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# Positive random walks and an identity for half-space SPDEs*

Shalin Parekh†

**Abstract**

The purpose of this article is to investigate the continuum limit of a distributional identity for half-space directed polymers given in [BBC20]. The limiting identity turns out to relate the multiplicative-noise half-space stochastic heat equation with Dirichlet boundary condition to the same equation with Robin boundary condition. The identity is related to conjectured Gaussian fluctuation behavior of subcritical half-space KPZ.

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†Columbia University, United States of America. E-mail: sp3577@columbia.edu

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# 1 Introduction and context

The present work will focus on three related objects: uniform measures on collections of nearest-neighbor *non-negative* paths (e.g., Brownian meander), directed polymers weighted by such measures, and multiplicative-noise stochastic partial differential equations (SPDE) in a half-space.

## 1.1 Half-space stochastic heat equations

We begin our discussion with SPDE's. The multiplicative-noise stochastic heat equation has been a frequent subject of research within stochastic analysis and mathematical physics in recent years. This equation arises naturally in the context of directed polymers and interacting particle systems, as a *weak scaling limit* [Cor12]. In spatial dimension one, the multiplicative-noise stochastic heat equation is also related to the so-called KPZ equation via the Hopf-Cole transform, and may be solved by the classical Itô-Walsh construction [Wal86] or by more modern techniques such as regularity structures [HL18]. In the present article, we consider the stochastic heat equation with multiplicative noise on a *half-line*:

$$\partial_T Z(T,X) = \frac{1}{2}\partial_X^2 Z(T,X) + Z(T,X)\cdot\xi(T,X), \qquad X\geq 0, T\geq 0, \tag{SHE}$$

where $\xi$ is a Gaussian space-time white noise on $\mathbb{R}_+\times\mathbb{R}_+$. Naturally one needs to impose boundary conditions at $X=0$ in order to make sense of this equation. In the present work we consider two types of boundary conditions, Robin and Dirichlet. First let us write the Robin boundary condition of parameter $A\in\mathbb{R}$:

$$\partial_X Z(T,0) = AZ(T,0). \tag{1.1}$$

This type of homogeneous boundary condition has been considered in [CS18, Par19, GPS20, BBCW18] in the context of interacting particle systems, and a robust solution theory has been developed in [GH19] using techniques of [Hai14]. This boundary condition transforms into a Neumann boundary condition for the half-space KPZ equation upon taking the logarithm. Next, we consider the Dirichlet boundary condition for the half-space SHE:

$$Z(T,0) = 0. \tag{1.2}$$

This type of boundary condition was considered, for instance, in [GLD12], in the context of directed polymers near an absorbing wall. Again, one can make sense of the equation using classical techniques of [Wal86] or more modern ones such as [Hai14]. Our main result compares these two types of boundary conditions; specifically it allows us to interchange information about the initial data with that of the boundary condition imposed on the SHE:

**Theorem 1.1.** *Fix $A\in\mathbb{R}$. Let $Z^{(A)}_{Rob}(T,X)$ denote the solution of* (SHE) *with Robin boundary parameter $A$ as in* (1.1) *and delta initial data $Z^{(A)}_{Rob}(0,X)=\delta_0(X)$. Let $Z^{(A)}_{Dir}(T,X)$ be the solution to* (SHE) *with Dirichlet boundary condition* (1.2) *and initial data $Z^{(A)}_{Dir}(0,X)=e^{B_X-(A+\frac{1}{2})X}$, where $B$ is a standard Brownian motion independent of $\xi$. Then for each $T\geq 0$ we have the following equality of distributions:*

$$Z^{(A)}_{Rob}(T,0) \overset{d}{=} \lim_{X\to 0}\frac{Z^{(A)}_{Dir}(T,X)}{X}. \tag{1.3}$$

This will be a consequence of Theorem 2.4, which in turn will use the work of [BBC20, Wu18] and Theorem 2.2 (our main technical result) as inputs. Let us now discuss the motivation for this result, the contexts in which it has arisen, and the methods used to prove it.

To give some motivation towards (1.3), we now explain it using the exact solvability framework developed in [BBC20], which is a crucial input to the proof of (1.3). Both the left and right sides of (1.3) have interpretations in terms of partition functions of a certain family of probabilistic models known as *directed polymers* (see Section 1.2). Specifically, the left side of (1.3) can be related to a polymer that is modeled on a Brownian motion which gets reweighted according to its local time at zero, whereas the right side can be related to a polymer that is modeled on a Brownian motion conditioned to remain positive. In [BBC20], the authors use certain nontrivial symmetries of Macdonald polynomials in order to obtain information about the large-scale behavior of *discrete* versions of these polymer models and others (which is similar in theme to, and builds on, older works of [BC14, COSZ14, OSZ14, IS04, BR01]). One particular result in that paper (Proposition 8.1) is a highly non-obvious identity in distribution for directed polymers with log-gamma weights, that effectively allows one to switch some of the bulk weights of the random environment with those on the boundary without changing the distribution of the associated partition function. Our main goal was to take the SPDE limit of that identity, which effectively gives Theorem 1.1 under the appropriate scaling. Hence our result can be viewed as a special case of more general algebraic principles that may be used to extract certain nontrivial symmetries in certain half-space models.

The right side of (1.3) equals $(\partial_X Z_{Dir}^{(A)})(T,0)$. It is not clear why this derivative should even exist in the first place, since the spatial regularity of $Z_{Dir}$ is much worse than $C^1$. One of our main technical results, given in Section 4, is that the limit in (1.3) is indeed well-defined (Corollary 4.3). In fact we will prove something stronger: the limit in the right side of (1.3) simultaneously exists for all $T \geq 0$ almost surely, and is Hölder 1/4− as a function of $T$ almost surely.

In order to convince the reader that (1.3) is at least plausible, let us verify formally that the expectations are the same on both sides of the equation. Let $P_{Rob}^{(A)}(T;X,Y)$ denote the Robin boundary heat kernel and let $P_{Dir}(T;X,Y)$ denote the Dirichlet boundary one, where by *heat kernel* we mean the fundamental solution of the heat equation with the associated boundary condition started from the delta measure at point $X$. Letting $P(T;X) = \frac{1}{\sqrt{2\pi T}} e^{-X^2/2T}$, one may verify directly that these kernels are given by the following explicit formulas for $T,X,Y \geq 0$:

$$P_{Rob}^{(A)}(T;X,Y) = P(T;X+Y) + P(T;X-Y) - 2A\int_0^\infty P(T;X+Y+Z)e^{-AZ}dZ,$$
$$P_{Dir}(T;X,Y) = \lim_{A\to\infty} P_{Rob}^{(A)}(T;X,Y) = P(T;X-Y) - P(T;X+Y).$$

By the Duhamel principle (see Definition 2.1) it holds that $\mathbb{E}[Z_{Rob}^{(A)}(T,X)] = P_{Rob}^{(A)}(T;0,X)$ and $\mathbb{E}[Z_{Dir}^{(A)}(T,X)] = \mathbb{E}[\int_0^\infty P_{Dir}(T;X,Y)e^{B_Y-(A+1/2)Y}dY] = \int_0^\infty P_{Dir}(T;X,Y)e^{-AY}dY$. One then formally interchanges an expectation and a derivative to obtain

$$\begin{aligned}\mathbb{E}[\partial_X\big|_{X=0} Z_{Dir}^{(A)}(T,X)] &= \int_0^\infty \partial_X\big|_{X=0} P_{Dir}(T;X,Y)e^{-AY}dY \\ &= 2\int_0^\infty \partial_Y P_{Dir}(T;0,Y)e^{-AY}dY = P_{Rob}^{(A)}(T;0,X) = \mathbb{E}[Z_{Rob}^{(A)}(T,0)],\end{aligned}$$

where we integrate by parts in the third equality. This shows at a purely formal level that the expectations on either side of (1.3) are the same.

Theorem 1.1 suggests a duality between the initial data of a solution to the half-space SHE and the boundary conditions one imposes on it. It may be interesting to see if more general versions of this hold. For example, could it be possible that the identity holds as a *process* in $T$ and not just in the one-point sense? Using this type of idea, one may potentially obtain useful information about objects of interest, such as the Neumann-boundary Kardar-Parisi-Zhang (KPZ) equation that was considered in [CS18]. It was conjectured in [Par19] that one has the almost-sure convergence

$$\lim_{T\to\infty} \frac{1}{T} \log Z^{(A)}_{Rob}(T,0) = \begin{cases} -\frac{1}{24}, & A \geq -1/2 \\ (A+1/2)^2 - \frac{1}{24}, & A \leq -1/2, \end{cases}$$

which would give the exact law of large numbers for Neumann-boundary KPZ. Unfortunately Theorem 1.1 alone is not enough to obtain this result. Nevertheless, it is plausible and even hopeful that a clever use of (1.3) (perhaps combined with some new ideas and techniques) could lead to quantitative results that are close to the above expression. Indeed, despite the fact that on the Robin side of (1.3) there is no *visible* phase transition at $A = -1/2$, the appearance of the term $A + 1/2$ on the *Dirichlet* side already indicates the presence of a nontrivial change in large-scale behavior at $A = -1/2$. Section 1.3 of [Par19] includes a further discussion of this. More than just computing the above limit, we are also interested in computing the limiting distribution of the fluctuations around the mean value. These should be of order $T^{1/2}$ and Gaussian in the $A < -1/2$ case, and they should be of order $T^{1/3}$ and random-matrix theoretic otherwise (with separate cases when $A = -1/2$ and $A > -1/2$). See for instance [Par19, BBCW18, BBCS18, BBC16].

The main technical difficulties in the present work are of an analytic nature: translating the discrete identity in [BBC20] to that of (1.3) required us to prove a general convergence result for directed polymers, stated below as Theorem 1.2. As we will now see, this involves the analysis of an interesting object in its own right: the Brownian meander.

### 1.2 Directed polymers weighted by positive random walks

This brings us to the method of proof of Theorem 1.1. As suggested above, it will be proved using an approximation via *directed polymers* with very specific weights, where a discrete version of this identity holds.

Directed polymers are natural probabilistic objects that were first introduced in [HH85, IS88]. They generalize directed first- and last-passage percolation and have deep connections to statistical mechanics and stochastic analysis. Specifically, we consider an environment $\{\omega_{i,j}\}_{(i,j)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}}$ consisting of i.i.d., mean-zero, finite-variance random variables. The standard deviation of the weights is referred to as the *inverse temperature*. One may define a partition function $Z^\omega(n,x)$ as a sum over all *directed* nearest-neighbor simple random walk paths $(i,\gamma_i)_{0\leq i\leq n}$ of length $n$ starting from $(0,x)$, of the product of all weights $e^{\omega_{i,\gamma_i}}$ along the path. Similarly, there is also a natural way to define *random* Markovian transition densities associated to this environment $\omega$, wherein a nearest-neighbor path $\gamma$ has probability proportional to the product of weights $e^{\omega_{i,\gamma_i}}$ along it. As is standard practice in statistical mechanics, one may then ask questions about the existence of infinite-volume limits of these path measures and their typical fluctuation scale, as well as the typical scale and shape of the fluctuations of the partition function itself [Com17].

Many seminal results in these directions have been proved, perhaps most notably that there is a phase transition which becomes apparent in high dimensions. Specifically, in spatial dimensions greater than two, there is a strictly positive critical value of the inverse temperature below which weak disorder holds, meaning that the fluctuations of a typical polymer path look like Brownian motion and one may construct infinite-length path measures [CY06, Com17]. Such polymers are said to exhibit *weak disorder.* In contrast, lower-dimensional polymers at any nonzero inverse temperature are known to be characterized by *strong disorder,* meaning that the path fluctuations are quite different and there is no sensible notion of an infinite volume Gibbs measure [Com17]. The results of [AKQ14a, AKQ14b] examined the partition function in a regime that lies *between* strong and weak disorder. Specifically, in spatial dimension one, they scaled the inverse temperature of the model like $n^{-1/4}$ and simultaneously applied diffusive scaling to the partition function, and there they observed that the fluctuations are governed by (SHE) and that the path measures themselves have a continuum analogue. Recent work of [CD20, CSZ18] has investigated the intermediate-disorder behavior in two spatial dimensions, where the scaling $n^{-1/4}$ is replaced by $(\log n)^{-1/2}$. In a different direction, [Wu18] extended the work of [AKQ14a] to the case of half-space polymers with Robin boundary condition.

We will be interested in the analogous half-space question of intermediate-disorder fluctuations of the directed polymer partition function associated to *uniform non-negative path measures*. Specifically, let

- $\mathbf{P}_x^n$ denote the uniform probability measure on the collection of all paths $(\gamma_i)_{0\le i\le n}$ such that $\gamma_0 = x$, $|\gamma_{i+1}-\gamma_i| = 1$ for $i<n$, and $\gamma_i \ge 0$ for all $i\le n$.
- $\omega_{i,j}$ be i.i.d. mean-zero, variance-one random variables that are uniformly bounded from below by a deterministic constant.
- $f_n$ be a sequence of functions bounded uniformly by a function growing at-worst exponentially fast near infinity such that $f_n(n^{1/2}\,\cdot)$ converges (as $n\to\infty$) to some function $f(\cdot)$ in the Hölder space $C^{\alpha}_{loc}(\mathbb{R}_+)$, for all $\alpha\in(0,1/2)$.

Letting $\mathbf{E}_x^n$ denote the expectation with respect to $\mathbf{P}_x^n$, and setting $S$ to be the canonical process associated to $\mathbf{P}_x^n$, one defines a directed-polymer partition function as follows:

$$Z_k^{\omega}(n,x) := \mathbf{E}_x^n\bigg[f_k(S_n)\prod_{i=0}^{n}(1+k^{-1/4}\omega_{i,S_i})\bigg].$$

Note that the expectation is taken only with respect to the random walk, *conditional* on the environment $\omega_{i,j}$, which is always assumed to be independent of the walk. We consider the rescaled partition function

$$\mathscr{Z}_n(T,X) := Z_n^{\omega}(nT, n^{1/2}X), \tag{1.4}$$

where the quantity on the right side is defined by linear interpolation between points of the lattice $L := \{(x,n)\in\mathbb{Z}^2_{\ge 0} : n-x\in 2\mathbb{Z}\}$.

In a manner analogous to [AKQ14a] we show that $\mathscr{Z}_n$ converges in law to a random continuous space-time field. The natural candidate for such a limit would be a continuum analogue of $Z_k^{\omega}(n,x)$, where the expectation $\mathbf{E}_x^n$ over positive *discrete* random walks is replaced by that of *continuous* ones. Indeed the limiting space-time field can be described as follows: it has the formal Feynman-Kac interpretation that takes as its input the so-called *Brownian meander* [DIM77, DI77] on a finite time interval, and exponentially weighs it by its integral against a space-time white noise field. More precisely, if $\mathscr{P}_t^T(X,Y)$ denotes the inhomogeneous Markov transition density at time $t$ of Brownian motion started from $X$ and conditioned to stay positive until time $T\ge t$, then this limiting space-time field $\mathscr{Z}$ necessarily solves the multiplicative-noise SPDE on the half-space that is given in Duhamel form by

$$\mathscr{Z}(T,X) = \int_{\mathbb{R}^+}\mathscr{P}_T^T(X,Y)f(Y)dY + \int_0^T\int_{\mathbb{R}^+}\mathscr{P}_{T-S}^T(X,Y)\mathscr{Z}(S,Y)\xi(dYdS), \tag{1.5}$$

where $\xi$ is a space-time white noise and $f$ is the limiting function from the third bullet point above. An important step towards proving Theorem 1.1 will be to show that a solution of (1.5) exists and makes sense even when $X = 0$, and then to show that it can in turn be related to the derivative of the solution of the Dirichlet-boundary SHE at the origin. This will all be done in Section 4; more specifically we will show that the solution of (1.5) equals

$$\mathscr{X}(T,X) = \frac{Z_{Dir}(T,X)}{2\Phi(X/\sqrt{T})-1}, \qquad T,X>0, \tag{1.6}$$

where $Z_{Dir}$ solves (SHE) with Dirichlet boundary condition (1.2) with the same initial data as $\mathscr{X}$, and $\Phi$ is the cdf of a standard normal variable so that $\mathscr{X}(T,0) = (2\pi T)^{1/2}\lim_{X\to 0}\frac{Z_{Dir}(T,X)}{X}$. We then have the following result.

**Theorem 1.2.** *The sequence of processes $\mathscr{X}_n$ defined in* (1.4) *converge in law to the solution of* (1.5) *as $n\to\infty$. The convergence occurs in the sense of finite-dimensional distributions. If we assume that the $\omega_{i,j}$ have $p>8$ moments, then distributional convergence holds when the space $C(\mathbb{R}_+\times\mathbb{R}_+)$ is equipped with the topology of uniform convergence on compact sets.*

This theorem will be proved in Section 5.2 in greater generality (where the distribution of the weights $\omega$ may vary with $n$), see Proposition 5.9 and Theorem 5.11. It is actually a simplified version of Theorem 2.2 which is the true input to proving Theorem 1.1. The main difficulty towards this result will be in obtaining the necessary estimates for the inhomogeneous transition densities (and their discrete analogues) appearing in (1.5).

Thus the proof of Theorem 1.2 will lead to some new technical results related to the uniform measures $\mathbf{P}_x^n$ and their continuum analogues. These will be collected in appendices at the end of the paper. To illustrate a few such results, we will prove a coupling result for such random walks in the nearest-neighbor case, and then we will use that coupling to show the following concentration property: there exist constants $c, C>0$ (independent of $n, x\geq 0$) such that for all $u>0$ and all $k\leq n$ one has that

$$\mathbf{P}_x^n\big(\sup_{0\leq i\leq k}|S_i - x| > u\big) \leq Ce^{-cu^2/k}.$$

We remind the reader that $S_i$ is the *conditioned* walk. The study of such random walks started with the invariance principle of [Ig74], further generalized in [Bol76]. Later, the study expanded considerably, with local limit theorems [Car05] and expansions to heavy-tailed increments [CC08]. We will see that some of the estimates we derive are similar in spirit to some of those works, but the intricate details are somewhat different. We will give proofs of many of these technical results because the highly specific estimates needed to prove Theorem 1.2 were not found in those references (since our random walk does not necessarily start at zero).

It should be noted that we work with a simplified version of the partition function as opposed to much of the previous literature: [AKQ14a, CSY03] and related works. There the partition function $Z_k^\omega(n,x)$ is defined with weights $e^{k^{-1/4}\omega_{i,S_i}}$ instead of the quantity $1+k^{-1/4}\omega_{i,S_i}$ that we have used in (1.4) above. The reason for this is that the latter object is mathematically simpler because it is already renormalized (has expectation exactly 1 rather than approximately 1), and hence leads to simpler proofs and less stringent moment restrictions. However, it should be noted that the exponential version is more natural from the physical point of view, and entire results such as [DZ16] have been devoted to finding the correct renormalization and phase transition behavior for that version as a function of the moment assumptions.

**Outline:** In Section 2, we prove Theorem 1.1 as Theorem 2.4, which uses [BBC20] and [Wu18] as important inputs. In Section 3, we will introduce and state some estimates about the transition densities associated to positive random walks, though the proofs are postponed to the appendices. In Section 4, we will develop the existence and uniqueness theory of the limiting SPDE (1.5) from Theorem 1.2, and as a corollary we prove that $\partial_X Z_{Dir}(T,0)$ exists. In Section 5, we prove Theorem 1.2 by using the estimates developed in the appendices. In the appendices we derive some elementary but powerful bounds related to the measures $\mathbf{P}^n_x$, which are crucial for the proofs in the main body.

## 2 Main results

In this section, we show how to prove Theorem 1.1. We denote non-negative reals as $\mathbb{R}_+$ and non-negative integers as $\mathbb{Z}_{\geq 0}$.

We will use the notion of mild solutions for SPDEs throughout this article. Thus for completeness, we begin by giving the formal definition of such a solution, although it is peripheral to the main goals of the section.

**Definition 2.1** (Mild Solution)**.** *Recall the Dirichlet-boundary heat kernel*

$$P_t^{Dir}(X,Y) := \frac{1}{\sqrt{2\pi t}}\big(e^{-(X-Y)^2/2t} - e^{-(X+Y)^2/2t}\big). \tag{2.1}$$

*Let $\xi$ be a space-time white noise defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let $\mu$ be an independent random Borel measure on $\mathbb{R}_+$. A continuous space-time process $Z_{Dir} = (Z_{Dir}(T,X))_{T,X\geq 0}$ is a mild solution of the Dirichlet-boundary SHE with initial data $\mu$ if $\mathbb{P}$-almost surely, for all $X,T \geq 0$ one has that*

$$Z_{Dir}(T,X) = \int_{\mathbb{R}_+} P_T^{Dir}(X,Y)\mu(dY) + \int_0^T \int_{\mathbb{R}_+} P_{T-S}^{Dir}(X,Y) Z_{Dir}(S,Y)\xi(dS,dY),$$

*where the integral against $\xi$ is meant to be interpreted in the Itô-Walsh sense [Wal86].*

The fact that this object exists will be established as a special case of the results in Section 4. The definition of the Robin boundary version $Z^{(A)}_{Rob}$ of (SHE) is very similar, but one replaces the Dirichlet heat kernel with the Robin boundary one throughout. We refer the reader to Section 4 of [Par19] for more details, including the existence/uniqueness of this Robin boundary version.

The proof of Theorem 1.1 will be obtained by approximating both $Z^{(A)}_{Dir}$ and $Z^{(A)}_{Rob}$ by the partition function of a directed polymer with log-gamma weights. For these weights we use a known identity that allows us to switch the boundary weights with those on the initial data without changing the distribution of the partition function along the boundary [BBC20] (Proposition 8.1). The approximation argument will strongly emulate the arguments given in [Wu18, AKQ14a] although there are new challenges that make the convergence result rather difficult and technical. These additional difficulties are a byproduct of the inhomogeneous Markov transition densities for random walks conditioned to stay above zero.

Let us explicitly state the Dirichlet-boundary approximation result now. For each $n \in \mathbb{N}$, let $\omega^n = \{\omega^n_{i,j}\}_{i\geq j\geq 0}$ denote a random environment indexed by the principal octant of $\mathbb{Z}^2$ with the following properties:

- The "bulk-environment" random variables $\{\omega^n_{i,j}\}_{i\geq j\geq 1}$ are i.i.d., and the "lower-boundary" random variables $\{\omega^n_{i,0}\}_{i\geq 0}$ are also i.i.d. These two collections are independent.
- For $j>0$ (the bulk variables) $\omega^n_{i,j}$ have finite second moment. Furthermore one has $\mathbb{E}[\omega^n_{i,j}] = 0$ and $\mathbb{E}[(\omega^n_{i,j})^2] = 1 + o(1)$ as $n \to \infty$.
- For $j = 0$ (at the lower boundary) $\log(1 + n^{-1/4}\omega^n_{i,j})$ has finite second moment; moreover there exist $\mu, \sigma \in \mathbb{R}$ such that $\mathbb{E}[\omega^n_{i,j}] = \mu n^{-1/4} + o(n^{-1/4})$, and $var(\omega^n_{i,j}) = \sigma^2 + o(1)$ as $n \to \infty$. We also assume $\omega_{i,0}$ have $2+\epsilon$ moments for some $\epsilon > 0$.

An upright path in $\mathbb{Z}^2$ is a function $\gamma = (\gamma_1, \gamma_2) : \{0, ..., n\} \to \mathbb{Z}^2$ such that both $\gamma_1$ and $\gamma_2$ are weakly increasing, and $\gamma_1(i) + \gamma_2(i) - i$ is constant in $i$. For $p \geq q \geq 0$ define the random partition function

$$Z_n(p,q) := \sum_{\gamma:(0,0)\to(p,q)} 2^{-\#\{i\leq p+q\,:\,\gamma_2(i)\neq 0\}} \prod_{i=0}^{p+q} (1 + n^{-1/4}\omega^n_{\gamma_1(i),\gamma_2(i)}),$$

where the sum is taken over all upright paths $\gamma$ from $(0,0)$ to $(p,q)$ that stay in the octant $\{(i,j) : i \geq j \geq 0\}$. As a convention, we also set $Z_n(p,q) = Z_n(p,0)$ for $q \leq 0$. Let $\Phi$ denote the cdf of a standard normal distribution. We define the rescaled processes

$$\mathscr{Z}_n(T,X) := \frac{1}{2\Phi\big(\frac{X+n^{-1/2}}{\sqrt{T}}\big) - 1} \cdot Z_n(nT + n^{1/2}X, nT - n^{1/2}X), \qquad T, X \geq 0$$

where we interpolate linearly between integer values of $Z_n$.

The following result is the primary technical contribution of this work.

**Theorem 2.2.** *In the above notations and assumptions, the sequence of processes $\mathscr{Z}_n$ converges in distribution (in the sense of finite-dimensional marginals, as $n \to \infty$) to the unique space-time process satisfying* (1.5) *(equivalently given by* (1.6)*) with initial data $\mathscr{Z}(0,X) = Z_{Dir}(0,X) = e^{\sigma B_X + (\mu - \frac{1}{2}\sigma^2)X}$, where $B$ is a standard Brownian motion independent of the space-time white noise $\xi$. If we assume that all weights $\omega^n_{i,j}$ have more than eight moments bounded independently of $n$, then distributional convergence holds when the space $C(\mathbb{R}_+ \times \mathbb{R}_+)$ is equipped with the topology of uniform convergence on compact sets.*

We will see that Theorem 2.2 is essentially equivalent to a more complicated version of Theorem 1.2, where the distribution of the weights $\omega$ depends on $n$ and the domain of the polymer paths has been changed from a quadrant to an octant of $\mathbb{Z}^2$, which makes the geometry more challenging to work with. Accordingly, the proof of this theorem will proceed in two steps: first by reducing the claim of the theorem to that of Theorem 1.2 with a specific initial data (which will be achieved in Section 5.1), and then proving Theorem 1.2 which is simpler thanks to known methods and is done in Section 5.2.

**Remark 2.3.** There are really two different regimes in which one should interpret Theorem 2.2. One regime is $X > 0$, where the result merely says that $Z_n(nT + n^{1/2}X, nT)$ converges to $Z_{Dir}(T,X)$. The more interesting regime is $X = 0$, in which case the theorem says that $(\pi nT/2)^{1/2} Z_n(nT,nT)$ converges in law to $\lim_{X\to 0} \frac{Z_{Dir}(T,X)}{2\Phi(X/\sqrt{T})-1}$, i.e.,

$$n^{1/2} Z_n(nT,nT) \xrightarrow{d} \lim_{X\to 0} \frac{Z_{Dir}(T,X)}{X}.$$

An advantage of our approach is that the proof will simultaneously cover both regimes. In fact, we will see that convergence even takes place in a parabolic Hölder space of the appropriate regularity provided that the weights have more than eight moments.

We now combine this result with the Robin boundary result of [Wu18] and the log-gamma identities of [BBC20] in order to obtain the following result, which clearly implies Theorem 1.1. In what follows, we denote by $\Gamma^{-1}(\theta,c)$ the inverse-gamma distribution of shape parameter $\theta$ and scale parameter $c$, i.e., the law of the random variable $cX$, where $X$ has pdf given by

$$f(x) = \frac{x^{-\theta-1}}{\Gamma(\theta)} e^{-1/x}, \qquad x > 0.$$

We will also write $\mathbb{E}[\Gamma^{-1}(\theta,c)] = \frac{c}{\theta-1}$ and $var(\Gamma^{-1}(\theta,c)) = \frac{c^2}{(\theta-1)^2(\theta-2)}$ to denote respectively the expectation and variance of such a random variable.

For $n \in \mathbb{N}$, let $\zeta_n^1 = \{\zeta_n^1(i,j)\}_{i\geq j\geq 0}$ and $\zeta_n^2 = \{\zeta_n^2(i,j)\}_{i\geq j\geq 0}$ be fields of independent random variables with the following distributions

$$\zeta_n^1(i,j) \sim \begin{cases} \Gamma^{-1}(2\sqrt{n},\ \frac{1}{2}\mathbb{E}[\Gamma^{-1}(2\sqrt{n},1)]^{-1}), & i \neq j \\ \\ \Gamma^{-1}(\sqrt{n}+A+\frac{1}{2},\ \frac{1}{2}\mathbb{E}[\Gamma^{-1}(2\sqrt{n},1)]^{-1}), & i = j \end{cases}$$

$$\zeta_n^2(i,j) \sim \begin{cases} \Gamma^{-1}(2\sqrt{n},\ \frac{1}{2}\mathbb{E}[\Gamma^{-1}(2\sqrt{n},1)]^{-1}), & j \neq 0 \\ \\ \Gamma^{-1}(\sqrt{n}+A+\frac{1}{2},\ \frac{1}{2}\mathbb{E}[\Gamma^{-1}(2\sqrt{n},1)]^{-1}), & j = 0. \end{cases}$$

Let $Z_n^1$ and $Z_n^2$ denote the associated partition functions, i.e.,

$$Z_n^\alpha := \sum_{\gamma:(0,0)\to(\lfloor nT\rfloor,\lfloor nT\rfloor)} \prod_{i=0}^{2\lfloor nT\rfloor} \zeta_n^\alpha(\gamma_1(i),\gamma_2(i)), \qquad \text{for } \alpha \in \{1,2\}. \tag{2.2}$$

Here the sum is taken over all upright paths $\gamma$ from $(0,0)$ to $(\lfloor nT\rfloor,\lfloor nT\rfloor)$ that stay in the octant $\{(i,j) : i \geq j \geq 0\}$.

**Theorem 2.4** (Joint with [BBC20, Wu18]). *With $Z_n^1$ and $Z_n^2$ defined in* (2.2)*, the following are true:*

1. *$\sqrt{n}Z_n^1$ converges in distribution as $n\to\infty$ to the left-hand side of* (1.3)*.*
2. *$\sqrt{n}Z_n^2$ converges in distribution as $n\to\infty$ to the right-hand side of* (1.3)*.*
3. *For every $n$, one has $Z_n^1 \stackrel{d}{=} Z_n^2$.*

*Proof.* Item *(1)* is proved as Theorem 5.1(B) of [Wu18] using techniques from [AKQ14a]. Item *(3)* is proved in Proposition 8.1 of [BBC20] by developing the theory of half-space Macdonald processes. Thus we only need to prove Item *(2)*, and this will be done using Theorem 2.2, in the special case where $X = 0$. As in Theorem 4.5 of [AKQ14a], we define a family of independent weights $\omega^n = \{\omega^n_{i,j}\}_{i\geq j\geq 0}$ according to the rule:

$$2\zeta_n^2(i,j) = 1 + (4n)^{-1/4}\omega^n_{i,j}, \qquad j > 0,$$

$$\zeta_n^2(i,0) = 1 + n^{-1/4}\omega^n_{i,0}.$$

There are now three things to verify, corresponding to the three bullet points preceding Theorem 2.2. Using the fact that

$$\mathbb{E}[\Gamma^{-1}(\theta,1)] = \frac{1}{\theta-1}, \qquad var(\Gamma^{-1}(\theta,1)) = \frac{1}{(\theta-1)^2(\theta-2)},$$

one gets the desired asymptotics on $\mathbb{E}[\omega_{i,j}]$ and on $\mathbb{E}[(\omega_{i,j})^2]$, with $\mu = -A$ and $\sigma^2 = 1$. This proves the corollary (and thus also Theorem 1.1). ☐

Once again we would like to emphasize the tremendous importance of [BBC20] as the primary input to proving the preceding theorem, and thus the main result (1.3). It may be interesting to explore more robust methods that might give a direct proof of (1.3) using purely stochastic analytic methods instead of exact solvability, but we have tried and this seems out of reach for us at the moment. With Theorem 2.4 in place, we will now shift the goals of the paper to the analytical and technical aspects focusing on the methods used to prove Theorem 2.2.

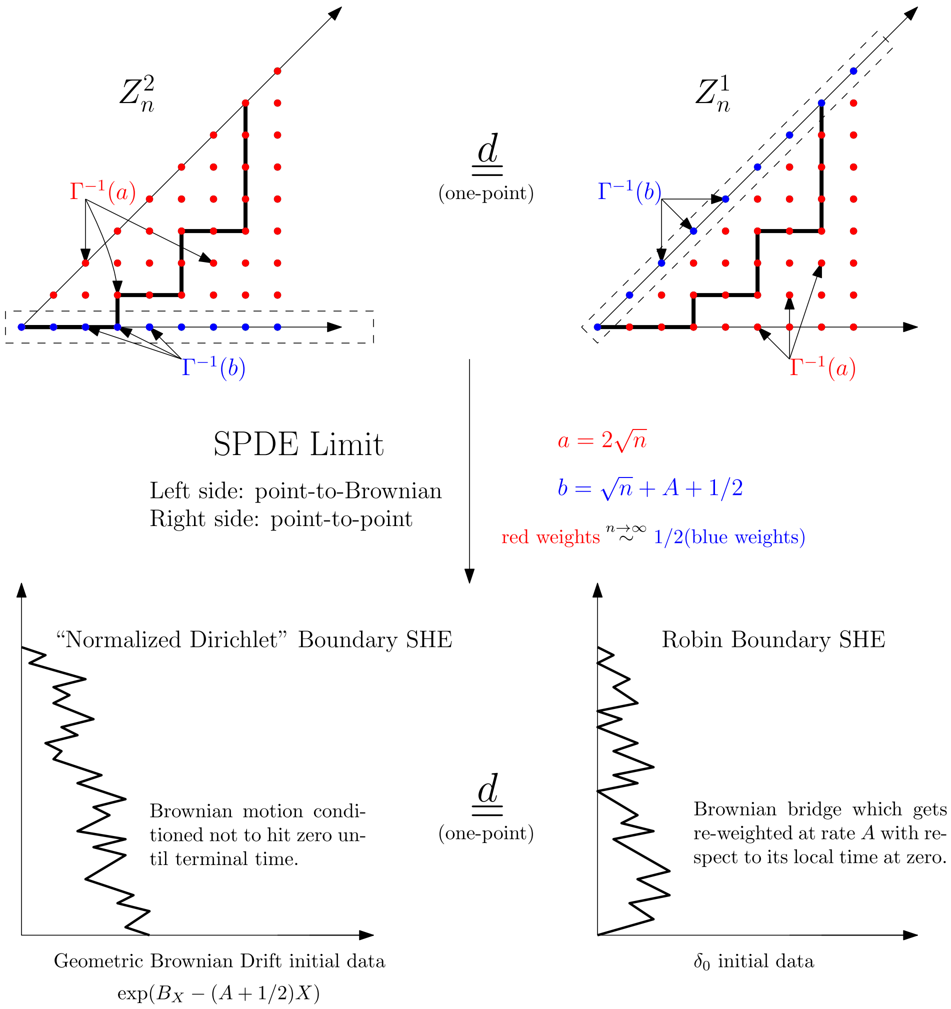


Figure 1: A graphical description of Theorem 2.4. The weight of a given path is the product of the weights along it, and the partition function $Z_n^\alpha$ for $\alpha \in \{1,2\}$ is given by summing the weights of all upright paths from $(0,0)$ to $(\lfloor nT \rfloor, \lfloor nT \rfloor)$ that stay in the octant. We have represented the SPDE limits by their respective (purely formal) Feynman-Kac representations.

Since the sum defining the partition function in the preceding results is over all upright paths that stay in the principal octant of $\mathbb{Z}^2$, it is natural to relate those quantities to reflecting random walk measures. However, if one does asymptotics in Corollary 2.4, she may verify that $\zeta_n^2(j,j) \to 1/2$ in probability as $n \to \infty$. What this means is that instead of pure reflection, our random walk path loses mass by a factor of $1/2$ each time it hits zero. Hence, it is clear that the analysis in proving Theorem 2.2 will involve taking a close look at these random walk measures, as well as directed polymers weighted by such measures, as suggested in the introduction.

More precisely, fix some $x \in \mathbb{Z}_{\geq 0}$, and define a sample space of non-negative random walk trajectories by

$$\Omega_x^n := \{(s_0, ..., s_n) \in \mathbb{Z}^n : |s_{i+1} - s_i| = 1, s_i \geq 0, s_0 = x\}.$$

Define a sub-probability measure $\mu_x^n$ and a probability measure $\mathbf{P}_x^n$ on $\Omega_x^n$ by

$$\mu_x^n(S) := 2^{-n}, \qquad \mathbf{P}_x^n(S) := \frac{1}{\#\Omega_x^n} = \frac{\mu_x^n(S)}{\mu_x^n(\Omega_x^n)}, \qquad \text{for all } S \in \Omega_x^n.$$

As an intermediate step in proving Theorem 2.2, we obtain the following result.

**Theorem 2.5.** *With the above notation, the following are true.*

1. *(Markov Property) Fix $n, x \geq 0$. Let $S = (S_k)_{k=0}^n$ denote the coordinate process associated to $\mathbf{P}_x^n$, i.e., $S$ is a $\Omega_x^n$-valued random variable with law $\mathbf{P}_x^n$. Then $(S_k)_{k=0}^n$ is a time-inhomogeneous Markov process, in fact conditionally on $(S_k)_{k=0}^K$ with $K < n$, the process $(S_{k+K})_{k=0}^{n-K}$ is distributed according to $\mathbf{P}_{S_K}^{n-K}$. One has explicit transition densities for $0 \leq i_1 < ... < i_k \leq n$:*

$$\mathbf{P}_x^n(S_{i_1} = s_1, ..., S_{i_k} = s_k) = \mathfrak{p}_{i_1}^n(x, s_1)\mathfrak{p}_{i_2-i_1}^{n-i_1}(s_1, s_2) \cdots \mathfrak{p}_{i_k-i_{k-1}}^{n-i_{k-1}}(s_{k-1}, s_k),$$

   *where $\mathfrak{p}_i^n$ is given in Definition 3.2 below.*

2. *(Mass) For every $x \in \mathbb{Z}_{\geq 0}$, the total mass of $\mu_x^n$ is asymptotically $(x+1)\sqrt{\frac{2}{\pi n}}$:*

$$\lim_{n\to\infty} n^{1/2}\mu_x^n(\Omega_x^n) = (x+1)\sqrt{2/\pi}.$$

3. *(Concentration) There exist $C, c > 0$ such that for every $x \geq 0$, every $0 \leq m \leq k \leq n$, and every $u > 0$ one has that*

$$\mathbf{P}_x^n\big(\sup_{m\leq i\leq k} |S_i - S_m| > u\big) \leq Ce^{-cu^2/(k-m)}.$$

4. *(Convergence of Transition Densities) Let $\mathfrak{p}_n^N$ be as in Item (1). One has the convergence*

$$(n/2)^{1/2}\mathfrak{p}_{2\lfloor tn\rfloor}^{2\lfloor Tn\rfloor}(2\lfloor n^{1/2}X/\sqrt{2}\rfloor, 2\lfloor n^{1/2}Y/\sqrt{2}\rfloor) \stackrel{n\to\infty}{\longrightarrow} \mathscr{P}_t^T(X, Y),$$

   *where $\mathscr{P}_t^T$ is the transition probability for a certain (inhomogeneous) Markov process defined in Definition 3.4 below. Moreover, for fixed $(t, T, X)$ the convergence in the $Y$-variable occurs in $L^p(\mathbb{R}_+, e^{aY}dY)$ for every $p \in [1, \infty)$.*

The first part of the theorem is elementary and the last part is a more local version of the results of [Ig74, Bol76]. The third part is new as far as we know, and the second part will simply follow from the local central limit theorem. All proofs may be found in the appendices, except for *(1)* which is proved in Section 3.

**Remark 2.6.** One can actually formulate an invariance principle for this family of measures. This was done in greater generality in [Ig74, Bol76]. Fix $X, T \geq 0$. For each $x, N \geq 0$, let $(S_n^{x,N})_{n=0}^N$ be distributed according to $\mathbf{P}_x^N$. Then the processes $(N^{-1/2}S_{Nt}^{N^{1/2}X,NT})_{t\in[0,T]}$ converge in law (with respect to the uniform topology on $C[0,T]$, as $N \to \infty$) to a time-inhomogeneous Markov process $B$ on $[0,T]$ whose transition densities $\mathscr{P}_t^T(X,Y)$ are given by the limit in Item (4). This limiting process $B$ may be interpreted as a standard Brownian motion conditioned to stay positive until time $T$; see Proposition 3.5. This invariance principle will be immediate from the results of Appendix A, but it will not be needed for the results above.

Let us now discuss the basic idea of the proof of Theorem 2.2 in the special case when $(T, X) = (1, 0)$ because this is enough to give the main idea. Denote by $\mathbf{E}_{KRW}$ the expectation with respect to a reflected random walk of length $2n$ that is started from $0$ and killed at the origin with probability $1/2$, i.e., the one whose transition density is equal to $p_n^{(1/2)}$ which is defined in Section 3 below. By rotating the picture appropriately, one rewrites the partition function appearing in Theorem 2.2 as a discrete Feynman-Kac formula for this killed walk:

$$\begin{aligned} Z_n &= \sum_{\gamma:(0,0)\to(n,n)} 2^{-\#\{i\le 2n\,:\,\gamma_2(i)>0\}} \prod_{i=0}^{2n}(1+n^{-1/4}\omega^n_{\gamma_1(i),\gamma_2(i)}) \\ &= \mathbf{E}_{KRW}\Big[z_0^n(S_{T_n})\prod_{i=0}^{T_n-1}(1+n^{-1/4}\hat\omega^n_{i,S_i})\cdot 1_{\{\text{survival}\}}\Big], \end{aligned} \tag{2.3}$$

where

- $\hat\omega^n_{i,j}$ is defined to be $\omega^n_{(n-\frac{i-j}{2}),(n-\frac{i+j}{2})}$ for all $i,j$.
- The expectation $\mathbf{E}_{KRW}$ is taken *only* with respect to the random walk $S$, i.e., **conditional** on the $\omega^n_{i,j}$ (which are always assumed to be independent of $S$).
- $T_n$ is the first time that $(i,S_i)$ hits the diagonal line $\{(2n-j,j): 0\le j\le 2n\}$.
- $z_0^n(x) := \prod_{i=0}^x(1+n^{-1/4}\omega^n_{i,0})$ can be thought of as a sort of "initial data" for the above discrete Feynman-Kac representation.
- $\{\text{survival}\}$ is the event that the random walk survives up to time $2n$ (or equivalently, up to time $T_n$).

Now, using Theorem 2.5(2) with $x=0$, one finds that $\mathbf{P}_{KRW}(\text{survival})\approx\sqrt{2/\pi n}$. Moreover, we can make the approximation $T_n\approx 2n$ for reasons justified later, see Proposition 5.8. This essentially reduces the octant geometry to that of a quadrant, thus reducing the theorem statement to that of Theorem 1.2, which is simpler as we see below. Combining this with the above gives

$$\begin{aligned} \sqrt{\frac{\pi n}{2}}Z_n &\approx \mathbf{E}_{KRW}\Big[z_0^n(S_{2n})\prod_{i=0}^{2n}(1+n^{-1/4}\hat\omega^n_{i,S_i})\,\Big|\,\text{survival}\Big] \\ &= \mathbf{E}_{KRW}\Big[z_0^n(S_{2n})\sum_{k=0}^{2n}n^{-k/4}\sum_{1\le i_1<\ldots<i_k\le 2n}\prod_{j=1}^k\hat\omega^n_{i_j,S_{i_j}}\,\Big|\,\text{survival}\Big]. \end{aligned} \tag{2.4}$$

In the notation of Theorem 2.5, the killed random walk conditioned to survive has law $\mathbf{P}^n_x$ and the associated Markov process has transition densities $\mathfrak{p}^N_n$. Using theorem 2.5*(1)*, the expectation in the preceding expression may be expanded as

$$\sum_{k=0}^{2n}n^{-k/4}\sum_{0\le i_1<\ldots<i_k\le 2n}\ \sum_{(x_1,\ldots,x_{k+1})\in\mathbb{Z}^{k+1}_{\ge 0}} z_0^n(x_{k+1})\prod_{j=1}^{k+1}\mathfrak{p}^{2n-i_{j-1}}_{i_j-i_{j-1}}(x_{j-1},x_j)\prod_{j=1}^k\hat\omega^n_{i_j,x_j}, \tag{2.5}$$

with $x_0:=0$, $i_0:=0$, and $i_{k+1}:=2n$. Recall that $\log(1+u)\approx u-\frac12u^2$, so by writing

$$\begin{aligned} z_0^n(x) = e^{\sum_0^x\log(1+n^{-1/4}\omega^n_{i,0})} &\approx e^{\sum_0^x\left(n^{-1/4}\omega^n_{i,0}-\frac12n^{-1/2}(\omega^n_{i,0})^2\right)} \\ &= e^{n^{-1/4}\sum_0^x(\omega^n_{i,0}-n^{-1/4}\mu)+n^{-1/2}\mu x-\frac{1}{2n^{1/2}}\sum_0^x(\omega^n_{i,0})^2}, \end{aligned} \tag{2.6}$$

one may convince herself (using Donsker's principle and the law of large numbers together with the third bullet point preceding Theorem 2.2) that as $n\to\infty$,

$$\big(z_0^n(n^{1/2}X)\big)_{X\ge 0}\xrightarrow{d}\big(e^{\sigma B_X+(\mu-\frac12\sigma^2)X}\big)_{X\ge 0} \tag{2.7}$$

for a Brownian motion $B$. Then taking the limit of (2.5) as $n \to \infty$ by using Theorem 2.5*(4)* (with some uniformity estimates), one obtains the Wiener-Itô chaos series

$$\sum_{k=0}^{\infty} \int_{0 \le t_1 < \ldots < t_k \le 1} \int_{\mathbb{R}_+^{k+1}} e^{\sigma B_{x_{k+1}} + (\mu - \frac{1}{2}\sigma^2) x_{k+1}} \\ \times \prod_{j=1}^{k+1} \mathscr{P}_{t_j - t_{j-1}}^{1-t_{j-1}} (x_{j-1}, x_j) dx_{k+1} \xi(dx_k, dt_k) \cdots \xi(dx_1, dt_1),$$

with the convention $x_0 = 0$, $t_0 = 0$, $t_{k+1} = 1$, and where the $\mathscr{P}_t^T$ are the conditional heat kernels from the limit in Theorem 2.5*(4)*, and $\xi$ is a space-time white noise. But (as we will see in Proposition 4.2 below) this chaos series is precisely equal to

$$\lim_{X \to 0} \frac{Z_{Dir}(1, X)}{2\Phi(X) - 1} = \sqrt{\pi/2} \lim_{X \to 0} \frac{Z_{Dir}(1, X)}{X},$$

where the initial data is $e^{\sigma B_X + (\mu - \frac{1}{2}\sigma^2)X}$, and $\Phi$ is the cdf of a standard normal, which implies that $\Phi(0) = 1/2$ and $\Phi'(0) = \sqrt{\pi/2}$ giving the equality above. This will complete the argument for Theorem 2.2. Note that no part of the argument relies on the finer details of the weights $\omega_{i,j}^n$ beyond their mean and variance.

## 3 Uniform measures on collections of positive paths

In this section we will introduce the inhomogeneous heat kernels $\mathfrak{p}_n^N$ associated to random walks conditioned to stay positive. We begin with an elementary discussion of the properties of these measures, and later we state technical estimates about these measures that will be necessary in subsequent sections, though their proofs are postponed to the appendices.

**Definition 3.1.** *For $n \in \mathbb{Z}_{\ge 0}$ and $x \in \mathbb{Z}$, let $p_n(x)$ denote the standard heat kernel on $\mathbb{Z}$ (i.e., the transition function for a discrete-time simple symmetric random walk started from zero). Then we define*

$$p_n^{(1/2)}(x, y) = p_n(x - y) - p_n(x + y + 2), \qquad n, x, y \ge 0.$$

The kernels $p_n^{(1/2)}$ have the following probabilistic interpretation. Consider a simple symmetric random walk $(S_n)_{n \ge 0}$ with $S_0 = 0$ on the integer lattice $\mathbb{Z}$. Impose the condition that this random walk gets killed, i.e., enters an auxiliary death state, at the first instance that it hits the value $-1$. Equivalently one can consider a random walk reflected at $0$ that dies independently with probability $1/2$ each time it attempts to move from site $0$ to site 1. Then $p_n^{(1/2)}(x, y)$ is the probability of the following event: the walk started from $x$ is at position $y$ at time $n$.

**Definition 3.2.** *We define the following quantity for integers $0 \le n \le N$*

$$\mathfrak{p}_n^N(x, y) := p_n^{(1/2)}(x, y) \frac{\psi(y; N - n)}{\psi(x; N)}, \qquad \textit{where} \qquad \psi(x; n) := \sum_{y \ge 0} p_n^{(1/2)}(x, y).$$

The probabilistic relevance of these kernels $\mathfrak{p}_n^N$ will be demonstrated shortly in Proposition 3.3. As in Theorem 2.5, let

$$\Omega_x^N := \{(s_0, ..., s_N) \in \mathbb{Z}_{\ge 0}^{N+1} : |s_{i+1} - s_i| = 1, s_0 = x\}.$$

Then denote by $\mathbf{P}_x^N$ the uniform probability measure on $\Omega_x^N$, and let $S$ denote the coordinate process associated to this measure (e.g., $S$ can be the identity map on $\Omega_x^N$). In plainer terms, $S$ is a simple symmetric random walk of length $N$ conditioned to stay non-negative throughout its course.

**Proposition 3.3.** *$S$ is an inhomogeneous Markov process on the discrete time interval $\{0, ..., N\}$. In fact, for $0 \le i_1 < ... < i_n \le N$ one has*

$$\begin{aligned}\mathbf{P}_x^N(S_{i_1} = s_1, ..., S_{i_n} = s_n) &= \mathfrak{p}_{i_1}^N(x, s_1)\mathfrak{p}_{i_2-i_1}^{N-i_1}(s_1, s_2)\cdots\mathfrak{p}_{i_n-i_{n-1}}^{N-i_{n-1}}(s_{n-1}, s_n) \\ &= p_{i_1}^{(1/2)}(x, s_1)p_{i_2-i_1}^{(1/2)}(s_1, s_2)\cdots p_{i_n-i_{n-1}}^{(1/2)}(s_{n-1}, s_n)\frac{\psi(s_n, N-i_n)}{\psi(x, N)}.\end{aligned}$$

*In particular, for $M < N$ the conditional law of $(S_{M+k})_{k=0}^{N-M}$ given $(S_k)_{k=0}^M$ is distributed according to $\mathbf{P}_{S_M}^{N-M}$.*

This proves Theorem 2.5*(1)* and shows that the $\mathfrak{p}_n^N(x, \cdot)$ are probability measures.

*Proof.* Write $S_{[0,M]}$ for the restriction of $S$ to $\{0, 1, ..., M\}$, and write $S^{[M,N]}$ for the restriction of $S$ to $\{M, ..., N\}$ shifted by $M$ places (so $S^{[M,N]}$ is defined on $\{0, ..., N-M\}$). For nearest-neighbor paths $s_1$ and $s_2$ of lengths $M$ and $N-M$, respectively, such that $s_1(M) = s_2(0)$ one computes that

$$\mathbf{P}_x^N(S^{[M,N]} = s_2 | S_{[0,M]} = s_1) = \frac{\mathbf{P}_x^N(S = s_1 * s_2)}{\mathbf{P}_x^N(S_{[0,M]} = s_1)} = \frac{\frac{1}{\#\Omega_x^N}}{\frac{\#\{\pi\in\Omega_x^N :\, \pi|_{[0,M]}=s_1\}}{\#\Omega_x^N}}$$

$$= \frac{1}{\#\{\pi \in \Omega_x^N : \pi|_{[0,M]} = s_1\}} = \frac{1}{\#\Omega_{s_1(M)}^{N-M}} = \mathbf{P}_{s_1(M)}^{N-M}(S = s_2),$$

where $s_1 * s_2$ denotes the concatenation of paths. This immediately implies that given $(S_k)_{k=0}^M$ the law of $(S_{M+k})_{k=0}^{N-M}$ is distributed according to $\mathbf{P}_{S_M}^{N-M}$. This also implies that $(S_k)_{k=0}^M$ and $(S_{M+k})_{k=0}^{N-M}$ are conditionally independent given $S_M$. Therefore, in order to prove the given formula for transition densities, it suffices to prove the claim for $n = 1$; then the claim for general $n$ follows from the conditional independence and induction (recall that $n$ is the number of indices $0 \le i_1 < ... < i_n \le N$ appearing in the transition formula).

To prove the formula for $n = 1$ it suffices by conditional independence to assume that $i_n = N$. Note that $\mathbf{P}_x^N$ is the probability measure associated to the killed random walk conditioned to survive, so that

$$\mathbf{P}_x^N(S_N = s) = \frac{p_N^{(1/2)}(x, s)}{\sum_{y\ge 0} p_N^{(1/2)}(x, y)} = p_N^{(1/2)}(x, s)\frac{1}{\psi(x, N)},$$

which proves the claim. □

Next we introduce the continuum analogues of the previously introduced measures. We will generally use capital letters to distinguish macroscopic variables from lowercase microscopic ones.

**Definition 3.4.** *Let $P_t(X) := e^{-X^2/2t}/\sqrt{2\pi t}$ denote the standard heat kernel on the whole line $\mathbb{R}$. Recall the Dirichlet boundary heat kernel*

$$P_t^{Dir}(X, Y) := P_t(X - Y) - P_t(X + Y).$$

*We then define the inhomogeneous kernel for $0 \le t \le T$ and $X, Y > 0$ :*

$$\mathscr{P}_t^T(X, Y) := \begin{cases} P_t^{Dir}(X, Y)\frac{2\Phi(Y/\sqrt{T-t})-1}{2\Phi(X/\sqrt{T})-1} & t < T \\ P_T^{Dir}(X, Y)\frac{1}{2\Phi(X/\sqrt{T})-1} & t = T, \end{cases}$$

*where* $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-u^2/2}du$ *is the cdf of a standard normal. For* $X = 0$*, one analogously defines the quantity for* $Y > 0$ *and* $T \geq t \geq 0$*:*

$$\mathscr{P}_t^T(0,Y) = \begin{cases} Y(T/t^3)^{1/2}e^{-Y^2/2t}\big(2\Phi(Y/\sqrt{T-t})-1\big) & t<T \\ (Y/T)e^{-Y^2/2T} & t=T, \end{cases}$$

*which is the limit of the previously defined* $\mathscr{P}_t^T(X,Y)$ *as* $X \to 0$*.*

We now discuss the relevance of these kernels as Markov transition densities. Specifically, for $X > 0$ define $\mathbf{W}_X^T$ to be the probability measure on $C([0,T],\mathbb{R}_+)$ obtained by conditioning Brownian motion on $[0,T]$ started from $X$ to stay strictly positive until time $T$.[1] We define $B$ to be the canonical process associated to $\mathbf{W}_X^T$. One can also define $\mathbf{W}_0^T$ as the weak limit of the $\mathbf{W}_X^T$ as $X \to 0$. The fact that this limiting measure exists is not difficult but not entirely trivial either (see the appendices). It is called the *Brownian meander* and has been studied extensively in [DIM77, DI77, CM81, Ig74] and subsequent papers on the subject.

**Proposition 3.5.** *Fix some* $T, X > 0$ *and let* $\mathbf{W}_X^T$ *be as defined above, and let* $B$ *denote the associated canonical process. Consider the kernels* $\mathscr{P}_t^T$ *defined before. Then for* $0 \leq t_1 < ... < t_n \leq T$ *and* $Y_1, ..., Y_n > 0$*,*

$$\begin{aligned}\mathbf{W}_X^T&(B_{t_1} \in dY_1,\ ...,\ B_{t_n} \in dY_n) \\ &= \mathscr{P}_{t_1}^T(X,Y_1)\mathscr{P}_{t_2-t_1}^{T-t_1}(Y_1,Y_2)\cdots\mathscr{P}_{t_n-t_{n-1}}^{T-t_{n-1}}(Y_{n-1},Y_n)\,dY_1\cdots dY_n\end{aligned}$$

*In particular, if* $T < S$ *then the conditional law of* $(B_{t+S})_{t\in[0,T-S]}$ *given* $(B_t)_{t\in[0,S]}$ *is equal to* $\mathbf{W}_{B_S}^{T-S}$*. The same statements hold true for* $X = 0$*.*

Before moving on to the proof, we remark that when $X \neq 0$ and $t_n \neq T$, the above formula for transition densities reduces to

$$P_{t_1}^{Dir}(X,Y_1)P_{t_2-t_1}^{Dir}(Y_1,Y_2)\cdots P_{t_n-t_{n-1}}^{Dir}(Y_{n-1},Y_n)\frac{2\Phi(Y_n/\sqrt{T-t_n})-1}{2\Phi(X/\sqrt{T})-1}\,dY_1\cdots dY_n.$$

When $t_n = T$ the numerator $2\Phi(Y_n/\sqrt{T-t_n})-1$ should be interpreted as 1. When $X = 0$ this expression becomes $0/0$, and one needs to take the limit, which gives the formula stated in the above proposition.

*Proof.* Assuming $X > 0$ the proof is analogous to that of Proposition 3.3. Basically one first shows that if $S < T$ then the conditional law of $(B_{t+S})_{t\in[0,T-S]}$ given $(B_t)_{t\in[0,S]}$ is equal to $\mathbf{W}_{B_S}^{T-S}$, and furthermore that $(B_{t+S})_{t\in[0,T-S]}$ and $(B_t)_{t\in[0,S]}$ are conditionally independent given $B_S$. This may be proven by a single computation using the basic properties of standard Brownian motion.

As in the proof of Proposition 3.3, this then reduces the claim to proving the formula for $n = 1$ and $t_n = T$. In turn, this follows by noticing that $\mathbf{W}_X^T$ is the same as Brownian motion killed at zero but conditioned to survive. Hence one finds that

$$\mathbf{W}_X^T(B_T \in dY) = \frac{P_T^{Dir}(X,Y)dY}{\int_0^\infty P_T^{Dir}(X,Z)dZ} = \frac{P_T^{Dir}(X,Y)dY}{2\Phi(X/\sqrt{T})-1},$$

which proves the claim. ☐

This concludes the introductory material on the subject, and we now state several technical estimates on these inhomogeneous heat kernels that are used heavily in the sequel. The proofs may be found in Appendix B.

[1] This is not the same as a 3D Bessel process, which is BM conditioned to stay positive for *all* time and is time-homogeneous.

**Proposition 3.6.** *Fix $\tau \geq 0$. Then for $n \geq 0$, define*

$$\mathscr{P}_n(t,T;X,Y) := (n/2)^{1/2}\mathfrak{p}^{2\lfloor Tn\rfloor}_{2\lfloor tn\rfloor}(2\lfloor n^{1/2}X/\sqrt{2}\rfloor, 2\lfloor n^{1/2}Y/\sqrt{2}\rfloor).$$

*Then for each fixed $X,T,t \geq 0$, as $n \to \infty$ the map $Y \mapsto \mathscr{P}_n(t,T;X,Y)$ converges pointwise and in $L^p(\mathbb{R}_+, e^{aY}dY)$ to $\mathscr{P}_t^T(X,Y)$ for all $p \geq 1$ and $a \geq 0$. Furthermore for all $X,T \geq 0$, the map $(t,Y) \mapsto \mathscr{P}_n(t,T;X,Y)$ converges pointwise and in $L^p(dt \otimes e^{aY}dY)$ to $\mathscr{P}_t^T(X,Y)$ for all $p \in [1,3)$ and $a \geq 0$ (as $n \to \infty$).*

We refer the reader to Proposition B.6 of the appendix for the proof. We remark that the factors of 2 appearing in the definition of $\mathscr{P}_n$ are only necessary due to the periodicity of the simple random walk.

**Proposition 3.7.** *Let $a, \tau > 0$ and let $\mathscr{P}_t^T$ be the kernels from Definition 3.4. Then there exists a constant $C = C(\tau, a)$ such that for all $X, Y \geq 0$, all $\theta \in [0,1/2]$, and all $s \leq t \leq T \leq \tau$ one has the following*

$$\int_{\mathbb{R}_+} \mathscr{P}_t^T(X,Z)e^{aZ}dZ \leq Ce^{aX}, \tag{3.1}$$

$$\int_{\mathbb{R}_+} \mathscr{P}_t^T(X,Z)^2e^{aZ}dZ \leq Ct^{-1/2}e^{aX}, \tag{3.2}$$

$$\int_{\mathbb{R}_+} \big(\mathscr{P}_t^T(X,Z) - \mathscr{P}_t^T(Y,Z)\big)^2 e^{aZ}dZ \leq Ct^{-\frac{1}{2}-\theta}e^{a(X+Y)}|X-Y|^{2\theta}, \tag{3.3}$$

$$\int_{\mathbb{R}_+} \big(\mathscr{P}_s^{T-t+s}(X,Z) - \mathscr{P}_t^T(X,Z)\big)^2 e^{aZ}dZ \leq Cs^{-\frac{1}{2}-\theta}e^{2aX}|t-s|^{\theta} \tag{3.4}$$

The proof may be found as the very last thing in Appendix B. We remark that these bounds will be the key behind the proofs of Section 4 below.

## 4 Existence of the derivative in Dirichlet SHE

Note that in order to prove the identity (1.3), one first needs to prove that the mild solution of $Z_{Dir}$ exists and that the limit on the right-hand side of (1.3) also exists. In this section we actually do something much stronger. We will prove that the mild solution of $Z_{Dir}$ and the aforementioned limits not only exist, but in fact one almost surely has the *simultaneous* existence of $\lim_{X\to 0}\frac{Z_{Dir}(T,X)}{X}$ for *all* $T \geq 0$, for a fixed initial data. Furthermore this limit is Hölder-continuous as a function of $T$.

All of this will essentially be proved in a single step by showing that for $X, T \geq 0$ the chaos series

$$\sum_{k=0}^{\infty}\int_{0\leq t_1<\ldots<t_k\leq T}\int_{\mathbb{R}_+^{k+1}} f(X_{k+1})\prod_{j=1}^{k+1}\mathscr{P}^{T-t_{j-1}}_{t_j-t_{j-1}}(X_{j-1},X_j)dX_{k+1}\xi_T(dX_k,dt_k)\cdots\xi_T(dX_1,dt_1),$$

converges uniformly over compact subsets of $(T,X) \in \mathbb{R}_+ \times \mathbb{R}_+$, where $t_0 := 0, X_0 := X$, $\xi_T(X,t) := \xi(X, T-t)$ for a space-time white noise $\xi$, and $f$ is some random initial data with at-worst exponential growth at infinity. Then we will show almost trivially that when $X, T > 0$ this chaos series equals $Z_{Dir}(T,X)/\big(2\Phi(X/\sqrt{T})-1\big)$, where $\Phi$ is the cdf of a standard normal and $Z_{Dir}$ satisfies the conditions of Definition 2.1. This would simultaneously prove existence of $Z_{Dir}$ and also the desired limit. This is because we know the above chaos series extends continuously to $X = 0$, which means $\lim_{X\to 0}\frac{Z_{Dir}(T,X)}{2\Phi(X/\sqrt{T})-1}$ exists, which is equivalent to showing that $\lim_{X\to 0}\frac{Z_{Dir}(T,X)}{X}$ exists (for all $T \geq 0$, a.s.).

In order to prove the uniform convergence of this chaos series, we are going to use the inhomogeneous heat kernel estimates stated at the end of Section 3. The proofs may be skipped without any effect on the readability of Section 5, although some ideas are similar to ones used there.

With this motivation, we now move on to the main results of this section. Given some possibly random initial data $f : \mathbb{R}_+ \to \mathbb{R}_+$, recall from (1.5) the following Duhamel-form SPDE:

$$\mathscr{Z}(T,X) = \int_{\mathbb{R}^+} \mathscr{P}_T^T(X,Y) f(Y) dY + \int_0^T \int_{\mathbb{R}^+} \mathscr{P}_{T-S}^T(X,Y) \mathscr{Z}(S,Y) \xi(dY dS), \tag{4.1}$$

where $\xi$ is a space-time white noise and so the above should be interpreted as an Itô integral. Since $\mathscr{Z}$ appears on both sides of this relation, it is not clear that a solution would even exist. Thus we have the following result, which will be proved by rigorously expanding (4.1) into the chaos series mentioned above.

**Theorem 4.1.** *Fix $a, \tau > 0$ and suppose that we have some random function-valued initial data $f$ satisfying*

$$\sup_{X \geq 0} e^{-aX} \mathbb{E}[f(X)^2] < \infty.$$

*Then, a unique solution to the SPDE* (4.1) *with initial data $f$ exists in the class of space-time functions $\mathscr{Z}(T,X)$ that satisfy*

$$\sup_{\substack{X \geq 0 \\ T \in [0,\tau]}} e^{-aX} \mathbb{E}[\mathscr{Z}(T,X)^2] < \infty.$$

*Furthermore, the solution $\mathscr{Z}$ may be constructed in such a way that its law is supported on the space of functions that are Hölder-continuous of exponent $1/2 - \epsilon$ in the $X$ variable and $1/4 - \epsilon$ in the $T$ variable on any compact subset of $(T,X) \in (0,\infty) \times [0,\infty)$ for any $\epsilon > 0$.*

*Proof.* This is adapted from the proofs given in [Par19, Section 4]. Informally, one argues as follows: define the following sequence of iterates:

$$u_0(T,X) = \int_{\mathbb{R}_+} \mathscr{P}_T^T(X,Y) f(Y) dY,$$

$$u_{n+1}(T,X) = \int_0^T \int_{\mathbb{R}^+} \mathscr{P}_{T-S}^T(X,Y) u_n(S,Y) \xi(dY dS).$$

In other words, $u_n$ is just the $n^{th}$ term of a chaos series given by the expansion of (4.1). Thus it is clear that the desired solution to (4.1) should be given by $\sum_{n \geq 0} u_n$. Hence, in order to formalize these ideas, we will show that the series $\sum_{n \geq 0} u_n$ converges in the appropriate Banach space of random space-time functions.

To this end, let us define a Banach space $\mathcal{B}$ of $C(\mathbb{R}_+)$-valued processes $u = (u(T,\cdot))_{T \in [0,\tau]}$ that are adapted to the natural filtration of $\xi$ and with norm given by

$$\|u\|_{\mathcal{B}}^2 := \sup_{\substack{X \geq 0 \\ T \in [0,\tau]}} e^{-aX} \mathbb{E}[u(T,X)^2].$$

Then define a sequence of functions $F_n : [0,\tau] \to \mathbb{R}$ for $n \geq 0$ by

$$F_n(T) := \sup_{\substack{X \geq 0 \\ S \in [0,T]}} e^{-aX} \mathbb{E}[u_n(S,X)^2],$$

where $u_n$ are the iterates defined above. By Itô isometry, it is clear that

$$\begin{aligned}\mathbb{E}[u_{n+1}(T,X)^2] &= \int_0^T \int_{\mathbb{R}_+} \mathscr{P}^T_{T-S}(X,Y)^2 \mathbb{E}[u_n(S,Y)^2] dY dS \\ &\leq \int_0^T \left[ \int_{\mathbb{R}_+} \mathscr{P}^T_{T-S}(X,Y)^2 e^{aY} dY \right] F_n(S) dS. \end{aligned} \tag{4.2}$$

Now by (3.2) we have that

$$\int_{\mathbb{R}_+} \mathscr{P}^T_{T-S}(X,Y)^2 e^{aY} dY \leq C(T-S)^{-1/2} e^{aX}, \qquad \forall T \in [0,\tau], X \geq 0, \tag{4.3}$$

where $C$ may depend on $a$ and $\tau$. Furthermore one notes that the $F_n$ are increasing functions of $T$, and therefore $T \mapsto \int_0^T (T-S)^{-1/2} F_n(S) dS$ is also increasing (which may be verified by making the substitution $S = TU$). Combining this fact with (4.2) and (4.3), one obtains

$$F_{n+1}(T) \leq C \int_0^T (T-S)^{-1/2} F_n(S) dS, \tag{4.4}$$

where $C$ does not depend on $n$. Now, we claim that $F_0(T) \leq C$ (with $C = C(a,\tau)$). Indeed, by Jensen's inequality and Fubini's theorem, one has

$$\mathbb{E}[u_0(T,X)^2] = \mathbb{E}\bigg[ \bigg( \int_{\mathbb{R}_+} \mathscr{P}^T_T(X,Y) f(Y) dY \bigg)^2 \bigg] \leq \int_{\mathbb{R}_+} \mathscr{P}^T_T(X,Y) \mathbb{E}[f(Y)^2] dY \leq Ce^{aX},$$

where in the last inequality we used (3.1) together with the assumption that $\mathbb{E}[f(X)^2] \leq Ce^{aX}$. This proves that $F_0 \leq C$, which means that one may iterate (4.4) to obtain

$$F_n(T) \lesssim C^n T^{n/2}/(n/2)!, \tag{4.5}$$

which implies that $\sum_{n\geq 0} \|u_n\|_{\mathcal{B}} < \infty$. This completes the proof of existence.

The proof of uniqueness is essentially the same. Indeed, if $\mathscr{Z}$ and $\mathscr{Z}'$ were two solutions in $\mathcal{B}$ that are started from the same initial data $f$, then an application of Itô's isometry reveals that

$$\mathbb{E}\big[(\mathscr{Z}(T,X) - \mathscr{Z}'(T,X))^2\big] = \int_0^T \int_{\mathbb{R}_+} \mathscr{P}^T_{T-S}(X,Y)^2 \mathbb{E}[(\mathscr{Z}(S,Y) - \mathscr{Z}'(S,Y))^2] dS dY.$$

Then one iterates as above and may obtain that the left-hand side is bounded above (uniformly in $T, X$) by $C^n T^{n/2}/(n/2)!$, and by letting $n \to \infty$ this tends to zero.

Now we address the Hölder regularity. Let $u_n$ be the iterates defined above. We know that $u_0$ is a smooth function of $(T,X) \in (0,\infty) \times [0,\infty)$ because it is the solution to the deterministic (i.e., noiseless) version of SPDE (4.1) which is just an inhomogeneous heat equation (e.g., one may simply differentiate $u_0$ under the integral sign). Thus, it suffices to prove that the function $\mathscr{Z}_0 := \mathscr{Z} - u_0 = \sum_{n\geq 1} u_n$ has the required Hölder regularity, so this is what we will do.

Henceforth fix an exponent $\gamma \in (0,1/2)$. For the spatial regularity, one computes that

$$\begin{aligned}\mathbb{E}[(u_{n+1}(T,X) - u_{n+1}(T,Y))^2] &= \int_0^T \int_{\mathbb{R}_+} \big( \mathscr{P}^T_{T-S}(X,Z) - \mathscr{P}^T_{T-S}(Y,Z) \big)^2 \mathbb{E}[u_n(S,Z)^2] dZ dS \\ &\leq \int_0^T \bigg[ \int_{\mathbb{R}_+} \big( \mathscr{P}^T_{T-S}(X,Z) - \mathscr{P}^T_{T-S}(Y,Z) \big)^2 e^{aZ} dZ \bigg] F_n(S) dS\end{aligned}$$

$$\begin{aligned}&\le C\int_0^T (T-S)^{\gamma-1}|X-Y|^{1-2\gamma}e^{a(X+Y)}F_n(S)dS\\&\overset{(4.5)}{\le} Ce^{a(X+Y)}|X-Y|^{1-2\gamma}\int_0^T (T-S)^{\gamma-1}\frac{C^nS^{n/2}}{(n/2)!}dS\\&\le C^{n+1}e^{a(X+Y)}|X-Y|^{1-2\gamma}\frac{T^{(n+2\gamma)/2}}{(n/2)!}\int_0^1(1-a)^{\gamma-1}a^{n/2}da.\end{aligned}$$

In the third line we used (3.3) with $\theta=\frac12-\gamma$, and in the final line we made a substitution $S=Ta$. Note that the final integral is bounded independently of $n$, so it may be absorbed into the constant (which will then depend on $\gamma$). Using hypercontractivity of the Ornstein-Uhlenbeck semigroup associated to the Gaussian noise $\xi$, we can bound the $p^{th}$ moments of elements of the homogeneous Wiener chaoses in terms of their second moments. Specifically, if $p\ge 2$ then Equation 7.2 of [Hai16] says that:

$$\begin{aligned}\mathbb{E}\big[\big|u_{n+1}(T,X)-u_{n+1}(T,Y)\big|^p\big]^{1/p}&\le (p-1)^{(n+1)/2}\mathbb{E}\big[(u_{n+1}(T,X)-u_{n+1}(T,Y))^2\big]^{1/2}\\&\le C^{(n+1)/2}p^{(n+1)/2}e^{a(X+Y)/2}\frac{T^{(n+1)/4}}{\sqrt{(n/2)!}}|X-Y|^{\frac12-\gamma}.\end{aligned}$$

Using Minkowski's inequality and summing over all $n$, we then obtain

$$\mathbb{E}\big[\big|\mathscr{Z}_0(T,X)-\mathscr{Z}_0(T,Y)\big|^p\big]^{1/p}\le\sum_{n\ge1}\mathbb{E}\big[\big|u_n(T,X)-u_n(T,Y)\big|^p\big]^{1/p}\le D(p,T)e^{a(X+Y)/2}|X-Y|^{\frac12-\gamma}.$$

Here $D(p,T):=\sum_n\frac{(CpT^{1/2})^{(n+1)/2}}{\sqrt{(n/2)!}}$, which is independent of $X,Y$ and increasing as a function of $T$. This is enough by Kolmogorov's criterion to ensure that $\mathscr{Z}_0$ is Hölder continuous of exponent $1/2-\gamma-\epsilon$ (on compact sets) in the spatial variable.

For the temporal regularity, one computes

$$\begin{aligned}&\mathbb{E}[(u_{n+1}(T,X)-u_{n+1}(S,X))^2]\\&=\mathbb{E}\bigg[\bigg(\int_0^T\int_{\mathbb{R}_+}\mathscr{P}^T_{T-U}(X,Z)u_n(U,Z)\xi(dZdU)-\int_0^S\int_{\mathbb{R}_+}\mathscr{P}^S_{S-U}(X,Z)u_n(U,Z)\xi(dZdU)\bigg)^2\bigg]\\&=\int_0^S\int_{\mathbb{R}_+}\big(\mathscr{P}^T_{T-U}(X,Z)-\mathscr{P}^S_{S-U}(X,Z)\big)^2\mathbb{E}[u_n(U,Z)^2]dZdU\\&\qquad+\int_S^T\int_{\mathbb{R}_+}\mathscr{P}^T_{T-U}(X,Z)^2\mathbb{E}[u_n(U,Z)^2]dZdU.\end{aligned}$$

Let us call the integrals in the final expression $I_1$ and $I_2$ respectively. As before, one has $\mathbb{E}[u_n(U,Z)^2]\le e^{aZ}F_n(U)\le e^{aZ}\frac{C^nU^{n/2}}{(n/2)!}$. Then one uses (3.4) with $\theta=\frac12-\gamma$ to bound the inner integral of $I_1$ by

$$\int_{\mathbb{R}_+}\big(\mathscr{P}^T_{T-U}(X,Z)-\mathscr{P}^S_{S-U}(X,Z)\big)^2e^{aZ}dZ\le Ce^{2aX}(S-U)^{\gamma-1}|T-S|^{\frac12-\gamma},$$

and one also uses (3.2) to bound the inner integral of $I_2$ as

$$\int_{\mathbb{R}_+}\mathscr{P}^T_{T-U}(X,Z)^2e^{aZ}dZ\le C(T-U)^{-1/2}e^{aX}.$$

Then one finally performs the integral over $U$ on the respective domains, and one can obtain (as in the spatial case) that $I_1+I_2\le C^{n+1}e^{2aX}T^{(n+1)/2}|T-S|^{\frac12-\gamma}/(n/2)!$. Then one uses hypercontractivity and sums over $n$ (exactly as in the spatial case), to get that

$$\mathbb{E}\big[\big|\mathscr{Z}_0(T,X)-\mathscr{Z}_0(S,X)\big|^p\big]^{1/p}\le D(p,T)e^{2aX}|T-S|^{\frac14-\frac{\gamma}{2}}.$$

Here $D(p,T)$ is an increasing function of $T$ (the same one as before), so it can be bounded from above on any compact set of $(T,X)$. This is enough to give Hölder regularity of $\frac{1}{4}-\frac{\gamma}{2}-\epsilon$ in time, by Kolmogorov's criterion. □

Next, we discuss the relationship of the $\mathscr{Z}$ that we have constructed in Theorem 4.1 with the Dirichlet-boundary SHE.

**Proposition 4.2.** *Any solution of the SPDE* (4.1) *must a.s. satisfy the following relation for all $T,X>0$*

$$\mathscr{Z}(T,X)\big(2\Phi(X/\sqrt{T})-1\big)=Z_{Dir}(T,X)$$

*where $Z_{Dir}$ solves the Dirichlet-boundary SHE as in Definition 2.1 with the same initial data $f$.*

*Proof.* One notes the following relation for $X>0$, which is immediate from Definition 3.4:

$$\mathscr{P}_t^T(X,Y)\big(2\Phi(X/\sqrt{T})-1\big)=\begin{cases} P_t^{Dir}(X,Y)\big(2\Phi(Y/\sqrt{T-t})-1\big), & t<T\\ P_T^{Dir}(X,Y), & t=T.\end{cases} \tag{4.6}$$

So suppose $\mathscr{Z}$ solves (4.1), and define

$$A(T,X):=\mathscr{Z}(T,X)\big(2\Phi(X/\sqrt{T})-1\big).$$

By multiplying both sides of (4.1) by $2\Phi(X/\sqrt{T})-1$ and applying (4.6), one has the relation

$$\begin{aligned}A(T,X)&=\int_{\mathbb{R}_+}P_T^{Dir}(X,Y)f(Y)dY+\int_0^T\int_{\mathbb{R}_+}P_{T-S}^{Dir}(X,Y)\left[\mathscr{Z}(S,Y)\big(2\Phi(Y/\sqrt{S})-1\big)\right]\xi(dY,dS)\\ &=\int_{\mathbb{R}_+}P_T^{Dir}(X,Y)f(Y)dY+\int_0^T\int_{\mathbb{R}_+}P_{T-S}^{Dir}(X,Y)A(S,Y)\xi(dY,dS),\end{aligned}$$

so that $A$ is indeed a mild solution to the Dirichlet-boundary SHE. □

One thing we have not addressed is the *uniqueness* of solutions to the Dirichlet-boundary SHE in some large enough class of random space-time functions. This can be obtained from Theorem 4.1 with minimal work, and with the same conditions on the initial data $f$, one can in fact obtain existence/uniqueness in the space of $\xi$-adapted space-time functions $A$ satisfying $\sup_{T\le\tau,\,X\ge0}\mathbb{E}[A(T,X)^2]<\infty$.

**Corollary 4.3.** *Consider any solution $Z_{Dir}$ of the Dirichlet-boundary SHE, started from any initial data $f$ satisfying the assumptions of Theorem 4.1. Then almost surely, for every $T>0$ the limit of $\frac{Z_{Dir}(T,X)}{X}$ exists as $X\to0$.*

*Proof.* Consider the solution $\mathscr{Z}$ to (4.1) started from initial data $f$. By the preceding proposition, we can couple this with the solution to the Dirichlet-boundary SHE in such a way so that

$$\mathscr{Z}(T,X)=\frac{Z_{Dir}(T,X)}{2\Phi(X/\sqrt{T})-1}$$

for all $X>0$ and $T\ge0$. But we know that $\mathscr{Z}$ extends continuously to $X=0$ by Theorem 4.1, hence we know that

$$\lim_{X\to0}\frac{Z_{Dir}(T,X)}{2\Phi(X/\sqrt{T})-1}$$

exists, and since $2\Phi(X/\sqrt{T})-1$ has nonzero derivative at $X=0$, the claim follows. □

## 5 Convergence of the partition function to SHE

In this section we use a discrete chaos expansion together with the methods of [AKQ14a, CSZ17a] and the heat kernel estimates of the previous sections in order to prove Theorem 2.2. The first step (Section 5.1) is to simplify the geometry of the region where our directed polymer lives, and then (in Section 5.2) we will prove the convergence result in the simpler domain.

As a notational convention, we will usually write $C$ for constants, and we will not generally specify when irrelevant terms are being absorbed into the constants. We will also write $C(a), C(a,p), C(a,p,K)$ whenever we want to specify exactly which parameters the constant depends on. This will not always be specified, though. This applies throughout the paper. Please be warned that we will freely use many different bounds from the appendices in the following proofs, so the reader may wish to skim those estimates first.

### 5.1 Reduction from the octant model to the quadrant model

In this section, we reduce the technicality of working with the partition function in an octant to working with it in a quadrant, which simplifies many computations. The dichotomy here is that the quadrant has a simple geometry that makes polymer-convergence results of the desired type quite straightforward; on the other hand, the octant has the advantage that one has nice identities such as those of Corollary 2.4(3) which fail for a quadrant. Hence, one viewpoint is simpler for technical computations while the other is well-adapted for exact solvability. The results of this section are specific to the case of our positive random walk measures; however, the general outline and arguments that will be given may be easily modified for other random walk measures, such as the reflecting walk, as long as the analogous heat kernel bounds hold. Thus this section may potentially prove useful to other works of a similar flavor.

In what follows, we fix a sequence $\omega^n = \{\omega^n_{i,j}\}_{i,j\geq 0}$ of i.i.d. random environments with $n \in \mathbb{N}$. As always, we denote by $\mathbb{E}$ (resp. $\mathbb{P}$) the expectation (resp. probability) with respect to the environment $\omega^n_{i,j}$ and we denote by $\mathbf{E}^n_x$ (resp. $\mathbf{P}^n_x$) the expectation (resp. probability) with respect to the positive random walk measures of Section 3. Furthermore $T_n$ will denote the first time that this random walk $(i, S_i)$, started from $(0,x)$ with $x \geq 0$, hits the diagonal line $\{(j, 2n-j) : j \geq 0\}$.

First we need an estimate on the variance of the discrete chaos series terms.

**Lemma 5.1.** *Let $\mathfrak{p}^N_n(x,y)$ be the positive random walk transition probabilities given in Definition 3.2. Then there exist constants $B, C, K > 0$ such that for all $x, n, k \geq 0$ and $a \geq 0$,*

$$\sum_{\substack{0\leq i_1<\ldots<i_k\leq n \\ (x_1,\ldots,x_k)\in\mathbb{Z}^k_{\geq 0}}} \mathfrak{p}^n_{i_1}(x,x_1)^2 \mathfrak{p}^{n-i_1}_{i_2-i_1}(x_1,x_2)^2 \cdots \mathfrak{p}^{n-i_{k-1}}_{i_k-i_{k-1}}(x_{k-1},x_k)^2 e^{ax_k} \leq B e^{ax+Ka^2n} C^k n^{k/2}/(k/2)!,$$

*where $(k/2)!$ is a shorthand for $\Gamma(1+k/2)$.*

*Proof.* We first state a bound, which is Proposition B.3 in the appendix: there exist constants $C, K > 0$ such that for all $x \geq 0$, all $N \geq n \geq 0$, all $a \geq 0$, and all $p \geq 1$ one has that

$$\sum_{y\geq 0} \mathfrak{p}^N_n(x,y)^p e^{ay} \leq C^p (n+1)^{-(p-1)/2} e^{ax+Ka^2n}.$$

Applying this $k$ times, one sees that

$$\sum_{(x_1,\ldots,x_k)\in\mathbb{Z}^k_{\geq 0}} \mathfrak{p}^n_{i_1}(x,x_1)^2 \mathfrak{p}^{n-i_1}_{i_2-i_1}(x_1,x_2)^2 \cdots \mathfrak{p}^{n-i_{k-1}}_{i_k-i_{k-1}}(x_{k-1},x_k)^2 e^{ax_k}$$

$$\leq C^k e^{ax+Ka^2n}(i_1+1)^{-1/2}(i_2-i_1+1)^{-1/2}\cdots(i_k-i_{k-1}+1)^{-1/2}.$$

Thus the desired sum is bounded above by

$$e^{ax+Ka^2n}\sum_{1\leq i_1<\ldots<i_k\leq n+1} i_1^{-1/2}(i_2-i_1)^{-1/2}\cdots(i_k-i_{k-1})^{-1/2}.$$

Now one recognizes that

$$n^{-k/2}\sum_{1\leq i_1<\ldots<i_k\leq n+1} i_1^{-1/2}(i_2-i_1)^{-1/2}\cdots(i_k-i_{k-1})^{-1/2}$$

$$=\frac{1}{n^k}\sum_{1\leq i_1<\ldots<i_k\leq n+1}\big(\frac{i_1}{n}\big)^{-1/2}\big(\frac{i_2}{n}-\frac{i_1}{n}\big)^{-1/2}\cdots\big(\frac{i_k}{n}-\frac{i_{k-1}}{n}\big)^{-1/2}, \tag{5.1}$$

which as a Riemann sum approximation is bounded above by (say) twice

$$\int_{0\leq t_1<\ldots<t_k\leq 1} t_1^{-1/2}(t_2-t_1)^{-1/2}\cdots(t_k-t_{k-1})^{-1/2}dt_1\cdots dt_k\leq B/(k/2)!,$$

where $B>0$. Hence the lemma is proved. □

Now we use the variance bound in conjunction with Doob's martingale inequality to get a bound on the expected supremum in the partition function.

**Lemma 5.2.** *Take a sequence $\omega^n=\{\omega^n_{i,j}\}$ of random environments with variance uniformly bounded above by 1. Furthermore let $\{z_0^n(x)\}_{x\geq 0}$ be some sequence of non-negative stochastic processes, independent of the $\omega^n$, with the property that $\mathbb{E}[z_0^n(x)^2]\leq Ke^{an^{-1/2}x}$ for some constants $K,a$ that are independent of $n$ and $x$. Then there exists a constant $C$ such that for all $n,x\geq 0$ one has that*

$$\mathbb{E}\Bigg[\sup_{0\leq k\leq n}\mathbf{E}_x^n\Big[z_0^n(S_n)\prod_{i=0}^k(1+n^{-1/4}\omega^n_{i,S_i})\Big]^2\Bigg]\leq Ce^{an^{-1/2}x}.$$

*Proof.* First we fix some $n\in\mathbb{N}$ and we note that the process

$$M_k^n:=\mathbf{E}_x^n\Big[z_0^n(S_n)\prod_{i=1}^k(1+n^{-1/4}\omega^n_{i,S_i})\Big]$$

is a $\mathbb{P}$-martingale in the $k$ variable with respect to the filtration $(\mathcal{F}_k^n)_{k\geq 0}$, where $\mathcal{F}_k^n$ is generated by $z_0^n$ and $\{\omega^n_{i,j}\}_{0\leq j\leq i\leq k}$. Therefore by Doob's martingale inequality it is clear that $\mathbb{E}[\sup_{0\leq k\leq n}(M_k^n)^2]\leq 4\mathbb{E}[(M_n^n)^2]$. This reduces our work to proving the claim without the supremum inside the expectation (and replacing $k$ by $n$ in the upper limit of the product). To do this, we set $x_0:=x$ and we write

$$\mathbb{E}\Bigg[\mathbf{E}_x^n\Big[z_0^n(S_n)\prod_{i=1}^n(1+n^{-1/4}\omega^n_{i,S_i})\Big]^2\Bigg]$$

$$=\mathbb{E}\Bigg[\bigg(\sum_{\substack{0\leq k\leq n\\ 0\leq i_1<\ldots<i_k\leq n\\ (x_1,\ldots,x_{k+1})\in\mathbb{Z}_{\geq 0}^{k+1}}} n^{-k/4}z_0^n(x_{k+1})\prod_{j=1}^k\mathfrak{p}^{n-i_{j-1}}_{i_j-i_{j-1}}(x_{j-1},x_j)\omega_{i_jx_j}\cdot\mathfrak{p}^{n-i_k}_{n-i_k}(x_k,x_{k+1})\bigg)^2\Bigg]$$

$$=\sum_{\substack{0\leq k\leq n\\ 0\leq i_1<\ldots<i_k\leq n\\ (x_1,\ldots,x_k)\in\mathbb{Z}_{\geq 0}^{k}}} n^{-k/2}\prod_{j=1}^k\mathfrak{p}^{n-i_{j-1}}_{i_j-i_{j-1}}(x_{j-1},x_j)^2\,\mathbb{E}\Bigg[\bigg(\sum_{x_{k+1}\in\mathbb{Z}_{\geq 0}} z_0^n(x_{k+1})\mathfrak{p}^{n-i_k}_{n-i_k}(x_k,x_{k+1})\bigg)^2\Bigg],$$

where $x_0 := x$ by convention. By Jensen we then have that

$$\Big(\sum_{x_{k+1}\geq 0} z_0^n(x_{k+1})\mathfrak{p}_{n-i_k}^{n-i_k}(x_k,x_{k+1})\Big)^2 \leq \sum_{x_{k+1}\geq 0} z_0^n(x_{k+1})^2\mathfrak{p}_{n-i_k}^{n-i_k}(x_k,x_{k+1}).$$

We know by assumption that $\mathbb{E}[z_0^n(x_{k+1})^2] \leq e^{an^{-1/2}x_{k+1}}$. Thus we find that the expectation of the last expression is bounded above by $Ce^{an^{-1/2}x_k}$ because of the inequality $\sum_{y\geq 0}\mathfrak{p}_n^N(x,y)e^{ay} \leq Ce^{ax+Ka^2n}$, which holds by Proposition B.1 in the appendix. Thus by Lemma 5.1 we have

$$\begin{aligned}
\mathbb{E}\bigg[\mathbf{E}_x^n\Big[z_0^n(S_n)\prod_{i=1}^n(1+n^{-1/4}\omega_{i,S_i}^n)\Big]^2\bigg] &\leq \sum_{\substack{0\leq k\leq n\\ 0\leq i_1<\ldots<i_k\leq n\\ (x_1,\ldots,x_k)\in\mathbb{Z}_{\geq 0}^k}} n^{-k/2}\prod_{j=1}^k \mathfrak{p}_{i_j-i_{j-1}}^{n-i_{j-1}}(x_{j-1},x_j)^2\\
&\quad\cdot Ce^{an^{-1/2}x_k}\\
&\leq \sum_{k=0}^n n^{-k/2}BC^{k+1}e^{an^{-1/2}x}n^{k/2}/(k/2)!\\
&\leq Be^{an^{-1/2}x}\sum_{k=0}^\infty C^{k+1}/(k/2)!.
\end{aligned}$$

This completes the proof. $\square$

We now introduce a class of Banach spaces that will be useful for describing convergence of initial data:

**Definition 5.3.** *Let $\alpha,\delta\in(0,1)$. A function $f:\mathbb{R}\to\mathbb{R}$ is said to be in the exponentially $\delta$-weighted $\alpha$-Hölder space $\mathscr{C}_{e(\delta)}^\alpha(\mathbb{R})$ if*

$$\sup_{x\in\mathbb{R}}\frac{|f(x)|}{e^{\delta|x|}} + \sup_{\substack{x,y\in\mathbb{R}\\ |x-y|\leq 1}}\frac{|f(x)-f(y)|}{e^{\delta|x|}|x-y|^\alpha} < \infty.$$

*We turn $\mathscr{C}_\delta^\alpha$ into a Banach space by defining the norm of $f$ to be the above quantity.*

A straightforward consequence of Arzela-Ascoli is that $\mathscr{C}_{e(\delta)}^\alpha$ embeds compactly into $\mathscr{C}_{e(\delta')}^{\alpha'}$ for $\alpha'<\alpha$ and $\delta<\delta'$. The key estimate of this section is as follows:

**Theorem 5.4** (Key Estimate). *Fix $\alpha\in(0,1)$. Suppose that $(z_0^n(x))_{x\in\mathbb{Z}_{\geq 0}}$ is a family of deterministic non-negative functions such that the linearly interpolated and rescaled family $z_0^n(n^{1/2}x)$ are bounded with respect to the norm of $\mathscr{C}_{e(\delta)}^\gamma$ for some $\gamma,\delta\in(0,1)$. Define the "error" random variable*

$$\mathcal{E}(x,n) := \sup_{k\in[n-n^\alpha,n]}\bigg|\mathbf{E}_x^n\Big[z_0^n(S_n)\prod_{i=1}^n\big(1+n^{-1/4}\omega_{i,S_i}^n\big) - z_0^n(S_k)\prod_{i=1}^k\big(1+n^{-1/4}\omega_{i,S_i}^n\big)\Big]\bigg|.$$

*Then $\sup_{x\geq 0} e^{-3an^{-1/2}x}\mathbb{E}[\mathcal{E}(x,n)]\to 0$ as $n\to\infty$.*

*Proof.* By the triangle inequality, we have $\mathcal{E}(x,n)\leq\mathcal{E}_1(x,n)+\mathcal{E}_2(x,n)$, where

$$\mathcal{E}_1(x,n) := \sup_{k\in[n-n^\alpha,n]}\bigg|\mathbf{E}_x^n\Big[z_0^n(S_n)\Big(\prod_{i=1}^n\big(1+n^{-1/4}\omega_{i,S_i}^n\big) - \prod_{i=1}^k\big(1+n^{-1/4}\omega_{i,S_i}^n\big)\Big)\Big]\bigg|,$$

$$\mathcal{E}_2(x,n) := \sup_{k\in[n-n^\alpha,n]}\bigg|\mathbf{E}_x^n\Big[\big(z_0^n(S_n)-z_0^n(S_k)\big)\prod_{i=1}^k\big(1+n^{-1/4}\omega_{i,S_i}^n\big)\Big]\bigg|.$$

We separately show that both of these satisfy the desired bound. Henceforth when we write $n^\alpha$ we actually mean $\lceil n^\alpha \rceil$.

First we consider $\mathcal{E}_1$. First we establish a martingale inequality. If $(M_k)_{k\geq 0}$ is a martingale defined on any probability space, then note that for $r \leq n$ one has that

$$\sup_{r\leq k\leq n} |M_n - M_k| \leq |M_n - M_r| + \sup_{r\leq k\leq n} |M_k - M_r|,$$

and by Doob's inequality one has that $\big\| \sup_{r\leq k\leq n} |M_k - M_r| \big\|_p \leq \frac{p}{p-1} \|M_n - M_r\|_p$, therefore one has that

$$\big\| \sup_{r\leq k\leq n} |M_n - M_k| \big\|_p \leq \|M_n - M_r\|_p + \frac{p}{p-1}\|M_n - M_r\|_p = \frac{2p-1}{p-1}\|M_n - M_r\|_p. \tag{5.2}$$

Now let us fix some $n \in \mathbb{N}$. Let us define a martingale

$$M_k^n := \mathbf{E}_x^n \Big[ z_0^n(S_n) \prod_{i=1}^k (1 + n^{-1/4}\omega_{i,S_i}^n) \Big].$$

This is a $\mathbb{P}$-martingale in the $k$ variable, for fixed $n \in \mathbb{N}$. Consequently, using (5.2) with $p = 2$ gives us

$$\mathbb{E}\big[ \sup_{k\in[n-n^\alpha,n]} (M_n - M_k)^2 \big] \leq 9\mathbb{E}[(M_n^n - M_{n-n^\alpha}^n)^2]. \tag{5.3}$$

Computing the right-hand side, one gets

$$\begin{aligned} \mathbb{E}[(M_n^n - M_{n-n^\alpha}^n)^2] &= \mathbb{E}\Bigg[ \mathbf{E}_x^n \Bigg[ z_0^n(S_n) \Bigg( \prod_{i=1}^n \big(1 + n^{-1/4}\omega_{i,S_i}^n\big) - \prod_{i=1}^{n-n^\alpha} \big(1 + n^{-1/4}\omega_{i,S_i}^n\big) \Bigg) \Bigg]^2 \Bigg] \\ &= \sum_{\substack{0\leq k\leq n-n^\alpha \\ 0\leq i_1<\ldots<i_k\leq n-n^\alpha \\ (x_1,\ldots,x_k)\in\mathbb{Z}_{\geq 0}^k}} n^{-k/2} \cdot \mathfrak{p}_{i_1}^n(x,x_1)^2 \prod_{j=1}^{k-1} \mathfrak{p}_{i_j-i_{j-1}}^{n-i_{j-1}}(x_j,x_{j+1})^2 \mathfrak{F}_n(i_k,x_k), \end{aligned} \tag{5.4}$$

where $\mathfrak{F}_n(i_k,x_k)$ is given by

$$\sum_{\substack{1\leq \ell\leq n^\alpha \\ 0\leq j_1<\ldots<j_\ell\leq n^\alpha \\ (u_1,\ldots,u_\ell)\in\mathbb{Z}_{\geq 0}^\ell}} n^{-\ell/2}\mathfrak{p}_{n-n^\alpha+j_1-i_k}^n(x_k,u_1)^2 \cdot \prod_{v=1}^{\ell-1} \mathfrak{p}_{j_v-j_{v-1}}^{n^\alpha-j_{v-1}}(u_j,u_{j+1})^2 \Bigg( \sum_{u_{\ell+1}\geq 0} z_0^n(u_{\ell+1})\mathfrak{p}_{n^\alpha-j_\ell}^{n^\alpha-j_\ell}(u_\ell,u_{\ell+1}) \Bigg)^2.$$

Note that the latter sum starts at $\ell = 1$ rather than $\ell = 0$ which is crucial. These expressions come from writing

$$\begin{aligned} &\prod_{i=1}^n \big(1 + n^{-1/4}\omega_{i,S_i}^n\big) - \prod_{i=1}^{n-n^\alpha} \big(1 + n^{-1/4}\omega_{i,S_i}^n\big) \\ &\qquad = \prod_{k=1}^{n-n^\alpha} (1 + n^{-1/4}\omega_{k,S_k}^n) \Bigg( \prod_{\ell=1}^{n^\alpha} (1 + n^{-1/4}\omega_{\ell+n-n^\alpha,S_{\ell+n-n^\alpha}}^n) \; - \; 1 \Bigg), \end{aligned}$$

and then expanding both products and taking expectations. The subtraction of $1$ from the second product is what causes the sum defining $\mathfrak{F}_n$ to start at $\ell = 1$ rather than $\ell = 0$.

By Jensen and the fact that $z_0^n(x) \leq Ce^{an^{-1/2}x}$ (with say $a = 2\delta$ where $\delta$ is the same as in the theorem statement) we then have that

$$\Big(\sum_{u_{\ell+1}\geq 0} z_0^n(u_{\ell+1})\mathfrak{p}^{n^\alpha-j_\ell}_{n^\alpha-j_\ell}(u_\ell,u_{\ell+1})\Big)^2 \leq \sum_{u_{\ell+1}\geq 0} z_0^n(u_{\ell+1})^2\mathfrak{p}^{n^\alpha-j_\ell}_{n^\alpha-j_\ell}(u_\ell,u_{\ell+1})\leq Ce^{an^{-1/2}u_\ell},$$

where we used Proposition B.1 in the last bound. Then by repeatedly applying Proposition B.3, note that $\mathfrak{F}_n(i_k,x_k)$ is bounded above by

$$\sum_{\ell=1}^{n^\alpha+1} n^{-\ell/2}C^\ell e^{an^{-1/2}x_k} \sum_{1\leq j_1<\ldots<j_\ell\leq n^\alpha+1} (n-n^\alpha+j_1-i_k)^{-1/2}(j_2-j_1)^{-1/2}\cdots(j_\ell-j_{\ell-1})^{-1/2}.$$

Consequently the entirety of (5.4) is bounded above, after again applying Proposition B.3 several more times, by

$$\sum_{\substack{0\leq k\leq n-n^\alpha\\ 1\leq i_1<\ldots<i_k\leq n-n^\alpha+1\\ 1\leq \ell\leq n^\alpha\\ 1\leq j_1<\ldots<j_\ell\leq n^\alpha+1}} n^{-(k+\ell)/2}C^{k+\ell}e^{an^{-1/2}x}i_1^{-1/2} \cdot\prod_{r=1}^{k-1}(i_r-i_{r-1})^{-1/2}(n-n^\alpha+j_1-i_k)^{-1/2}\prod_{v=1}^{\ell-1}(j_v-j_{v-1})^{-1/2}.$$

We rewrite that as $e^{an^{-1/2}x}$ multiplied by

$$\sum_{\substack{0\leq k\leq n-n^\alpha\\ 1\leq i_1<\ldots<i_k\leq n-n^\alpha+1\\ 1\leq \ell\leq n^\alpha\\ 1\leq j_1<\ldots<j_\ell\leq n^\alpha+1}} n^{-\ell(1-\alpha)/2}n^{-k}n^{-\ell\alpha}C^{k+\ell}(\frac{i_1}{n})^{-1/2} \cdot\prod_{r=1}^{k-1}(\frac{i_r}{n}-\frac{i_{r-1}}{n})^{-1/2}(\frac{n-n^\alpha+j_1-i_k}{n^\alpha})^{-1/2}\prod_{v=1}^{\ell-1}(\frac{j_v}{n^\alpha}-\frac{j_{v-1}}{n^\alpha})^{-1/2}.$$

Except for the factor $n^{-\ell(1-\alpha)/2}$ we recognize a Riemann sum approximation for

$$\sum_{\substack{k\geq 0\\ \ell\geq 1}} C^{k+\ell}\int_{0\leq t_1<\ldots<t_k\leq 1}\int_{t_k\leq s_1<\ldots<s_\ell\leq 1} t_1^{-1/2}\cdots(t_k-t_{k-1})^{-1/2}(s_1-t_k)^{-1/2}\cdots(s_\ell-s_{\ell-1})^{-1/2}d\mathbf{t}d\mathbf{s}.$$

This series may be bounded by

$$\sum_{\substack{k\geq 0\\ \ell\geq 1}} C^{k+\ell}/\big((k+\ell)/2\big)!$$

which converges absolutely to a constant independently of $n$. Since $\ell\geq 1$ in all expressions above, the left over factor $n^{-\ell(1-\alpha)/2}$ is at worst $n^{-(1-\alpha)/2}$. Summarizing the bounds, we showed that $\mathbb{E}[\mathcal{E}_1(n,x)]$ is bounded above by at worst $Ce^{an^{-1/2}x}n^{-(1-\alpha)/2}$ which implies the desired result on $\mathcal{E}_1$.

Now we consider $\mathcal{E}_2(x,n)$. Since $z_0^n$ is bounded in $\mathscr{C}^\gamma_{e(\delta)}$ we have the following bound with $C$ independent of $x,y,n$:

$$|z_0^n(n^{1/2}x)-z_0^n(n^{1/2}y)|\leq C|x-y|^\gamma e^{\delta(x+y)}.$$

Using positivity of $B_k^n := \prod_1^k (1+n^{-1/2}\omega^n_{i,S_i})$ we then find that

$$\begin{aligned}
\mathcal{E}_2(x,n) &\le \sup_{k\in[n-n^\alpha,n]} \mathbf{E}_x^n\Big[\big|z_0^n(S_n)-z_0^n(S_k)\big|B_k^n\Big] \\
&\le Cn^{-\gamma/2} \sup_{k\in[n-n^\alpha,n]} \mathbf{E}_x^n\Big[|S_n-S_k|^\gamma e^{\delta n^{-1/2}(S_n+S_k)}B_k^n\Big] \\
&= Cn^{-\gamma/2} \sup_{k\in[n-n^\alpha,n]} \mathbf{E}_x^n\Big[\mathbf{E}_{S_k}^{n-k}[|\tilde S_{n-k}-\tilde S_0|^\gamma e^{\delta n^{-1/2}\tilde S_{n-k}}]e^{\delta n^{-1/2}S_k}B_k^n\Big],
\end{aligned}$$

where the final equality follows from the Markov property of the positive random walk $S$. Now we recognize that

$$\mathbf{E}_y^N[|S_N-S_0|^\gamma e^{\delta S_N}] \le \mathbf{E}_y^N[|S_N-S_0|^{2\gamma}]^{1/2}\mathbf{E}_y^N[e^{2\delta S_N}]^{1/2} \le CN^{\gamma/2}e^{\delta y+K\delta^2 N},$$

where $C,K$ are independent of $y,N$, by Propositions A.9 and B.1. Consequently we find for $k\in[n-n^\alpha,n]$ that

$$\mathbf{E}_{S_k}^{n-k}[|\tilde S_{n-k}-\tilde S_0|^\gamma e^{\delta n^{-1/2}\tilde S_{n-k}}] \le C(n-k)^{\gamma/2}e^{\delta n^{-1/2}S_k} \le Cn^{\alpha\gamma/2}e^{\delta n^{-1/2}S_k}.$$

Combining our bounds, we find that

$$\mathcal{E}_2(x,n) \le Cn^{-(1-\alpha)\gamma/2} \sup_{k\in[n-n^\alpha,n]} \mathbf{E}_x^n[e^{2\delta n^{-1/2}S_k}B_k^n]. \tag{5.5}$$

Now for any $\lambda>0$, $(e^{\lambda S_k})_k$ is a $\mathbf{P}_x^n$-submartingale because $(S_k)$ is a submartingale (Lemma A.2) and since $x\mapsto e^{\lambda x}$ is increasing and convex for any $\lambda$. Thus letting $\mathcal{G}_k$ denote the filtration generated by the first $k$ steps of the $n$-step positive random walk $S$, we find

$$\mathbf{E}_x^n[e^{\lambda S_k}B_k^n] \le \mathbf{E}_x^n[\mathbf{E}_x^n[e^{\lambda S_n}|\mathcal{G}_k]B_k^n] = \mathbf{E}_x^n[e^{\lambda S_n}B_k^n],$$

for all $k\le n, \lambda>0$ because $B_k^n$ is $\mathcal{G}_k$-measurable. Setting $\lambda=2\delta n^{-1/2}$, this means that

$$\mathbb{E}\Big[\sup_{k\le n}\mathbf{E}_x^n[e^{2\delta n^{-1/2}S_k}B_k^n]^2\Big] \le \mathbb{E}\Big[\sup_{k\le n}\mathbf{E}_x^n[e^{2\delta n^{-1/2}S_n}B_k^n]^2\Big] \le Ce^{2\delta n^{-1/2}x}, \tag{5.6}$$

where we used Lemma 5.2 in the last bound, with $z_0^n(x) := e^{2\delta n^{-1/2}x}$. Combining (5.5) and (5.6) gives the required result. ☐

Next we give some Kolmogorov-type moment conditions that ensure tightness of the sequence $z_0^n$ of initial data in $\mathscr{C}^\alpha_{e(\delta)}$.

**Proposition 5.5.** *Suppose that $\{z^n\}_{n\ge1}$ is a family of random functions on $\mathbb{R}$ that satisfies the following moment conditions for some constants $a,p,\beta,C$ independent of $n,x,y$.*

- $\mathbb{E}[|z^n(x)-z^n(y)|^p] \le C|x-y|^{p\beta/2}e^{a(|x|+|y|)}$.
- *there exist positive integrable random variables $D(n)$ such that $\sup_n \mathbb{E}[D(n)]<\infty$ and $z^n(x)\le D(n)e^{a|x|}$.*

*Then assuming $p>1/\beta$, there exist $\delta>a$ and $\alpha<\beta-p^{-1}$ such that $(z^n)$ is tight with respect to the topology of $\mathscr{C}^\alpha_{e(\delta)}$.*

Before the proof, we remark that when we apply this result, the functions will be defined on $\mathbb{R}_+$ as opposed to all of $\mathbb{R}$ and thus the absolute values on $x,y$ are unnecessary. Furthermore, the $z^n$ appearing in the proposition statement will actually be rescaled and linearly interpolated functions $z_0^n(n^{1/2}x)$.

*Proof.* Recall from earlier that $\mathscr{C}^\alpha_{e(\delta)}$ embeds *compactly* into $\mathscr{C}^{\alpha'}_{e(\delta')}$ whenever $\delta' > \delta$ and $\alpha' < \alpha$. Therefore to prove the lemma, it suffices to show that if the two inequalities in the lemma statement hold uniformly over a family $\mathcal{F}$ of real-valued functions, then there exist $\alpha, \delta$ such that

$$\lim_{a\to\infty} \sup_{z\in\mathcal{F}} \mathbb{P}(\|z\|_{\mathscr{C}^\alpha_{e(\delta)}} > a) = 0.$$

We actually show something stronger, namely that under the given assumptions, there exists $C > 0$ such that for all $a > 0$

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\|z^n\|_{\mathscr{C}^\alpha_{e(\delta)}} > a) \le Ca^{-1}. \tag{5.7}$$

To prove this, for a function $z$ we write $\|z\|_{\mathscr{C}^\alpha_{e(\delta)}} = \|z\|_\delta + [z]_{\alpha,\delta}$ where $\|z\|_\delta := \sup_{x\in\mathbb{R}} \frac{|z(x)|}{e^{\delta|x|}}$ and $[z]_{\alpha,\delta} := \sup_{x\in\mathbb{R}} e^{-\delta|x|} \sup_{|y-x|\le 1} \frac{|z(x)-z(y)|}{|x-y|^\alpha}$.

To prove (5.7), the following fact will be useful to us: For any $\gamma \in (0,1)$, the $\gamma$-Hölder seminorm $[f]_\gamma$ of a function $f : [0,1] \to \mathbb{R}$ is equivalent (as a seminorm) to the quantity given by $\sup_{v\in\mathbb{N},1\le k\le 2^v} 2^{\gamma v}|f(k2^{-v}) - f((k-1)2^{-v})|$. This is proved as an intermediate step in the standard proof of the classical Kolmogorov-Chentsov criterion.

The exact choices of $\alpha, \delta$ will be specified later, but for now let them denote generic constants. Now to prove (5.7) let us write for a function $z$,

$$\begin{aligned}
\|z\|_\delta &\le \sup_{v\in\mathbb{Z}} e^{-\delta|v|}\big(|z(v)| + \sup_{x\in[v,v+1]} |z(x) - z(v)|\big) \\
&\le \sup_{v\in\mathbb{Z}} e^{-\delta|v|}\left(|z(v)| + \sup_{x\in[v,v+1]} \frac{|z(x) - z(v)|}{|x-v|^\alpha}\right) \\
&\lesssim \sup_{v\in\mathbb{Z}} e^{-\delta|v|}\left(|z(v)| + \sup_{r\in\mathbb{N},1\le k\le 2^r} 2^{\alpha r}|z(v + k2^{-r}) - z(v + (k-1)2^{-r})|\right),
\end{aligned}$$

where $\lesssim$ denotes the absorption of some universal constant which can depend on $\alpha, \delta$ but not on the function $z$. Likewise let us note that

$$[z]_{\alpha,\delta} \lesssim \sup_{v\in\mathbb{Z}} e^{-\delta|v|} \sup_{r\in\mathbb{N},1\le k\le 2^r} 2^{\alpha r}|z(v + k2^{-r}) - z(v + (k-1)2^{-r})|.$$

Consequently we find that

$$\|z\|_{\mathscr{C}^\alpha_\delta} \lesssim A(z,\delta) + B(z,\alpha,\delta),$$

where

$$\begin{aligned}
A(z,\delta) &:= \sup_{v\in\mathbb{Z}} e^{-\delta|v|}|z(v)|, \\
B(z,\alpha,\delta) &:= \sup_{v\in\mathbb{Z}} e^{-\delta|v|} \sup_{r\in\mathbb{N},1\le k\le 2^r} 2^{\alpha r}|z(v + k2^{-r}) - z(v + (k-1)2^{-r})|.
\end{aligned}$$

Now, with $z^n$ uniformly satisfying the bounds given in the lemma statement, let us bound these terms $A(z^n,\delta)$ and $B(z^n,\alpha,\delta)$ individually to obtain (5.7). We will do this by using the hypotheses in the lemma. Note that by a brutal union bound and Markov's inequality followed by the hypothesis $z^n \le D(n)e^{an^{-1/2}x}$, we have

$$\begin{aligned}
\mathbb{P}(A(z^n,\delta) > a) &\le \sum_{v\in\mathbb{Z}} \mathbb{P}(|h^\epsilon(v)| > e^{\delta|v|}a) \\
&\le \sum_{v\in\mathbb{Z}} a^{-1}e^{-\delta|v|}\mathbb{E}[z^n(v)] \\
&\le \sup_j \mathbb{E}[D(j)] \cdot a^{-1} \sum_{v\in\mathbb{Z}} e^{-(\delta-a)|v|},
\end{aligned}$$

The series converges to a finite value independent of $n$ as long as $\delta$ is chosen larger than $a$. Next we control $B$, which will also just use a brutal union bound and Markov's inequality:

$$\begin{aligned}\mathbb{P}(B(z^n,\alpha,\delta)>a) &\le \sum_{\substack{v\in\mathbb{Z}\\ r\in\mathbb{N}\\ 1\le k\le 2^r}} \mathbb{P}(2^{\alpha r}|z^n(v+k2^{-r})-z^n(v+(k-1)2^{-r})|>e^{\delta|v|}a)\\ &\le \sum_{\substack{v\in\mathbb{Z}\\ r\in\mathbb{N}\\ 1\le k\le 2^r}} a^{-p}2^{\alpha pr}e^{-\delta|v|p}\mathbb{E}\big|z^n(n+k2^{-r})-z^n(v+(k-1)2^{-r})\big|^p\\ &\le a^{-p}\sum_{\substack{v\in\mathbb{Z}\\ r\in\mathbb{N}\\ 1\le k\le 2^r}} 2^{(\alpha-\beta)pr}e^{(2a-\delta)p|v|}\\ &= a^{-p}\sum_{\substack{v\in\mathbb{Z}\\ r\in\mathbb{N}}} 2^{\big[1+(\alpha-\beta)p\big]r}e^{(2a-\delta)p|v|}\end{aligned}$$

The double series converges to a finite value independent of $n$ so long as $\delta,\alpha$ are chosen so as to satisfy $\delta>2a$ and $1+(\alpha-\beta)p<0$. This is permissible so long as $p>\beta^{-1}$. □

**Lemma 5.6.** *Let $(X_n)_{n\ge 0}$ be a non-negative $L^1$ supermartingale. Then*

$$\mathbb{P}\big(\sup_n X_n>a\big)\le \frac{\mathbb{E}[X_0]}{a}.$$

*Proof.* We apply Doob-Meyer decomposition to write $X=M-A$, where $M$ is a martingale with $M_0=X_0$, and $A_0$ is a non-decreasing process with $A_0=0$. Then $M$ is a positive martingale and $X\le M$. Doob's first martingale inequality then gives

$$\mathbb{P}\big(\sup_{n\le N}X_n>a\big)\le \mathbb{P}\big(\sup_{n\le N}M_n>a\big)\le\frac{\mathbb{E}[M_N]}{a}=\frac{\mathbb{E}[M_0]}{a}.$$

Since $M_0=X_0$, letting $N\to\infty$ gives the claim because the right side does not depend on $N$ and the left side approaches $\mathbb{P}\big(\sup_n X_n>a\big)$ by monotone convergence. □

**Proposition 5.7.** *For each $n\in\mathbb{N}$, let $\{\omega^n_{i,0}\}_{i\ge1}$ be a family of i.i.d. random variables such that $\omega^n_{i,0}$ has finite $p^{th}$ moment, with $p>2$. Also assume that $1+n^{-1/4}\omega^n_{i,0}>0$ a.s. and that $\sup_n\mathbb{E}[|\omega^n_{1,0}|^p]<\infty$. Furthermore assume that $\mathbb{E}[\omega^n_{i,0}]=\mu n^{-1/4}+o(n^{-1/4})$ and var$(\omega^n_{i,0})=\sigma^2+o(1)$ as $n\to\infty$. Define $z^n_0(x):=\prod_{i=1}^x(1+n^{-1/4}\omega^n_{i,0})$. Then $z^n_0$ satisfies the first two conditions of Proposition 5.5:*

- *$\mathbb{E}[|z^n_0(x)-z^n_0(y)|^p]\le Cn^{-p/4}|x-y|^{p/2}e^{an^{-1/2}(x+y)}$ for some constants $C,a$ independent of $n,x,y$.*
- *with the same $a$, there exist square-integrable random variables $D(n)$ such that $\sup_n\mathbb{E}[D(n)^2]<\infty$ and $z^n_0(x)\le D(n)e^{an^{-1/2}x}$ for all $n,x$ almost surely.*

*Proof.* Before proving either bullet point, we prove a preliminary bound. By Taylor expanding $u^p$ near $u=1$ we see $(1+n^{-1/4}\omega^n_{i,0})^p=1+pn^{-1/4}\omega^n_{i,0}+\frac12(p^2-p)n^{-1/2}(\omega^n_{i,0})^2+o(n^{-1/2})$, which has expectation roughly $1+n^{-1/2}(p\mu+\frac{p^2-p}{2}\sigma^2)+o(n^{-1/2})$. For some $a=a(p)$ this is bounded above by $1+an^{-1/2}$, and so we see that

$$\mathbb{E}[z^n_0(x)^p]=\prod_{i=1}^x\mathbb{E}[(1+n^{-1/4}\omega^n_{i,0})^p]\le(1+n^{-1/2}a)^x\le e^{an^{-1/2}x},\tag{5.8}$$

since $1+v \le e^v$. With this preliminary bound in mind, we proceed to the proof of the first bullet point. It suffices to prove the claim when $y=0$ (i.e., $z_0^n(y)=1$), by independence of the multiplicative increments of $z_0^n$. Let us begin by writing

$$\mathbb{E}[|z_0^n(x)-1|^p] \le 2^p\left(\mathbb{E}\left[\left|z_0^n(x)-\frac{z_0^n(x)}{\mathbb{E}[z_0^n(x)]}\right|^p\right]+\mathbb{E}\left[\left|\frac{z_0^n(x)}{\mathbb{E}[z_0^n(x)]}-1\right|^p\right]\right).$$

Let us denote these expectations on the right side as $E_1$ and $E_2$, respectively. We bound each of these separately. For $E_1$, one notes by using (5.8) that

$$\begin{aligned}E_1 = \mathbb{E}[z_0^n(x)^p]\left|1-\frac{1}{\mathbb{E}[z_0^n(x)]}\right|^p &\le e^{an^{-1/2}x}\left|1-e^{-an^{1/2}x}\right|^p\\ &\le e^{an^{-1/2}x}\left(an^{-1/2}x\right)^p = a^p e^{an^{-1/2}x}n^{-p/2}x^p,\end{aligned}$$

where we used $\mathbb{E}[z_0^n(x)] \le \mathbb{E}[z_0^n(x)^p]^{1/p} \le e^{an^{-1/2}x}$ (by (5.8)) in the first inequality, and we used $1-e^{-v}\le v$ in the second one. Finally, note that $u^pe^u \le Cu^{p/2}e^{2u}$ for some $C>0$ independent of $u$, and applying this with $u=an^{-1/2}x$ already gives the desired bound on $E_1$.

Now we bound $E_2$. This is the difficult part, and one needs to somehow exploit cancellations that occur at the quadratic scale (e.g., via a Burkholder-type inequality). To do this, first note that the process $M_x^n := \frac{z_0^n(x)}{\mathbb{E}[z_0^n(x)]}$ is a martingale in the $x$-variable (for fixed $n$). Define $\zeta_i^n := \frac{1+n^{-1/4}\omega_{i,0}^n}{\mathbb{E}[1+n^{-1/4}\omega_{i,0}]}$. Then Burkholder-Davis-Gundy says

$$E_2 \le C\mathbb{E}\left[\left(\sum_{i=1}^x (M_i^n-M_{i-1}^n)^2\right)^{p/2}\right] = C\mathbb{E}\left[\left(\sum_{i=1}^x (\zeta_1^n)^2\cdots(\zeta_{i-1}^n)^2(\zeta_i^n-1)^2\right)^{p/2}\right] \tag{5.9}$$

Now, using the given conditions, $|\zeta_i^n-1| \le C(n^{-1/4}|\omega_{i,0}^n|+n^{-1/2})$, so the square is bounded by $C(n^{-1/2}(\omega_{i,0}^n)^2+n^{-1})$. Writing $\|A\|_p := \mathbb{E}[|A|^p]^{1/p}$, we notice by the triangle inequality and independence of $\zeta_i^n$ that

$$\left\|\sum_{i=1}^x(\zeta_1^n)^2\cdots(\zeta_{i-1}^n)^2(\zeta_i^n-1)^2\right\|_{p/2} \le Cn^{-1/2}\sum_{i=1}^x\|(\zeta_1^n)^2\|_{p/2}\cdots\|(\zeta_{i-1}^n)^2\|_{p/2}\|(\omega_{i,0}^n)^2+n^{-1/2}\|_{p/2}.$$

Now, it holds that $\|(\zeta_1^n)^2\|_{p/2} \le e^{2an^{-1/2}/p}$, by (5.8) (with $x=1$). Hence each term of the sum can be bounded above by $e^{2an^{-1/2}x/p}$. The contribution of the $n^{-1/2}$ term next to $(\omega_{i,0}^n)^2$ is then seen to be negligible, so we disregard it. Hence the the entire sum may be bounded by $Cn^{-1/2}xe^{2an^{-1/2}x/p}$, which, combined with (5.9) and the fact that $\|(\omega_{i,0}^n)^2\|_{p/2}$ is bounded independently of $n$ by assumption, completes the proof.

Now we prove the second bullet point. Note that $\frac{z_0^n(x)^p}{\mathbb{E}[z_0^n(x)^p]}$ is a positive martingale in the $x$-variable. Let $D(n) := \sup_{x\ge 0} z_0^n(x)/\mathbb{E}[z_0^n(x)^p]^{1/p}$. Then it is clear from Lemma 5.6 that $\mathbb{P}(D(n)^p > a) \le a^{-1}$, so that $\mathbb{P}(D(n)>a) \le a^{-p}$. If $p>2$, then this easily implies that $\sup_n \mathbb{E}[D(n)^2]<\infty$. But (5.8) tells us that $\mathbb{E}[z_0^n(x)^p]^{1/p} \le Ce^{an^{-1/2}x}$, so we are done. ☐

Next, we finally prove the octant-quadrant reduction theorem, i.e., that we can replace $T_n$ with $2n$ as discussed in the proof sketch at the end of Section 2. Let us reformulate the main notational conventions here:

- $S$ is a simple symmetric random walk of length $n$ started from $x$ and conditioned to stay positive throughout its course (i.e., the canonical process associated to the measures $\mathbf{P}_x^n$). We assume $n-x$ is even.

- $\hat\omega^n_{i,j}$ is defined to be $\omega^n_{(n-\frac{i-j}{2}),(n-\frac{i+j}{2})}$ for all $i,j$ of the same parity, where $\omega^n_{i,j}$ is a family of random environments satisfying the conditions of the three bullet points before Theorem 2.2, but now the bulk random variables are indexed by all pairs $(i,j)$ with $|i|\geq j$.
- $T_n$ is the first time that $n-i=S_i$.
- $z_0^n(x):=\prod_{i=0}^x(1+n^{-1/4}\bar\omega^n_{i,0})$, where the $\bar\omega_{i,0}$ have $p>2$ moments.

We remark that all conditions of Theorem 5.4 are *almost* satisfied by this environment. The only caveat is that the sequence of initial data is not deterministic, however by Propositions 5.5 and 5.7 and Skorohod's Lemma (and the fact that $z_0^n$ are independent of the bulk weights) we may choose a probability space on which $z_0^n\to z_0$ almost surely with respect to the topology of $\mathscr{C}^\alpha_{e(\delta)}$ for some choice of $\alpha,\delta\in(0,1)$. Here $z_0(x)$ is a geometric Brownian motion with the appropriate diffusion and drift coefficients. Note that a.s. convergence is stronger than a.s. boundedness in that norm which is the condition required in Theorem 5.4. Thus there is no loss of generality in assuming that the initial data are in fact deterministic.

**Proposition 5.8** (Octant-Quadrant Reduction)**.** *In the notation of the bullet points immediately above, we define the following random variable for $n,x\geq 0$:*

$$\mathscr{E}(x,n):=\mathbf{E}^n_x\bigg[z_0^n(S_n)\prod_{i=0}^{n-1}(1+n^{-1/4}\hat\omega^n_{i,S_i})\bigg]-\mathbf{E}^n_x\bigg[z_0^n(S_{T_n})\prod_{i=0}^{T_n-1}(1+n^{-1/4}\hat\omega^n_{i,S_i})\bigg].$$

*Let $x_n$ be a sequence of non-negative integers such that $x_n\leq Cn^{1/2}$ for some $C>0$. Then $\mathscr{E}(x_n,n)\to 0$ in probability.*

*Proof.* First we will show that $\sum_n\mathbf{P}^n_{x_n}(T_n\leq n-n^{2/3})<\infty$. By Borel-Cantelli, this would imply that all $\mathbf{P}^n_{x_n}$ may be coupled to the same probability space in such a way that one almost surely has $T_n>n-n^{2/3}$ for large enough $n$. Then the result follows immediately from Theorem 5.4 by taking $\alpha=2/3$ in the definition of $\mathcal{E}(x,n)$. Note that the choice of exponent $2/3$ is arbitrary and could be replaced by any $\alpha>1/2$.

To prove that $\sum_n\mathbf{P}^n_{x_n}(T_n\leq n-n^{2/3})<\infty$, one first notes that the event $\{T_n\leq n-n^{2/3}\}$ can only happen if $\sup_{i\leq n}S_i\geq n^{2/3}$. But by Theorem A.7, we know that there are universal constants $C,c,c'>0$ so that

$$\mathbf{P}^n_{x_n}\big(\sup_{i\leq n}S_i\geq n^{2/3}\big)\leq Ce^{-c(n^{2/3}-x_n)^2/n}\leq Ce^{-c(n^{2/3}-Cn^{1/2})^2/n}\leq Ce^{-c'n^{1/3}}.$$

The right side is summable as a function of $n$, completing the proof. $\square$

Note that by equations (2.3) and (2.4) and the surrounding discussion (but replacing $n$ above by $2n$), the above proposition reduces the proof of Theorem 2.2 to that of Theorem 1.2 but with varying weights, so this is what we focus on now.

### 5.2 Convergence for the quadrant model

In this section we finally complete the main goals of the paper. Unless otherwise stated, we always implicitly assume the following:

- All families $\{\omega^n_{i,j}\}$ of i.i.d. weights satisfy the assumptions that were stated in the bullet points before Theorem 2.2.

With the reduction (Proposition 5.8) finished, we define a partition function in the quadrant that is modified to take parity into account. Specifically, given $(n,x)$ in the lattice $L := \{(n,x) \in \mathbb{Z}^2_{\geq 0} : n - x \equiv 0 \pmod 2\}$ we define

$$
\begin{aligned}
Z_k(n,x) &:= \mathbf{E}^n_x\Big[z^k_0(S_n)\prod_{i=1}^n (1 + k^{-1/4}\omega^k_{n-i,S_i})\Big] \\
&= \sum_{r=0}^n k^{-r/4} \sum_{\substack{1\leq i_1<\ldots<i_r\leq n \\ (x_1,\ldots,x_{r+1})\in\mathbb{Z}^{r+1}_{\geq 0}}} \prod_{j=1}^r \mathfrak{p}^{n-i_{j-1}}_{i_j - i_{j-1}}(x_{j-1},x_j)\omega^k_{n-i_j,x_j} \cdot \big(z^k_0(x_{r+1})\mathfrak{p}^{n-i_r}_{n-i_r}(x_r,x_{r+1})\big),
\end{aligned}
\tag{5.10}
$$

with $i_0 := 0, x_0 := x$, and $z^k_0(x) = \prod_{i=0}^x (1 + k^{-1/4}\omega^k_{i,0})$ (in fact $z^k_0$ can be any sequence of functions converging weakly and also satisfying the two bullet points of Proposition 5.7). Consider the following family of diffusively rescaled processes

$$\mathscr{Z}_n(T,X) := Z_n(nT, n^{1/2}X), \qquad T,X \geq 0, \tag{5.11}$$

where we interpolate linearly between points of the lattice $L$. We will now show that $\mathscr{Z}_n$ converge in law as $n \to \infty$ with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}_+ \times \mathbb{R}_+$ to the solution of (4.1). The first step for doing this is proving tightness in the appropriate Hölder space. This part is not necessary if one is only interested in following the minimal logical flow for the proof of Theorem 1.1, and thus some of the proofs are not included. As always we denote $\|X\|_p := \mathbb{E}[|X|^p]^{1/p}$.

**Proposition 5.9** (Tightness)**.** *Let $\mathscr{Z}_n$ be defined as in* (5.11)*, and assume that (for each $k$), the i.i.d. weights $\{\omega^k_{i,j}\}_{i,j}$ have $p > 8$ moments, bounded independently of $k$. Then for every $a \geq 0$, $\theta \in [0,1)$, and compact set $K \subset [0,\infty)^2$ there exists $C = C(a,p,\theta,K) > 0$ such that one has the following estimates uniformly over all pairs of space-time points $(T,X),(S,Y) \in K$:*

$$\|\mathscr{Z}_n(T,X)\|_p \leq C, \tag{5.12}$$

$$\|\mathscr{Z}_n(T,X) - \mathscr{Z}_n(T,Y)\|_p \leq C|X-Y|^{\theta/2}, \tag{5.13}$$

$$\|\mathscr{Z}_n(T,X) - \mathscr{Z}_n(S,X)\|_p \leq C|T-S|^{\theta/4}. \tag{5.14}$$

*In particular, the laws of the $\mathscr{Z}_n$ are tight with respect to the topology of uniform convergence on compact subsets of $C(\mathbb{R}_+ \times \mathbb{R}_+)$.*

The restriction $p > 8$ is only necessary to obtain tightness in the Hölder space. Using more elegant arguments, this may be extended to $p \geq 6$ (see Appendix B of [AKQ14a]). The one-point convergence result will only require two moments though.

*Proof.* Note that the functions $Z_k$ defined in (5.10) satisfy the following Duhamel-form relation

$$Z_k(n,x) = \sum_{y\geq 0} \mathfrak{p}^n_n(x,y)z^k_0(y) + k^{-1/4}\sum_{i=0}^{n-1}\sum_{y\geq 0}\mathfrak{p}^n_{n-i}(x,y)Z_k(i,y)\omega^k_{i,y}. \tag{5.15}$$

Define the martingale $M_r(x,n,k) := k^{-1/4}\sum_{i=0}^{r-1}\sum_{y\geq 0}\mathfrak{p}^n_{n-i}(x,y)Z_k(i,y)\omega^k_{i,y}$. This is a martingale in the $r$-variable, with respect to the filtration $\mathcal{F}^k_r := \sigma(\{\omega^k_{i,j}\}_{1\leq i\leq r; j\geq 0})$. This is because $Z_k(i,y)$ is $\mathcal{F}^k_r$-measurable, and $\mathcal{F}^k_r$ is independent of the mean-zero random variables $\omega^k_{r,y}$ with $y \geq 0$. Applying Burkholder-Davis-Gundy and then Minkowski's inequality to $M_r(x,n,k)$ shows that

$$
\begin{aligned}
\|M_r(x,n,k)\|^2_p &\leq C\Bigg\| k^{-1/2}\sum_{i=0}^{r-1}\Big[\sum_{y\geq 0}\mathfrak{p}^n_{n-i}(x,y)Z_k(i,y)\omega^k_{i,y}\Big]^2\Bigg\|_{p/2} \\
&\leq Ck^{-1/2}\sum_{i=0}^{r-1}\Bigg\|\sum_{y\geq 0}\mathfrak{p}^n_{n-i}(x,y)Z_k(i,y)\omega^k_{i,y}\Bigg\|^2_p.
\end{aligned}
\tag{5.16}
$$

Next, we notice that since the $\omega^k_{i,y}$ are independent of $Z_k(i,y)$, another application of Burkholder-Davis-Gundy (or in this case, its more elementary version for independent sums, the Marcinkiewicz-Zygmund inequality) shows that

$$\Big\| \sum_{y\geq 0} \mathfrak{p}^n_{n-i}(x,y) Z_k(i,y)\omega^k_{i,y}\Big\|_p^2 \leq C\sum_{y\geq 0}\mathfrak{p}^n_{n-i}(x,y)^2 \|Z_k(i,y)\|_p^2 \|\omega^k_{i,y}\|_p^2. \tag{5.17}$$

Since $p\leq p_0$ and the $p_0^{th}$ moments of $\omega^k_{i,y}$ are bounded independently of $k,i,y$ it follows that $\|\omega^k_{i,y}\|_p^2$ may be absorbed into the constant. Combining (5.15),(5.16),(5.17), one finds that

$$\|Z_k(n,x)\|_p^2 \leq C\bigg(\sum_{y\geq 0}\mathfrak{p}^n_n(x,y)\|z^k_0(y)\|_p\bigg)^2 + Ck^{-1/2}\sum_{i=0}^{n-1}\sum_{y\geq 0}\mathfrak{p}^n_i(x,y)^2\|Z_k(n-i,y)\|_p^2. \tag{5.18}$$

Now, we note that $\|z^k_0(y)\|_p \leq e^{ak^{-1/2}y}$ by (5.8). Hence, $\sum_y \mathfrak{p}^n_n(x,y)\|z^k_0(y)\|_p$ may be bounded above by $Ce^{ak^{-1/2}x+Ka^2k^{-1}n}$ by Proposition B.1. After this, we set $x_0:=x$ and $i_0:=0$ and we iterate (5.18). Then we get

$$\begin{aligned}\|Z_k(n,x)\|_p^2 &\leq C\sum_{r=0}^n k^{-r/2}\sum_{\substack{0\leq i_1<\ldots<i_r<n\\ (x_1,\ldots,x_r)\in\mathbb{Z}^r_{\geq 0}}}\prod_{j=1}^r \mathfrak{p}^{n-i_j}_{n-i_{j-1}}(x_{i-1},x_i)^2\cdot e^{ak^{-1/2}x_r+Ka^2n/k}\\ &\overset{\text{Lemma 5.1}}{\leq} Ce^{ak^{-1/2}x+Ka^2n/k}\sum_{r=0}^n C^k k^{-r/2}n^{r/2}/(r/2)!\\ &\leq Ce^{ak^{-1/2}x+Bn/k},\end{aligned} \tag{5.19}$$

where we use $\sum_r C^k k^{-r/2}n^{r/2}/(r/2)!\leq e^{C^2n/k}$ and then rename $B:=Ka^2+C^2$. Now replace $x$ by $n^{1/2}X$, $n$ by $nT$, and $k$ by $n$. This will give $\|\mathscr{Z}_n(T,X)\|_p^2\leq Ce^{aX+BT}$. But $e^{aX+BT}$ can be bounded from above on any compact set, proving (5.12).

The proofs of (5.13) and (5.14) use similar ideas (e.g., Burkholder-Davis-Gundy, convexity inequalities like Minkowski and Jensen, and the recursive relations satisfied by $Z_k$) and will be left out. Now we need to argue tightness from these estimates. But this is a direct corollary of the Kolmogorov continuity criterion (two-parameter version), Prokhorov's theorem, and the Arzela-Ascoli Theorem. Note that the condition $p>8$ is needed to obtain a positive exponent in Kolmogorov's criterion. □

Now that we have proved tightness, we only need to obtain convergence of finite-dimensional marginals of $\mathscr{Z}_n$ to those of SPDE (4.1). Thanks to the Cramer-Wold device (and linearity of integration with respect to space-time white noise) this will not be more difficult than just proving convergence of *one-point* marginals. This can be done by using the convergence result in Proposition 3.6 together with the machinery developed in the papers [AKQ14a, CSZ17a].

Specifically, we will use Theorem 2.3 of [CSZ17a], which in turn was inspired by the results of Section 4 in [AKQ14a]. We state this result in a version that is adapted to our own context. Throughout, we will fix $T>0$ and we will denote $\Delta_k(T):=\{(t_1,...,t_k):0<t_1<...<t_k<T, t_i\in\mathbb{R}\}$. Also denote by $\Delta_k^n(T):=\{(\frac{t_1}{n},...,\frac{t_k}{n}):0<t_1<...<t_k<Tn, t_i\in\mathbb{Z}\}$, and let $(\mathbb{R}^d)_n:=(n^{-1/2}\mathbb{Z})^d$. Then define

$$\mathcal{L}^n_k:=\Delta^n_k(T)\times(\mathbb{R}^k)_n,$$

and equip $\mathcal{L}^n_k$ with the $\sigma$-finite measure that assigns mass $n^{-3/2}=n^{-1}\cdot n^{-1/2}$ to each distinct space-time point $(\frac{t}{n},\frac{x}{\sqrt{n}})$. We denote by $L^2(\mathcal{L}^n_k)$ the $L^2$-space associated to this measure.

**Theorem 5.10** (Theorem 2.3 of [CSZ17a]). *For each $n \in \mathbb{N}$, let $\{\omega^n_{i,j}\}_{i,j\geq 0}$ be a family of random weights with mean zero and $var(\omega^n_{i,j}) = \sigma^2 + o(1)$ (as $n \to \infty$). Let $\{F^n_k\}_{n,k\in\mathbb{N}}$ be a family of functions defined on $\mathcal{L}^n_k$. Suppose that $F_k : \Delta_k(T) \times \mathbb{R}^k \to \mathbb{R}$ is a family of continuous functions such that $\|F^n_k - F_k\|_{L^2(\mathcal{L}^n_k)} \to 0$ as $n \to \infty$, for every $k \in \mathbb{N}$. Furthermore assume that*

$$\sup_n \sum_{k\geq 0} \|F^n_k\|^2_{L^2(\mathcal{L}^n_k)} < \infty.$$

*Then define random variables*

$$X_n := \sum_{k\geq 0} n^{-3k/4} \sum_{(\vec{t},\vec{x})\in\mathcal{L}^n_k} F^n_k(\vec{t},\vec{x})\omega_{(nt_1),(n^{1/2}x_1)} \cdots \omega_{(nt_k),(n^{1/2}x_k)}.$$

*Then $X_n$ converges in distribution as $n \to \infty$ to the random variable*

$$\sum_{k=0}^{\infty} \sigma^k \int_{\Delta_k(T)} \int_{\mathbb{R}^k_+} F_k(t_1,...,t_k;x_1,...,x_k)\xi(dx_1dt_1)\cdots\xi(dx_kdt_k),$$

*where $\xi$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$.*

We refer the reader to Section 4 of [AKQ14a] for an explanation of the scaling exponent $n^{-3k/4}$. With this result in place, we are now ready to prove the main result of this section, which is a generalization of Theorem 1.2 to the case where the weights $\omega$ vary with $n$.

**Theorem 5.11.** *Let $\mathscr{Z}_n$ be as defined in* (5.11)*. Then the finite-dimensional marginals of $\mathscr{Z}_n$ converge to those of SPDE* (4.1)*. More precisely, if $F \subset \mathbb{R}_+ \times \mathbb{R}_+$ is finite, then $(Z_n(T,X))_{(T,X)\in F}$ converges in law to $(\mathscr{Z}(T,X))_{(T,X)\in F}$ where $\mathscr{Z}$ solves* (4.1) *with initial data $\mathscr{Z}(0,X) = e^{\sigma B_X + (\mu-\sigma^2/2)X}$ for a standard Brownian motion $B$.*

*Proof.* Using the discussion at the end of Section 2 (more specifically, equations (2.6) and (2.7)), we know that $z^n_0(n^{1/2}X)$ converges in law to a geometric Brownian motion with drift, specifically $e^{\sigma B_X+(\mu-\sigma^2/2)X}$. We exploit Skorohod's lemma to couple all of the $z^n_0$ to the same probability space in such a way so that this convergence occurs a.s. uniformly on compact sets.

Fix $x, t > 0$. In our case, we set

$$F^n_k(t_1,...,t_k;x_1,...,x_k) := \sum_{x_{k+1}\in n^{-1/2}\mathbb{Z}_{\geq 0}} z^n_0(n^{1/2}x_{k+1}) \prod_{j=1}^{k+1} \mathscr{P}_n(t_j - t_{j-1}, T - t_{j-1}; x_{j-1}, x_j),$$

$$F_k(t_1,...,t_k;x_1,...,x_k) := \int_{\mathbb{R}_+} e^{B_{x_{k+1}}-(A+1/2)x_{k+1}} \prod_{j=1}^{k+1} \mathscr{P}^{T-t_{j-1}}_{t_j-t_{j-1}}(x_{j-1},x_j)\, dx_{k+1},$$

where $\mathscr{P}^T_t$ was given in Definition 3.4, $\mathscr{P}_n$ was defined in Proposition 3.6 and where $(x_0,t_0) := (x,t)$. The condition that

$$\sup_n \sum_{k\geq 0} \|F^n_k\|^2_{L^2(\mathcal{L}^n_k)} < \infty,$$

follows quite simply from Lemma 5.1. Also the condition that $\|F^n_k - F_k\|_{L^2(\mathcal{L}^n_k)} \to 0$ as $n \to \infty$, follows by inducting on the last statement in Proposition 3.6.

By Theorem 5.10, we conclude that the one-point marginals of $\mathscr{Z}_n$ converge to those of the solution of (4.1). The proof for multi-point marginals is similar, but one defines a new family $\tilde{F}^n_k$ by taking linear combinations of the $F^n_k$ that are defined above, then one applies the Cramer-Wold device to make the conclusion. ☐

Note that (via Proposition 5.8) this result also implies Theorem 2.2, thus completing the main goals of the paper. One thing that we have not yet explained the normalization $\big(2\Phi\big(\frac{X+n^{-1/2}}{\sqrt{T}}\big)-1\big)^{-1}$ appearing in Theorem 2.2. This is an easy consequence of the fact that (by the local central limit theorem and (A.2)), the asymptotic mass of the measures $\mu^{nT}_{n^{1/2}X}$ appearing in Theorem 2.5 is equal to $2\Phi\big(\frac{X+n^{-1/2}}{\sqrt{T}}\big)-1+o(n^{-1/2})$.

## A Preliminary estimates and concentration of measure

The purpose of this appendix is to gather estimates for the simple symmetric random walk conditioned to stay positive. The results and proofs are classical in spirit, and the literature on such measures is extensive [Ig74, Bol76, Car05, CC08, DIM77] etc. However, we will only give a brief exposition of those selected estimates that apply to our nearest-neighbor weights, many of which we could not find in the above references, and might be applicable to other models.

We recall the uniform positive random walk measures $\mathbf{P}^n_x$ and the three associated quantities ($\mathfrak{p}^N_n, p^{(1/2)}_n$, and $\psi$) that were defined in Section 3. The main goal of this appendix will be to prove the following concentration inequality for the measures $\mathbf{P}^n_x$:

$$\mathbf{P}^n_x\big(\sup_{1\le j\le k}|S_j-x|>u\big)\le Ce^{-cu^2/k},$$

where $C,c$ are independent of $n,x,k$ with $k\le n$. This will in turn allow us to prove various $L^p$ moment bounds that are used in Section 5. The methods used in proving these results will be coupling arguments and martingale techniques, some of which might be useful in and of themselves. More specifically, the main key will be to notice that for fixed $n\in\mathbb{N}$, the process

$$M^n_k:=\frac{S_k+1}{\psi(S_k,n-k)},\qquad\qquad 0\le k\le n,$$

is a $\mathbf{P}^n_x$-martingale with respect to the $k$-variable. Moreover we will use the fact that $(S_k)$ is itself a submartingale. First we state a few preliminary lemmas.

**Lemma A.1.** *Let $\psi(x,N)$ be as in Definition 3.2. Then there exists a constant $C>0$ such that for all $x,N\ge 0$ one has*

$$\frac{x+1}{x+1+C\sqrt{N}}\le\psi(x,N)\le 1\wedge\left(\frac{C(x+1)}{\sqrt{N}}\right).$$

*Furthermore for each $x\ge 0$ one has that*

$$\lim_{N\to\infty}\sqrt{N}\psi(x,N)=(x+1)\sqrt{2/\pi}.$$

Note that this already proves Theorem 2.5*(2)*. Furthermore note that the upper and lower bounds on $\psi$ are strong enough to give an upper and lower envelope on $\psi$, i.e.,

$$C^{-1}\frac{x+1}{x+1+\sqrt{N}}\le\psi(x,N)\le C\frac{x+1}{x+1+\sqrt{N}}.\tag{A.1}$$

This is because $1\wedge w\le\frac{2w}{1+w}$. We now proceed to the proof.

*Proof.* First we prove the upper bound. Let $p_N$ denote the standard heat kernel on the whole line $\mathbb{Z}$. Using Definitions 3.1 and 3.2 and the fact that $p_N$ is symmetric and sums to 1, it holds that

$$\psi(x,N)=\sum_{y\ge 0}\big(p_N(x-y)-p_N(x+y+2)\big)=p_N(x+1)+p_N(0)+2\sum_{1\le u\le x}p_N(u).\tag{A.2}$$

Now, we use the simple uniform bound $p_N \le CN^{-1/2}$ to see that the right side of the last expression is bounded above by $2C(x+1)N^{-1/2}$. On the other hand, it is obvious that $\psi(x,N) \le 1$ for all $x, N$. So, we have obtained the desired upper bound.

Next, we prove the lower bound. We consider two different cases: $x \le 2\sqrt{N}$ and $x > 2\sqrt{N}$.

First we consider the case $x > 2\sqrt{N}$. One may apply Hoeffding's inequality for the simple random walk to deduce that

$$\psi(x,N) = p_N(x+1) + p_N(0) + 2\sum_{1\le u\le x} p_N(u) \ge \sum_{-x\le u\le x} p_N(u) \ge 1 - 2e^{-(x+1)^2/2N}.$$

Now set $q := \frac{(x+1)^2}{2N}$. Then $q \ge 2$, so $q+2 \le e^q$, and thus $\frac{1}{1-2e^{-q}} \le 1 + \frac{2}{q}$. This means that $\psi(x,N)^{-1} \le 1 + \frac{N}{2(x+1)^2}$. But since $x+1 \ge \sqrt{N}$, it follows that $\frac{N}{(x+1)^2} \le \frac{\sqrt{N}}{x+1}$. Hence we obtain $\psi(x,N) \ge \frac{x+1}{x+1+0.5\sqrt{N}}$, whenever $x > 2\sqrt{N}$.

Now we consider the case $x \le 2\sqrt{N}$. The local central limit theorem tells us that $p_N(u) \ge \frac{c}{\sqrt{N}} e^{-2u^2/N} \ge \frac{c}{\sqrt{N}} e^{-8}$, for some $c > 0$ and all $u, N$ with $u \le 2\sqrt{N}$. Hence

$$\psi(x,N) = p_N(x+1) + p_N(0) + 2\sum_{1\le u\le x} p_N(u) \ge \sum_{0\le u\le x} p_N(u) \ge \frac{ce^{-8}}{\sqrt{N}}(x+1).$$

Now one simply notes that $\frac{ce^{-8}}{\sqrt{N}} \ge \frac{1}{x+1+c^{-1}e^8\sqrt{N}}$. This proves the lower bound.

Finally, we prove the last statement about the limit. For this, let us write

$$\psi(x,N) = p_N(x+1) + p_N(0) + 2\sum_{1\le u\le x} p_N(u)$$

The local limit theorem tells us that for each $u$, the quantity $\sqrt{N}p_N(u)$ oscillates back and forth between $\sqrt{2/\pi}$ and zero (depending on the parity of $N$) as $N$ becomes large. This already implies that $N^{1/2}$ times the right side converges to $\big(1+x\big)\sqrt{2/\pi}$. □

**Lemma A.2** (Monotonicity)**.** *Fix $n \in \mathbb{N}$. Then $\psi(x,n)$ is an increasing function of $x$. Thus, $\mathfrak{p}_1^n(x,x+1) \ge 1/2 \ge \mathfrak{p}_1^n(x,x-1)$ for all $x, n \ge 0$. Furthermore $\mathfrak{p}_1^n(x,x+1)$ is a decreasing function of $x$, and $\mathfrak{p}_1^n(x,x-1)$ is an increasing function of $x$.*

The proof is straightforward.

**Proposition A.3** (Coupling lemma for Positive Walks)**.** *Fix $n \in \mathbb{N}$ and $x \ge 0$. There exists a coupling $\mathbf{Q}^n_{x,x+1}$ of the measures $\mathbf{P}^n_x$ and $\mathbf{P}^n_{x+1}$ that is supported on pairs $(\gamma,\gamma')$ of paths such that $|\gamma'_i - \gamma_i| \le 1$ for all $i \le n$. More generally, for fixed $n \in \mathbb{N}$, the measures $\{\mathbf{P}^n_x\}_{x\ge 0}$ may all be coupled together in such a way that the coordinate processes associated to neighboring values of $x$ are never more than distance $1$ apart.*

*Proof.* Let $\{U_i\}_{i=1}^n$ be a sequence of i.i.d. Uniform$[0,1]$ random variables. We make an inductive construction as follows. Let $S_0 = x$ and $S'_0 = x+1$.

Suppose that $S_0, ..., S_k$ and $S'_0, ..., S'_k$ have been constructed in such a way that $|S_i - S'_i| = 1$ for all $k$. If $S'_k = S_k + 1$, we define

$$(S_{k+1}, S'_{k+1}) := \begin{cases} (S_k - 1, S'_k - 1), & U_{k+1} > \mathfrak{p}_1^{n-k}(S_k, S_k+1) \\ (S_k + 1, S'_k - 1), & \mathfrak{p}_1^{n-k}(S_k, S_k+1) > U_{k+1} > \mathfrak{p}_1^{n-k}(S'_k, S'_k+1) \\ (S_k + 1, S'_k + 1), & U_{k+1} < \mathfrak{p}_1^{n-k}(S'_k, S'_k+1) \end{cases}.$$

We know by lemma A.2 that one of these cases must hold. Similarly, if $S'_k = S_k - 1$, then we define $(S_{k+1}, S'_{k+1})$ in a symmetric fashion. This completes the inductive step.

A close look at this construction reveals that for $x_1, ..., x_n \geq 0$ one has

$$P(S_1 = x_1, S_2 = x_2, ..., S_n = x_n) = \mathfrak{p}_1^n(x, x_1) \prod_{j=1}^{n-1} \mathfrak{p}_1^{n-j}(x_j, x_{j+1}),$$

$$P(S_1' = x_1, S_2' = x_2, ..., S_n' = x_n) = \mathfrak{p}_1^n(x+1, x_1) \prod_{j=1}^{n-1} \mathfrak{p}_1^{n-j}(x_j, x_{j+1}).$$

By Proposition 3.3, $S$ is distributed as $\mathbf{P}_x^n$ and $S'$ is distributed as $\mathbf{P}_{x+1}^n$.

The proof of the more general statement is very similar. One simply uses a uniform coupling together with the Lemma A.2, and the argument is a straightforward generalization of the one given above for two values of $x$. ☐

**Proposition A.4** (Martingales for Positive Walks)**.** *Fix $x, n, k \geq 0$ with $k \leq n$. Let $S$ be distributed according to $\mathbf{P}_x^n$. For $i \leq k$ define a function $f^{(k,n)}(x, i) := \mathbf{E}_x^{n-i}[S_{k-i}]$. Then the process*

$$M_i = M_i^{(k,n)} := f^{(k,n)}(S_i, i), \qquad 0 \leq i \leq k$$

*is a martingale with respect to the natural filtration of $S$. Furthermore it has bounded increments*

$$|M_{i+1} - M_i| \leq 2, \qquad 0 \leq i \leq k-1.$$

*In the special case when $k = n$, one has the explicit form $f^{(n,n)}(x, i) = -1 + \frac{x+1}{\psi(x, n-i)}$*

*Proof.* We suppress the superscript $(k, n)$ on $f$ from now on. Letting $\mathcal{F}_k$ denote the natural filtration of $S$, it is a consequence of the Markov property that $f(S_i, i) = \mathbf{E}_x^n[S_k|\mathcal{F}_i]$, which shows that $M$ is a martingale in the $i$-variable for fixed $x, n, k$.

To prove that it has bounded increments, first note that

$$f(x, i) - f(x+1, i) = \mathbf{E}_x^{n-i}[S_{k-i}] - \mathbf{E}_{x+1}^{n-i}[S_{k-i}].$$

By the coupling lemma (Proposition A.3), this is bounded in absolute value by 1. Consequently, one finds that

$$|f(x \pm 1, k+1) - f(x, k)| = \Big| f(x \pm 1, k+1) - \sum_{y \in \{x-1, x+1\}} \mathfrak{p}_1^{n-k}(x, y) f(y, k+1) \Big|$$

$$\leq \sum_{y \in \{x-1, x+1\}} \mathfrak{p}_1^{n-k}(x, y) \big| f(x \pm 1, k+1) - f(y, k+1) \big| = \mathfrak{p}_1^{n-k}(x, x \mp 1) \cdot 2 \leq 2,$$

which gives the desired result.

For the final statement, if $k = n$ one may compute $\mathbf{E}_x^{n-i}[S_{n-i}] = \sum_{y \geq 0} y \mathfrak{p}_{n-i}^{n-i}(x, y) = \psi(x, n-i)^{-1} \sum_{y \geq 0} y p_{n-i}^{(1/2)}(x, y)$. Now the claim follows from the fact that $y \mapsto y+1$ is a unipotent eigenfunction of the semigroup $p^{(1/2)}$, i.e., $\sum_{y \geq 0} (y+1) p_n^{(1/2)}(x, y) = x+1$ for every $n, x \geq 0$. ☐

**Lemma A.5.** *Let $b \geq 0$. There exists a constant $C = C(b) > 0$ such that for all $n \geq 0$ and all $x, y, z \geq 0$ one has*

$$p_n^{(1/2)}(x, y) \leq C \left[ \frac{1}{\sqrt{n+1}} \wedge \frac{x+1}{n+1} \right] e^{-bn^{-1/2}|x-y|}.$$

$$\big| p_n^{(1/2)}(x, y) - p_n^{(1/2)}(x, z) \big| \leq C \left[ \frac{1}{n+1} \wedge \frac{x+1}{(n+1)^{3/2}} \right] |z-y| e^{-bn^{-1/2}\left(|x-y| \wedge |x-z|\right)}.$$

The proof will be omitted, because it is fairly standard and follows the same train of estimates given in works such as [DT16]. The main point is to note that the standard heat kernel on $\mathbb{Z}$ satisfies $\sum_{x\in\mathbb{Z}} p_n(x)z^x = (z+z^{-1})^n 2^{-n}$, so we can use Cauchy's integral formula to write it as follows:

$$p_n(x) = \frac{1}{2\pi i}\oint_C z^{-x-1}2^{-n}(z+z^{-1})^n dz.$$

Then we choose the contour $C$ cleverly (specifically a circle of radius $e^{bn^{-1/2}}$ centered at the origin) and finally use the fact that $p_n^{(1/2)}$ can be written in terms of $p_n$ via Definition 3.1. See Appendix A of [DT16] for details on obtaining bounds in this way.

**Lemma A.6.** *There exists a constant $C>0$ such that for all $x\geq 0$ and all $n\geq k\geq 1$ one has that*

$$\mathbf{E}_x^n[S_k] \leq x + Ck^{1/2}.$$

*Proof.* We consider two cases, $k>n/2$ and $k\leq n/2$.

*Case 1.* $k>n/2$. First, we claim that $\mathbf{E}_x^n[S_k]\leq \mathbf{E}_x^n[S_n]$. In fact, it is even true that $S$ forms a $\mathbf{P}_x^n$-submartingale and thus $\mathbf{E}_x^n[S_k]$ is an increasing function of $k$ for every $n$. This follows immediately from Lemma A.2 after noticing that $\mathbf{E}_x^n[S_{k+1}|\mathcal{F}_k] = S_k + (2\mathfrak{p}_1^{n-k}(S_k,S_k+1)-1)\geq S_k$. Now, from the preceding proposition, we know that $M_k := \frac{S_k+1}{\psi(S_k,n-k)}$ forms a martingale. Thus, we see that

$$\mathbf{E}_0^n[S_n+1] = \mathbf{E}_x^n[M_0] = \frac{x+1}{\psi(x,n)} \leq x+1+Cn^{1/2},$$

where we applied the lower bound of Lemma A.1 in the final bound. Since $k>n/2$, we see that $n^{1/2}\leq 2^{1/2}k^{1/2}$, which gives the desired bound in this case.

*Case 2.* $k\leq n/2$. First we use the coupling lemma (Proposition A.3) to see that $\mathbf{E}_x^n[S_k]\leq 1+\mathbf{E}_{x-1}^n[S_k]$. Iterating this $x$ times shows that

$$\mathbf{E}_x^n[S_k]\leq x+\mathbf{E}_0^n[S_k].$$

Thus we only need to show that $\mathbf{E}_0^n[S_k]\leq Ck^{1/2}$. To prove this, let us write $\mathbf{E}_0^n[S_k] = \sum_{y\geq 0}\mathfrak{p}_k^n(0,y)y$. Now we write $\mathfrak{p}_k^n(0,y) = p_k^{(1/2)}(0,y)\frac{\psi(y,n-k)}{\psi(0,n)}$. By Lemma A.1 we know $\frac{1}{\psi(0,n)}\leq C\sqrt{n}$. Furthermore we also know from the same lemma that $\psi(y,n-k)$ is bounded above by $1\wedge(Cy(n-k)^{-1/2})$, which is in turn bounded above by $1\wedge(Cyn^{-1/2})$ since $k\leq n/2$. Moreover, we also know from Lemma A.5 that $p_k^{(1/2)}(0,y)\leq \frac{C}{k+1}e^{-y/\sqrt{k}}$. Thus, we find that

$$\mathbf{E}_0^n[S_k]\leq \frac{C}{k+1}\bigg[\sum_{0\leq y\leq\sqrt{n}} e^{-y/\sqrt{k}}n^{1/2}(n^{-1/2}y^2) + \sum_{y\geq\sqrt{n}} e^{-y/\sqrt{k}}(n^{1/2}y)\bigg]. \tag{A.3}$$

Let us refer to the two sums inside the square brackets on the right side as $J_1$ and $J_2$, respectively.

First we bound $J_1$. Now, we use the bound $\sum_{r\geq 0} r^2\alpha^r\leq\frac{2}{(1-\alpha)^3}$ (valid for $\alpha<1$) and we see that

$$J_1\leq C\sum_{y\geq 0}y^2 e^{-y/\sqrt{k}}\leq\frac{C}{(1-e^{-1/\sqrt{k}})^3}\leq Ck^{3/2}.$$

In the last bound, we used the elementary bound $(1-e^{-q})^{-1}\leq 1+q^{-1}$ (which in turn implies $(1-e^{-q})^{-3}\leq 2^3(1+q^{-3})$) with $q=k^{-1/2}$.

Next, we bound $J_2$. Using the bound $\sum_{r\geq s} r\alpha^r\leq C\big[\frac{\alpha^s}{(1-\alpha)^2}+\frac{s\alpha^s}{1-\alpha}\big]$, we see that

$$J_2 = n^{1/2}\sum_{y\geq\sqrt{n}} e^{-y/\sqrt{k}}y\leq n^{1/2}\bigg[\frac{e^{-\sqrt{n/k}}}{(1-e^{-1/\sqrt{k}})^2}+\frac{n^{1/2}e^{-\sqrt{n/k}}}{1-e^{-1/\sqrt{k}}}\bigg].$$

Now

$$n^{1/2}e^{-\sqrt{n/k}} = k^{1/2}(n/k)^{1/2}e^{-\sqrt{n/k}} \le k^{1/2}\sup_{u>0} ue^{-u} = Ck^{1/2}.$$

Similarly, one finds that $ne^{-\sqrt{n/k}} \le Ck$. We also note that $(1-e^{-q})^{-1} \le 1+q^{-1}$, and thus $(1-e^{-q})^{-2} \le 2+2q^{-2}$. Taking $q = k^{-1/2}$ and then combining the last few expressions, one finally gets $J_2 \le Ck^{3/2}$.

Combining the bounds of $J_1$ and $J_2$ with (A.3), we obtain the desired bound. □

Finally we have our concentration theorem, the main result of this appendix.

**Theorem A.7** (Concentration). *As before, let $S = (S_k)_{0\le k\le n}$ denote the canonical process associated to $\mathbf{P}^n_x$. Then there exist $C, c > 0$ such that for every $x \ge 0$, every $0 \le k \le n$, and every $u > 0$ one has that*

$$\mathbf{P}^n_x\big(\sup_{0\le i\le k} |S_i - x| > u\big) \le Ce^{-cu^2/k}.$$

In other words, on time scales of length $k$, the path measure $\mathbf{P}^n_x$ concentrates on spatial scales of order $\sqrt{k}$ around $x$. The idea of the proof is to exploit the martingales from Proposition A.4 and apply well-known concentration inequalities for bounded-increment martingales. The Gaussian decay constant $c$ will be obtained as $1/32$, which is not sharp (presumably $c = 1/2$ should be possible, but we do not have a proof).

*Proof.* Throughout this proof, $x, n$, and $k$ will be **fixed**. Let us write

$$\mathbf{P}^n_x\big(\sup_{0\le i\le k} |S_i - x| > u\big) = \mathbf{P}^n_x\big(\sup_{0\le i\le k} S_i > x+u\big) + \mathbf{P}^n_x\big(\inf_{0\le i\le k} S_i < x-u\big).$$

Let us refer to the terms on the right side as $p_1, p_2$ respectively.

First we bound $p_2$. Recall from Lemma A.2 that $\mathfrak{p}^n_1(x, x+1) \ge 1/2 \ge \mathfrak{p}^n_1(x, x-1)$ for all $n, x \ge 0$. This trivially shows that $S$ is a submartingale, which directly gives the claim for $p_2$ by Azuma's inequality [Azu67] for submartingales, with $c = 1/2$.

Now we will bound $p_1$, which is more difficult. Letting $M = (M^{(n,k)}_i)^k_{i=0}$ denote the martingale from Proposition A.4, it is clear that $S_k = M_k$. Furthermore $M_0 = f(x, 0) = \mathbf{E}^n_x[S_k] \le Ck^{1/2} + x$ by Proposition A.6. Since the increments of $M$ are bounded above by 2, we may apply Azuma's inequality again to see that

$$\begin{aligned}\mathbf{P}^n_x(S_k > x+u) = \mathbf{P}^n_x\big(M_k > x+u\big) &\le \mathbf{P}^n_x\big(M_k - M_0 > u - Ck^{1/2}\big)\\ &\le e^{-(u-Ck^{1/2})^2/8k} \le Ce^{-u^2/16k}.\end{aligned}$$

In the last inequality, we used the fact that $(u-Ck^{1/2})^2 \ge \frac{1}{2}u^2 - C^2k$. This, in turn, is because $(a+b)^2 \le 2(a^2+b^2)$. Combining the bounds on $p_1$ and $p_2$ shows that

$$\mathbf{P}^n_x(|S_k - x| > u) \le Ce^{-u^2/16k}. \tag{A.4}$$

Since $(S_i)$ is a submartingale (Lemma A.2) and since $x \mapsto e^{\lambda x}$ is increasing and convex it follows that the process $(e^{\lambda S_i})^n_{i=0}$ is a $\mathbf{P}^n_x$-submartingale as well. Thus, we may apply Doob's martingale inequality to see that

$$p_1 \le Ce^{-\lambda(x+u)}\mathbf{E}^n_x[e^{\lambda S_k}] = Ce^{-\lambda(x+u)}\bigg(1 + \int_0^\infty \lambda e^{\lambda y}\mathbf{P}^n_x(S_k > y)du\bigg).$$

Now we split the integral as $\int_0^x$ plus $\int_x^\infty$. We use the crude bound $\mathbf{P}^n_x(S_k > y) \le 1$ for the integral over $[0, x]$, and we use the bound (A.4) for the other. This gives

$$p_1 \le Ce^{-\lambda u} + Ce^{-\lambda u}\int_x^\infty \lambda e^{\lambda(y-x)-(y-x)^2/16k}dy \le C(e^{-\lambda u} + \lambda k^{1/2}e^{4\lambda^2 k - \lambda u}).$$

Setting $\lambda = \frac{u}{8k}$ gives a bound of $C(e^{-u^2/8k} + uk^{-1/2}e^{-u^2/16k})$. Now one simply notes that $r \le Ce^{r^2/32}$, so that $uk^{-1/2} \le Ce^{u^2/32k}$. This gives the desired bound on $p_1$, where the constant appearing in the theorem statement is $c := 1/32$. $\square$

We now give a slightly generalized version of the concentration theorem.

**Corollary A.8.** *In the same setting as the previous theorem, there exist $C, c > 0$ such that for every $x \ge 0$, every $0 \le m \le k \le n$, and every $u > 0$ one has that*

$$\mathbf{P}_x^n\big( \sup_{m \le i \le k} |S_i - S_m| > u\big) \le Ce^{-cu^2/(k-m)}.$$

*Here, $C, c$ are the same as in the previous theorem.*

*Proof.* Define

$$g(k,n,x,u) := \mathbf{P}_x^n\big( \sup_{0 \le i \le k} |S_i - x| > u\big).$$

By the Markov property (conditioning on the first $m$ steps), we have that

$$\mathbf{P}_x^n\big( \sup_{m \le i \le k} |S_i - S_m| > u\big) = \mathbf{E}_x^n\Big[ g(k-m, n-m, S_m, u)\Big].$$

But Theorem A.7 tells us that $g(k,n,x,u) \le Ce^{-cu^2/k}$ independently of $x, n$. $\square$

**Corollary A.9.** *Let $p > 0$. There exists a constant $C = C_p > 0$ such that for every $x \ge 0$ and every $0 \le k \le m \le n$, one has*

$$\mathbf{E}_x^n\big[|S_k - S_m|^p\big] \le C|k-m|^{p/2}.$$

*Proof.* Let us write

$$\mathbf{E}_x^n\big[|S_k - S_m|^p\big] = \int_0^\infty pu^{p-1}\mathbf{P}_x^n(|S_k - S_m| > u)du.$$

By Corollary A.8, this is bounded above by

$$C\int_0^\infty pu^{p-1}e^{-cu^2/(k-m)}du = Cp(k-m)^{p/2}\int_0^\infty v^{p-1}e^{-cv^2}dv = C_p(k-m)^{p/2},$$

where we made a substitution $y = (k-m)^{-1/2}u$ in the first equality. $\square$

By Arzela-Ascoli, the preceding corollary clearly implies tightness of the diffusively rescaled process mentioned in Remark 2.6. Indeed we can use this to easily recover classical results such as [Ig74, BJD06] in this nearest-neighbor case, for instance by showing that any subsequential limit has the same finite-dimensional marginal distributions as $\mathbf{W}_X^T$ which in turn can be shown e.g. by Proposition B.6 below.

## B Heat kernel estimates for conditioned walks

We now prove various estimates for the heat kernels $\mathfrak{p}_n^N$ defined in Section 3. Not much motivation will be given here, but the content of Sections 4 and 5 has illustrated the applicability of these estimates. The methods used in proving these bounds will be elementary bounds together with the results of Appendix A (specifically Propositions A.5 and A.3, and Theorem A.7).

**Proposition B.1.** *There exist constants $C, K > 0$ such that for all $x \ge 0$, all $N \ge n \ge 0$, and all $a \ge 0$ one has that*

$$\sum_{y \ge 0} \mathfrak{p}_n^N(x,y)e^{ay} \le Ce^{ax + Ka^2n}.$$

*Proof.* In the notation of Appendix A, let us write

$$\sum_{y\geq 0}\mathfrak{p}_n^N(x,y)e^{ay}=\mathbf{E}_x^N[e^{aS_n}]=1+\int_0^\infty ae^{au}\mathbf{P}_x^N(S_n>u)du.$$

Now we split the integral as $\int_0^x+\int_x^\infty$. For the integral over $[0,x]$ we use the crude bound $\mathbf{P}_x^N(S_n>u)\leq 1$. For the integral over $[x,\infty)$, we use the result of Theorem A.7. This will give

$$\mathbf{E}_x^N[e^{aS_n}]\leq e^{ax}+C\int_x^\infty ae^{au}e^{-c(u-x)^2/n}du\leq e^{ax}+C\cdot an^{1/2}e^{ax+\frac{a^2n}{4c}}.$$

Since $an^{1/2}\leq e^{a^2n}$, this gives the result with $K:=1+\frac{1}{4c}$. □

We remark that $c=1/32$ from the proof of Theorem A.7, so we can obtain $K=9$ in the preceding proposition. Conjecturally, the optimal value of $K$ should be $1/2$, as is the case for the simple random walk (as seen from $\cosh(a)\leq e^{\frac{1}{2}a^2}$).

**Lemma B.2.** *Fix $b>0$. There exists $C=C(b)>0$ such that for all $x\geq 0$ and all $N\geq n\geq 0$ one has that*

$$\mathfrak{p}_n^N(x,y)\leq\frac{C}{\sqrt{n+1}}e^{-b|x-y|/\sqrt{n}}.$$

We remark that this bound is fairly strong, and many of our estimates could have been derived from this result rather than from the concentration theorem (but only in a weaker form because the decay is merely exponential rather than Gaussian).

*Proof.* We consider four different cases.

*Case 1.* $x\geq\sqrt{N}$. Then, one has $\frac{\psi(y,N-n)}{\psi(x,N)}\leq\frac{1}{\psi(x,N)}\leq C$ by Lemma A.1. Thus it holds that $\mathfrak{p}_n^N(x,y)\leq Cp_n^{(1/2)}(x-y)\leq C(n+1)^{-1/2}e^{-b|x-y|/\sqrt{n}}$. The final inequality comes from the first bound of Lemma A.5.

*Case 2.* $n<N/2$ and $y\leq x$. Then one has

$$\begin{aligned}\mathfrak{p}_n^N(x,y)&\leq Cp_n^{(1/2)}(x,y)\left[\frac{x+1+\sqrt{N}}{x+1}\right]\left[\frac{y+1}{y+1+\sqrt{N-n}}\right]\\&\leq C(n+1)^{-1/2}e^{-b|x-y|/\sqrt{n}}\left[\frac{x+1+\sqrt{N}}{x+1}\right]\left[\frac{x+1}{x+1+\sqrt{N-n}}\right]\\&\leq C(n+1)^{-1/2}e^{-b|x-y|/\sqrt{n}}\left[\frac{N}{N-n}\right]^{1/2}.\end{aligned}$$

We used (A.1) in the first line and we used Lemma A.5 and that $y\mapsto\frac{y+1}{y+1+\sqrt{N-n}}$ is monotone increasing in the second line. Then we canceled the $x+1$ and used the fact that $x\mapsto\frac{x+1+\sqrt{N}}{x+1+\sqrt{N-n}}$ is monotone decreasing in the last line. Since $n<N/2$ it follows that $\left[\frac{N}{N-n}\right]^{1/2}\leq 2^{1/2}$ so that term may be absorbed into $C$.

*Case 3.* $n<N/2$ and $y\geq x$. Then

$$\begin{aligned}\mathfrak{p}_n^N(x,y)&\leq Cp_n^{(1/2)}(x,y)\left[\frac{x+1+\sqrt{N}}{x+1}\right]\left[\frac{y+1}{y+1+\sqrt{N-n}}\right]\\&\leq Cp_n^{(1/2)}(x,y)\left[\frac{x+1+\sqrt{N}}{x+1}\right]\left[\frac{y+1}{x+1+\sqrt{N-n}}\right]\\&\leq C\left[\frac{N}{N-n}\right]^{1/2}p_n^{(1/2)}(x,y)\frac{y+1}{x+1}\end{aligned}$$

$$= C\left[\frac{N}{N-n}\right]^{1/2} p_n^{(1/2)}(x,y)\left[\frac{y-x}{x+1} \quad + \quad 1\right]$$
$$\leq C\left[\frac{y-x}{n+1} + C(n+1)^{-1/2}\right]e^{-b|x-y|/\sqrt{n}}. \tag{B.1}$$

Here we noted $y \geq x$ in the second line, and we used the fact that $x \mapsto \frac{x+1+\sqrt{N}}{x+1+\sqrt{N-n}}$ is monotone decreasing in the third line. In the final line, we used $\left[\frac{N}{N-n}\right]^{1/2} \leq 2^{1/2}$ (since $n < N/2$) and also the first bound of Lemma A.5. Now, we know that the bound (B.1) is true for all $b$, in particular it is true with $b$ replaced by $b+1$, after perhaps making the constant bigger. Thus we see that

$$\frac{|x-y|}{n+1}e^{-(b+1)|x-y|/\sqrt{n}} \leq \frac{1}{\sqrt{n+1}}e^{-b|x-y|/\sqrt{n}}\left[\frac{|x-y|}{\sqrt{n}}e^{-|x-y|/\sqrt{n}}\right].$$
$$\leq \frac{1}{\sqrt{n+1}}e^{-b|x-y|/\sqrt{n}}\sup_{u>0} ue^{-u} = \frac{C}{\sqrt{n+1}}e^{-b|x-y|/\sqrt{n}}.$$

*Case 4.* $x \leq \sqrt{N}$ and $n \geq N/2$. Since $x \leq \sqrt{N} \leq \sqrt{2n}$, we can apply Lemmas A.1 and A.5 to see that

$$\mathfrak{p}_n^N(x,y) \leq Cp_n^{(1/2)}(x,y)\frac{x+1+\sqrt{N}}{x+1}$$
$$\leq C\frac{x+1}{n+1}e^{-b|x-y|/\sqrt{n}} \cdot \frac{2\sqrt{2n}+1}{x+1} \leq C(n+1)^{-1/2}e^{-b|x-y|/\sqrt{n}}.$$

This completes the proof of all cases. ☐

**Proposition B.3.** *There exist constants $C, K > 0$ such that for all $x \geq 0$, all $N \geq n \geq 0$, all $a \geq 0$, and all $p \geq 1$ one has that*

$$\sum_{y\geq 0}\mathfrak{p}_n^N(x,y)^p e^{ay} \leq C^p(n+1)^{-(p-1)/2}e^{ax+Ka^2n}.$$

*Proof.* Using Lemma B.2 with $b=0$, one finds that

$$\mathfrak{p}_n^N(x,y)^p = \mathfrak{p}_n^N(x,y)^{p-1}\mathfrak{p}_n^N(x,y) \leq \frac{C^{p-1}}{(n+1)^{(p-1)/2}}\mathfrak{p}_n^N(x,y).$$

Then the claim follows immediately from Proposition B.1. ☐

We now bound space-time differences of the heat kernels $\mathfrak{p}_n^N$.

**Lemma B.4.** *There exists a constant $C > 0$ such that for all $x, y, z \geq 0$ one has that*

$$\left|\mathfrak{p}_n^N(x,y) - \mathfrak{p}_n^N(x,z)\right| \leq \frac{C}{n+1}\left[\frac{N+1}{N-n+1}\right]^{1/2}|y-z|.$$

*Proof.* Without loss of generality, assume $y \geq z$. It suffices to prove the bound in the case $y = z+1$. In the general case, one simply adds the bound $y-z$ times. Let us write

$$\left|\mathfrak{p}_n^N(x,z+1) - \mathfrak{p}_n^N(x,z)\right| = \left|\frac{p_n^{(1/2)}(x,z+1)\psi(z+1,N-n) - p_n^{(1/2)}(x,z)\psi(z,N-n)}{\psi(x,N)}\right|$$
$$\leq |p_n^{(1/2)}(x,z+1) - p_n^{(1/2)}(x,z)|\frac{\psi(z+1,N-n)}{\psi(x,N)} + p_n^{(1/2)}(x,z)\frac{|\psi(z+1,N-n)-\psi(z,N-n)|}{\psi(x,N)}.$$

Let us call the two terms of the last expression $I_1, I_2$ respectively. From here, one considers two cases ($x \leq \sqrt{N}$ and $x \geq \sqrt{N}$) and bound $I_1, I_2$ separately each time. The arguments are similar to the ones above, so the proof is not included. ☐

**Proposition B.5.** *Fix $p \geq 1$. There exists a constant $C = C(p) > 0$ such that for all $x, y \geq 0$, all $N \geq n \geq m \geq 0$, and all $a \geq 0$ one has that*

$$\sum_{z \geq 0} \left|\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\right|^{2p} e^{az} \leq C e^{a(x+y)+Ka^2 n}\big(n^{\frac{1}{2}-\frac{3}{2}p} + a^p n^{\frac{1}{2}-p}\big)|x-y|^p, \tag{B.2}$$

$$\sum_{z \geq 0} \left|\mathfrak{p}_m^{N-n+m}(x,z) - \mathfrak{p}_n^N(x,z)\right|^{2p} e^{az} \leq C e^{2ax+Ka^2 n}\big(m^{\frac{1}{2}-\frac{3}{2}p} + a^p m^{\frac{1}{2}-p}\big)|n-m|^{p/2}. \tag{B.3}$$

*In the spatial bound* (B.2)*, the constant $C$ grows at worst exponentially in $p$.*

We remark that in the special case that $p = 1$ and $a \leq Cn^{-1/2}$, one has that $n^{\frac{1}{2}-\frac{3}{2}p} + a^p n^{\frac{1}{2}-p} \leq Cn^{-1}$ and similarly for $m$. This is the case in which this bound will most often be applied.

*Proof.* We first start out by proving an auxiliary bound:

$$\sum_{z \geq 0} \big(\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\big)^2 e^{az} \leq C e^{a(x+y)+Ka^2 n}\big(n^{-1} + an^{-1/2}\big)\left[\frac{N+1}{N-n+1}\right]^{1/2}|x-y|. \tag{B.4}$$

Let us prove this. The coupling lemma (A.3) and the preceding lemma will be key here. First, by the coupling lemma, we know that $\mathbf{P}_x^N$ and $\mathbf{P}_y^N$ may be coupled in such a way so that the respective coordinate processes (call them $(S_n^x)_{n=0}^N$ and $(S_n^y)_{n=0}^N$) are never a distance more than $|y-x|$ apart (i.e., $\sup_{n \leq N} |S_n^x - S_n^y| \leq |x-y|$ a.s.). Let $E$ denote the expectation with respect to the coupled measure. Now, by writing $\big(\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\big)^2 = \mathfrak{p}_n^N(x,z)\big(\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\big) - \mathfrak{p}_n^N(y,z)\big(\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\big)$ we may write

$$\begin{aligned}
&\sum_{z \geq 0}\big(\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\big)^2 e^{az} \\
&\quad = \mathbf{E}_x^N[(\mathfrak{p}_n^N(x,S_n) - \mathfrak{p}_n^N(y,S_n))e^{aS_n}] - \mathbf{E}_y^N[(\mathfrak{p}_n^N(x,S_n) - \mathfrak{p}_n^N(y,S_n))e^{aS_n}] \\
&\quad = E[(\mathfrak{p}_n^N(x,S_n^x) - \mathfrak{p}_n^N(y,S_n^x))e^{aS_n^x}] - E[(\mathfrak{p}_n^N(x,S_n^y) - \mathfrak{p}_n^N(y,S_n^y))e^{aS_n^y}] \\
&\quad = E[(\mathfrak{p}_n^N(x,S_n^x) - \mathfrak{p}_n^N(x,S_n^y))e^{aS_n^x}] + E[\mathfrak{p}_n^N(x,S_n^y)(e^{aS_n^x} - e^{aS_n^y})] \\
&\qquad + E[(\mathfrak{p}_n^N(y,S_n^y) - \mathfrak{p}_n^N(y,S_n^x))e^{aS_n^y}] + E[\mathfrak{p}_n^N(y,S_n^x)(e^{aS_n^y} - e^{aS_n^x})].
\end{aligned}$$

Let us refer to the terms in the last expression as $J_1, J_2, J_3, J_4$, respectively. Since $J_1$ and $J_3$ occupy symmetric roles, it suffices to bound $J_1$ and then the analogous bound for $J_3$ automatically follows. The same thing happens for $J_2$ and $J_4$. With this understanding, we will only prove the desired bound for $J_1$ and $J_2$.

Let us start by bounding $J_1$. By Lemma B.4, we see that

$$\begin{aligned}
|\mathfrak{p}_n^N(x,S_n^x) - \mathfrak{p}_n^N(x,S_n^y)| &\leq \frac{C}{n+1}\left[\frac{N+1}{N-n+1}\right]^{1/2}|S_n^x - S_n^y| \\
&\leq \frac{C}{n+1}\left[\frac{N+1}{N-n+1}\right]^{1/2}|x-y|.
\end{aligned}$$

Here we applied the coupling in the second inequality. Applying the definition of $J_1$ and then Proposition B.1, we therefore obtain that

$$J_1 \leq \frac{C}{n+1}\left[\frac{N+1}{N-n+1}\right]^{1/2}|x-y|E[e^{aS_n^x}] \leq \frac{C}{n+1}\left[\frac{N+1}{N-n+1}\right]^{1/2}|x-y|e^{ax+Ka^2 n}.$$

This already gives the desired bound on $J_1$. As discussed, the analogous bound on $J_3$ is obtained in an identical fashion, but one will get $e^{ay}$ instead of $e^{ax}$. The final bound on $J_1 + J_3$ is then obtained by noting that $e^{ax} + e^{ay} \leq 2e^{a(x+y)}$.

Now we bound $J_2$. First note that $|e^u - e^v| \le |u-v|e^{u\vee v}$ for all $u,v \in \mathbb{R}$. Thus $|e^{aS_n^y} - e^{aS_n^x}| \le a|S_n^y - S_n^x|e^{a(S_n^y \vee S_n^x)} \le a|y-x|e^{a(S_n^y+S_n^x)}$. By Cauchy-Schwarz, we in turn bound $E[e^{a(S_n^y+S_n^x)}] \le \mathbf{E}_x^N[e^{2aS_n}]^{1/2}\mathbf{E}_y^N[e^{2aS_n}]^{1/2} \le Ce^{a(x+y)+Ka^2n}$, by Proposition B.3. Now, we also know from Lemma B.2 that $\mathfrak{p}_n^N(x,S_n^y) \le C(n+1)^{-1/2}$. Using these facts, we find that

$$J_2 \le Ca|y-x|E[\mathfrak{p}_n^N(x,S_n^y)e^{a(S_n^y+S_n^x)}] \le Can^{-1/2}|x-y|e^{a(y+x)+Ka^2n}.$$

Already this proves the required bound on $J_2$. The analogous bound on $J_4$ follows immediately. This completes the proof of (B.4).

Now let us prove the spatial estimate (B.2). For $m \le n$, we use the semigroup property to write $\mathfrak{p}_n^N(x,z) = \sum_{y\ge 0}\mathfrak{p}_m^N(x,y)\mathfrak{p}_{n-m}^{N-m}(y,z)$ and then using Jensen's inequality, we find that

$$\begin{aligned}\big|\mathfrak{p}_n^N(x,z) - \mathfrak{p}_n^N(y,z)\big|^{2p} &= \Big|\sum_{w\ge 0}\big(\mathfrak{p}_m^N(x,w) - \mathfrak{p}_m^N(y,w)\big)\mathfrak{p}_{n-m}^{N-m}(w,z)\Big|^{2p} \\ &\le \bigg(\sum_{w\ge 0}\big(\mathfrak{p}_m^N(x,w) - \mathfrak{p}_m^N(y,w)\big)^2\mathfrak{p}_{n-m}^{N-m}(w,z)\bigg)^p\end{aligned}$$

Denoting by $I$ the left-hand side of (B.2), we then find by Minkowski's inequality that

$$\begin{aligned}I^{1/p} &\le \Bigg(\sum_{z\ge 0}\bigg[\sum_{w\ge 0}\big(\mathfrak{p}_m^N(x,w) - \mathfrak{p}_m^N(y,w)\big)^2\mathfrak{p}_{n-m}^{N-m}(w,z)e^{az/p}\bigg]^p\Bigg)^{1/p} \\ &\overset{\text{Minkowski}}{\le} \sum_{w\ge 0}\bigg[\sum_{z\ge 0}\big(\mathfrak{p}_m^N(x,w) - \mathfrak{p}_m^N(y,w)\big)^{2p}\mathfrak{p}_{n-m}^{N-m}(w,z)^pe^{az}\bigg]^{1/p} \\ &= \sum_{w\ge 0}\big(\mathfrak{p}_m^N(x,w) - \mathfrak{p}_m^N(y,w)\big)^2\bigg[\sum_{z\ge 0}\mathfrak{p}_{n-m}^{N-m}(w,z)^pe^{az}\bigg]^{1/p} \\ &\overset{Prop.B.3}{\le} C\sum_{w\ge 0}\big(\mathfrak{p}_m^N(x,w) - \mathfrak{p}_m^N(y,w)\big)^2(n-m)^{\frac{1-p}{2p}}e^{(aw+Ka^2(n-m))/p} \\ &\overset{\text{(B.4)}}{\le} C(n-m)^{\frac{1-p}{2p}}\big(m^{-1}+am^{-1/2}\big)\bigg[\frac{N+1}{N-m+1}\bigg]^{1/2}|x-y|e^{\big(a(x+y)+Ka^2n\big)/p}.\end{aligned}$$

Setting $m := n/2$ then gives (B.2), because $\big[\frac{N+1}{N-\frac12 n+1}\big]^{1/2} \le \big[\frac{N+1}{\frac12 N+1}\big]^{1/2} \le 2^{1/2}$. Note that the constant $C$ does not depend on $p$, which also proves the final sentence given in the theorem statement after noting that $\big(n^{-1}+an^{-1/2}\big)^p \le 2^p(n^{-p}+a^pn^{-p/2})$.

We move on to the temporal estimate (B.3). The main idea is to use Jensen's inequality together with the spatial estimate. Specifically, we start off by writing

$$\begin{aligned}\big|\mathfrak{p}_m^{N-n+m}(x,z) - \mathfrak{p}_n^N(x,z)\big|^{2p} &= \Big|\mathfrak{p}_n^{N-n+m}(x,z) - \sum_{y\ge 0}\mathfrak{p}_{n-m}^N(x,y)\mathfrak{p}_m^{N-n+m}(y,z)\Big|^{2p} \\ &= \Big|\sum_{y\ge 0}\mathfrak{p}_{n-m}^N(x,y)\big(\mathfrak{p}_m^{N-n+m}(x,z) - \mathfrak{p}_m^{N-n+m}(y,z)\big)\Big|^{2p} \\ &\overset{\text{Jensen}}{\le} \sum_{y\ge 0}\mathfrak{p}_{n-m}^N(x,y)\big|\mathfrak{p}_m^{N-n+m}(x,z) - \mathfrak{p}_m^{N-n+M}(y,z)\big|^{2p}.\end{aligned}$$

Next, we multiply by $e^{az}$, then sum over $z$, and interchange the sum over $z$ with the sum over $y$. Letting $J$ denote the left-hand side of (B.3), this gives

$$\begin{aligned} J &\le \sum_{y\ge 0} \mathfrak{p}^N_{n-m}(x,y) \sum_{z\ge 0} \big|\mathfrak{p}^{N-n+m}_m(x,z) - \mathfrak{p}^{N-n+m}_m(y,z)\big|^{2p} e^{az} \\ &\le C^p \sum_{y\ge 0} \mathfrak{p}^N_{n-m}(x,y) e^{a(x+y)+Ka^2m}\big(m^{\frac12-\frac32 p} + a^p m^{\frac12 -p}\big)|y-x|^p \\ &= C^p e^{ax+Ka^2m}\big(m^{\frac12-\frac32 p} + a^p m^{\frac12-p}\big)\mathbf{E}^N_x[|S_{n-m}-x|^p e^{aS_{n-m}}]. \end{aligned}$$

All that is left to do is to show that one has $\mathbf{E}^N_x[|S_{n-m}-x|^p e^{aS_{n-m}}] \le Ce^{ax+Ka^2(n-m)}|n-m|^{p/2}$. This is an easy consequence of the concentration theorem. Indeed, for any $k\le N$ one may write

$$\mathbf{E}^N_x[|S_k-x|^p e^{aS_k}] \le \mathbf{E}^N_x[|S_k-x|^{2p}]^{1/2}\mathbf{E}^N_x[e^{2aS_k}]^{1/2},$$

and then the claim follows immediately from Propositions B.1 and Corollary A.9. □

Next we prove a strong convergence result for the discrete kernels $\mathfrak{p}^N_n$ to the continuous ones $\mathscr{P}^T_t$ from Definition 3.4, from which we can easily obtain estimates for the continuous kernels as well. In the case of Brownian meander at terminal time ($X=0$ and $t=T$), the following result is weaker than the local convergence result of [Car05], but we actually need it for all $(t,T)$ so we give an original proof.

**Proposition B.6.** *Fix $\tau\ge 0$. Then for $n\ge 0$, define*

$$\mathscr{P}_n(t,T;X,Y) := (n/2)^{1/2}\mathfrak{p}^{2\lfloor Tn\rfloor}_{2\lfloor tn\rfloor}(2\lfloor n^{1/2}X/\sqrt{2}\rfloor, 2\lfloor n^{1/2}Y/\sqrt{2}\rfloor).$$

*Then for each fixed $X,T,t\ge 0$, the map $Y\mapsto \mathscr{P}_n(t,T;X,Y)$ converges pointwise and in $L^p(\mathbb{R}_+, e^{aY}dY)$ to $\mathscr{P}^T_t(X,Y)$ for all $p\ge 1$ and $a\ge 0$ (as $n\to\infty$).*

*Furthermore for all $X,T\ge 0$, the map $(t,Y)\mapsto\mathscr{P}_n(t,T;X,Y)$ converges pointwise and in $L^p(dt\otimes e^{aY}dY)$ to $\mathscr{P}^T_t(X,Y)$ for all $p\in[1,3)$ and $a\ge 0$ (as $n\to\infty$).*

From now on, we will abbreviate quantities such as $\mathfrak{p}^{2\lfloor Tn\rfloor}_{2\lfloor tn\rfloor}(2\lfloor n^{1/2}X/\sqrt{2}\rfloor, 2\lfloor n^{1/2}Y/\sqrt{2}\rfloor)$ by just writing $\mathfrak{p}^{2nT}_{2nt}((2n)^{1/2}X,(2n)^{1/2}Y)$ instead. We hope that this abuse of notation will not cause any confusion, but in reality one should keep in mind that all quantities are only defined with *even* integers. The reason for this is the periodicity of the simple random walk: $\mathfrak{p}^N_n(x,y)$ vanishes if $n$ and $x-y$ have different parity. If it were not for this parity consideration, we could take a limit of the simpler quantity $n^{1/2}\mathfrak{p}^{\lfloor nT\rfloor}_{\lfloor nt\rfloor}(\lfloor n^{1/2}X\rfloor, \lfloor n^{1/2}Y\rfloor)$.

*Proof.* First, let us prove pointwise convergence. Letting $p_n$ denote the standard heat kernel on all $\mathbb{Z}$, we recall that

$$p^{(1/2)}_n(x,y) = p_n(x-y) - p_n(x+y+2).$$

$$\psi(x,n) = p_n(0) + p_n(x+1) + 2\sum_{1\le y\le x} p_n(y) = \sum_{-x\le y\le x+1} p_n(y).$$

Let $F_n$ denote the cdf associated to $p_n$, so that $\psi(x,n) = F_n(x+1)-F_n(-x) = F_n(x)+F_n(x+1)-1$. By uniformity of convergence of cdf's in the central limit theorem we know that $F_n(n^{1/2}x)$ converges uniformly (on $\mathbb{R}$) to $\Phi(x)$, where $\Phi$ is the cdf of a standard normal. From this it is clear that $\psi(n,n^{1/2}x) = F_n(n^{1/2}x)+F_n(n^{1/2}x+1)-1$ converges uniformly to $2\Phi(x)-1$ (because $\Phi$ has no atoms). In turn, one deduces that $\psi(2nT,(2n)^{1/2}X) = \psi(2nT,(2nT)^{1/2}X/\sqrt{T})$ converges to $2\Phi(X/\sqrt{T})-1$. From here, completing the proof of pointwise convergence is easy using the local central limit theorem (though notice that $X=0$ requires a separate proof) as done in earlier proofs.

Now we will fix $t, T, X$, and we will address convergence in $L^p(\mathbb{R}_+, e^{aY}dY)$. The main idea is simply to use dominated convergence in conjunction with Lemma B.2. Specifically, that lemma (applied with $b = 2at^{1/2}/p$) tells us that

$$\mathscr{P}_n(t,T;X,Y) \le Ct^{-1/2}e^{-2a|X-Y|/p}. \tag{B.5}$$

Here $C$ is a constant independent of $Y$ (but it will depend on $t, a, p$). Letting $p \ge 1$, it is then clear from (B.5) that for *fixed* $X, T, t$, the sequence of maps

$$Y \mapsto \mathscr{P}_n(t,T;X,Y)^p e^{aY}$$

is dominated (uniformly in $n$) by a function that is integrable on $\mathbb{R}_+$. This is enough to guarantee by dominated convergence that

$$\int_{\mathbb{R}_+} |\mathscr{P}_n(t,T;X,Y) - \mathscr{P}_t^T(X,Y)|^p e^{aY}dY \to 0.$$

Similarly, one uses (B.5) in conjunction with the dominated convergence theorem to obtain convergence in $L^p(\mathbb{R}_+\times\mathbb{R}_+, dt\otimes e^{aY}dY)$ of $(Y,t)\mapsto \mathscr{P}_n(t,T;X,Y)$. This argument only works for $p\in[1,3)$, since the singularity of $\int_{\mathbb{R}_+} t^{-p/2}e^{-pt^{-1/2}|X-Y|}dY \sim t^{-(p-1)/2}$ fails to be absolutely integrable near $t=0$, if $p \ge 3$. □

**Proposition B.7.** *Let $a, \tau > 0$ and let $\mathscr{P}_t^T$ be the kernels from Definition 3.4. Then there exists a constant $C = C(\tau, a)$ such that for all $X, Y \ge 0$, all $\theta \in [0, 1/2]$, and all $s \le t \le T \le \tau$ one has the following*

$$\int_{\mathbb{R}_+} \mathscr{P}_t^T(X,Z)e^{aZ}dZ \le Ce^{aX}, \tag{B.6}$$

$$\int_{\mathbb{R}_+} \mathscr{P}_t^T(X,Z)^2e^{aZ}dZ \le Ct^{-1/2}e^{aX}, \tag{B.7}$$

$$\int_{\mathbb{R}_+} \big(\mathscr{P}_t^T(X,Z) - \mathscr{P}_t^T(Y,Z)\big)^2 e^{aZ}dZ \le Ct^{-\frac{1}{2}-\theta}e^{a(X+Y)}|X-Y|^{2\theta}, \tag{B.8}$$

$$\int_{\mathbb{R}_+} \big(\mathscr{P}_s^{T-t+s}(X,Z) - \mathscr{P}_t^T(X,Z)\big)^2 e^{aZ}dZ \le Cs^{-\frac{1}{2}-\theta}e^{2aX}|t-s|^{\theta} \tag{B.9}$$

*Proof.* The claims follow from the $L^1$ and $L^2$ convergence in Proposition B.6. More specifically, (B.6) follows from Proposition B.1 and convergence in $L^1(\mathbb{R}_+, e^{aY}dY)$. Next, (B.7) follows from Proposition B.3 and convergence in $L^2(e^{aY}dY)$. Expressions (B.8) and (B.9) with $\theta = 1/2$ follow immediately from Proposition B.5 and convergence in $L^2(e^{aY}dY)$. The appearance of the terms $Ka^2n$ in the exponent will be absorbed into the constant because $a$ effectively becomes replaced by $n^{-1/2}a$. The $\theta = 0$ cases of (B.8) and (B.9) follow immediately from (B.7) and the fact that $e^{aX} + e^{aY} \le 2e^{a(X+Y)}$. The proofs for general $\theta$ then follow easily by geometric interpolation (i.e., $\min\{a,b\} \le a^\theta b^{1-\theta}$ for all $a, b \ge 0$ and all $\theta \in [0,1]$). □

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