$ of $(\mathscr{E}, \mathscr{F})$ is given as follows.

Proposition 3.10. *It holds*

$$\mathscr{F}_{\mathrm{e}} = \left\{ f : f < \infty, \mathfrak{m}\text{-a.e.}, f|_{\mathfrak{R}^+} \in \mathscr{F}^+_{\mathrm{e}}, f|_{\mathfrak{R}^3} \in \mathscr{F}^3_{\mathrm{e}}, \widetilde{f|_{\mathfrak{R}_+}}(\boldsymbol{0}) = \widetilde{f|_{\mathfrak{R}^3}}(\boldsymbol{0}) \right\}.$$

Proof. Denote the family on the right hand side by $\mathscr{G}$. Take an arbitrary function $f \in \mathscr{F}_{\mathrm{e}}$. Let $\mathscr{C}$ given by (3.5) be a special standard core of $(\mathscr{E}, \mathscr{F})$. By [14, Theorem 2.1.7], there exist $f_n \in \mathscr{C}$ which are $\mathscr{E}$-Cauchy and converge to f, $\mathscr{E}$-q.e. as $n \to \infty$. Then it follows from Proposition 3.7 that ${}^+f_n := f_n|_{\mathfrak{R}_+}$ (resp. ${}^3f_n := f_n|_{\mathfrak{R}_3}$) is $\mathscr{E}^+$-Cauchy (resp. $\mathscr{E}^3$-Cauchy) and ${}^+f_n \to f|_{\mathfrak{R}^+}$, $\mathscr{E}^+$-q.e. (resp. ${}^3f_n \to f|_{\mathfrak{R}^3}$, $\mathscr{E}^3$-q.e.). Particularly, $f|_{\mathfrak{R}^+} \in \mathscr{F}^+_{\mathrm{e}}, f|_{\mathfrak{R}^3} \in \mathscr{F}^3_{\mathrm{e}}$ and $\widetilde{f|_{\mathfrak{R}_+}}(\mathbf{0}) = \lim_{n\to\infty} {}^+f_n(\mathbf{0}) = \lim_{n\to\infty} {}^3f_n(\mathbf{0}) = \widetilde{f|_{\mathfrak{R}^3}}(\mathbf{0})$. This yields $\mathscr{F}_{\mathrm{e}} \subset \mathscr{G}$.

To the contrary, take $f \in \mathscr{G}$. Then ${}^+f := f|_{\mathfrak{R}_+}$ admits an approximation sequence ${}^+f_n \in \mathscr{C}^+$ with ${}^+f_n \to {}^+f$ pointwisely and ${}^3f := f|_{\mathfrak{R}^3}$ admits an approximation sequence ${}^3f_n \in \mathscr{C}^3$ with ${}^+f_n \to {}^3f$, $\mathscr{E}^3$-q.e., where $\mathscr{C}^+$ and $\mathscr{C}^3$ are given in (3.5). It follows that

$$\lim_{n\to\infty} {}^+f_n(\mathbf{0}) = \widetilde{{}^+f}(\mathbf{0}) = \widetilde{{}^3f}(\mathbf{0}) = \lim_{n\to\infty} {}^3f_n(\mathbf{0}).$$

In the case that a subsequence $\{n_k : k \geq 1\}$ of $\{n : n \geq 1\}$ exists such that ${}^+f_{n_k}(\mathbf{0}) \neq 0$ for all k (resp. ${}^3f_{n_k}(\mathbf{0}) \neq 0$ for all k),

$$f_k(x) := \begin{cases} \dfrac{{}^3f_{n_k}(\mathbf{0})}{{}^+f_{n_k}(\mathbf{0})} {}^+f_{n_k}(x), & x \in \mathfrak{R}_+, \\ {}^3f_{n_k}(x), & x \in \mathfrak{R}^3, \end{cases} \quad \left(\text{resp. } f_k(x) := \begin{cases} {}^+f_{n_k}(x), & x \in \mathfrak{R}_+, \\ \dfrac{{}^+f_{n_k}(\mathbf{0})}{{}^3f_{n_k}(\mathbf{0})} {}^3f_{n_k}(x), & x \in \mathfrak{R}^3, \end{cases} \right)$$

defines a function in $\mathscr{C}$, which is an approximation sequence of f. This leads to $f \in \mathscr{F}_{\mathrm{e}}$. Otherwise we can assume without loss of generality that ${}^+f_n(\mathbf{0}) = {}^3f_n(\mathbf{0}) = 0$ for all n. Then $f_n := {}^+f_n$ on $\mathfrak{R}_+$ and $f_n := {}^3f_n$ on $\mathfrak{R}^3$ gives an approximation sequence of f and it yields $f \in \mathscr{F}_{\mathrm{e}}$ as well. That completes the proof. □

The following corollary is a straightforward consequence of Proposition 3.10.

Corollary 3.11. *$(\mathscr{E}, \mathscr{F})$ is recurrent if and only if both $(\mathscr{E}^+, \mathscr{F}^+)$ and $(\mathscr{E}^3, \mathscr{F}^3)$ are recurrent. Otherwise $(\mathscr{E}, \mathscr{F})$ is transient.*

Proof. It suffices to note that $(\mathscr{E}, \mathscr{F})$ is recurrent, if and only if $1 \in \mathscr{F}_{\mathrm{e}}$ and $\mathscr{E}(1, 1) = 0$. □

3.5 Generator of $(\mathscr{E}, \mathscr{F})$

The generator of $(\mathcal{E}^+, \mathcal{F}^+)$ is $\mathcal{A}^+ := \frac{1}{2}\frac{d^2}{dm_+ d\mathtt{s}_+}$ with the domain (see e.g. [13])

$$\mathcal{D}(\mathcal{A}^+) := \left\{ f \in \mathcal{F}^+ : \frac{df}{d\mathtt{s}_+} \ll m_+, \frac{d^2 f}{dm_+ d\mathtt{s}_+} \in L^2(\mathbb{R}_+, m_+) \right\}.$$

Particularly, $C_c^\infty(\mathbb{R}_+) \subset \mathcal{D}(\mathcal{A}^+)$. Then the generator of $(\mathscr{E}^+, \mathscr{F}^+)$ is $\mathscr{A}^+ := \iota_+^* \mathcal{A}^+$, where $(\iota_+^*\mathcal{A}^+)f := \mathcal{A}^+(f \circ \iota_+)$ for $f \in \mathcal{D}(\mathscr{A}^+) := \{g : g \circ \iota_+ \in \mathcal{D}(\mathcal{A}^+)\}$. On the other hand, the generator $\mathcal{A}^3 := \mathcal{A}_\gamma$ of $(\mathcal{E}^3, \mathcal{F}^3)$ is given by (2.7). Analogically the generator of $(\mathscr{E}^3, \mathscr{F}^3)$ is $\mathscr{A}^3 := \iota_3^* \mathcal{A}^3$, where $(\iota_3^*\mathcal{A}^3)f := \mathcal{A}^3(f \circ \iota_3)$ for $f \in \mathcal{D}(\mathscr{A}^3) := \{g : g \circ \iota_3 \in \mathcal{D}(\mathcal{A}^+)\}$.

Denote the generator of $(\mathscr{E}, \mathscr{F})$ by $\mathscr{A}$ with the domain $\mathcal{D}(\mathscr{A})$. Set $C_c^\infty(E) := \mathscr{C}$ defined by (3.5) with $\mathscr{C}^+ := C_c^\infty(\mathbb{R}_+) \circ \iota_+^{-1}$ and $\mathscr{C}^3 := C_c^\infty(\mathbb{R}^3) \circ \iota_3^{-1}$. It is straightforward to verify that $C_c^\infty(E) \subset \mathcal{D}(\mathscr{A})$ and for all $f \in C_c^\infty(E)$,

$$\mathscr{A} f|_{\mathfrak{R}_+} = \mathtt{p} \cdot \mathscr{A}^+(f|_{\mathfrak{R}_+}), \quad \mathscr{A} f|_{\mathfrak{R}^3} = \mathscr{A}^3(f|_{\mathfrak{R}^3}).$$

Define another operator on $L^2(E,\mathfrak{m})$ as follows:

$$\begin{aligned}&\mathcal{D}(\mathscr{G}) := \left\{f\in\mathscr{F} : f|_{\mathfrak{R}_+}\in\mathcal{D}(\mathscr{A}^+), f|_{\mathfrak{R}^3}\in\mathcal{D}(\mathscr{A}^3)\right\},\\ &\mathscr{G}f|_{\mathfrak{R}_+} := \mathtt{p}\cdot\mathscr{A}^+(f|_{\mathfrak{R}_+}),\quad \mathscr{G}f|_{\mathfrak{R}^3} := \mathscr{A}^3(f|_{\mathfrak{R}^3}),\quad \forall f\in\mathcal{D}(\mathscr{G}).\end{aligned}$$

Clearly $C_c^\infty(E)\subset\mathcal{D}(\mathscr{G})$ and $\mathscr{G}|_{C_c^\infty(E)} = \mathscr{A}|_{C_c^\infty(E)}$. Note that $\mathscr{G}$ is not self-adjoint on $L^2(E,\mathfrak{m})$, since $\mathcal{D}(\mathscr{G})\subsetneqq\{f\in L^2(E,\mathfrak{m}) : f|_{\mathfrak{R}_+}\in\mathcal{D}(\mathscr{A}^+), f|_{\mathfrak{R}^3}\in\mathcal{D}(\mathscr{A}^3)\}\subset\mathcal{D}(\mathscr{G}^*)$ where $\mathscr{G}^*$ is the adjoint operator of $\mathscr{G}$. Particularly, $\mathscr{A}\neq\mathscr{G}$. Furthermore, we have the following.

Proposition 3.12. *The following hold for $\mathscr{A}$ and $\mathscr{G}$:*

(1) $\mathscr{A}$ is a self-adjoint extension of $\mathscr{G}$ on $L^2(E,\mathfrak{m})$.
(2) When $\rho+1/\rho\in L^1(\mathbb{R}_+)$ and $\gamma>0$, $f\in\mathcal{D}(\mathscr{G})$ if and only if $f\in\mathcal{D}(\mathscr{A})$ and $\mathfrak{m}_+(\mathscr{A}f|_{\mathfrak{R}_+}) = \mathfrak{m}_3(\mathscr{A}f|_{\mathfrak{R}^3}) = 0$.

Proof. (1) Take $f\in\mathcal{D}(\mathscr{G})$. Then $f\in\mathscr{F}$ and for any $g\in\mathscr{F}$,

$$\begin{aligned}\mathscr{E}(f,g) &= \mathtt{p}\cdot\mathscr{E}^+(f|_{\mathfrak{R}_+}, g|_{\mathfrak{R}_+}) + \mathscr{E}^3(f|_{\mathfrak{R}^3}, g|_{\mathfrak{R}^3})\\ &= (-\mathtt{p}\cdot\mathscr{A}^+(f|_{\mathfrak{R}_+}), g|_{\mathfrak{R}_+})_{L^2(E,\mathfrak{m})} + (-\mathscr{A}^3(f|_{\mathfrak{R}^3}), g|_{\mathfrak{R}^3})_{L^2(E,\mathfrak{m})}\\ &= (-\mathscr{G}f, g)_{L^2(E,\mathfrak{m})},\end{aligned}$$

where the second equality is due to $f|_{\mathfrak{R}_+}\in\mathcal{D}(\mathscr{A}^+)$ and $f|_{\mathfrak{R}^3}\in\mathcal{D}(\mathscr{A}^3)$. Hence we can conclude that $f\in\mathcal{D}(\mathscr{A})$ and $\mathscr{A}f = \mathscr{G}f$.

(2) When $\rho+1/\rho\in L^1(\mathbb{R}_+)$ and $\gamma>0$, the constant functions belong to both $\mathscr{F}^+$ and $\mathscr{F}^3$. Take $f\in\mathcal{D}(\mathscr{G})$. It follows from the first assertion that $\mathscr{A}f=\mathscr{G}f$, which yields $\mathscr{A}f|_{\mathfrak{R}_+} = \mathscr{G}f|_{\mathfrak{R}_+} = \mathtt{p}\cdot\mathscr{A}^+(f|_{\mathfrak{R}_+})$. Thus $\mathfrak{m}_+(\mathscr{A}f|_{\mathfrak{R}_+}) = \mathtt{p}\cdot\mathfrak{m}_+\left(\mathscr{A}^+(f|_{\mathfrak{R}_+})\right) = -\mathtt{p}\cdot\mathscr{E}^+(f|_{\mathfrak{R}_+},1) = 0$. Analogically we can obtain $\mathfrak{m}_3(\mathscr{A}f|_{\mathfrak{R}^3}) = 0$. To the contrary, let $f\in\mathcal{D}(\mathscr{A})$ such that $\mathfrak{m}_+(\mathscr{A}f|_{\mathfrak{R}_+}) = \mathfrak{m}_3(\mathscr{A}f|_{\mathfrak{R}^3}) = 0$. Take arbitrary $g^+\in\mathscr{F}^+$, define a function g on E by letting $g|_{\mathfrak{R}_+} := g^+$ and $g|_{\mathfrak{R}^3}\equiv g^+(\mathbf{0})$. Clearly $g\in\mathscr{F}$ and it follows that

$$\begin{aligned}\mathscr{E}(f,g) &= (-\mathscr{A}f, g)_{L^2(E,\mathfrak{m})}\\ &= (-\mathscr{A}f|_{\mathfrak{R}_+}, g^+)_{L^2(\mathfrak{R}_+,\mathfrak{m}_+)} + g^+(\mathbf{0})\cdot\mathfrak{m}_3(\mathscr{A}f|_{\mathfrak{R}^3})\\ &= (-\mathscr{A}f|_{\mathfrak{R}_+}, g^+)_{L^2(\mathfrak{R}_+,\mathfrak{m}_+)}.\end{aligned}$$

On the other hand, $\mathscr{E}(f,g) = \mathtt{p}\cdot\mathscr{E}^+(f|_{\mathfrak{R}_+}, g^+) + g^+(\mathbf{0})\cdot\mathscr{E}^3(f|_{\mathfrak{R}^3},1) = \mathtt{p}\cdot\mathscr{E}^+(f|_{\mathfrak{R}_+}, g^+)$. These yield for all $g^+\in\mathscr{F}^+$,

$$\mathscr{E}^+(f|_{\mathfrak{R}_+}, g^+) = \left(-\frac{1}{\mathtt{p}}\cdot\left(\mathscr{A}f|_{\mathfrak{R}_+}\right), g^+\right)_{L^2(\mathfrak{R}_+,\mathfrak{m}_+)}.$$

Consequently, $f|_{\mathfrak{R}_+}\in\mathcal{D}(\mathscr{A}^+)$ and $\mathscr{A}^+(f|_{\mathfrak{R}_+}) = \frac{1}{\mathtt{p}}\cdot\left(\mathscr{A}f|_{\mathfrak{R}_+}\right)$. Analogically we can obtain that $f|_{\mathfrak{R}^3}\in\mathcal{D}(\mathscr{A}^3)$ and $\mathscr{A}^3(f|_{\mathfrak{R}^3}) = \mathscr{A}f|_{\mathfrak{R}^3}$. Eventually $f\in\mathcal{D}(\mathscr{G})$. That completes the proof. □

Remark 3.13. Consider the recurrent case $\rho+1/\rho\in L^1(\mathbb{R}_+)$ and $\gamma>0$. For any $f\in\mathcal{D}(\mathscr{A})$, $\mathfrak{m}(\mathscr{A}f) = -\mathscr{E}(f,1) = 0$. This yields $\mathfrak{m}_+(\mathscr{A}f|_{\mathfrak{R}_+}) + \mathfrak{m}_3(\mathscr{A}f|_{\mathfrak{R}^3}) = 0$. Consequently, $f\in\mathcal{D}(\mathscr{G})$ is also equivalent to $f\in\mathcal{D}(\mathscr{A})$ and $\mathfrak{m}_+(\mathscr{A}f|_{\mathfrak{R}_+}) = 0$ (or $\mathfrak{m}_3(\mathscr{A}f|_{\mathfrak{R}^3}) = 0$).

Let $L^2(E)$ be the L^2-space on E endowed with the Lebesgue measures on $\mathfrak{R}_+$ and $\mathfrak{R}^3$ respectively. The notation $\mathbf{\Delta}$ denotes the Laplacian operator acting on $C_c^\infty(E\setminus\{\mathbf{0}\})$, i.e. for any $f\in C_c^\infty(E\setminus\{\mathbf{0}\})$, $\mathbf{\Delta}f(x) := \Delta f(x)$ for $x\in\mathfrak{R}^3\setminus\{\mathbf{0}\}$ and $\mathbf{\Delta}f(r) := f''(r)$ for $r\in\mathfrak{R}_+\setminus\{\mathbf{0}\}$. Recall that $h_{\rho,\gamma}$ is defined by (3.4). The following result is obvious by Lemma 2.2 and Example 3.2.

Proposition 3.14. *Take a constant $\alpha \in \mathbb{R}$ and let $\rho(r) := \mathrm{e}^{-2\alpha r}/\pi$. Set $h_{\alpha,\gamma} := h_{\rho,\gamma}$. Then the following operator*

$$\mathcal{D}(\mathscr{L}_{\alpha,\gamma}) := \{f \in L^2(E) : f/h_{\alpha,\gamma} \in \mathcal{D}(\mathscr{A})\},$$
$$\mathscr{L}_{\alpha,\gamma} f := h_{\alpha,\gamma} \cdot \mathscr{A}\left(\frac{f}{h_{\alpha,\gamma}}\right) + \left(\frac{\alpha^2}{2} \cdot f|_{\mathfrak{R}_+} + \frac{\gamma^2}{2} f|_{\mathfrak{R}^3}\right), \quad f \in \mathcal{D}(\mathscr{L}_{\alpha,\gamma}),$$

is a self-adjoint extension of $\mathbf{\Delta}$ (acting on $C_c^\infty(E \setminus \{\mathbf{0}\})$) on $L^2(E)$. Furthermore, if $(\alpha_1, \gamma_1) \neq (\alpha_2, \gamma_2)$, then $\mathscr{L}_{\alpha_1,\gamma_1} \neq \mathscr{L}_{\alpha_2,\gamma_2}$.

3.6 Existence of transition density

The following proposition states the existence of the transition density of M, which is the foundation of the study in §5.

Proposition 3.15. *The dBMV M satisfies the absolute continuity condition in the following sense: For any $x \in E$ and $t > 0$, it holds $P_t(x, \cdot) \ll \mathfrak{m}$, where $\{P_t(x, \cdot) = \mathbb{P}_x(M_t \in \cdot) : t \geq 0\}$ denotes the semigroup of M. Particularly there exists a density function $\{p(t, x, y) : t > 0, x, y \in E\}$ such that $P_t(x, dy) = p(t, x, y)\mathfrak{m}(dy)$.*

Proof. By [14, Theorem 4.2.4], it suffices to show that any $\mathfrak{m}$-polar set is polar (with respect to M). Let B be such a nearly Borel $\mathfrak{m}$-polar set and set

$$\varphi(x) := \mathbb{E}_x\left(\mathrm{e}^{-\sigma_B}; \sigma_B < \infty\right), \quad \forall x \in E,$$

where $\sigma_B := \inf\{t > 0 : M_t \in B\}$. Proposition 3.7 indicates $B \subset \mathfrak{R}^3 \setminus \{\mathbf{0}\}$ and the definition of $\mathfrak{m}$-polar set tells us $\varphi = 0$, $\mathfrak{m}$-a.e. We need to show $\varphi(x) = 0$ for every $x \in E$ to conclude that B is polar. Clearly φ is q.e. finely continuous (see e.g. [14, Theorem 4.2.5]) and hence $\varphi(x) = 0$ for q.e $x \in E$ by applying [14, Lemma 4.1.5]. Particularly $\varphi(x) = 0$ for all $x \in \mathfrak{R}_+$ due to Proposition 3.7. Let $M^{3,\mathbf{0}}$ be the part process of M on $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$. Note that M^3 satisfies the absolute continuity condition as mentioned in Remark 2.3, and thus so does $M^{3,\mathbf{0}}$. Fix $x \in \mathfrak{R}^3 \setminus \{\mathbf{0}\}$. Then

$$\varphi(x) = \mathbb{E}_x\left(\mathrm{e}^{-\sigma_B}; \sigma_B < \sigma_{\{\mathbf{0}\}}\right) + \mathbb{E}_x\left(\mathrm{e}^{-\sigma_B}; \sigma_B \geq \sigma_{\{\mathbf{0}\}}\right).$$

Since $M^{3,\mathbf{0}}$ satisfies the absolute continuity condition, it follows that B is polar with respect to $M^{3,\mathbf{0}}$ and $\mathbb{E}_x\left(\mathrm{e}^{-\sigma_B}; \sigma_B < \sigma_{\{\mathbf{0}\}}\right) = 0$. On $\{\sigma_B \geq \sigma_{\{\mathbf{0}\}}\}$, $\sigma_B = \sigma_B \circ \theta_{\sigma_{\{\mathbf{0}\}}} + \sigma_{\{\mathbf{0}\}}$. By denoting the filtration of M by $(\mathcal{F}_t)_{t\geq 0}$ and using the strong Markovian property, we have

$$\begin{aligned}
\mathbb{E}_x\left(\mathrm{e}^{-\sigma_B}; \sigma_B \geq \sigma_{\{\mathbf{0}\}}\right) &= \mathbb{E}_x\left(\mathrm{e}^{-\sigma_B \circ \theta_{\sigma_{\{\mathbf{0}\}}}} \cdot \mathrm{e}^{-\sigma_{\{\mathbf{0}\}}}; \sigma_B \geq \sigma_{\{\mathbf{0}\}}\right) \\
&\leq \mathbb{E}_x\left(\mathrm{e}^{-\sigma_B} \circ \theta_{\sigma_{\{\mathbf{0}\}}}; \sigma_B \geq \sigma_{\{\mathbf{0}\}}\right) \\
&\leq \mathbb{E}_x\left(\mathbb{E}_x\left(\mathrm{e}^{-\sigma_B} \circ \theta_{\sigma_{\{\mathbf{0}\}}} | \mathcal{F}_{\sigma_{\{\mathbf{0}\}}}\right)\right) \\
&= \mathbb{E}_x\left(\mathbb{E}_{M_{\sigma_{\{\mathbf{0}\}}}}\left(\mathrm{e}^{-\sigma_B}\right)\right) = \mathbb{E}_x\left(\mathbb{E}_{\mathbf{0}}\left(\mathrm{e}^{-\sigma_B}\right)\right) \\
&= \mathbb{E}_x\left(\varphi(\mathbf{0})\right) = 0.
\end{aligned}$$

Consequently $\varphi(x) = 0$, which eventually yields that B is polar with respect to M. That completes the proof. □

4 Signed radial process of dBMV

Let $(\mathscr{E}, \mathscr{F})$ be in Theorem 3.5 and M be its associated dBMV. This section is devoted to obtaining the expression of the signed radial process induced by M. Define a map $u : E \mapsto \mathbb{R}$ as follows:

$$u(x) := \begin{cases} |x|, & x = (x_1, x_2, x_3, 0) \in \mathfrak{R}^3, \\ -r, & x = (0, 0, 0, r) \in \mathfrak{R}_+, \end{cases}$$

and let $Y_t := u(M_t)$ for any $t \geq 0$. Then $Y := (Y_t)_{t\geq 0}$ is the so-called *signed radial process* of M. Set $\ell^{(\mathtt{p})}_{\rho,\gamma} := \mathfrak{m} \circ u^{-1}$, which is a fully supported Radon measure on $\mathbb{R}$. In practise, one can easily obtain

$$\ell^{(\mathtt{p})}_{\rho,\gamma}(dr) = \frac{\mathrm{e}^{-2\gamma|r|}}{\pi} dr|_{(0,\infty)} + \mathtt{p} \cdot \rho(-r) dr|_{(-\infty,0)}. \tag{4.1}$$

Hereafter we will write $\ell^{(\mathtt{p})}_{\rho,\gamma}$ as ℓ for short if no confusions caused.

Proposition 4.1. *$Y = (Y_t)_{t\geq 0}$ is an ℓ-symmetric diffusion process on $\mathbb{R}$. It is associated with the regular Dirichlet form on $L^2(\mathbb{R}, \ell)$:*

$$\begin{aligned} \mathscr{F}^Y &= \{f \in L^2(\mathbb{R}, \ell) : f' \in L^2(\mathbb{R}, \ell)\}, \\ \mathscr{E}^Y(f, g) &= \frac{1}{2} \int_{\mathbb{R}} f'(x) g'(x) \ell(dx), \quad f, g \in \mathscr{F}^Y, \end{aligned} \tag{4.2}$$

where f' stand for the weak derivative of f for all $f \in \mathscr{F}^Y$.

Proof. To prove Y is a Markov process, we apply [22, Theorem (13.5)]. It suffices to show that for any bounded $f \in \mathcal{E}^u(\mathbb{R})$, there exists $g \in \mathcal{E}^u(\mathbb{R})$ such that

$$P_t(f \circ u) = g \circ u, \tag{4.3}$$

where $\mathcal{E}^u(\mathbb{R})$ is the family of all universally measurable functions on $\mathbb{R}$ and P_t is the semigroup of M. By the rotational invariance of M^3, it is not hard to find that for any $x, y \in E$ with $u(x) = u(y) =: r$,

$$\int_E f(u(\cdot)) \mathbb{P}_x(M_t \in \cdot) = \int_E f(u(\cdot)) \mathbb{P}_y(M_t \in \cdot).$$

This implies $P_t(f \circ u)(x) = P_t(f \circ u)(y)$. Set $g(r) := P_t(f \circ u)(x)$, which is a well-defined function on $\mathbb{R}$ since u is surjective. The universal measurability of g is derived as follows. Since u is continuous, it follows that $f \circ u \in \mathcal{E}^u(E)$ and thus $P_t(f \circ u) \in \mathcal{E}^u(E)$. For any set $A \in \mathcal{B}(\mathbb{R})$, let $B_+ := g^{-1}(A) \cap (0, \infty)$ and $B_- := g^{-1}(A) \cap (-\infty, 0]$. We have

$$\iota_3(B_+ \times S^2) \cup \iota_+(-B_-) = (P_t(f \circ u))^{-1}(A) \in \mathcal{E}^u(E).$$

Hence $\iota_3(B_+ \times S^2) = (P_t(f \circ u))^{-1}(A) \cap (\mathfrak{R}^3 \setminus \{\mathbf{0}\}) \in \mathcal{E}^u(E)$ and $\iota_+(-B_-) = (P_t(f \circ u))^{-1}(A) \cap \mathfrak{R}_+ \in \mathcal{E}^u(E)$. This leads to $B_+, B_- \in \mathcal{E}^u(\mathbb{R})$ by the continuity of ι_+, ι_3. Therefore, $g^{-1}(A) = B_+ \cup B_- \in \mathcal{E}^u(\mathbb{R})$.

By applying [22, Theorem (13.5)], we can conclude that Y is a Markov process and its transition semigroup is

$$P^Y_t f := g,$$

where f, g are in (4.3). Moreover, for any two functions f_1, f_2, we have

$$\begin{aligned} &(P^Y_t f_1, f_2)_\ell \\ &\quad = ((P^Y_t f_1) \circ u, f_2 \circ u)_{\mathfrak{m}} = (P_t(f_1 \circ u), f_2 \circ u)_{\mathfrak{m}} = (f_1 \circ u, P_t(f_2 \circ u))_{\mathfrak{m}} = (f_1, P^Y_t f_2)_\ell. \end{aligned}$$

This leads to the symmetry of Y. By means of Yosida approximation, we can easily obtain that Y is associated with the Dirichlet form on $L^2(\mathbb{R}, \ell)$:

$$\begin{aligned} \mathscr{F}^Y &= \{f : f \circ u \in \mathscr{F}\}, \\ \mathscr{E}^Y(f, f) &= \mathscr{E}(f \circ u, f \circ u), \quad f \in \mathscr{F}^Y. \end{aligned}$$

A simple computation gives the expression (4.2) of $(\mathscr{E}^Y, \mathscr{F}^Y)$. The regularity of $(\mathscr{E}^Y, \mathscr{F}^Y)$ is also clear by virtue of [19, Corollary 3.11]. That completes the proof. □

Remark 4.2. By the expression of $(\mathscr{E}^Y, \mathscr{F}^Y)$, one can easily figure out (see e.g. [19]) that Y is an irreducible diffusion on $\mathbb{R}$, whose speed measure is ℓ and scale function $\mathtt{s}^Y$ is as follows: For $r \geq 0$,

$$\mathtt{s}^Y(r) = \begin{cases} \dfrac{\pi e^{2\gamma r} - \pi}{2\gamma}, & \text{when } \gamma \neq 0, \\ \pi r, & \text{when } \gamma = 0; \end{cases}$$

and for $r < 0$,

$$\mathtt{s}^Y(r) = -\int_r^0 \frac{1}{\mathtt{p} \cdot \rho(-s)} ds.$$

Not surprisingly, Y is recurrent if and only if $1/\rho \notin L^1(\mathbb{R}_+)$ and $\gamma \geq 0$. Otherwise Y is transient. A straightforward computation yields that the infinities $\pm\infty$ are not approachable in finite time (cf. (2.14)) and hence Y is conservative. This leads to the conservativeness of M.

On the other hand, the symmetrizing measures of Y are unique up to a constant in the sense that if a non-trivial measure μ is a symmetric measure of Y then $\mu = c \cdot \ell$ for some constant c (see e.g. [24]). This yields that for different $\mathtt{p}$, Y is different and therefore so is M.

From now on we impose the following condition on ρ:

(ACP) ρ is absolutely continuous and $\rho(r) > 0$ for all $r \in \mathbb{R}_+$.

Note that **(ACP)** indicates (3.2). Denote the family of probability measures for Y by $\{\mathbb{P}^Y_r : r \in \mathbb{R}\}$. We next prove that Y is a semi-martingale with the quadratic variation process $\langle Y \rangle_t = t$ and figure out the associated SDE for Y. The symmetric semi-martingale local time of Y at 0 is denoted by $L^0_t(Y)$, that is,

$$L^0_t(Y) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y_s) d\langle Y \rangle_s = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y_s) ds, \quad \forall t \geq 0.$$

The main result of this section is as follows, and in its proof the celebrated Fukushima's decomposition is employed.

Theorem 4.3. *Assume that **(ACP)** holds. The signed radial process Y is a semi-martingale whose quadratic variation process is $\langle Y \rangle_t = t$. Furthermore for any $r \in \mathbb{R}$, $Y = (Y_t)_{t \geq 0}$ under the probability measure $\mathbb{P}^Y_r$ is the unique solution to the following well-posed SDE (that is, this SDE has weak solutions and the pathwise uniqueness holds for it.):*

$$\begin{aligned} dY_t &= dW_t + b(Y_t)dt + \frac{1 - \pi \mathtt{p} \rho(0)}{1 + \pi \mathtt{p} \rho(0)} \cdot dL^0_t(Y), \\ Y_0 &= r, \end{aligned} \tag{4.4}$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, b is defined by

$$b(r) := \begin{cases} -\gamma, & r \geq 0, \\ \dfrac{-\rho'(-r)}{2\rho(-r)}, & r < 0, \end{cases} \tag{4.5}$$

and $L^0(Y) = (L^0_t(Y))_{t \geq 0}$ is the symmetric semi-martingale local time of Y at 0.

Proof. We first show that Y is a semi-martingale. Take $f(r) := r \in \mathscr{F}^Y_{\mathrm{loc}}$ and consider the Fukushima's decomposition for f:

$$f(Y_t) - f(Y_0) = M^f_t + N^f_t.$$

The martingale part M^f is determined by its energy measure $\mu_{\langle f\rangle}$ and for any $g\in C_c^\infty(\mathbb{R})$ (see [14, Theorem 5.5.2]),

$$\int g d\mu_{\langle f\rangle}=2\mathscr{E}^Y(fg,f)-\mathscr{E}^Y(f^2,g)=\int gd\ell.$$

It follows that $\mu_{\langle f\rangle}=\ell$ and hence M^f has the same distribution as standard Brownian motion. For the zero-energy part N^f, we note

$$-\mathscr{E}^Y(f,g)=-\frac{1}{2}\int_{\mathbb{R}}g'(r)\ell(dr)=\frac{1-\pi\mathtt{p}\rho(0)}{2\pi}\cdot g(0)-\int_{-\infty}^0 g(r)\frac{\rho'(-r)}{2\rho(r)}\ell(dr)-\gamma\int_0^\infty g(r)\ell(dr).$$

Thus [14, Corollary 5.5.1] yields that N^f is of bounded variation, and its associated signed smooth measure is

$$\mu_{N^u}=\frac{1-\pi\mathtt{p}\rho(0)}{2\pi}\cdot\delta_0+b(r)\ell(dr).$$

Eventually, we can conclude

$$Y_t-Y_0=W_t+\int_0^t b(Y_s)ds+\frac{1-\pi\mathtt{p}\rho(0)}{2\pi}\cdot l_t^0,\quad t\geq 0,\tag{4.6}$$

where (W_t) is a certain standard Brownian motion and $l^0:=(l^0_t)_{t\geq 0}$ is the the local time of Y at 0, i.e. is the positive continuous additive functional of M having Revuz measure δ_0. Particularly, Y is a semi-martingale and $\langle Y\rangle_t=t$.

Since $\rho(r)>0$ for all $r\geq 0$, it is straightforward to verify that $b\in L^1_{\mathrm{loc}}(\mathbb{R})$. The well-posedness of (4.4) is concluded by e.g. [17, Theorem 7.1]. It suffices to note that [17, Lemma 4.3] yields

$$L_t^0(Y)=\frac{1+\pi\mathtt{p}\rho(0)}{2\pi}\cdot l^0_t.$$

Therefore (4.6) implies that Y is a weak solution to (4.4). That completes the proof. □

The weight parameter $\mathtt{p}$ appearing in the symmetric measure $\mathfrak{m}^{(\mathtt{p})}$ plays a role of so-called “skew” constant. When $\rho\equiv 1$ and $\gamma=0$, Y is noting but the well-known skew Brownian motion with the skew constant $\frac{1}{1+\pi\mathtt{p}}$, which behaves like a Brownian motion except for the sign of each excursion is chosen by using an independent Bernoulli random variable of the parameter $\frac{1}{1+\pi\mathtt{p}}$. For general ρ and γ, Y is called a *general skew Brownian motion* in a recent work [17]. The non-skew case means that the last term in (4.4) disappears, i.e. $\mathtt{p}=\frac{1}{\pi\rho(0)}$, and clearly the following corollary holds.

Corollary 4.4. *When $\mathtt{p}=\frac{1}{\pi\rho(0)}$, Y is the unique solution to the SDE*

$$dY_t=dW_t+b(Y_t)dt,$$

where W is a standard Brownian motion and b is defined by (4.5).

We end this section with a remark for the condition **(ACP)**. Firstly, $\rho>0$ is only employed to conclude the well-posedness of (4.4). In fact when $\rho(0)=0$ (for example $\rho(r):=|r|^\alpha$ for a constant $0<\alpha<1$), the other derivation still works and Y is a weak solution to

$$dY_t=dW_t+b(Y_t)dt+dL^0_t(Y).\tag{4.7}$$

Note that b is independent of $\mathtt{p}$ and for different $\mathtt{p}$, Y is different as we explained in Remark 4.2. These yield that (4.7) has infinite weak solutions. The point 0 where $L^0(Y)$ locates is usually called a *barrier* for (4.7), as in the reduced case $b\equiv 0$ (though this will not happen in (4.4) because $b\equiv 0$ implies that ρ is constant, which contradicts to

$\rho(0) = 0$), the solution to (4.7) is nothing but the reflecting Brownian motion on $\mathbb{R}_+$. At this time, Y runs on $\mathbb{R}_+$ and cannot go across the barrier 0 to reach the left axis. However the presence of b in (4.7) leads to infinite solutions Y, which are all irreducible as stated in Remark 4.2. That means Y starting from everywhere can reach every point of $\mathbb{R}$, and the barrier 0 is definitely fake for it. In [17, §7.2], this kind of barriers are called *pseudo barriers* and we also refer more discussions about the equations (4.4) and (4.7) to [17]. Secondly, the condition **(ACP)** can be weaken to

(BV) ρ is cadlag locally of bounded variation on $\mathbb{R}_+$.

Under **(BV)**, let ν_ρ be the signed Radon measure on $(0,\infty)$ induced by ρ and set $\mu_\rho := \hat{\mu}_\rho \circ u^{-1}$ on $(-\infty, 0)$ with

$$\hat{\mu}_\rho(dr) := \frac{\nu_\rho(dr)}{\rho(r)+\rho(r-)}, \quad r>0,$$

where $\rho(r-)$ is the left limit of ρ at $r>0$. As stated in [17, Lemma 5.2], Y is still a semi-martingale with $\langle Y\rangle_t = t$ and a weak solution to the SDE:

$$dY_t = dW_t - \gamma 1_{(0,\infty)}(Y_t)dt + \frac{1-\pi \mathrm{p}\rho(0)}{1+\pi\mathrm{p}\rho(0)}\cdot dL^0_t(Y) + \int_{r\in(-\infty,0)} \mu_\rho(dr) dL^r_t(Y), \tag{4.8}$$

where $(L^r_t)_{t\geq 0}$ is the symmetric local time of Y at $r<0$. It is worth noting that **(BV)** is the weakest assumption for the derivation of (4.8), whose well-posedness holds under the following assumption with the convention $\rho(0-) := \rho(0)$ (see e.g. [17, Lemma 6.3]):

(P) $\rho(r), \rho(r-) > 0$ for all $r\in\mathbb{R}_+$.

Clearly **(ACP)** implies **(BV)** and **(P)**. If we denote the absolute continuous part and singular part of μ_ρ by $b_\rho(r)dr$ and κ_ρ respectively, i.e. $\mu_\rho(dr) = b_\rho(r)dr + \kappa_\rho(dr)$, then the last term on the right hand side of (4.8) is equal to

$$\int_{r\in(-\infty,0)} b_\rho(r) dr dL^r_t(Y) + \int_{r\in(-\infty,0)} \kappa_\rho(dr) dL^r_t(Y) = b_\rho(Y_t)dt + \int_{r\in(-\infty,0)} \kappa_\rho(dr) dL^r_t(Y)$$

by applying the occupation times formula. The condition **(ACP)** indicates $\kappa_\rho = 0$ and meanwhile (4.8) reduces to (4.4).

5 Short-time heat kernel estimate for dBMVs

In this section, utilizing the SDE characterization for the radial process of M derived in Theorem 4.3, we establish the two-sided short-time heat kernel estimate for M, i.e., for $t\leq T$ with an arbitrary $0<T<\infty$. As in Proposition 3.15, $p(t,x,y)$ denotes the transition density of M with respect to $\mathfrak{m}$. Recall that $|\cdot|$ denotes the Euclidean distance on $\mathfrak{R}_+$ as well as on $\mathfrak{R}^3$, and by slightly abusing the notation,

$$|x-y| := |x-\mathbf{0}| + |y-\mathbf{0}|, \quad x\in\mathfrak{R}^3,\, y\in\mathfrak{R}_+. \tag{5.1}$$

Before we introduce the main result of this section, we restate the following definition for Kato class functions (see, for instance, [6]):

Definition 5.1 (Kato class $\mathbf{K}_{n,d}$)**.** *Given $d\in\mathbb{N}$, we say a function $f:\mathbb{R}^n\to\mathbb{R}$ is in Kato class $\mathbf{K}_{n,d}$ if*

$$\lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^n}\int_{|x-y|<r}\frac{|f(y)|}{|x-y|^{d-2}}dy = 0, \quad \textit{for } d\geq 3,$$

$$\lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^n}\int_{|x-y|<r}\log\left(|x-y|^{-1}\right)|f(y)|\,dy = 0, \quad \textit{for } d=2,$$

and

$$\sup_{x\in\mathbb{R}^n}\int_{|x-y|\le 1}|f(y)|\,dy<\infty,\quad \textit{for } d=1.$$

Remark 5.2. We point out that in Definition 5.1, it is *not* necessary that $n=d$. Also, $L^q(\mathbb{R}^d)\subset\left\{f:|f|^2\in\mathbf{K}_{d,d}\right\})$ for all $q\in(d,+\infty]$.

Recall that Proposition 3.15 states the existence of the transition density $p(t,x,y)$ of M but in the sense of almost everywhere. The main result of this section below claims its continuity and obtains its short time estimates. Note that every function g defined on $\mathbb{R}_+$ is regarded as the one on $\mathbb{R}$ by imposing $g|_{(-\infty,0)}=0$ if no confusions caused.

Theorem 5.3. *Assume* (3.3)*,* ***(ACP)*** *and that*

$$\frac{\rho'}{\rho}\in\{f:\,|f|^2\in\mathbf{K}_{1,1}\}. \tag{5.2}$$

Then the transition density $p(t,x,y)$ of the distorted Brownian motion with varying dimension M with respect to $\mathfrak{m}$ is jointly continuous on $(0,\infty)\times E\times E$. Furthermore, for any fixed $0<T<\infty$, there exist positive constants C_i, $1\le i\le 12$, depending on ρ,γ,p,T such that $p(t,x,y)$ satisfies the following estimates: When $t\in(0,T]$,

(i) *For $x,y\in\mathfrak{R}_+$,*

$$\frac{C_1}{\sqrt{t}\rho(|y|)}\mathrm{e}^{-\frac{C_2|x-y|^2}{t}}\le p(t,x,y)\le\frac{C_3}{\sqrt{t}\rho(|y|)}\mathrm{e}^{-\frac{C_4|x-y|^2}{t}}; \tag{5.3}$$

(ii) *For $x\in\mathfrak{R}_+$ and $y\in\mathfrak{R}^3$,*

$$\frac{C_5}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-\frac{C_6|x-y|^2}{t}}\le p(t,x,y)\le\frac{C_7}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-\frac{C_8|x-y|^2}{t}}; \tag{5.4}$$

(iii) *For $x,y\in\mathfrak{R}^3$,*

$$\frac{C_9}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-\frac{C_{10}(|x|+|y|)^2}{t}}+q(t,x,y)\le p(t,x,y)\le\frac{C_{11}}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-\frac{C_{12}(|x|+|y|)^2}{t}}+q(t,x,y), \tag{5.5}$$

where

$$q(t,x,y)=\sqrt{\frac{2\pi}{t^3}}|x||y|\mathrm{e}^{-\frac{\gamma^2}{2}t+\gamma(|x|+|y|)-\frac{|x-y|^2}{2t}},\quad t>0,\,x,y\in\mathfrak{R}^3, \tag{5.6}$$

is the transition density of killed distorted Brownian motion $M^{3,\mathbf{0}}$ (see the statement after Definition 3.3) with respect to $\mathfrak{m}_3$.

Remark 5.4. Remark 5.2 yields that if $\rho'/\rho\in L^q(\mathbb{R}_+)$ for some $q\in(1,+\infty]$, then (5.2) holds. An example satisfying all these assumptions is given in Example 3.2, i.e. $\rho(r):=\mathrm{e}^{-2\alpha r}/\pi$ for a constant $\alpha\in\mathbb{R}$. In this case, $\rho'/\rho\equiv-2\alpha\in L^\infty(\mathbb{R}_+)$.

The proof will be divided into several steps. To accomplish it, we prepare a lemma concerning the short-time heat kernel estimate for the signed radial process Y.

Lemma 5.5. *Assume the same assumptions as Theorem 5.3 hold. Set a measure on $\mathbb{R}$*

$$\widehat{\ell}(dr):=\frac{2}{1+\kappa}dr|_{(-\infty,0)}+\frac{2}{1-\kappa}dr|_{(0,\infty)}$$

with $\kappa:=(1-\pi\mathrm{p}\rho(0))\,/\,(1+\pi\mathrm{p}\rho(0))$. Then the signed radial process Y has a jointly continuous transition density function $\widehat{p}^Y(t,r_1,r_2)$ with respect to $\widehat{\ell}$, i.e. $\mathbb{P}^Y_{r_1}(Y_t\in dr_2)=\widehat{p}^Y(t,r_1,r_2)\widehat{\ell}(dr_2)$ for all $t>0$ and $r_1,r_2\in\mathbb{R}$, and $\widehat{p}^Y$ is jointly continuous on $(0,\infty)\times\mathbb{R}\times\mathbb{R}$. Furthermore, for every $T\ge 0$, there exist constants $C_i>0$, $13\le i\le 16$, such that the following estimate holds:

$$\frac{C_{13}}{\sqrt{t}}\mathrm{e}^{-\frac{C_{14}|r_1-r_2|^2}{t}}\le\widehat{p}^Y(t,r_1,r_2)\le\frac{C_{15}}{\sqrt{t}}\mathrm{e}^{-\frac{C_{16}|r_1-r_2|^2}{t}},\quad 0<t\le T,\,r_1,r_2\in\mathbb{R}. \tag{5.7}$$

Proof. The idea of the proof to the estimate (5.7) is refereed to, for instance, [25, Theorem A]. Note that $-1<\kappa<1$. Let Z be the skew Brownian motion

$$dZ_t = dW_t + \kappa\cdot dL^0_t(Z),$$

where W is a certain one-dimensional standard Brownian motion and $L^0_t(Z)$ is the symmetric semimartingale local time of Z at 0. Clearly Z is symmetric with respect to $\widehat{\ell}$ (see e.g. [15]) and the transition density function $p^Z(t,r_1,r_2)$ of Z with respect to $\widehat{\ell}$ is explicitly known as follows: (see e.g. [21, III.(1.16)]):

$$\begin{aligned} &p^Z(t,r_1,r_2)\\ &\quad= \frac{1-\kappa}{2}\left[g_t(r_2-r_1)+\kappa g_t(r_2+r_1)\right]\mathbf{1}_{\{r_1>0,r_2>0\}} + \frac{1-\kappa^2}{2}g_t(r_2-r_1)\mathbf{1}_{\{r_1\ge 0,r_2\le 0\}} \\ &\quad+ \frac{1+\kappa}{2}\left[g_t(r_2-r_1)-\kappa g_t(r_2+r_1)\right]\mathbf{1}_{\{r_1<0,r_2<0\}} + \frac{1-\kappa^2}{2}g_t(r_2-r_1)\mathbf{1}_{\{r_1\le 0,r_2\ge 0\}}, \end{aligned} \tag{5.8}$$

where $g_t(r)=\mathrm{e}^{-r^2/2t}/\sqrt{2\pi t}$. Clearly p^Z is jointly continuous on $(0,\infty)\times\mathbb{R}\times\mathbb{R}$ and smooth at $r_1,r_2\neq 0$. One can verify directly that for $t>0$ and $r_1\neq 0$,

$$\mathbb{R}\ni r_2\mapsto \nabla_{r_1}p^Z(t,r_1,r_2) \tag{5.9}$$

is continuous, and for some constants $c>0$ and $0<\alpha<\beta$ the following inequalities hold for $t\in(0,T]$:

$$p^Z(t,r_1,r_2)\le ct^{-1/2}\exp(-\alpha|r_1-r_2|^2/t),\quad r_1,r_2\in\mathbb{R}, \tag{5.10}$$

and

$$|\nabla_{r_1}p^Z(t,r_1,r_2)|\le ct^{-1}\exp(-\beta|r_1-r_2|^2/t),\quad r_1\neq 0, r_2\in\mathbb{R}. \tag{5.11}$$

The diffusion process Y can be obtained from Z through a drift perturbation (i.e. Girsanov transform) induce by b given by (4.5). Note that (5.2) leads to $|b|^2\in\mathbf{K}_{1,1}$. We now set $k_0(t,r_1,r_2)=p^Z(t,r_1,r_2)$, and then inductively define

$$k_n(t,r_1,r_2) := \int_0^t\int_{\mathbb{R}} k_{n-1}(t-s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(s,r_3,r_2)dr_3ds,\quad \text{for } n\ge 1. \tag{5.12}$$

Before we proceed with our proof, we first record the following computation: Since $|b|^2\in\mathbf{K}_{1,1}$, it holds

$$\begin{aligned} &\left(\int_{\mathbb{R}}|b(r_3)|^2\mathrm{e}^{-2(\beta-\alpha)|r_2-r_3|^2/T}\,dr_3\right)^{1/2}\\ =&\left(\sum_{i=0}^\infty\int_{i\le|r_3-r_2|\le i+1}|b(r_3)|^2\mathrm{e}^{-2(\beta-\alpha)|r_2-r_3|^2/T}\,dr_3\right)^{1/2}\\ \le&\left(\sum_{i=0}^\infty \mathrm{e}^{-2(\beta-\alpha)i^2/T}\int_{i\le|r_3-r_2|\le i+1}|b(r_3)|^2\,dr_3\right)^{1/2}\overset{(5.2)}{<}\infty. \end{aligned}$$

Thus we set that for some $0<c_1<\infty$,

$$\left(\int_{\mathbb{R}}|b(r_3)|^2\mathrm{e}^{-2(\beta-\alpha)|r_2-r_3|^2/T}\,dr_3\right)^{1/2}<c_1. \tag{5.13}$$

Assuming that for some $n\ge 1$, for all $j=0,\cdots,n-1$, there exist $c_2>0$ and $0<c_3<1/2$ such that

$$|k_j(t,r_1,r_2)|\le c_2\cdot c_3^j\cdot t^{-1/2}\exp(-\alpha|r_1-r_2|^2/t),\quad 0<t\le T,\ r_1,r_2\in\mathbb{R}.$$

Clearly (5.10) tells us that this holds for $n=1$. In view of (5.10) and (5.11), by induction we have on $0<s<t\le T$,

$$\begin{aligned}
&|k_n(t,r_1,r_2)|\\
&\quad=\left|\int_0^t\int_{0\neq r_3\in\mathbb{R}}k_{n-1}(t-s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(s,r_3,r_2)dr_3ds\right|\\
&\quad\le c\cdot c_2\cdot c_3^{n-1}\cdot\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{t-s}}\mathrm{e}^{-\frac{\alpha|r_1-r_3|^2}{t-s}}\cdot|b(r_3)|\cdot\frac{1}{s}\mathrm{e}^{-\frac{\beta|r_3-r_2|^2}{s}}dr_3ds\\
&\quad=c\cdot c_2\cdot c_3^{n-1}\int_0^t\frac{1}{s^{3/4}}\frac{1}{(t-s)^{1/4}}\int_{\mathbb{R}}\frac{1}{(t-s)^{1/4}}\mathrm{e}^{-\frac{\alpha|r_1-r_3|^2}{t-s}}|b(r_3)|\frac{1}{s^{1/4}}\mathrm{e}^{-\frac{\beta|r_3-r_2|^2}{s}}dr_3ds\\
&\quad\le c\cdot c_2\cdot c_3^{n-1}\int_0^t\frac{1}{s^{3/4}}\frac{1}{(t-s)^{1/4}}ds\cdot\left(\int_{\mathbb{R}}\frac{1}{\sqrt{t-s}}\mathrm{e}^{-\frac{2\alpha|r_1-r_3|^2}{t-s}}\cdot\frac{1}{\sqrt{s}}\mathrm{e}^{-\frac{2\alpha|r_3-r_2|^2}{s}}dr_3\right)^{1/2}\\
&\qquad\times\left(\int_{\mathbb{R}}|b(r_3)|^2\mathrm{e}^{-\frac{2(\beta-\alpha)|r_2-r_3|^2}{T}}dr_3\right)^{1/2}.
\end{aligned}$$

A straightforward computation yields that $\int_0^t 1/\left(s^{3/4}(t-s)^{1/4}\right)ds$ is bounded by a constant independent of t. It follows from the Chapman-Kolmogorov equation for Gaussian densities and (5.13) that for some constant $c_4>0$,

$$|k_n(t,r_1,r_2)|\le c_4\cdot c_3^{n-1}\frac{1}{t^{1/4}}\mathrm{e}^{-\frac{\alpha|r_1-r_2|^2}{t}}.$$

Therefore, by choosing $t_0\in(0,T]$ small enough such that $t_0^{1/4}<(c_2c_3)/c_4$, it holds for all $n\in\mathbb{N}$:

$$|k_n(t,r_1,r_2)|\le c_2\cdot c_3^n\cdot t^{-1/2}\exp(-\alpha|r_1-r_2|^2/t),\quad 0<t\le t_0,\ r_1,r_2\in\mathbb{R}. \tag{5.14}$$

It then follows that $\sum_{n=0}^\infty k_n(t,r_1,r_2)$ converges locally uniformly on $(t,r_1,r_2)\in(0,t_0]\times\mathbb{R}\times\mathbb{R}$. From here using the exact same argument in [20, Lemma 3.17], one can see that $\sum_{n=0}^\infty k_n(t,r_1,r_2)$ is absolutely convergent for $(t,r_1,r_2)\in(0,T]\times\mathbb{R}\times\mathbb{R}$ and indeed the transition density of Y with respect to $\widehat{\ell}$, i.e.

$$\widehat{p}^Y(t,r_1,r_2)=\sum_{n=0}^\infty k_n(t,r_1,r_2),\quad 0<t\le T,\ r_1,r_2\in\mathbb{R}.$$

Furthermore, it holds for some constant $c_6>0$ such that

$$\widehat{p}^Y(t,r_1,r_2)\le c_6t^{-1/2}\exp\left(-\alpha|r_1-r_2|^2/t\right),\quad 0<t\le T,\ r_1,r_2\in\mathbb{R}. \tag{5.15}$$

By a standard chain argument (see, e.g., [20, pp. 36-37]), it is not hard to see that the same Gaussian type lower bound holds.

Finally let us prove the joint continuity of $\widehat{p}^Y$ on $(0,\infty)\times\mathbb{R}\times\mathbb{R}$. We first show it for $t\in(0,t_0]$. When $t\le 0$, we write $k_n(t,r_1,r_2)=0$ for all $n\ge 0$ for convenience and (5.12) becomes

$$k_n(t,r_1,r_2):=\int_0^{t_0}\int_{\mathbb{R}}k_{n-1}(t-s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(s,r_3,r_2)dr_3ds. \tag{5.16}$$

For the sake of the local uniform convergence of $\sum_{n=0}^\infty k_n(t,r_1,r_2)$, it suffices to show the joint continuity of k_n. To accomplish this, we utilize induction and clearly $k_0=p^Z$ is jointly continuous on $(0,t_0]\times\mathbb{R}\times\mathbb{R}$. Assume this holds for k_{n-1}. Take two arbitrary constants $R>0$ and $\delta<t_0/2$ and we turn to derive the joint continuity of k_n for $\delta\le t\le t_0$ and $0\le|r_1|,|r_2|\le R$. To do this, fix a small constant $\varepsilon<\delta/2$ and split the integrand on the right hand side of (5.16) into three parts by multiplying $I_+(s) := 1_{(0,\varepsilon]}(s)$, $I_\varepsilon(s) := 1_{(\varepsilon,t-\varepsilon)}(s)$ and $I_-(s) := 1_{[t-\varepsilon,t]}(s)$ respectively. Denote

$$J_{\pm,\varepsilon}(t,r_1,r_2) := \int_0^{t_0}\int_{\mathbb{R}} I_{\pm,\varepsilon}(s)k_{n-1}(t-s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(s,r_3,r_2)dr_3ds. \tag{5.17}$$

Then $k_n(t,r_1,r_2) = J_+(t,r_1,r_2) + J_\varepsilon(t,r_1,r_2) + J_-(t,r_1,r_2)$. Due to the joint continuity of k_{n-1} and the continuity of (5.9), the integrands on the right hand side of (5.17) are jointly continuous at (t,r_1,r_2) for almost every (s,r_3) (except for $s=\varepsilon, t-\varepsilon$ or t, and $r_3=0$). Now we show the joint continuity of $J_{\pm,\varepsilon}$ for $\delta\le t\le t_0$ and $0\le|r_1|,|r_2|\le R$ by utilizing dominated convergence theorem respectively. We derive the upper bound for the integrand in J_ε firstly. Note that for $s\in(\varepsilon,t-\varepsilon)$, (5.11) and (5.14) yield

$$|k_{n-1}(t-s,r_1,r_3)|\lesssim\frac{1}{\sqrt{\varepsilon}}$$

and

$$\begin{aligned}\left|\nabla_{r_3}p^Z(s,r_3,r_2)\right| &\lesssim \frac{1}{\varepsilon}\exp(-\beta|r_3-r_2|^2/t_0)\\ &\le \frac{1}{\varepsilon}\left(1_{\{r:|r|\le 2R\}}(r_3) + 1_{\{r:|r|>2R\}}(r_3)\cdot\exp(-\frac{\beta|r_3|^2}{4t_0})\right),\end{aligned}$$

where the last inequality holds since for $|r_3|>2R>R\ge|r_2|$, $|r_3-r_2|\ge|r_3|/2$. Obviously

$$\frac{1}{\varepsilon^{3/2}}b(r_3)\left(1_{\{r:|r|\le 2R\}}(r_3) + 1_{\{r:|r|>2R\}}(r_3)\cdot\exp(-\frac{\beta|r_3|^2}{4t_0})\right)\in L^1([0,t_0]\times\mathbb{R})$$

due to $|b|^2\in\mathbf{K}_{1,1}$. This yields the joint continuity of J_ε. To treat J_-, we utilize substitution as follows

$$\begin{aligned}J_-(t,r_1,r_2) &= \int_{t-\varepsilon}^t\int_{\mathbb{R}} k_{n-1}(t-s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(s,r_3,r_2)dr_3ds\\ &= \int_0^\varepsilon\int_{\mathbb{R}} k_{n-1}(s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(t-s,r_3,r_2)dr_3ds.\end{aligned}$$

Analogically $|k_{n-1}(s,r_1,r_3)|\lesssim s^{-1/2}\in L^1([0,t_0])$ and

$$\begin{aligned}\left|\nabla_{r_3}p^Z(t-s,r_3,r_2)\right| &\lesssim \frac{2}{\delta}\exp(-\beta|r_3-r_2|^2/t_0)\\ &\le \frac{2}{\delta}\left(1_{\{r:|r|\le 2R\}}(r_3) + 1_{\{r:|r|>2R\}}(r_3)\cdot\exp(-\frac{\beta|r_3|^2}{4t_0})\right).\end{aligned}$$

Hence J_- is jointly continuous. Finally, in the following we use generalized dominated convergence theorem (see, e.g., [12, §2.3, Exercise 20]) to establish the continuity of J_+. Since $s\le\varepsilon<\delta/2$ and $\delta\le t\le t_0$, it follows from (5.14) that

$$\begin{aligned}|k_{n-1}(t-s,r_1,r_3)| &\lesssim \sqrt{\frac{2}{\delta}}\exp\left(-\alpha|r_1-r_3|^2/t_0\right)\\ &\le \sqrt{\frac{2}{\delta}}\left(1_{\{r:|r|\le 2R\}}(r_3) + 1_{\{r:|r|>2R\}}(r_3)\cdot\exp(-\frac{\alpha|r_3|^2}{4t_0})\right) =: g(r_3).\end{aligned}$$

Then for $s\le\varepsilon$,

$$\left|k_{n-1}(t-s,r_1,r_3)\cdot b(r_3)\cdot\nabla_{r_3}p^Z(s,r_3,r_2)\right|\lesssim g(r_3)|b(r_3)|\cdot|\nabla_{r_3}p^Z(s,r_3,r_2)|. \tag{5.18}$$

For notation convenience, in this proof we denote by

$$h_{r_2}(s,r_3) := g(r_3)b(r_3)\nabla_{r_3}p^Z(s,r_3,r_2),$$

and

$$f_{t,r_2}(s,r_3) := k_{n-1}(t-s,r_1,r_3)b(r_3)\nabla_{r_3}p^Z(s,r_3,r_2).$$

In addition, we set

$$h_{r_2}(s,r_3) := 0, \quad f_{t,r_2}(s,r_3) := 0 \quad \text{when } r_2 = 0, \text{ or } r_3 = 0, \text{ or } r_3 = \pm r_2. \tag{5.19}$$

For an arbitrary pair of fixed (t^*, r_2^*) with $\delta \le t^* \le t_0$ and $r_2^* \in \mathbb{R}$, the joint continuity of $f_{t,r_2}(s,r_3)$ at $t = t^*, r_2 = r_2^*$ is equivalent to:

$$\lim_{\substack{t\to t^*\\ r_2\to r_2^*}} \int_{\mathbb{R}}\int_0^{\varepsilon} f_{t,r_2}(s,r_3)\, dsdr_3 = \int_{\mathbb{R}}\int_0^{\varepsilon} f_{t^*,r_2^*}(s,r_3)\, dsdr_3. \tag{5.20}$$

Next we show (5.20) using generalized dominated convergence theorem. Towards this, we need to verify the following conditions (i)-(iv):

(i). $|f_{t,r_2}(s,r_3)| \le |h_{t,r_2}(s,r_3)|$, for all $0 < s \le \varepsilon$, and all $r_3 \in \mathbb{R}$.

(ii).

$$\lim_{\substack{t\to t^*\\ r_2\to r_2^*}} f_{t,r_2}(s,r_3) = f_{t^*,r_2^*}(s,r_3), \quad \text{for all } 0 < s \le \varepsilon \text{ and all } r_3 \ne 0, \pm r_2^*. \tag{5.21}$$

(iii).

$$\lim_{r_2\to r_2^*} h_{r_2}(s,r_3) = h_{r_2^*}(s,r_3), \quad \text{for all } 0 < s \le \varepsilon \text{ and all } r_3 \ne 0, \pm r_2^*. \tag{5.22}$$

(iv).

$$\lim_{r_2\to r_2^*} \int_{\mathbb{R}}\int_0^{\varepsilon} h_{r_2}(s,r_3)\, dsdr_3 = \int_{\mathbb{R}}\int_0^{\varepsilon} h_{r_2^*}(s,r_3)\, dsdr_3. \tag{5.23}$$

(i) obviously holds in view of (5.18). (5.21) and (5.22) are also both obviously true. Finally, to verify (iv), i.e.,

$$\lim_{r_2\to r_2^*} \int_{\mathbb{R}} g(r_3)b(r_3)\int_0^{\varepsilon} \nabla_{r_3}p^Z(s,r_3,r_2)\, dsdr_3 = \int_{\mathbb{R}} g(r_3)b(r_3)\int_0^{\varepsilon} \nabla_{r_3}p^Z(s,r_3,r_2^*)\, dsdr_3.$$

We first observe that for a.e. r_3 (in fact, except at $r_3 = \pm r_2$ or 0), the mapping

$$r_2 \mapsto \int_0^{\varepsilon} |\nabla_{r_3}p^Z(s,r_3,r_2)|ds \tag{5.24}$$

is continuous. Furthermore, for $r_3 \ne 0$ with $r_2 + r_3 \ne 0$ and $r_2 \ne r_3$, it follows from (5.8) that

$$|\nabla_{r_3}p^Z(s,r_3,r_2)| \le 4\left(\left|-\frac{r_2-r_3}{s}g_s(r_2-r_3)\right| + \left|\frac{r_2+r_3}{s}g_s(r_2+r_3)\right|\right). \tag{5.25}$$

By computation we have

$$|r_2-r_3|\int_0^{\varepsilon} \frac{g_s(r_2-r_3)}{s}ds = \frac{1}{\sqrt{2\pi}}\int_{\frac{|r_2-r_3|^2}{\varepsilon}}^{\infty} \frac{\mathrm{e}^{-\tilde{s}/2}}{\sqrt{\tilde{s}}}d\tilde{s} \le 1,$$

where for the " $=$ " we use the substitution $\tilde{s} := |r_2-r_3|^2/s$. Similarly,

$$|r_2+r_3|\int_0^{\varepsilon} \frac{g_s(r_2+r_3)}{s}ds = \frac{1}{\sqrt{2\pi}}\int_{\frac{|r_2+r_3|^2}{\varepsilon}}^{\infty} \frac{\mathrm{e}^{-\tilde{s}/2}}{\sqrt{\tilde{s}}}d\tilde{s} \le 1,$$

with the substitution $\tilde{s} := |r_2+r_3|^2/s$. Plugging the above upper bounds back into (5.25), we get

$$|\nabla_{r_3}p^Z(s,r_3,r_2)| \le 8, \quad \text{for all } (r_2,r_3) \text{ with } r_3 \ne 0, \text{ or } \pm r_2.$$

Finally, in order to claim (5.23) using dominated convergence theorem, we observe that $g(r_3)b(r_3) \in L^1(\mathbb{R})$. This fact together with (5.19) establishes (5.23). Now with all (i)-(iv) having been verified, the joint continuity of J_+ is the consequence of generalized dominated convergence theorem (see, e.g., [12, §2.3, Exercise 20]). By letting $\delta \downarrow 0$ and $R \uparrow \infty$, we eventually conclude the joint continuity of k_n on $(0, t_0] \times \mathbb{R} \times \mathbb{R}$. This leads to the joint continuity of $\widehat{p}^Y$ on $(0, t_0] \times \mathbb{R} \times \mathbb{R}$. For $t_0 \le t \le 2t_0$, note that

$$\widehat{p}^Y(t, r_1, r_2) = \int_{\mathbb{R}} \widehat{p}^Y(t_0, r_1, r_3) \widehat{p}^Y(t - t_0, r_3, r_2) \widehat{\ell}(dr_3)$$

and $\widehat{p}^Y$ is bounded by (5.15). Then the dominated convergence theorem indicates the joint continuity of $\widehat{p}^Y$ for $t \in [t_0, 2t_0]$. By repeating this argument, one can eventually conclude the joint continuity of $\widehat{p}^Y$ on $(0, \infty) \times \mathbb{R} \times \mathbb{R}$. This completes the proof. ☐

Remark 5.6. Clearly,

$$\widetilde{p}^Y(t, r_1, r_2) := \frac{2}{1+\kappa} \widehat{p}^Y(t, r_1, r_2) \cdot 1_{\{r_2 < 0\}} + \frac{2}{1-\kappa} \widehat{p}^Y(t, r_1, r_2) \cdot 1_{\{r_2 > 0\}}$$

is the transition density function of Y with respect to the Lebesgue measure. Note that $\widetilde{p}^Y$ is not continuous at $r_1 = 0$ or $r_2 = 0$, unless $\kappa = 0$, i.e. $\pi \mathtt{p} \rho(0) = 1$. It is easy to figure out that $\widetilde{p}^Y$ satisfies the same Gaussian type estimate as (5.7) for $r_1, r_2 \neq 0$.

The following corollary shows the joint continuity of the transition density function of Y with respect to its symmetric measure ℓ.

Corollary 5.7. *Let ℓ be the symmetric measure* (4.1) *of Y. Then Y has a jointly continuous transition density function $p^Y(t, r_1, r_2)$ with respect to ℓ, i.e. $\mathbb{P}^Y_{r_1}(Y_t \in dr_2) = p^Y(t, r_1, r_2)\ell(dr_2)$ for all $t > 0$ and $r_1, r_2 \in \mathbb{R}$, and p^Y is jointly continuous on $(0, \infty) \times \mathbb{R} \times \mathbb{R}$.*

Proof. It suffices to note that

$$p^Y(t, r_1, r_2) := \frac{2}{(1+\kappa)\mathtt{p}\rho(-r_2)} \widehat{p}^Y(t, r_1, r_2) \cdot 1_{\{r_2 < 0\}} + \frac{2\pi e^{2\gamma|r_2|}}{1-\kappa} \widehat{p}^Y(t, r_1, r_2) \cdot 1_{\{r_2 \ge 0\}},$$

which is continuous at $r_2 = 0$. This completes the proof. ☐

Now we turn to the proof of Theorem 5.3. Lemma 5.5 yields the two-sided estimates on $\widehat{p}(t, x, y)$ when $x, y \in \mathfrak{R}_+$ since $i_+(M_t) = -Y_t$ when $M_t \in \mathfrak{R}_+$, and hence (5.3) can be concluded.

Proof of (5.3). Fix $x, y \in \mathfrak{R}_+$. Take arbitrary $0 \le a < b$, we have

$$\begin{aligned} \int_{y \in \mathfrak{R}_+, a \le |y| \le b} p(t, x, y) \mathfrak{m}(dy) &= \mathbb{P}_x(M_t \in \mathfrak{R}_+, a \le |M_t| \le b) = \mathbb{P}^Y_{-|x|}(a \le -Y_t \le b) \\ &= \frac{2}{1+\kappa} \int_{-b}^{-a} \widehat{p}^Y(t, -|x|, r) dr = \frac{2}{1+\kappa} \int_a^b \widehat{p}^Y(t, -|x|, -r) dr. \end{aligned}$$

It follows from $p(t, x, y) = p(t, x, \iota_+(|y|))$ and $\mathfrak{m}(dy)|_{\mathfrak{R}_+} = \mathtt{p} \cdot m_+ \circ \iota_+^{-1}(dy) = \mathtt{p}\rho(|y|)d|y|$ that

$$\int_{y \in \mathfrak{R}_+, a \le |y| \le b} p(t, x, y) \mathfrak{m}(dy) = \mathtt{p} \int_{a \le |y| \le b} p(t, x, \iota_+(|y|)) \rho(|y|) d|y|$$

and thus

$$p(t, x, y) = \frac{2}{(1+\kappa)\mathtt{p}\rho(|y|)} \widehat{p}^Y(t, -|x|, -|y|). \tag{5.26}$$

Since $||x| - |y|| = |x - y|$, Lemma 5.5 immediately yields that

$$\frac{c_1}{\sqrt{t}} e^{-c_2|x-y|^2/t} \le \widehat{p}^Y(t, -|x|, -|y|) \le \frac{c_3}{\sqrt{t}} e^{-c_4|x-y|^2/t}, \qquad t \in (0, T] \text{ and } x, y \in \mathfrak{R}_+.$$

Therefore the desired result (5.3) follows from (5.26). ☐

To prove (5.4), the crucial fact is that starting from $x \in \mathfrak{R}_+$ (resp. $y \in \mathfrak{R}^3$), M must pass through the origin $\mathbf{0}$ before reaching $y \in \mathfrak{R}^3$ (resp. $x \in \mathfrak{R}_+$). As usual $\sigma_{\{\mathbf{0}\}} := \inf\{t > 0 : M_t = \mathbf{0}\}$ denotes the first hitting time of $\{\mathbf{0}\}$ relative to M.

Proof of (5.4). Consider $x \in \mathfrak{R}_+$ and $y \in \mathfrak{R}^3$. We first note that in this case by the symmetry of $p(t,x,y)$ in x and y,

$$p(t,x,y) = p(t,y,x) = \int_0^t \mathbb{P}_y(\sigma_{\{\mathbf{0}\}} \in ds) p(t-s, \mathbf{0}, x).$$

By the rotational invariance of 3-dimensional distorted Brownian motion X^3, $\mathbb{P}_y(\sigma_{\{\mathbf{0}\}} \in ds)$ only depends on $|y|$, therefore so does $y \mapsto p(t,x,y)$. For all $0 \le a < b$ and $x \in \mathfrak{R}_+$, using polar coordinates we have

$$\begin{aligned} \frac{2}{1-\kappa} \int_a^b \widehat{p}^Y(t, -|x|, r) dr &= \mathbb{P}_{-|x|}(a \le Y_t \le b) \\ &= \mathbb{P}_x(M_t \in \mathfrak{R}^3 \text{ with } a \le |M_t| \le b) \\ &= \int_{y \in \mathfrak{R}^3 : a \le |y| \le b} p(t,x,y) \mathfrak{m}(dy). \end{aligned} \tag{5.27}$$

Note that $\mathfrak{m}(dy)|_{\mathfrak{R}^3} = \left(h_{\rho,\gamma}(y)^2 |y|^2 d|y| d\sigma\right) \circ \iota_3^{-1}$, where σ is the surface measure on the sphere S^2, and the density function $h_{\rho,\gamma}(y)^2 = \frac{\mathrm{e}^{-2\gamma|y|}}{4\pi^2|y|^2}$ only depends on $|y|$ as well. Thus the last term in (5.27) is equal to

$$\int_{a \le |y| \le b} p(t,x,y) h_{\rho,\gamma}(y)^2 |y|^2 d|y| \int_{S^2} d\sigma = \int_{a \le |y| \le b} 4\pi \cdot p(t,x,y) h_{\rho,\gamma}(y)^2 |y|^2 d|y|.$$

This yields

$$\frac{2}{1-\kappa} \widehat{p}^Y(t, -|x|, |y|) = 4\pi |y|^2 p(t,x,y) h_{\rho,\gamma}(y)^2. \tag{5.28}$$

By Lemma 5.5 we can obtain that

$$\frac{c_1}{\sqrt{t}} \mathrm{e}^{-c_2(|x|+|y|)^2/t} \le \widehat{p}^Y(t, -|x|, |y|) \le \frac{c_3}{\sqrt{t}} \mathrm{e}^{-c_4(|x|+|y|)^2/t}. \tag{5.29}$$

In view of (5.1), $|x-y| = |x| + |y|$ since $x \in \mathfrak{R}_+$ and $y \in \mathfrak{R}^3$. Eventually (5.4) can be concluded from (5.28) and (5.29). □

Next we study the case that both x and y are in $\mathfrak{R}^3$. To continue, we first establish the explicit density function (5.6) for 3-dimensional distorted Brownian motion $M^{3,\mathbf{0}}$ killed upon hitting $\mathbf{0}$, for any time $t > 0$. Denote this transition density function by $q(t,x,y)$. In other words, for any non-negative function $f \ge 0$ on $\mathfrak{R}^3 \setminus \{\mathbf{0}\}$,

$$\int_{\mathfrak{R}^3 \setminus \{\mathbf{0}\}} q(t,x,y) f(y) \mathfrak{m}(dy) = \mathbb{E}_x \left(f(M_t); t < \sigma_{\{\mathbf{0}\}} \right).$$

For $x = \mathbf{0}$ or $y = \mathbf{0}$, we make the convention $q(t,x,y) := 0$. The following result is the key ingredient of (5.5).

Lemma 5.8. *It holds that for $x, y \in \mathfrak{R}^3 \setminus \{\mathbf{0}\}$ and $t > 0$,*

$$q(t,x,y) = \mathrm{e}^{-\frac{\gamma^2}{2}t} \cdot \frac{\mathrm{e}^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{\frac{3}{2}}} \cdot \frac{1}{h_{\rho,\gamma}(x) h_{\rho,\gamma}(y)}.$$

Proof. Theorem 2.6 tells us that $M^{3,\mathbf{0}}$ is identified with $\iota_3({}_hW^\gamma)$, and it suffices to note that the transition semigroup of ${}_hW^\gamma$ is defined by (2.8). □

Now we have a position to complete the proof of (5.5).

Proof of (5.5). Consider $x, y \in \mathfrak{R}^3$. Note that starting from x, before hitting $\mathbf{0}$, M has the same distribution as that for $\iota_3(X^3)$, where X^3 is the 3-dimensional dBM appearing in §2. Thus $q(t,x,y)$ gives the probability density that M starting from x hits y at time t without hitting $\mathbf{0}$. As a consequence,

$$\bar{q}(t,x,y) := p(t,x,y) - q(t,x,y), \quad x, y \in \mathfrak{R}^3. \tag{5.30}$$

is the probability density for M starting from x hits $\mathbf{0}$ before ending up at y at time t, and this yields

$$\bar{q}(t,x,y) = \int_0^t p(t-s,\mathbf{0},y)\mathbb{P}_x(\sigma_{\{\mathbf{0}\}} \in ds).$$

As mentioned in the proof of (5.4), $p(t-s,\mathbf{0},y)$ is a function in y depending only on $|y|$. Therefore so is $y \mapsto \bar{q}(t,x,y)$. Since $\bar{q}(t,x,y) = \bar{q}(t,y,x)$, $x \mapsto \bar{q}(t,x,y)$ also depends only on $|x|$. For any $b > a \geq 0$, it follows that

$$\begin{aligned}\mathbb{P}_x\left(\sigma_{\{\mathbf{0}\}} < t,\, M_t \in \mathfrak{R}^3 \text{ with } a \leq |M_t| \leq b\right) &= \int_{a\leq|y|\leq b} \bar{q}(t,x,y)\mathfrak{m}(dy) \\ &= 4\pi \int_{a\leq|y|\leq b} \bar{q}(t,x,y)h_{\rho,\gamma}(y)^2|y|^2 d|y|.\end{aligned}$$

On the other hand, $\mathbb{P}_x\left(\sigma_{\{\mathbf{0}\}} < t,\, M_t \in \mathfrak{R}^3 \text{ with } a \leq |M_t| \leq b\right)$ is also equal to

$$\mathbb{P}^Y_{|x|}\left(\sigma_{\{\mathbf{0}\}} < t, a \leq |Y_t| \leq b\right) = \frac{2}{1-\kappa}\int_0^t \left(\int_a^b \widehat{p}^Y(t-s,0,r)dr\right)\mathbb{P}^Y_{|x|}\left(\sigma_{\{\mathbf{0}\}} \in ds\right).$$

This yields $4\pi|y|^2\bar{q}(t,x,y)h_{\rho,\gamma}(y)^2 = \frac{2}{1-\kappa}\int_0^t \widehat{p}^Y(t-s,0,|y|)\mathbb{P}^Y_{|x|}\left(\sigma_{\{\mathbf{0}\}} \in ds\right)$ and it follows from Lemma 5.5 that

$$\begin{aligned}4\pi|y|^2\bar{q}(t,x,y)h_{\rho,\gamma}(y)^2 &\leq \int_0^t \frac{\bar{c}_1}{\sqrt{t-s}}\mathrm{e}^{-\bar{c}_2|y|^2/(t-s)}\mathbb{P}^Y_{|x|}\left(\sigma_{\{\mathbf{0}\}} \in ds\right) \\ &\leq \bar{c}_3\int_0^t \widehat{p}^Y(t-s,0,-c_4|y|)\mathbb{P}^Y_{|x|}\left(\sigma_{\{\mathbf{0}\}} \in ds\right) \\ &= \bar{c}_3\,\widehat{p}^Y(t,|x|,-\bar{c}_4|y|) \\ &\leq \frac{\bar{c}_5}{\sqrt{t}}\mathrm{e}^{-\bar{c}_6(|x|+|y|)^2/t}.\end{aligned}$$

By a piece of similar argument, one can show that

$$4\pi|y|^2\bar{q}(t,x,y)h_{\rho,\gamma}(y)^2 \geq \frac{\bar{c}_7}{\sqrt{t}}\mathrm{e}^{-\bar{c}_8(|x|+|y|)^2/t}.$$

In other words, we have

$$\frac{\bar{c}_7}{4\pi|y|^2\sqrt{t}}\mathrm{e}^{-\bar{c}_8(|x|+|y|)^2/t} \leq \bar{q}(t,x,y)h_{\rho,\gamma}(y)^2 \leq \frac{\bar{c}_5}{4\pi|y|^2\sqrt{t}}\mathrm{e}^{-\bar{c}_6(|x|+|y|)^2/t}.$$

Since $h_{\rho,\gamma}(y)^2 = \frac{\mathrm{e}^{-2\gamma|y|}}{4\pi^2|y|^2}$, this yields

$$\frac{\pi\bar{c}_7}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-[\bar{c}_8(|x|+|y|)^2/t]} \leq \bar{q}(t,x,y) \leq \frac{\pi\bar{c}_5}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-[\bar{c}_6(|x|+|y|)^2/t]}. \tag{5.31}$$

Combining (5.31) with Lemma 5.8, also in view of (5.30), we get for $x, y \in \mathfrak{R}^3$ that

$$\frac{\pi\bar{c}_7}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-[\bar{c}_8(|x|+|y|)^2/t]} + q(t,x,y) \leq p(t,x,y) \leq \frac{\pi\bar{c}_5}{\sqrt{t}}\mathrm{e}^{2\gamma|y|-[\bar{c}_6(|x|+|y|)^2/t]} + q(t,x,y).$$

This completes the proof of (5.5). □

Finally we prove the joint continuity of $p(t,x,y)$.

Proof of joint continuity. Clearly, (5.26) and (5.28) tell us

$$p(t,x,y)=\begin{cases}\dfrac{2}{(1+\kappa)\mathrm{p}\rho(|y|)}\widehat{p}^Y(t,-|x|,-|y|), & x,y\in\mathfrak{R}_+,\\ \dfrac{2\pi}{1-\kappa}\mathrm{e}^{2\gamma|y|}\widehat{p}^Y(t,-|x|,|y|), & x\in\mathfrak{R}_+, y\in\mathfrak{R}^3.\end{cases}$$

Since $p(t,x,y)=p(t,y,x)$, it is straightforward to verify the joint continuity for $x\in\mathfrak{R}_+, y\in E$ and $x\in E, y\in\mathfrak{R}^3$. Now consider the case $x,y\in\mathfrak{R}^3$. Note that q is jointly continuous by its explicit expression (5.6). It suffices to obtain the joint continuity of $\bar{q}(t,x,y)=p(t,x,y)-q(t,x,y)$. To accomplish this, take $0\le a<b$ and we have

$$\mathbb{P}_x(M_t\in\mathfrak{R}^3, a\le|M_t|\le b)=\int_{a\le|y|\le b}q(t,x,y)\mathfrak{m}(dy)+\int_{a\le|y|\le b}\bar{q}(t,x,y)\mathfrak{m}(dy).$$

The left hand side is also equal to

$$\mathbb{P}^Y_{|x|}(a\le Y_t\le b)=\frac{2}{1-\kappa}\int_a^b\widehat{p}^Y(t,|x|,r)dr.$$

A straightforward computation yields

$$\int_{a\le|y|\le b}q(t,x,y)\mathfrak{m}(dy)=\frac{\mathrm{e}^{-\frac{\gamma^2t}{2}}}{(2\pi t)^{3/2}}|x|\mathrm{e}^{\gamma|x|-\frac{|x|^2}{2t}}\int_a^b|y|\mathrm{e}^{-\gamma|y|-\frac{|y|^2}{2t}}\chi\left(\frac{|x||y|}{t}\right)d|y|,$$

where $\chi(a):=\int_0^\pi\mathrm{e}^{a\cos\theta}\sin\theta d\theta$ is clearly a continuous function in $a\in\mathbb{R}$. Hence we can obtain for $x,y\in\mathfrak{R}^3$,

$$\bar{q}(t,x,y)=\frac{2\pi}{1-\kappa}\mathrm{e}^{2\gamma|y|}\widehat{p}^Y(t,|x|,|y|)-\frac{\pi\mathrm{e}^{-\frac{\gamma^2t}{2}}}{(2\pi t)^{3/2}}|x|\mathrm{e}^{\gamma|x|-\frac{|x|^2}{2t}}|y|\mathrm{e}^{\gamma|y|-\frac{|y|^2}{2t}}\chi\left(\frac{|x||y|}{t}\right). \quad (5.32)$$

The joint continuity of $\bar{q}$ is obvious by this explicit expression. This completes the proof. $\square$

Remark 5.9. Note that $\bar{q}(t,x,y)$ in (5.32) depends only on $|x|$ and $|y|$. It is worth pointing out that for $x,y\in\mathfrak{R}^3$, $p(t,x,y)$ does not depend on $|x|$ or $|y|$ only, since neither does $q(t,x,y)$.

References

[1] S Albeverio, F Gesztesy, R Høegh-Krohn, and H Holden, *Solvable models in quantum mechanics*, second ed., AMS Chelsea Publishing, Providence, RI, 2005. MR2105735

[2] Sergio Albeverio, Raphael Høegh-Krohn, and Ludwig Streit, *Energy forms, Hamiltonians, and distorted Brownian paths*, J. Math. Phys. **18** (1977), no. 5, 907–917. MR0446236

[3] Zhen-Qing Chen and Masatoshi Fukushima, *Symmetric Markov processes, time change, and boundary theory*, Princeton University Press, Princeton, NJ, 2012. MR2849840

[4] Zhen-Qing Chen and Masatoshi Fukushima, *One-point reflection*, Stochastic Process. Appl. **125** (2015), no. 4, 1368–1393. MR3310351

[5] Zhen-Qing Chen and Shuwen Lou, *Brownian motion on some spaces with varying dimension*, Ann. Probab. **47** (2019), no. 1, 213–269. MR3909969

[6] Zhen-Qing Chen and Zhongxin Zhao, *Diffusion processes and second order elliptic operators with singular coefficients for lower order terms*, Math. Ann. **302** (1995), 323–357. MR1336339

[7] M Cranston, L Koralov, S Molchanov, and B Vainberg, *Continuous model for homopolymers*, J. Funct. Anal. **256** (2009), no. 8, 2656–2696. MR2502529

[8] M Cranston, L Koralov, S Molchanov, and B Vainberg, *A solvable model for homopolymers and self-similarity near the critical point*, Random Oper. Stochastic Equations **18** (2010), no. 1, 73–95. MR2606477

[9] Michael Cranston and Stanislav Molchanov, *On the critical behavior of a homopolymer model*, Sci. China Math. **62** (2019), no. 8, 1463–1476. MR3984384

[10] K Bruce Erickson, *Continuous extensions of skew product diffusions*, Probab. Theory Related Fields **85** (1990), no. 1, 73–89. MR1044301

[11] Patrick J Fitzsimmons and Liping Li, *On the Dirichlet form of three-dimensional Brownian motion conditioned to hit the origin*, Sci. China Math. **62** (2019), no. 8, 1477–1492. MR3984385

[12] Gerald B Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999. MR1681462

[13] Masatoshi Fukushima, *On general boundary conditions for one-dimensional diffusions with symmetry*, J. Math. Soc. Japan **66** (2014), no. 1, 289–316. MR3161402

[14] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda, *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter & Co., Berlin, 2011. MR2778606

[15] J M Harrison and L A Shepp, *On skew Brownian motion*, Ann. Probab. **9** (1981), no. 2, 309–313. MR0606993

[16] Tero Kilpeläinen, *Weighted Sobolev Spaces and Capacity*, Ann. Acad. Sci. Fenn. Ser. A I Math. **19** (1994), no. 1, 95–113. MR1246890

[17] Liping Li, *On general skew Brownian motions*, arXiv:1812.08415.

[18] Liping Li and Xiaodan Li, *Dirichlet forms and polymer models based on stable processes*, Stochastic Process. Appl. **130** (2020), no. 10, 5940–5972. MR4140023

[19] Liping Li and Jiangang Ying, *On symmetric linear diffusions*, Trans. Amer. Math. Soc. **371** (2019), no. 8, 5841–5874. MR3937312

[20] Shuwen Lou, *Brownian motion with drift on spaces with varying dimension*, Stochastic Process. Appl. **129** (2019), no. 6, 2086–2129. MR3958425

[21] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, 1999. MR1725357

[22] Michael Sharpe, *General theory of Markov processes*, Academic Press, Inc., Boston, MA, 1988. MR0958914

[23] Masayoshi Takeda, *Criticality and subcriticality of generalized Schrödinger forms*, Illinois J. Math. **58** (2014), no. 1, 251–277. MR3331850

[24] Jiangang Ying and Minzhi Zhao, *The uniqueness of symmetrizing measure of Markov processes*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2181–2185. MR2596057

[25] Qi S Zhang, *Gaussian bounds for the fundamental solutions of* $\nabla(A\nabla u)+B\nabla u-u_t=0$, Manuscripta Math. **93** (1997), no. 3, 381–390. MR1457736