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Giant component of the soft random geometric graph*

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Abstract

Consider a 2-dimensional soft random geometric graph $G(\lambda, s, \phi)$, obtained by placing a Poisson(λs^2) number of vertices uniformly at random in a square of side s, with edges placed between each pair x, y of vertices with probability $\phi(\|x - y\|)$, where $\phi : \mathbb{R}_+ \to [0, 1]$ is a finite-range connection function. This paper is concerned with the asymptotic behaviour of the graph $G(\lambda, s, \phi)$ in the large-s limit with (λ, ϕ) fixed. We prove that the proportion of vertices in the largest component converges in probability to the percolation probability for the corresponding random connection model, which is a random graph defined similarly for a Poisson process on the whole plane. We do not cover the case where λ equals the critical value $\lambda_c(\phi)$.

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1 Introduction and statement of results

Let $\phi : [0, \infty) \to [0, 1]$ be a nonincreasing function. Given a locally finite point set $\mathcal{X} \subset \mathbb{R}^2$, let $G(\mathcal{X}, \phi)$ denote the graph with vertex set $\mathcal{X}$, where for each $\{x, y\} \subset \mathcal{X}$, the edge xy is included with probability $\phi(\|y - x\|)$, independently of the other pairs. Here $\| \cdot \|$ is the Euclidean norm.

Given $\lambda > 0$, let $\mathcal{H}_\lambda$ denote a homogeneous Poisson process of intensity λ in $\mathbb{R}^2$. Given also $s > 0$, set $B(s) := [-s/2, s/2]^2$ and let $\mathcal{H}_{\lambda,s}$ denote the restriction of $\mathcal{H}_\lambda$ to $B(s)$, which is a homogeneous Poisson process of intensity λ in $B(s)$. We are interested in the graphs $G(\mathcal{H}_\lambda, \phi)$ and $G(\mathcal{H}_{\lambda,s}, \phi)$, which are known as the *random connection model* [7] and *soft random geometric graph* [10] respectively, with connection function ϕ.

Let $\mathcal{H}_\lambda^{\mathbf{o}}$ denote the point process $\mathcal{H}_\lambda \cup \{\mathbf{o}\}$, where $\mathbf{o}$ is the origin in $\mathbb{R}^2$. For $k \in \mathbb{N}$, let $\pi_k(\phi, \lambda)$ denote the probability that the component of $G(\mathcal{H}_\lambda^{\mathbf{o}}, \phi)$ containing the origin is of order k. The *percolation probability* $\theta(\phi, \lambda)$ is the probability that $\mathbf{o}$ lies in an infinite component of the graph $G(\mathcal{H}_\lambda^{\mathbf{o}}, \phi)$, that is,

$$\theta(\phi, \lambda) := 1 - \sum_{k=1}^{\infty} \pi_k(\phi, \lambda).$$

A standard coupling argument shows that $\theta(\phi, \lambda)$ is nondecreasing in λ. The *critical value* (continuum percolation threshold) $\lambda_c(\phi)$ is defined by

$$\lambda_c(\phi) := \inf\{\lambda > 0 : \theta(\phi, \lambda) > 0\}. \tag{1.1}$$

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It is known (see [7]) that $0 < \lambda_c < \infty$, provided $0 < \int_{\mathbb{R}^2} \phi(\|x\|)dx < \infty$.

Let $j \in \mathbb{N}$. For any finite graph G let $L_j(G)$ denote the order of its jth-largest component, that is, the jth-largest of the orders of its components, or zero if it has fewer than j components. In this paper we prove the following results about convergence in probability of $s^{-2}L_j(G(\mathcal{H}_{\lambda,s},\phi))$ for $j = 1, 2$, as $s \to \infty$ with ϕ, λ fixed.

Theorem 1.1. Suppose $\lambda > 0$ with $\theta(\phi, \lambda) = 0$. Then

$$s^{-2}L_1(G(\mathcal{H}_{\lambda,s},\phi)) \xrightarrow{P} 0 \qquad \text{as } s \to \infty. \tag{1.2}$$

Theorem 1.2. Suppose $\sup\{r > 0 : \phi(r) > 0\} \in (0,\infty)$ and $\lambda > \lambda_c(\phi)$. Then as $s \to \infty$ we have that $s^{-2}L_1(G(\mathcal{H}_{\lambda,s},\phi)) \xrightarrow{P} \lambda\theta(\phi,\lambda)$ and $s^{-2}L_2(G(\mathcal{H}_{\lambda,s},\phi)) \xrightarrow{P} 0$.

These theorems do not address the case with $\lambda = \lambda_c(\phi)$, unless we know $\theta(\phi, \lambda_c(\phi)) = 0$ (if we had this, we could apply Theorem 1.1 when $\lambda = \lambda_c(\phi)$). Theorem 1.2 also does not address the case of ϕ having unbounded range (recall we are assuming ϕ is nonincreasing).

In the case with $\phi = \mathbf{1}_{[0,1]}$, these results were already proved in [8], but the method here provides an alternative and possibly shorter proof (the proof in [8] relies on a lengthy RSW argument from [7], as well as not working for general ϕ). When $\phi = \mathbf{1}_{[0,1]}$ it is known [7] that $\theta(\phi, \lambda_c(\phi)) = 0$ since this case is equivalent to a Boolean model. Proving (or disproving) that $\theta(\phi, \lambda_c(\phi)) = 0$ for general ϕ remains an open problem so far as we know.

In the case where $\phi = p\mathbf{1}_{[0,1]}$ for some fixed $p \in (0, 1]$, Lichev et al. [4] have very recently extended the first part of our Theorem 1.2 (concerning the largest component) from convergence in probability to almost sure convergence. They also extend the second part of Theorem 1.2 (regarding the second largest comonent) by showing that $L_2(G(\mathcal{H}_{\lambda,s})) = \Theta((\log s)^2)$ in probability as $s \to \infty$ along the sequence $(n^{1/2})_{n\in\mathbb{N}}$ (for $p = 1$, this result was known from [8, Theorem 10.18]).

2 Proof of theorems

2.1 Preliminaries

We introduce some further notation that will be used in proving Theorems 1.1 and 1.2.

For $\mathcal{X} \subset \mathbb{R}^2$ and $x, y \in \mathbb{R}^2$ we sometimes write $\mathcal{X}^x$ for $\mathcal{X} \cup \{x\}$ and $\mathcal{X}^{x,y}$ for $\mathcal{X} \cup \{x, y\}$.

Let $\phi : [0,\infty) \to [0, 1]$ be nonincreasing. We shall view ϕ as fixed from now on, and for any locally finite $\mathcal{X} \subset \mathbb{R}^2$ write simply $G(\mathcal{X})$ instead of $G(\mathcal{X}, \phi)$. For $x \in \mathcal{X}$, let $\mathcal{C}_x(\mathcal{X})$ denote the vertex set of the component of $G(\mathcal{X})$ containing x (these components are often called *clusters*). Also given bounded $A \subset \mathbb{R}^d$, we write $\mathcal{C}_A(\mathcal{X})$ for $\cup_{x\in\mathcal{X}\cap A}\mathcal{C}_x(\mathcal{X})$. We shall refer to $\mathcal{C}_A(\mathcal{X})$ as a *set-cluster*; it is a finite union of clusters, associated with the set A.

Given $x \in \mathbb{R}^2$ and $r > 0$, we write $D_r(x)$ for the disk $\{y \in \mathbb{R}^2 : \|y - x\| \le r\}$ and D_r for $D_r(\mathbf{o})$. Also let $S_r := B(2r) = [-r, r]^2$, and let $\mathbf{e} := (1, 0)$.

Given $A, B \subset \mathbb{R}^2$, and locally finite $\mathcal{X} \subset \mathbb{R}^2$, we write $\{A \leftrightarrow B \text{ in } G(\mathcal{X})\}$ for the event that there exist $x \in A \cap \mathcal{X}$ and $y \in B \cap \mathcal{X}$ such that there is a path in $G(\mathcal{X})$ from x to y. We write $\{A \leftrightarrow \infty \text{ in } G(\mathcal{X})\}$ for the intersection over all $n \ge 1$ of events $\{A \leftrightarrow \mathbb{R}^2 \setminus D_n \text{ in } G(\mathcal{X})\}$

Next, we assemble some known facts which will be used later.

We say a real-valued function f, defined on graphs $(\mathcal{V}, E)$, with $\mathcal{V} \subset \mathbb{R}^2$ locally finite, is *increasing* if $f(\mathcal{V}, E) \le f(\mathcal{V}', E')$ whenever $\mathcal{V} \subset \mathcal{V}'$ and $E \subset E'$. We say f is *decreasing* if $-f$ is increasing. Given $\lambda > 0$ and given ϕ, we say E is an increasing (resp. decreasing) event on $G(\mathcal{H}_\lambda)$ if $\mathbf{1}_E$ is an increasing (resp. decreasing) function of $G(\mathcal{H}_\lambda)$.

Lemma 2.1 (Harris-FKG inequality)**.** Suppose f, g are measurable bounded increasing real-valued functions defined on graphs $(\mathcal{V}, E)$ with $\mathcal{V} \subset \mathbb{R}^2$ locally finite. Then

$$\mathbb{E}[f(G(\mathcal{H}_\lambda))g(G(\mathcal{H}_\lambda))] \geq \mathbb{E}[f(G(\mathcal{H}_\lambda))]\mathbb{E}[g(G(\mathcal{H}_\lambda))].$$

The same inequality holds if f and g are both decreasing.

Proof. See [2], where measurability issues are also dealt with. □

Corollary 2.2 (Square Root trick)**.** Let $\lambda > 0$, $k \in \mathbb{N}$, $\varepsilon \in (0, 1)$. Suppose for $i = 1, \ldots, k$ we have increasing events A_i defined on $G(\mathcal{H}_\lambda)$, such that $\mathbb{P}[\cup_{i=1}^k A_i] > 1 - \varepsilon$.

Then $\max_{1 \leq i \leq k} \mathbb{P}[A_i] > 1 - \varepsilon^{1/k}$.

Proof. Set $M = \max_{1 \leq i \leq k} \mathbb{P}[A_i]$. The events A_i^c are all decreasing, so by Lemma 2.1,

$$\varepsilon > \mathbb{P}[\cap_{i=1}^k A_i^c] \geq \prod_{i=1}^k \mathbb{P}[A_i^c] \geq (1 - M)^k,$$

so that $1 - M < \varepsilon^{1/k}$ and $M > 1 - \varepsilon^{1/k}$. □

Given $\lambda > 0$, let $N_\infty(\phi, \lambda)$ be the number of infinite components of the graph $G(\mathcal{H}_\lambda)$. It is not hard to show that if $\theta(\phi, \lambda) = 0$, then $\mathbb{P}[N_\infty(\phi, \lambda) = 0] = 1$. The next preliminary result concerns *uniqueness of the infinite cluster*, in the other case, where $\theta(\phi, \lambda) > 0$.

Lemma 2.3. Suppose $\theta(\phi, \lambda) > 0$. Then $\mathbb{P}[N_\infty(\phi, \lambda) = 1] = 1$.

Proof. See [7]. □

Another useful fact is the *Mecke formula* for the random connection model. Let $s, \lambda > 0$ and suppose $f(x, G) \in \mathbb{R}_+$ is defined for all pairs (x, G) where G is a finite graph with vertex set $\mathcal{V}(G) \subset B(s)$ and $x \in \mathcal{V}(G)$. Then whenever the following expectations are defined we have

$$\mathbb{E} \sum_{x \in \mathcal{H}_{\lambda,s}} f(x, G(\mathcal{H}_{\lambda,s})) = \lambda \int_{B(s)} \mathbb{E}[f(x, G(\mathcal{H}^x_{\lambda,s}))]dx \tag{2.1}$$

and moreover if $g(x, y, G) \in \mathbb{R}_+$ is defined whenever additionally $y \in \mathcal{V}(G)$ then

$$\mathbb{E} \sum \sum_{x,y \in \mathcal{H}_{\lambda,s}, x \neq y} g(x, y, G(\mathcal{H}_{\lambda,s})) = \lambda^2 \int_{B(s)} \int_{B(s)} \mathbb{E}[g(x, y, G(\mathcal{H}^{x,y}_{\lambda,s}))]dydx. \tag{2.2}$$

The Mecke formulae (2.1) and (2.2) can be derived by conditioning on $\mathcal{H}_{\lambda,s}$ and using the usual Mecke formulae from e.g. [3].

Also of use to us is the following *sequential construction* of set-clusters in $G(\mathcal{H}_{\lambda,s})$. Let $\lambda, s > 0$ and $A \subset B(s)$ (typically a disk). The set-cluster $\mathcal{C}_A(\mathcal{H}_{\lambda,s}) := \cup_{x \in \mathcal{H}_\lambda \cap A} \mathcal{C}_x(\mathcal{H}_{\lambda,s})$ can be created as follows:

First generate $\mathcal{H}_\lambda \cap A$. Denote the points so created as *active points* and let the initial intensity function of *unexplored points*, i.e. Poisson points in $B(s)$ that are not yet generated, be $g_0 := \lambda \mathbf{1}_{B(s) \setminus A}(\cdot)$.

Next, choose an active point x_1 and generate a Poisson process with intensity function $h_1 := g_0(\cdot)\phi(\cdot - x_1)$, representing the previously unexplored points of $\mathcal{H}_{\lambda,s}$ that are connected directly to x_1. Label all the new points as 'active', and change the status of x_1 from 'active' to 'finished'. Also change the intensity of unexplored points from $g_0(\cdot)$ to $g_1 := g_0(\cdot)(1 - \phi(\cdot - x_1))$.

Then pick a new active point x_2 and repeat the above, using the new intensity of unexplored points. Keep repeating until we run out of active points, then stop.

This algorithm terminates almost surely for the following reason. At the nth stage (if we get that far), the new active point selected is denoted x_n and the intensity function of new points generated is $h_n := g_{n-1}\phi(\cdot - x_n)$. Since $0 \leq \phi \leq 1$ we have $h_n \leq g_{n-1}$ pointwise. Moreover, the intensity function g_n of unexplored points at this stage is given by $g_n = g_{n-1}(1 - \phi(\cdot - x_n))$, so that $g_n + h_n = g_{n-1}$ pointwise. Hence by a backward recursion, the total intensity function of the first n Poisson point processes generated is

$$\begin{aligned} h_1 + \cdots + h_n &\leq h_1 + \cdots + h_{n-1} + g_{n-1} \\ &= h_1 + \cdots + h_{n-2} + g_{n-2} \\ &= \cdots = h_1 + g_1 = g_0, \end{aligned}$$

and therefore the expected number of points generated in the first n steps is bounded above by $\int_{\mathbb{R}^2} g_0(x)dx$, which is finite and independent of n. Hence by monotone convergence the total number of points generated in all steps has finite mean, so it is almost surely finite.

We shall refer to the above procedure as *growing the set-cluster* $\mathcal{C}_A(\mathcal{H}_{\lambda,s})$ *sequentially*. This method is described in detail for the case $\phi = \mathbf{1}_{[0,1]}$ in [9], and for the general random connection model in [6].

2.2 The subcritical case

Proof of Theorem 1.1. Suppose $\lambda > 0$ with $\theta(\phi, \lambda) = 0$.

Let $\varepsilon > 0$. Let $N_s := \sum_{x \in \mathcal{H}_{\lambda,s}} \mathbf{1}\{|\mathcal{C}_x(\mathcal{H}_{\lambda,s})| \geq \varepsilon s^2\}$. If $L_1(G(\mathcal{H}_{\lambda,s})) \geq \varepsilon s^2$, then $N_s \geq \varepsilon s^2$. Hence by Markov's inequality and the Mecke formula,

$$\begin{aligned} \mathbb{P}[L_1(G(\mathcal{H}_{\lambda,s})) \geq \varepsilon s^2] \leq (\varepsilon s^2)^{-1}\mathbb{E}[N_s] &= (\varepsilon s^2)^{-1} \int_{B(s)} \mathbb{P}[|\mathcal{C}_x(\mathcal{H}^x_{\lambda,s})| \geq \varepsilon s^2]\lambda dx \\ &\leq (\varepsilon s^2)^{-1} \int_{B(s)} \mathbb{P}[|\mathcal{C}_x(\mathcal{H}^x_{\lambda})| \geq \varepsilon s^2]\lambda dx \\ &= \lambda\varepsilon^{-1} \sum_{k \geq \varepsilon s^2} \pi_k(\phi, \lambda), \end{aligned}$$

which tends to zero as $s \to \infty$. Therefore $s^{-2}L_1(G(\mathcal{H}_{\lambda,s})) \xrightarrow{P} 0$. □

2.3 Renormalization

From now on we assume ϕ is nonincreasing with $\inf\{r > 0 : \phi(r) = 0\} = 1$. We shall prove Theorem 1.2 only for this case, since simple scaling arguments then yield the general finite-range case in the statement of the theorem.

Lemma 2.4. Suppose $\lambda > \lambda_c(\phi)$. Then $\mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)] \to 1$ as $K \to \infty$.

Proof. Let E be the event that $G(\mathcal{H}_\lambda)$ has at least one infinite component. Then $\mathbb{P}[E] > 0$ since $\lambda > \lambda_c(\phi)$.

For $0 < K < L < \infty$, define also the events $E_K := \{D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)\}$, and $E_{K,L} := \{D_K \leftrightarrow D_L^c \text{ in } G(\mathcal{H}_\lambda)\}$. Then $E_K \uparrow E$ as $K \uparrow \infty$, and therefore it suffices to prove that $\mathbb{P}[E] = 1$. This follows from the ergodic property of the random connection model [7, Lemmas 2.6 and 2.8], but to make this paper more self-contained we provide a proof from first principles.

Suppose $\mathbb{P}[E] = p \in (0, 1)$. Note that $E_{K,L} \downarrow E_K$ as $L \to \infty$. Thus given $\varepsilon > 0$, we can find $K, L \in (0, \infty)$ with $1 < K < L$ such that $\mathbb{P}[E \setminus E_K] < \varepsilon$ and $\mathbb{P}[E_{K,L} \setminus E_K] < \varepsilon$.

Let $E'_K = \{D_K(3L\mathbf{e}) \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)\}$ and let $E'_{K,L} = \{D_K(3L\mathbf{e}) \leftrightarrow D_L(3L\mathbf{e})^c$ in $G(\mathcal{H}_\lambda)\}$. Then $E'_{K,L}$ is independent of $E_{K,L}$, and by translation invariance,

$$p = \mathbb{P}[E] \leq \mathbb{P}[E_K \cap E'_K] + 2\varepsilon \leq \mathbb{P}[E_{K,L} \cap E'_{K,L}] + 2\varepsilon = \mathbb{P}[E_{K,L}]^2 + 2\varepsilon \leq (p + \varepsilon)^2 + 2\varepsilon,$$

and since ε is arbitrarily small, we have $p \le p^2$, a contradiction. □

Given $\lambda, K, L, M \in (0, \infty)$ with $L > K, M > 2K$, define the following events:

- $U_{K,L,\lambda}$ is the event that there is a unique component of $G(\mathcal{H}_\lambda \cap D_{L+1})$ that meets both D_K and $\mathbb{R}^2 \setminus D_L$.
- $F_{K,M,\lambda} = \{D_K \leftrightarrow D_K(M\mathbf{e}) \text{ in } G(\mathcal{H}_\lambda \cap D_{3M})\}$.

Proposition 2.5. Suppose $\lambda > \lambda_c(\phi)$ and let $\varepsilon \in (0, 1)$. There exist finite constants $K > 0$ and $M > 3K$ such that (i) $\mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)] > 1 - \varepsilon$, and (ii) $\mathbb{P}[U_{K,M/3,\lambda}] > 1 - \varepsilon$, and (iii) $\mathbb{P}[F_{K,M,\lambda}] > 1 - \varepsilon$.

We shall use this to establish the limiting behaviour of $s^{-2}L_1(G(\mathcal{H}_{\lambda,s}))$ for $\lambda > \lambda_c(\phi)$. The point is that we can use it to compare $G(\mathcal{H}_\lambda)$ with a finite-range dependent percolation process on the lattice $\mathbb{Z}^2$.

To prepare for the proof of Proposition 2.5, fix $\lambda > \lambda_c(\phi)$ and $\varepsilon \in (0, 1)$. Set $\mu = (\lambda_c(\phi) + \lambda)/2$, so $\mu \in (\lambda_c(\phi), \lambda)$. We claim that we can (and do) choose $\eta > 0$ such that for any two distinct points $x, y \in [0, 3]^2$,

$$\mathbb{P}[\{x\} \leftrightarrow \{y\} \text{ in } G(\mathcal{H}^{x,y}_{\lambda-\mu} \cap [0, 3]^2)] \ge \eta. \tag{2.3}$$

To see this, partition $[0, 3]^2$ into squares (boxes) Q_i of side $1/9$. Let $i(x)$ (resp. $i(y)$) be the index of the box containing x (resp. y). With strictly positive probability, $\mathcal{H}_{\lambda-\mu} \cap Q_i \neq \varnothing$ for each i, and selecting one point $X_i \in \mathcal{H}_{\lambda-\mu} \cap Q_i$ for each i, we have connections between X_i and X_j for each pair of neighbouring boxes (i, j), and also between x and $X_{i(x)}$ and between y and $X_{i(y)}$, justifying the claim.

Next, we choose $\nu \in \mathbb{N}$ such that $(1 - \eta)^{\nu/9} < \varepsilon/3$. Then take

$$\delta := (\varepsilon/3)e^{-25\lambda\nu}. \tag{2.4}$$

Proof of Proposition 2.5. We adapt an argument in [1]. Let μ, η, ν and δ be as given above and let $\varepsilon_1 = (\delta/3)^{32}$. Using Lemma 2.4, choose $K \in (0, \infty)$ such that $\mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\mu)] > 1 - \varepsilon_1$. Since $\varepsilon_1 < \varepsilon$ and $\mu < \lambda$ this yields (i) at once. Since $\varepsilon_1 < \varepsilon$, we claim that we can (and do) choose $n_1 \in \mathbb{N}$ with $n_1 > K$ such that

$$\mathbb{P}[U_{K,n,\lambda}] > 1 - \varepsilon, \quad \forall n \in [n_1, \infty). \tag{2.5}$$

We leave the proof of this claim, using Lemma 2.3, as an exercise.

Now for integer $n \ge n_1$, and for $0 \le \alpha \le \beta \le n + 1$, define the event

$$E_n(\alpha, \beta) := \{D_K \leftrightarrow [n, n + 1] \times [\alpha, \beta] \text{ in } G(\mathcal{H}_\mu \cap S_{n+1})\}.$$

Since $\mathbb{P}[D_K \leftrightarrow S_n^c \text{ in } G(\mathcal{H}_\mu \cap S_{n+1})] \ge \mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\mu)] > 1 - \varepsilon_1$, using the Square Root trick we can deduce that

$$\mathbb{P}[E_n(0, n + 1)] > 1 - \varepsilon_1^{1/8}. \tag{2.6}$$

Next, observe that for fixed n, we have that as a function of α, $\mathbb{P}[E_n(0, \alpha)] - \mathbb{P}[E_n(\alpha, n + 1)]$ increases continuously from a value of $-\mathbb{P}[E_n(0, n + 1)]$ at $\alpha = 0$ to a value of $+\mathbb{P}[E_n(0, n + 1)]$ at $\alpha = n + 1$. Therefore we can and do choose $\alpha_n \in (0, n + 1)$ such that $\mathbb{P}[E_n(0, \alpha_n)] = \mathbb{P}[E_n(\alpha_n, n + 1)]$. Since $E_n(0, n + 1) = E_n(0, \alpha_n) \cup E_n(\alpha_n, n + 1)$, by (2.6) and a further application of the Square Root trick we obtain that

$$\mathbb{P}[E_n(\alpha_n, n + 1)] = \mathbb{P}[E_n(0, \alpha_n)] > 1 - \varepsilon_1^{1/16}. \tag{2.7}$$

By yet another application of the Square Root trick we obtain that

$$\max(\mathbb{P}[E_n(0, \alpha_n/2)], \mathbb{P}[E_n(\alpha_n/2, \alpha_n)]) > 1 - \varepsilon_1^{1/32}$$

so we can and do choose y_n, with either $y_n = \alpha_n/4$ or $y_n = 3\alpha_n/4$, such that

$$\mathbb{P}[E_n(y_n - \alpha_n/4, y_n + \alpha_n/4)] > 1 - \varepsilon_1^{1/32}. \tag{2.8}$$

Set $n_2 = 3n_1$. We claim that there exists integer $N \geq n_2$ such that $\alpha_{3N} < 4\alpha_N$. Indeed, if this were not true then we would have for all $k \geq 1$ that $\alpha_{3^k n_2} \geq 4^k \alpha_{n_2}$, but since $\alpha_n \leq n+1$ for all n, this would imply $3^k(n_2+1) \geq 4^k \alpha_{n_2}$ so that $(4/3)^k \leq (n_2+1)/\alpha_{n_2}$ for all k, which is not true (since $\alpha_{n_2} > 0$), justifying the claim.

Choose (deterministic) integer $N \geq n_2$ such that $\alpha_{3N} < 4\alpha_N$. Then by (2.7) and (2.8), setting $\varepsilon_2 := \varepsilon_1^{1/32}$ we have

$$\min(\mathbb{P}[E_N(\alpha_N, N+1)], \mathbb{P}[E_{3N}(y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4)]) > 1 - \varepsilon_2. \tag{2.9}$$

Now set $\mathbf{x} = (2N, y_{3N})$ (we use bold face to indicate certain fixed 2-vectors such as $\mathbf{o}$ and $\mathbf{e}$). Let $S_{N+1}(\mathbf{x}) := S_{N+1} + \mathbf{x} = [N-1, 3N+1] \times [y_{3N} - N - 1, y_{3N} + N + 1]$. Define the vertical blocks (see Figure 1)

$$\begin{aligned} I &:= [3N, 3N+1] \times [y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4], \\ J^+ &:= [3N, 3N+1] \times [y_{3N} + \alpha_N, y_{3N} + N + 1], \\ J^- &:= [3N, 3N+1] \times [y_{3N} - N - 1, y_{3N} - \alpha_N]. \end{aligned}$$

Let A^+ be the event $\{D_K(\mathbf{x}) \leftrightarrow J^+ \text{ in } G(\mathcal{H}_\mu \cap S_{N+1}(\mathbf{x}))\}$, and let A^- be the event $\{D_K(\mathbf{x}) \leftrightarrow J^- \text{ in } G(\mathcal{H}_\mu \cap S_{N+1}(\mathbf{x}))\}$. Then $\mathbb{P}[A^+] = \mathbb{P}[A^-] = \mathbb{P}[E_N(\alpha_N, N+1)]$. Also $E_{3N}(y_{3N} - \alpha_{3N}/4, y_{3N} + \alpha_{3N}/4) = \{D_K \leftrightarrow I \text{ in } G(\mathcal{H}_\mu \cap S_{3N+1})\}$. By (2.9) and the union bound,

$$\mathbb{P}[A^+ \cap A^- \cap \{D_K \leftrightarrow I \text{ in } G(\mathcal{H}_\mu \cap S_{3N+1})\}] > 1 - 3\varepsilon_2 = 1 - \delta.$$

Set $M = \|\mathbf{x}\|$. Then $M \geq 2N \geq 3n_1$. By (2.5), $\mathbb{P}[U_{K,M/3,\lambda}] > 1 - \varepsilon$ so we have (ii). The proof is then completed by the following 'gluing lemma'. □

Lemma 2.6. If $\mathbb{P}[A^+ \cap A^- \cap \{D_K \leftrightarrow I \text{ in } G(\mathcal{H}_\mu \cap S_{3N+1})\}] > 1 - \delta$, then $\mathbb{P}[F_{K,M,\lambda}] > 1 - \varepsilon$, where we take $M = \|\mathbf{x}\|$.

Remark The proof below is not needed for the special case $\phi = \mathbf{1}_{[0,1]}$, since in this case the lemma is immediate because $A^+ \cap A^- \cap \{D_K \leftrightarrow I \text{ in } G(\mathcal{H}_\mu \cap S_{3N+1})\}$ implies $F_{K,M,\mu}$.

Proof of Lemma 2.6. Divide $\mathbb{R}^2$ into half-open rectilinear squares Q_i of side 1 and for each i let Q_i^+ (respectively Q_i^{++}) be the half-open square of side 3 (resp. 5) with the same centre. We shall define a random variable Z taking values in $\mathbb{Z}_+ \cup \{+\infty\}$, as follows.

Grow the set-cluster $\mathcal{C} := \mathcal{C}_{D_K}(\mathcal{H}_\mu \cap S_{3N+1})$ sequentially. Let $\mathcal{P}$ be the point process of unexplored points of $\mathcal{H}_\mu \cap S_{N+1}(\mathbf{x})$ at the end of this procedure, i.e. the points of $\mathcal{H}_\mu \cap S_{N+1}(\mathbf{x})$ that either lie outside S_{3N+1} (since $S_{N+1}(\mathbf{x})$ is conceivably not entirely contained in S_{3N+1}) or are not connected by an edge to any point of $\mathcal{C}$. If $\mathcal{C} \cap D_K(\mathbf{x}) \neq \varnothing$ then set $Z = +\infty$.

Next, assuming $Z < \infty$, grow the set-cluster $\mathcal{C}_{D_K(\mathbf{x})}(\mathcal{P})$ sequentially but do not continue the exploration from any points created that lie in $\cup_{\{i:\mathcal{C} \cap Q_i \neq \varnothing\}} Q_i^+$; leave these points as 'active'. We denote this second set-cluster by $\mathcal{C}'$. Let $\mathcal{I} := \{i : \mathcal{C} \cap Q_i \neq \varnothing, \mathcal{C}' \cap Q_i^+ \neq \varnothing\}$ and let $Z = |\mathcal{I}|$, as illustrated in Figure 1.

Define the event $E := \{D_K \leftrightarrow I \text{ in } G(\mathcal{H}_\mu \cap S_{3N+1})\} = \{\mathcal{C} \cap I \neq \varnothing\}$.

Recall that ν was defined just after (2.3). Suppose $\nu < Z < \infty$. Then we can find $i_1, \ldots i_{\lceil \nu/9 \rceil} \in \mathcal{I}$ such that the squares $Q_{i_1}^+, \ldots, Q_{i_{\lceil \nu/9 \rceil}}^+$ are disjoint. Then sprinkling an independent Poisson process $\mathcal{H}_{\lambda-\mu}$ on top of $\mathcal{H}_\mu$ in these squares, for each square we have a chance at least η to join $\mathcal{C}$ to $\mathcal{C}'$ via the sprinkled points in that square. Since we may assume $\mathcal{H}_\lambda = \mathcal{H}_\mu \cup \mathcal{H}_{\lambda-\mu}$, setting $F' = \{D_K \leftrightarrow D_K(\mathbf{x}) \text{ in } G(\mathcal{H}_\lambda \cap (S_{3N+1} \cup S_{N+1}(\mathbf{x})))\}$ we obtain that

$$\mathbb{P}[(F')^c | E \cap \{Z > \nu\}] \leq (1-\eta)^{\nu/9} < \varepsilon/3. \tag{2.10}$$

Note that if $Z = +\infty$ then F' must occur so this case is included in (2.10).

Now suppose $Z \leq \nu$. If also event E occurs then it is not possible to find paths in $S_{N+1}(\mathbf{x})$ both from $D_K(\mathbf{x})$ to J^+, and from $D_K(\mathbf{x})$ to J^-, with neither path passing through $\cup_{\{i:\mathcal{C}\cap Q_i \neq \varnothing\}} Q_i^+$ (see Figure 1, and also [1, Figure 2.2]; note $I \cap J^+ = I \cap J^- = \varnothing$ since $\alpha_{3N}/4 < \alpha_N$). Therefore we have not yet achieved event $A^+ \cap A^-$ at this stage.

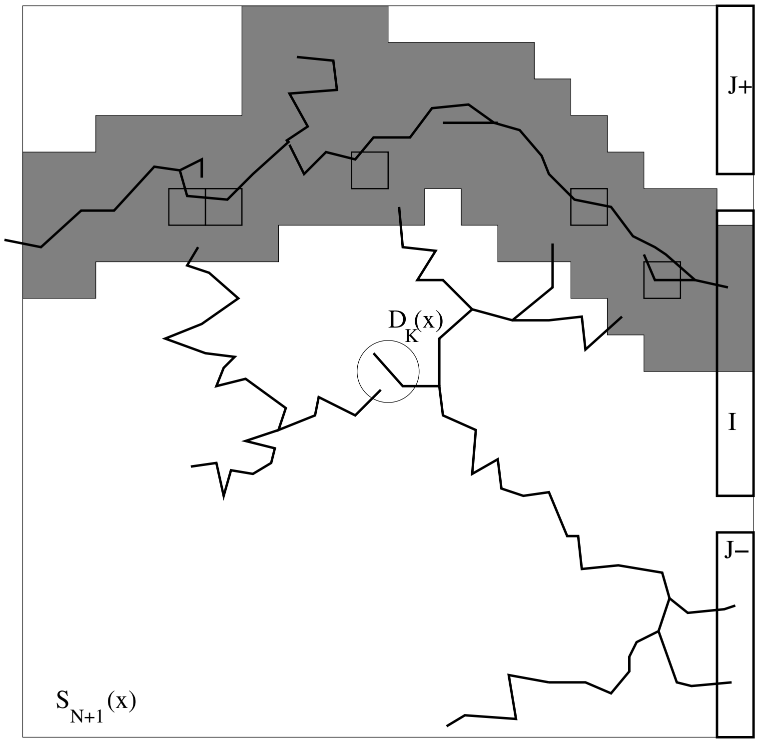

Figure 1: The top cluster is the part of $\mathcal{C}$ within $S_{N+1}(\mathbf{x})$. The other two clusters shown are $\mathcal{C}'$, which is grown within the square $S_{N+1}(\mathbf{x})$ but without further exploration from the points created in the shaded region, which is $\cup_{i:\mathcal{C}\cap Q_i \neq \varnothing} Q_i^+$. In this case $Z = 5$; the five squares Q_i for which $i \in \mathcal{I}$ are shown. It is not possible for $\mathcal{C}'$ to reach both J^+ and J^- by this stage.

At the next stage sample all of the new Poisson points (not part of $\mathcal{C}$ or $\mathcal{C}'$) in the union of squares Q_i^{++}, $i \in \mathcal{I}$. With probability at least $e^{-25\mu\nu}$ no new Poisson points are generated at this stage, and if this is the case then $A^+ \cap A^-$ does not occur because the cluster $\mathcal{C}'$ dies out without having reached J^+ and J^-. Thus we have

$$\mathbb{P}[(A^+ \cap A^-)^c | E \cap \{Z \leq \nu\}] \geq \exp(-25\mu\nu) \geq \exp(-25\lambda\nu).$$

Therefore if $\mathbb{P}[A^+ \cap A^- \cap E] > 1 - \delta$ we have

$$\delta > \mathbb{P}[(A^+ \cap A^-)^c \cap E \cap \{Z \leq \nu\}] \geq \exp(-25\lambda\nu)\mathbb{P}[E \cap \{Z \leq \nu\}],$$

so that $\mathbb{P}[E \cap \{Z \leq \nu\}] < \varepsilon/3$ by (2.4). Also by assumption $\mathbb{P}[E^c] < \delta < \varepsilon/3$. Combined with (2.10), this shows that

$$\mathbb{P}[(F')^c] \leq \mathbb{P}[E^c] + \mathbb{P}[(F')^c | E \cap \{Z > \nu\}] + \mathbb{P}[E \cap \{Z \leq \nu\}] < \varepsilon. \tag{2.11}$$

Since $(S_{3N+1} \cup S_{N+1}(\mathbf{x})) \subset D_{6N} \subset D_{3\|\mathbf{x}\|}$, we have $F' \subset \{D_K \leftrightarrow D_K(\mathbf{x}) \text{ in } G(\mathcal{H}_\lambda \cap D_{3\|\mathbf{x}\|})\}$. By rotation invariance, with $M := \|\mathbf{x}\|$, we thus have $\mathbb{P}[F_{K,M,\lambda}] > 1 - \varepsilon$, as required. □

2.4 Connection probability

Given $s > 0$, let V_s, W_s be independent uniformly distributed points in $B(s)$, independent of $\mathcal{H}_\lambda$. We shall characteraize the giant component of $G(\mathcal{H}_{\lambda,s})$ in terms of those vertices which are path-connected to a fixed disk centred at $\mathbf{o}$. For this, the following is useful.

Proposition 2.7. Suppose $\lambda > \lambda_c(\phi)$. Let $\varepsilon > 0$. Then there exists $K > 0$ such that $\mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)] > 1 - \varepsilon$, and

$$\liminf_{s\to\infty} \mathbb{P}[\{V_s\} \leftrightarrow D_K \text{ in } G(\mathcal{H}_{\lambda,s}^{V_s})] \geq \theta(\phi, \lambda) - \varepsilon. \tag{2.12}$$

Proof. Assume $\lambda > \lambda_c(\phi)$. Let $\varepsilon_1 \in (0, \varepsilon/3)$ be chosen such that if $(X_x)_{x\in\mathbb{Z}^2}$ is a 7-dependent Bernoulli random field on $\mathbb{Z}^2$ with $\mathbb{P}[X_x = 1] > 1 - 5\varepsilon_1$ for all $x \in \mathbb{Z}^d$, then for all $s > 0$ and all $x \in \mathbb{Z}^2 \cap B(s)$, there is a lattice path of 1's from x to $\mathbf{o}$ in $\mathbb{Z}^2 \cap B(s)$ with probability greater than $1 - \varepsilon/3$. The proof that such an ε_1 exists is standard, using e.g. [5] and a Peierls argument (e.g. [8, Theorem 9.8]).

Using Proposition 2.5, choose K, M such that $0 < K < M/3$, and

$$\min(\mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)], \mathbb{P}[U_{K,M/3,\lambda}], \mathbb{P}[F_{K,M,\lambda}]) > 1 - \varepsilon_1.$$

For each $x, y \in \mathbb{Z}^2$ with $\|x - y\| = 1$, let U_x denote the event that there is a unique component of $G(\mathcal{H}_\lambda \cap D_{(M/3)+1}(Mx))$ that meets both $D_K(Mx)$ and $\mathbb{R}^2 \setminus D_{M/3}(Mx)$. Let $F_{xy} := \{D_K(Mx) \leftrightarrow D_K(My) \text{ in } G(\mathcal{H}_\lambda \cap D_{3M}(Mx))\}$.

By translation and rotation invariance of $\mathcal{H}_\lambda$, $\mathbb{P}[U_x] > 1 - \varepsilon_1$ for each x, and $\mathbb{P}[F_{xy}] > 1 - \varepsilon_1$ for each (x, y).

For each $x \in \mathbb{Z}^2$, let us set $X_x = 1$ if event U_x occurs, and also F_{xy} occurs for each of the four $y \in \mathbb{Z}^2$ with $\|y - x\| = 1$; otherwise set $X_x = 0$. Then by the union bound $\mathbb{P}[X_x = 1] \geq 1 - 5\varepsilon_1$. Also X_x is determined by $\mathcal{H}_\lambda|_{D_{3M}(Mx)}$, so $(X_x, x \in \mathbb{Z}^2)$ is a 7-dependent Bernoulli random field. For example, if $x, y \in \mathbb{Z}^2$ with $\|x - y\|_\infty > 7$, then $D_{3M}(Mx) \cap D_{3M}(My) = 0$ so X_x and X_y are independent. By the choice of ε_1, for any $x \in B(s/M) \cap \mathbb{Z}^2$ there is a lattice path in $\mathbb{Z}^2 \cap B(s/M)$ from x to $\mathbf{o}$ of sites z with $X_z = 1$, with probability at least $1 - \varepsilon/3$.

Since $\mathbb{P}[D_{M+K} \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)] \geq \mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)] > 1 - \varepsilon_1$, by a similar argument to (2.5) we can (and do) take $M_1 > M + K$ such that $\mathbb{P}[U_{M+K,M_1,\lambda}] > 1 - \varepsilon_1$.

Let x_V be the closest point in $\mathbb{Z}^2 \cap B(s/M)$ to $M^{-1}V_s$. Consider the following events:

- $A_1 := \{V_s \in B(s - 4M_1) \setminus D_{3M_1}\}$. Provided s is large enough $\mathbb{P}[A_1] > 1 - \varepsilon/3$.
- A_2 is the event that there is a lattice path from x_V to $\mathbf{o}$ within $\mathbb{Z}^2 \cap B(s/M)$ with $X_x = 1$ for all sites x in the path. By the previous discussion, $\mathbb{P}[A_2] > 1 - \varepsilon/3$.
- A_3 is the event that there is a unique component in $G(\mathcal{H}_\lambda^{V_s} \cap D_{M_1+1}(V_s))$ that meets both $D_{M+K}(V_s)$ and $\mathbb{R}^2 \setminus D_{M_1}(V_s)$. Then $\mathbb{P}[A_3] > 1 - \varepsilon/3$.
- A_4 is the event that $\mathcal{C}_{V_s}(\mathcal{H}_\lambda^{V_s}) \cap D_{M_1}(V_s)^c \neq \varnothing$. Then $\mathbb{P}[A_4] \geq \theta(\phi, \lambda)$.

By the union bound $\mathbb{P}[\cap_{i=1}^4 A_i] \geq \theta(\phi, \lambda) - \varepsilon$, for all large enough s. Therefore it suffices to prove that if $\cap_{i=1}^4 A_i$ occurs, then $\{V_s\} \leftrightarrow D_K$ in $G(\mathcal{H}_{\lambda,s}^{V_s})$.

To see this, suppose $\cap_{i=1}^4 A_i$ occurs. Then we have $\|Mx_V - V_s\| \leq M$ so using A_1 we have $\mathrm{dist}(Mx_V, B(s)^c) \geq 2M_1 - M > (M/3) + 1$. Since $X_{x_V} = 1$ by A_2, there is a unique component of $G(\mathcal{H}_\lambda \cap D_{(M/3)+1}(Mx_V))$ that meets both $D_K(Mx_V)$ and $D_{(M/3)}(Mx_V)^c$, and since $D_{(M/3)+1}(Mx_V) \subset B(s)$, this extends to a unique component of $G(\mathcal{H}_{\lambda,s})$ that meets both $D_K(Mx_V)$ and $D_{M/3}(Mx_V)^c$. We denote this component by $\mathcal{C}$.

By A_2, the component $\mathcal{C}$ includes a vertex in D_K. Choose such a vertex and denote it by z.

Next, using A_1 observe that $\|V_s - z\| \geq \|V_s\| - \|z\| \geq 3M_1 - K \geq 2M_1$. Also $D_K(Mx_V) \subset D_{M+K}(V_s)$, so $\mathcal{C}$ meets both $D_{M+K}(V_s)$ and $D_{M_1}(V_s)^c$. Therefore using A_3 and A_4 we have that V_s is connected to $\mathcal{C}$, and thus $\{V_s\} \leftrightarrow D_K$ in $G(\mathcal{H}^{V_s}_{\lambda,s})$ as required. □

2.5 Proof of the giant component phenomenon

We now write $L_{i,s}$ for $L_i(G(\mathcal{H}_{\lambda,s}))$ (we are thinking of λ and ϕ as fixed with $\sup\{r : \phi(r) > 0\} = 1$). For convenience, we re-state Theorem 1.2, which we are now ready to prove.

Theorem 2.8. If $\lambda > \lambda_c(\phi)$, then $s^{-2}L_{1,s} \xrightarrow{P} \lambda\theta(\phi,\lambda)$ and $s^{-2}L_{2,s} \xrightarrow{P} 0$ as $s \to \infty$.

Proof. Assume $\lambda > \lambda_c(\phi)$. Let $\varepsilon > 0$ and using Proposition 2.7, choose $K > 0$ such that $\mathbb{P}[D_K \leftrightarrow \infty \text{ in } G(\mathcal{H}_\lambda)] > 1 - \varepsilon$, and (2.12) holds. Consider the sum

$$N_s := \sum_{x \in \mathcal{H}_{\lambda,s}} \mathbf{1}\{\{x\} \leftrightarrow D_K \text{ in } G(\mathcal{H}_{\lambda,s})\}.$$

Let V_s, W_s be as in Section 2.4. By the Mecke formula, $\mathbb{E}N_s = \lambda s^2 \mathbb{P}[\{V_s\} \leftrightarrow D_K \text{ in } G(\mathcal{H}^{V_s}_{\lambda,s})]$. Then using (2.12), and writing just θ for $\theta(\phi,\lambda)$, we deduce that

$$\liminf_{s\to\infty} s^{-2}\mathbb{E}N_s \geq \lambda(\theta - \varepsilon). \tag{2.13}$$

Next, let $N'_s = \sum_{x\in\mathcal{H}_{\lambda,s}} \mathbf{1}\{|\mathcal{C}_x(\mathcal{H}_{\lambda,s})| \geq s^{1/2}\}$ (here $|\cdot|$ represents number of elements). Using the Mecke formula (2.1) we have that $\mathbb{E}[N'_s] = \lambda s^2 \mathbb{P}[|\mathcal{C}_{V_s}(\mathcal{H}^{V_s}_{\lambda,s})| \geq s^{1/2}]$, and hence

$$\lim_{s\to\infty} s^{-2}\mathbb{E}N'_s = \lambda\theta. \tag{2.14}$$

Also $\mathbb{E}[N'_s(N'_s - 1)] = \lambda^2 s^4 \mathbb{P}[|\mathcal{C}_{V_s}(\mathcal{H}^{V_s,W_s}_{\lambda,s})| \geq s^{1/2}, |\mathcal{C}_{W_s}(\mathcal{H}^{V_s,W_s}_{\lambda,s})| \geq s^{1/2}|]$ by (2.2), so that

$$\lim_{s\to\infty} s^{-4}\mathbb{E}[N'_s(N'_s - 1)] = \lambda^2\theta^2.$$

Thus $s^{-2}N'_s \to \lambda\theta$ in L^2 and hence in probability, as $s \to \infty$.

Since $(N_s - N'_s)^+ \leq \mathcal{H}_\lambda(D_{K+s^{1/2}})$ we have that $s^{-2}\mathbb{E}[(N_s - N'_s)^+] \to 0$ as $s \to \infty$. Hence by (2.13) and (2.14),

$$\limsup_{s\to\infty} \mathbb{E}[s^{-2}(N'_s - N_s)^+] = \limsup_{s\to\infty} \mathbb{E}[s^{-2}(N'_s - N_s)] \leq \lambda\varepsilon.$$

Hence by Markov's inequality $\limsup_{s\to\infty} \mathbb{P}[s^{-2}(N'_s - N_s) \geq \varepsilon^{1/2}] \leq \lambda\varepsilon^{1/2}$, and hence

$$\begin{aligned}\limsup_{s\to\infty} \mathbb{P}[s^{-2}N_s \leq \lambda\theta - 2\varepsilon^{1/2}] \leq \limsup_{s\to\infty} \Big(&\mathbb{P}[s^{-2}N'_s \leq \lambda\theta - \varepsilon^{1/2}] \\ &+ \mathbb{P}[s^{-2}(N_s - N'_s) \leq -\varepsilon^{1/2}]\Big) \leq \lambda\varepsilon^{1/2}.\end{aligned}$$

As at (2.5), we can and do choose $M_2 > K$ so $\mathbb{P}[U_{K,M_2}] > 1 - \varepsilon$. If $(s/2) > M_2 + 1$ and U_{K,M_2} occurs then $L_{1,s} \geq N_s - \mathcal{H}_\lambda(D_{M_2})$, since all $x \in \mathcal{H}_{\lambda,s} \setminus D_{M_2}$ that are path-connected to D_K lie in the same component of $G(\mathcal{H}_{\lambda,s})$. Therefore

$$\mathbb{P}[s^{-2}L_{1,s} \leq \lambda\theta - 3\varepsilon^{1/2}] \leq \mathbb{P}[s^{-2}N_s \leq \lambda\theta - 2\varepsilon^{1/2}] + \mathbb{P}[s^{-2}\mathcal{H}_\lambda(D_{M_2}) \geq \varepsilon^{1/2}] + \mathbb{P}[U^c_{K,M_2}]$$

so that

$$\limsup_{s\to\infty} \mathbb{P}[s^{-2}L_{1,s} \leq \lambda\theta - 3\varepsilon^{1/2}] \leq \lambda\varepsilon^{1/2} + \varepsilon. \tag{2.15}$$

Conversely, note that if $s^2\lambda(\theta+\varepsilon) > s^{1/2}$ then

$$\mathbb{P}[s^{-2}L_{1,s} \geq \lambda(\theta+\varepsilon)] \leq \mathbb{P}[s^{-2}N'_s \geq \lambda(\theta+\varepsilon)]$$

which tends to zero. Combined with (2.15) this shows that $s^{-2}L_{1,s} \xrightarrow{P} \lambda\theta$.

If $s^2\lambda(\theta+\varepsilon) > s^{1/2}$ and $L_{1,s} + L_{2,s} \geq s^2\lambda(\theta+\varepsilon)$ then either $N'_s \geq s^2\lambda(\theta+\varepsilon)$ or $L_{1,s} + s^{1/2} \geq s^2\lambda(\theta+\varepsilon)$. Hence $\mathbb{P}[s^{-2}(L_{1,s}+L_{2,s}) > \lambda(\theta+\varepsilon)] \to 0$. Combined with (2.15) this shows that $s^{-2}(L_{1,s}+L_{2,s}) \xrightarrow{P} \lambda\theta$ and hence by Slutsky's theorem, $s^{-2}L_{2,s} \xrightarrow{P} 0$. □

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