Electron. Commun. Probab. **27** (2022), article no. 31, 1–10.
https://doi.org/10.1214/22-ECP471
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Local and uniform moduli of continuity of chi–square processes*

Michael B. Marcus† Jay Rosen‡

Abstract

Let $\{\eta_i(t), t \in [0,1]\}_{i=1}^k$ be independent copies of $\eta = \{\eta(t), t \in [0,1]\}$, a mean zero continuous Gaussian process. Let

$$Y_k := Y_k(t) = \sum_{i=1}^k \eta_i^2(t), \qquad t \in [0,1].$$

This paper shows how exact local (at 0) and uniform moduli of continuity (on [0,1]) of Y_k can be obtained from the exact local and uniform moduli of continuity of η.

Keywords: chi–square processes; moduli of continuity.
MSC2020 subject classifications: 60F15; 60G15; 60G17.
Submitted to ECP on June 2, 2021, final version accepted on May 20, 2022.

1 Introduction

Let $\eta = \{\eta(t); t \in [0,1]\}$ be a mean zero continuous Gaussian process with covariance $U = \{U(s,t), s,t \in [0,1]\}$, with $U(0,0) > 0$. Let $\{\eta_i; i = 1,\dots,k\}$ be independent copies of η and set,

$$Y_k(t) = \sum_{i=1}^k \eta_i^2(t), \qquad t \in [0,1]. \tag{1.1}$$

The stochastic process $Y_k = \{Y_k(t), t \in [0,1]\}$ is referred to as a chi–square process of order k with kernel U.

Chi-square processes appear naturally as limiting processes in various statistical models. See e.g. [9, 3, 2, 1, 7] and the references therein. Nevertheless our interest in them is primarily that they are simple examples of permanental processes that are easier to analyze and therefore provide a template for more general results about permanental processes.

Let $\{K(s,t), s,t \in T\}$ be a kernel, that need not be symmetric, with the property that for all $\mathbf{t}_n = (t_1,\dots,t_n)$ in T^n, the matrix $\mathcal{K}(\mathbf{t_n}) = \{K(t_i,t_j), i,j \in [1,n]\}$ determines an n-dimensional random variable random variable $(X_\alpha(t_1),\dots,X_\alpha(t_n))$ with Laplace transform,

$$E\left(e^{-\sum_{i=1}^n s_i X_\alpha(t_i)}\right) = \frac{1}{|I + \mathcal{K}(\mathbf{t_n})S(\mathbf{s_n})|^\alpha}, \tag{1.2}$$

*Research of Jay Rosen was partially supported by grants from the Simons Foundation.

†253 West 73rd. St., Apt. 2E, New York, NY 10023, USA. E-mail: mbmarcus@optonline.net

‡Department of Mathematics, College of Staten Island, CUNY, Staten Island, NY. E-mail: jrosen30@optimum.net

where $S(\mathbf{s_n})$ is a diagonal matrix with positive entries $\mathbf{s_n} = (s_1, \ldots, s_n)$, and $\alpha > 0$. It follows from the Kolmogorov Extension Theorem that $\{K(s,t), s, t \in T\}$ determines a stochastic process which we denote by $X_\alpha = \{X_\alpha(t), t \in T\}$ and refer to as an α-permanental process.

Clearly Y_k is a permanental process when $\mathcal{K}(\mathbf{t_n}) = \{E(Y_{t_i}, Y_{t_j}), i, j \in [1, n]\}$ and $\alpha = k/2$. The results obtained in this paper are used in our more general study of laws of the iterated logarithm for k/2 permanental processes [6].

In this paper we show that when the Gaussian process η has a local or uniform modulus of continuity the related $k/2$ chi-square process has a closely related local or uniform modulus of continuity.

Theorem 1.1. *Let $\phi(t)$ be a non-negative function function on $[0, \delta]$ for some $\delta > 0$. If*

$$\limsup_{t\to 0} \frac{\eta(t) - \eta(0)}{\phi(t)} = 1 \qquad a.s., \tag{1.3}$$

then for all integers $k \geq 1$,

$$\limsup_{t\to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)} = 2Y_k^{1/2}(0) \qquad a.s. \tag{1.4}$$

When $k = 1$ this is particularly simple. Since η is symmetric it follows from (1.3) that,

$$\liminf_{t\to 0} \frac{\eta(t) - \eta(0)}{\phi(t)} = -1 \qquad a.s. \tag{1.5}$$

Therefore, writing $Y_1(t) - Y_1(0) = (\eta(t) - \eta(0))(\eta(t) + \eta(0))$ and using the continuity of η, we see that

$$\limsup_{t\to 0} \frac{Y_1(t) - Y_1(0)}{\phi(t)} = 2\left(\eta(0) \vee -\eta(0)\right) \qquad a.s., \tag{1.6}$$

which is (1.4).

A result similar to Theorem 1.1 for the limiting behavior of chi–square sequences at infinity is given in [5, Lemma 6.5].

Set

$$\sigma^2(u,v) = E(\eta(u) - \eta(v))^2 \quad \text{and} \quad \widetilde{\sigma}^2(x) = \sup_{|u-v|\leq x} \sigma^2(u,v). \tag{1.7}$$

Theorem 1.2. *Assume that $\inf_{t\in[0,1]} U(t,t) > 0$ and,*

$$\lim_{x\to 0} \widetilde{\sigma}^2(x) \log 1/x = 0. \tag{1.8}$$

Let $\varphi(t)$ be a non-negative function function on $[0, 1]$. Then if

$$\lim_{h\to 0} \sup_{\substack{|u-v|\leq h \\ u,v\in\Delta}} \frac{\eta(u) - \eta(v)}{\varphi(|u-v|)} = 1 \qquad a.s., \tag{1.9}$$

for all intervals $\Delta \subset [0, 1]$, it follows that for all intervals $\Delta \subset [0, 1]$ and all integers $k \geq 1$,

$$\lim_{h\to 0} \sup_{\substack{|u-v|\leq h \\ u,v\in\Delta}} \frac{Y_k(u) - Y_k(v)}{\varphi(|u-v|)} = 2 \sup_{u\in\Delta} Y_k^{1/2}(u), \qquad a.s. \tag{1.10}$$

Necessary and sufficient conditions for the existence of functions ϕ and φ that satisfy (1.3) and (1.9), and what they are, are given in [4, Theorem 7.1.4]. However, they are very abstract. When $\sigma^2(u, v)$ satisfies mild regularity conditions, ϕ and φ are nice functions. An extensive treatment of Gaussian processes satisfying (1.3) and (1.9) is given in [4, Chapter 7]. It should be clear that not all Gaussian processes satisfy (1.3) and (1.9). For example, if a Gaussian process is continuously differentiable

$$\limsup_{t\to 0} \frac{\eta(t)-\eta(0)}{t} = \eta'(0), \tag{1.11}$$

and

$$\lim_{h\to 0} \sup_{\substack{|u-v|\le h\\ u,v\in\Delta}} \frac{\eta(u)-\eta(v)}{u-v} = \sup_{u\in\Delta} \eta'(u). \tag{1.12}$$

When η is a continuous Gaussian process with stationary increments, $\sigma^2(u,v)$ in (1.7) can be written as $\sigma^2(u-v,0)$. In this case if $\widetilde{\sigma}^2(x)$ is asymptotic to an increasing function at 0, then (1.9) implies (1.8). We discuss this further in Remark 2.3.

It is remarkable that the moduli functions ϕ and φ do not depend on k. This indicates that the extremes of the the increments of η take place on a very sparse set of points.

2 Proofs

Proof of Theorem 1.1. Let $\eta_i(t)$, $i=1,\dots,k$, be independent copies of $\eta(t)$. We write,

$$\begin{aligned} \eta_i^2(t)-\eta_i^2(0) &= (\eta_i(t)-\eta_i(0))(\eta_i(t)+\eta_i(0)) \\ &= (\eta_i(t)-\eta_i(0))(2\eta_i(0)+(\eta_i(t)-\eta_i(0))) \\ &= 2(\eta_i(t)-\eta_i(0))\eta_i(0)+(\eta_i(t)-\eta_i(0))^2. \end{aligned} \tag{2.1}$$

By (1.3)

$$\begin{aligned} &\limsup_{t\to 0} \frac{\sum_{i=1}^k(\eta_i(t)-\eta_i(0))^2}{\phi(t)} \\ &\qquad \le \sum_{i=1}^k \limsup_{t\to 0} \frac{|\eta_i(t)-\eta_i(0)|}{\phi(t)} \lim_{t\to 0} |\eta_i(t)-\eta_i(0)| = 0. \end{aligned} \tag{2.2}$$

Consequently, using (1.1) we see that,

$$\limsup_{t\to 0} \frac{Y_k(t)-Y_k(0)}{\phi(t)} = \limsup_{t\to 0} \frac{2\sum_{i=1}^k(\eta_i(t)-\eta_i(0))\eta_i(0)}{\phi(t)}. \tag{2.3}$$

Write,

$$\begin{aligned} &(\eta_i(t)-\eta_i(0))\eta_i(0) \\ &\quad = \left(\eta_i(t)-\frac{U(0,t)}{U(0,0)}\eta_i(0)\right)\eta_i(0) - \left(\frac{U(0,0)-U(0,t)}{U(0,0)}\right)\eta_i^2(0). \end{aligned} \tag{2.4}$$

We show below that

$$\limsup_{t\to 0} \frac{|U(0,0)-U(0,t)|}{U(0,0)\phi(t)} = 0. \tag{2.5}$$

Consequently,

$$\limsup_{t\to 0} \frac{Y_k(t)-Y_k(0)}{\phi(t)} = \limsup_{t\to 0} \frac{2\sum_{i=1}^k \left(\eta_i(t)-\frac{U(0,t)}{U(0,0)}\eta_i(0)\right)\eta_i(0)}{\phi(t)}. \tag{2.6}$$

The inovation in this proof is to recognize that

$$\sum_{i=1}^k \left(\eta_i(t)-\frac{U(0,t)}{U(0,0)}\eta_i(0)\right)\eta_i(0) \tag{2.7}$$

is actually a one dimensional real valued Gaussian process. Let $\{\xi_i(t), t \in [0,1]\}$, $i = 1, \ldots, k$, be independent copies of a mean zero Gaussian process $\{\xi(t), t \in [0,1]\}$, and set $\vec{\xi}(t) = (\xi_1(t), \ldots, \xi_k(t))$. Let $\vec{v} \in R^k$ with $\|\vec{v}\|_2 = 1$. Computing the covariances we see that,

$$\{(\vec{v} \cdot \vec{\xi}(t)), t \in [0,1]\} \overset{law}{=} \{\xi(t), t \in [0,1]\}. \tag{2.8}$$

(This relationship is used by P. Revesz in [8, Theorem 18.1] to obtain LILs for Brownian motion in R^k.)

Therefore, since $(\eta_i(t) - (U(0,t)/U(0,0))\eta_i(0))$ and $\eta_i(0)$ are independent for $i = 1, \ldots, k$, we see that,

$$\begin{aligned} &\left\{ \left(\vec{\eta}(t) - \frac{U(0,t)}{U(0,0)}\vec{\eta}(0)\right) \cdot \frac{\vec{\eta}(0)}{\|\vec{\eta}(0)\|_2}, t \in [0,1]\right\} \\ &\qquad \overset{law}{=} \left\{ \left(\eta(t) - \frac{U(0,t)}{U(0,0)}\eta(0)\right), t \in [0,1]\right\}, \end{aligned} \tag{2.9}$$

where $\vec{\eta}(t) = (\eta_1(t), \ldots, \eta_k(t))$ and

$$\|\vec{\eta}(t)\|_2 = \left(\sum_{i=1}^k \eta_i^2(t)\right)^{1/2} = Y_k^{1/2}(t). \tag{2.10}$$

Consequently, (2.6) implies that

$$\limsup_{t\to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)\|\vec{\eta}(0)\|_2} \overset{law}{=} \limsup_{t\to 0} \frac{2\left(\eta(t) - \frac{U(0,t)}{U(0,0)}\eta(0)\right)}{\phi(t)}. \tag{2.11}$$

Using (2.4) again and (2.5) we see that this implies that,

$$\limsup_{t\to 0} \frac{Y_k(t) - Y_k(0)}{\phi(t)\|\vec{\eta}(0)\|_2} \overset{law}{=} \limsup_{t\to 0} \frac{2(\eta(t) - \eta(0))}{\phi(t)} = 2, \tag{2.12}$$

where the last equality uses (1.3). Using (2.10) we obtain (1.4).

To obtain (2.5) we first note that it follows from (1.3) that,

$$\phi(t) = (E(\eta(t) - \eta(0))^2)^{1/2} h(t), \tag{2.13}$$

for some function h such that $\lim_{t\downarrow 0} h(t) = \infty$. To see this, suppose that it is false. Then there exists a sequence $\{t_k\}$, with $\lim_{k\to\infty} t_k = 0$, such that $\sup_k h(t_k) \le M$. Therefore, if (1.3) holds, we would have,

$$\sup_k \frac{\eta(t_k) - \eta(0)}{(E(\eta(t_k) - \eta(0))^2)^{1/2}} \le M \qquad a.s. \tag{2.14}$$

This is not possible because $\{\eta(t_k) - \eta(0)/(E(\eta(t_k) - \eta(0))^2)^{1/2}\}$ is a sequence of standard normal random variables.

Since,

$$U(0,0) - U(0,t) = E\left((\eta(t) - \eta(0))\,\eta(0)\right) \le E\left((\eta(t) - \eta(0))^2\right)^{1/2} U^{1/2}(0,0), \tag{2.15}$$

we have,

$$\frac{U(0,0) - U(0,t)}{\phi(t)} \le \frac{U^{1/2}(0,0)}{h(t)}. \tag{2.16}$$

Using the fact that $\lim_{t\downarrow 0} h(t) = \infty$ we get (2.5). □

Proof of Theorem 1.2. Note that (1.9) implies that $\{\eta(t), t \in [0,1]\}$ and therefore $\{\eta^2(t), t \in [0,1]\}$ are uniformly continuous on $[0,1]$, which in turn implies that for all $k \geq 1$, $\{Y_k(t); t \in [0,1]\}$ is uniformly continuous on $[0,1]$.

To show,

$$\lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \geq 2\sup_{t\in\Delta} Y_k^{1/2}(t), \qquad a.s. \tag{2.17}$$

it suffices to show that for any $d \in \Delta$,

$$\lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \geq 2Y_k^{1/2}(d), \qquad a.s. \tag{2.18}$$

This is because, (2.18) holding almost surely implies that for any countable dense set $\Delta' \subset \Delta$,

$$\lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \geq 2\sup_{d\in\Delta'} Y_k^{1/2}(d), \qquad a.s. \tag{2.19}$$

which implies (2.17).

Let $u, v, d \in \Delta$. We write,

$$\begin{aligned} \eta_i^2(u)-\eta_i^2(v) &= (\eta_i(u)-\eta_i(v))(\eta_i(u)+\eta_i(v)) \\ &= (\eta_i(u)-\eta_i(v))(2\eta_i(d)+(\eta_i(u)-\eta_i(d))+(\eta_i(v)-\eta_i(d))). \end{aligned} \tag{2.20}$$

Using the line above we see that we can write $Y_k(u)-Y_k(v)$ as a sum of three terms, two of which contain the product of two differences of η_i. We show below that these two terms, in the limit, are negligible with respect to the uniform modulus of $Y_k(u)-Y_k(v)$. To this end, using (2.10), we note that

$$\begin{aligned} &\lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)Y_k^{1/2}(d)} \\ &\geq \lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{2\sum_{i=1}^k(\eta_i(u)-\eta_i(v))\eta_i(d)}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)} \\ &\quad - \lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{2\sum_{i=1}^k(\eta_i(u)-\eta_i(v))(\eta_i(u)-\eta_i(d))}{\|\vec\eta(d)\|_2\varphi(|u-v|)}. \end{aligned} \tag{2.21}$$

It follows from (1.9) that,

$$\begin{aligned} &\lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{\sum_{i=1}^k(\eta_i(u)-\eta_i(v))(\eta_i(u)-\eta_i(d))}{\varphi(|u-v|)} \\ &\leq \lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{\sum_{i=1}^k|\eta_i(u)-\eta_i(v)|\sup_{u\in\Delta}|\eta_i(u)-\eta_i(d)|}{\varphi(|u-v|)} \\ &\leq \sum_{i=1}^k \sup_{u\in\Delta}|\eta_i(u)-\eta_i(d)| := \Delta^*. \end{aligned} \tag{2.22}$$

So that,

$$\begin{aligned} &\lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)Y_k^{1/2}(d)} \\ &\geq \lim_{h\to 0} \sup_{\substack{|u-v|\leq h\\ u,v\in\Delta}} \frac{2\sum_{i=1}^k(\eta_i(u)-\eta_i(v))\eta_i(d)}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)} - \frac{2\Delta^*}{\|\vec\eta(d)\|_2}. \end{aligned} \tag{2.23}$$

We now use our critical relationshp (2.8) to replace the sum in (2.23) by the difference of a real valued Gaussian process. To do this we write

$$\eta_i(u)-\eta_i(v)=V(v,d)\eta_i(u)-V(u,d)\eta_i(v)+G_i(u,v), \tag{2.24}$$

where, $V(u,v)=U(u,v)/U(d,d)$, and

$$G_i(u,v)=(1-V(v,d))\eta_i(u)-(1-V(u,d))\eta_i(v). \tag{2.25}$$

In this notation,

$$\begin{aligned}\frac{\sum_{i=1}^k(\eta_i(u)-\eta_i(v))\eta_i(d)}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)}&-\frac{\sum_{i=1}^k G_i(u,v)\eta_i(d)}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)}\\ &=\frac{\sum_{i=1}^k\left(V(v,d)\eta_i(u)-V(u,d)\eta_i(v)\right)\eta_i(d)}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)}.\end{aligned} \tag{2.26}$$

Note that for all $u,v\in[0,1]$,

$$E\left((V(v,d)\eta_i(u)-V(u,d)\eta_i(v))\eta_i(d)\right)=0.$$

This shows that $\eta_i(d)$ is independent of $\{V(v,d)\eta_i(u)-V(u,d)\eta_i(v);u,v\in[0,1]\}$. Therefore by (2.8),

$$\begin{aligned}&\Big\{\sum_{i=1}^k(V(v,d)\eta_i(u)-V(u,d)\eta_i(v))\frac{\eta_i(d)}{\|\vec\eta(d)\|_2};\quad u,v\in[0,1]\Big\}\\ &\quad\overset{law}{=}\Big\{V(v,d)\eta(u)-V(u,d)\eta(v);\quad u,v\in[0,1]\Big\}.\end{aligned} \tag{2.27}$$

It follows that,

$$\begin{aligned}&\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{\sum_{i=1}^k(\eta_i(u)-\eta_i(v))\eta_i(d)}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)}\\ &\qquad+\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{\sum_{i=1}^k|G_i(u,v)||\eta_i(d)|}{\|\vec\eta(d)\|_2\,\varphi(|u-v|)}\\ &\qquad\overset{law}{\ge}\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{V(v,d)\eta(u)-V(u,d)\eta(v)}{\varphi(|u-v|)}.\end{aligned} \tag{2.28}$$

Using (2.24) we write,

$$\begin{aligned}&\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{V(v,d)\eta(u)-V(u,d)\eta(v)}{\varphi(|u-v|)}\\ &\quad\ge\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{(\eta(u)-\eta(v))}{\varphi(|u-v|)}-\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{|G(u,v)|}{\varphi(|u-v|)}\\ &\quad=1-\lim_{h\to0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{|G(u,v)|}{\varphi(|u-v|)},\end{aligned} \tag{2.29}$$

where

$$G(u,v)=(1-V(v,d))\eta(u)-(1-V(u,d))\eta(v) \tag{2.30}$$

and we use (1.9) for the last line in (2.29).

It follows from (2.28) and (2.29) that,

$$\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{\sum_{i=1}^{k}(\eta_i(u)-\eta_i(v))\eta_i(d)}{\|\vec{\eta}(d)\|_2\,\varphi(|u-v|)}\overset{law}{\ge} 1-\mathcal{H}, \tag{2.31}$$

where,

$$\mathcal{H}=\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{\sum_{i=1}^{k}|G_i(u,v)||\eta_i(d)|}{\|\vec{\eta}(d)\|_2\,\varphi(|u-v|)}+\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{|G(u,v)|}{\varphi(|u-v|)}.$$

We now show that $\mathcal{H}=o(\Delta)$ almost surely. Using the Schwartz inequality followed by the triangle inequality we note that,

$$\frac{\sum_{i=1}^{k}|G_i(u,v)||\eta_i(d)|}{\|\vec{\eta}(d)\|_2\,\varphi(|u-v|)}\le\frac{\sum_{i=1}^{k}|G_i(u,v)|}{\varphi(|u-v|)}.$$

Therefore,

$$\mathcal{H}\le\sum_{i=0}^{k}\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{|G_i(u,v)|}{\varphi(|u-v|)},\qquad a.s., \tag{2.32}$$

where for notational convenience we have set $G_0=G$.

Using the notation in (1.7), it follows from (1.9) that we can write,

$$\varphi(h)=\widetilde{\sigma}(h)g(h),\ \text{where necessarily,}\ \lim_{h\to 0}g(h)=\infty. \tag{2.33}$$

This is shown by a minor modification of the argument used to prove (2.13). Note that for any sequence $h_k\to 0$ we can find sequences $\{u_k\}$, $\{v_k\}$ in Δ, with $|u_k-v_k|\le h_k$ such that $\widetilde{\sigma}(h_k)\le 2\sigma(u_k,v_k)$. Now, suppose that $\limsup_{h\to 0}g(h)=M$. Then by (1.9) we would have,

$$\sup_{k\to\infty}\frac{\eta(u_k)-\eta(v_k)}{\sigma(u_k,v_k)}\le 4M\qquad a.s.$$

This is not possible because each term $(\eta(u_k)-\eta(v_k))/\sigma(u_k,v_k)$ is a standard normal random variable.

We show in Lemma 2.1 below that,

$$|G(u,v)|\le\frac{\sigma(d,v)}{U^{1/2}(d,d)}|\eta(u)-\eta(v)|+\frac{\sigma(u,v)}{U^{1/2}(d,d)}|\eta(v)|. \tag{2.34}$$

Using this and then (1.9) and (2.33) we have,

$$\begin{aligned}\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{|G(u,v)|}{\varphi(|u-v|)}&\le\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{|\eta(u)-\eta(v)|\sigma(d,v)}{U^{1/2}(d,d)\,\varphi(|u-v|)}\\&\quad+\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{\sigma(u,v)}{U^{1/2}(d,d)\,\varphi(|u-v|)}\eta(v)\\&\le\sup_{d,v\in\Delta}\frac{\sigma(d,v)}{U^{1/2}(d,d)}+\lim_{h\to 0}\frac{1}{g(h)}\sup_{v\in\Delta}\frac{|\eta\,(v)\,|}{U^{1/2}(d,d)}=\frac{\widetilde{\sigma}(|\Delta|)}{U^{1/2}(d,d)},\end{aligned} \tag{2.35}$$

where $\widetilde{\sigma}(|\Delta|)$ is defined in (1.7). This shows that

$$\mathcal{H}\le(k+1)\frac{\widetilde{\sigma}(|\Delta|)}{U^{1/2}(d,d)}. \tag{2.36}$$

We now use (2.23), (2.31) and (2.35) to see that,

$$\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \tag{2.37}$$
$$\ge 2Y_k^{1/2}(d)\left(1-(k+1)\frac{\widetilde{\sigma}(|\Delta|)}{U^{1/2}(d,d)}\right)-2\Delta^*, \qquad a.s.,$$

where $d\in\Delta$ and Δ^* is defined in (2.22).

Suppose that $|\Delta|=1/n$. We show in Lemma 2.2 below that,

$$P\left(\Delta^*\ge k((1+2C)\widetilde{\sigma}^2(1/n)\log n)^{1/2}\right)\le\frac{2k}{n^C}. \tag{2.38}$$

Now let $\Delta(d,n)\subseteq\Delta$, be an interval of size $1/n$ that contains d. It follows from (2.37) and (2.38) applied to $\Delta(d,n)$ and $\Delta^*(d,n)$ that,

$$\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \tag{2.39}$$
$$\ge \lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta(d,n)}}\frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)}$$
$$\ge 2Y_k^{1/2}(d)\left(1-(k+1)\frac{\widetilde{\sigma}(1/n)}{U^{1/2}(d,d)}\right)-2k((1+2C)\widetilde{\sigma}^2(1/n)\log n)^{1/2},$$

except, possibly, on a set of measure $2k/n^C$. Taking $n\to\infty$, and using (1.8), gives (2.18) and consequently (2.17), which is the lower bound in (1.10).

We now obtain the upper bound in (1.10). Let $U=\inf_{t\in[0,1]}U(t,t)$ and note that $U>0$. Set $\Delta_{m,n}=\Delta\cap[\frac{m-1}{n},\frac{m+1}{n}]$. Analogous to (2.39) we have,

$$\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta_{m,n}}}\frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \tag{2.40}$$
$$\le 2Y_k^{1/2}(m/n)\left(1+(k+1)\frac{\widetilde{\sigma}(2/n)}{U^{1/2}}\right)+k((1+2C)\widetilde{\sigma}^2(2/n)\log n)^{1/2},$$

except, possibly, on a set of measure $2k/n^C$. The proof of (2.40) proceeds in essentially the same way as the proof of (2.39). In the proof of the lower bound in (2.39) we subtract several terms. In proving the upper bound in (2.40) we add these terms. It then follows that,

$$\lim_{h\to 0}\sup_{\substack{|u-v|\le h\\ u,v\in\Delta}}\frac{Y_k(u)-Y_k(v)}{\varphi(|u-v|)} \tag{2.41}$$
$$\le 2\sup_{m=1,\dots,n-1}Y_k^{1/2}(m/n)\left(1+(k+1)\frac{\widetilde{\sigma}(2/n)}{U^{1/2}}\right)+k((1+2C)\widetilde{\sigma}^2(2/n)\log n)^{1/2}$$
$$\le 2\sup_{v\in\Delta}Y_k^{1/2}(v)\left(1+(k+1)\frac{\widetilde{\sigma}(2/n)}{U^{1/2}}\right)+k((1+2C)\widetilde{\sigma}^2(2/n)\log n)^{1/2},$$

except possibly on a set of measure $2k/n^{C-1}$. Taking the limit as $n\to\infty$ gives the upper bound in (1.10). □

Lemma 2.1.

$$|G(u,v)|\le\frac{\sigma(d,v)}{U^{1/2}(d,d)}|\eta(u)-\eta(v)|+\frac{\sigma(u,v)}{U^{1/2}(d,d)}|\eta(v)|. \tag{2.42}$$

Proof. We write,

$$G(u,v)=(1-V(v,d))(\eta(u)-\eta(v))+(V(u,d)-V(v,d))\eta(v).$$

Note that,

$$\begin{aligned}|(V(u,d)-V(v,d))| &= \frac{|\,(E(\eta(u)-\eta(v))\eta(d))\,|}{U(d,d)}\\ &\leq \frac{\left(E(\eta(u)-\eta(v))^2E\eta^2(d)\right)^{1/2}}{U(d,d)}\\ &\leq \frac{\sigma(u,v)}{U^{1/2}(d,d)},\end{aligned}$$

and,

$$\begin{aligned}(1-V(v,d)) &= \frac{(E(\eta(d)-\eta(v))\eta(d))}{U(d,d)}\\ &\leq \frac{\left(E(\eta(d)-\eta(v))^2E\eta^2(d)\right)^{1/2}}{U(d,d)}\\ &\leq \frac{\sigma(d,v)}{U^{1/2}(d,d)}.\end{aligned}$$

□

Lemma 2.2.

$$P\left(\Delta^*\geq k((1+2C)\widetilde{\sigma}^2(|\Delta|)\log 1/|\Delta|)^{1/2}\right)\leq 2k|\Delta|^C; \tag{2.43}$$

(see (2.22)).

Proof. Let $\mathbf{a}$ be the median of $\sup_{u\in\Delta}(\eta_1(u)-\eta_1(d))$. It follows from [4, Lemma 5.4.3] that,

$$P\left(\sup_{u\in\Delta}|\eta_1(u)-\eta_1(d)|\geq \mathbf{a}+\widetilde{\sigma}(|\Delta|)t\right)\leq 2e^{-t^2/2}. \tag{2.44}$$

Since by [4, (7.113)],

$$\mathbf{a}=o(\widetilde{\sigma}^2(|\Delta|)\log 1/|\Delta|)^{1/2}, \tag{2.45}$$

we see that

$$P\left(\sup_{u\in\Delta}|\eta_1(u)-\eta_1(d)|\geq ((1+2C)\widetilde{\sigma}^2(|\Delta|)\log 1/|\Delta|)^{1/2}\right)\leq 2|\Delta|^C, \tag{2.46}$$

and since $\Delta^*=\sum_{i=1}^k\sup_{u\in\Delta}|\eta_i(u)-\eta_i(d)|$, see (2.22), this gives (2.43). □

Remark 2.3. When η has stationary increments, $\sigma^2(u,v)$ in (1.7) can be written as $\sigma^2(u-v)$. For $x\in[0,\delta]$ define,

$$\overline{\sigma}^2(x)=\mu\left(s:\sigma^2(x)\leq x\right),$$

where μ is Lebesgue measure. Clearly, $\overline{\sigma}^2(x)$ is an increasing function. It is called the increasing rearrangement of $\sigma^2(x)$. (See [4, Section 6,4] for more details.) We show in [4, (6.138)] that when η is continuous a.s.,

$$\lim_{x\to 0}\overline{\sigma}^2(x)\log(1/x)=0.$$

This shows that if $\widetilde{\sigma}^2(x)$ is asymptotic to an increasing function at 0, then (1.9), which implies that η is continuous a.s., implies (1.8).

References

[1] Aronowich M. and Adler R., *Extrema and level crossings of chi-square processes*, Advances in Applied Probability 18 (1986) 901–920. MR0867092

[2] Peng Liu and Lanpeng Ji, *Extremes of locally stationary chi-square processes with trend*, Stochastic Processes and their Applications 127 (2017) 497–525. MR3583761

[3] Peng Liu and Lanpeng Ji, *Extremes of Chi-square Processes with Trend*, arXiv:1407.6501v2 (2021).

[4] M. B. Marcus and J. Rosen, *Markov Processes, Gaussian Processes and Local Times*, Cambridge University Press, New York, (2006). MR2250510

[5] M. B. Marcus and J. Rosen, *Asymptotic Properties of Permanental Sequences*, Springer Briefs in Prob. and Math. Stat., Switzerland, (2021). MR4269776

[6] M. B. Marcus and J. Rosen, *Law of the iterated logarithm for $k/2$–permanental processes and the local times of associated Markov processes*, in preparation.

[7] Piterbarg V. I., *High excursions for nonstationary generalized chi-square processes*, Stochastic Processes and their Applications 53 (1994) 307–337. MR1302916

[8] P. Revesz, *Random Walk in Random and Non-random Environments*, Cambridge University Press, New York, (1990). MR1082348

[9] Charles-Elie Rabier and Alan Genz, *The supremum of Chi–Square processes, Methodology and Computing in Applied Probability*, Springer Verlag, (2014), 16 (3), 715–729. MR3239816