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# Erratum to: Coalescence estimates for the corner growth model with exponential weights*

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**Abstract**

We fix a mistake in the previously published paper Electron. J. Probab. 25: 1–31 (2020). The corrected version of the paper can also be found at arXiv:1911.03792.

**Keywords:** coalescence exit time; fluctuation exponent; geodesic; last-passage percolation; Kardar-Parisi-Zhang; random growth model.
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We made a mistake in the proof of Theorem 4.1, and we gratefully thank Manan Bhatia for pointing this out to us. In the original paper, the mistake appeared in (4.9), and we provide correct proof for it here.

Recall the beginning part of the proof from the original paper, and we continue from equation (4.8), with a weaker estimate:

$$\widetilde{\mathbb{P}}\big\{\exists z \text{ outside } [\![0, v_N]\!] \text{ such that } |\mathbf{Z}^{0\to z}| < \lfloor arN^{2/3}\rfloor\big\} \le Cr^{-3}. \tag{4.8}$$

Here $\widetilde{\mathbb{P}}$ is the modified environment defined above (4.5), and $\mathbf{Z}^{0\to z}$ is the exit time for the geodesic, which is define in the text between (3.4) and (3.5). Note the upper bound above is weaker than the one stated in (4.8) of the original paper, but it is enough for showing (4.5).

We treat the case $1 \le \mathbf{Z}^{0\to z} < \lfloor arN^{2/3}\rfloor$ of (4.8). The same arguments give the analogous bound for the case $-\lfloor arN^{2/3}\rfloor < \mathbf{Z} \le -1$. Start by perturbing the endpoint $v_N = \big(\lfloor N(1-\rho)^2\rfloor, \lfloor N\rho^2\rfloor\big)$ to a new point $w_N$ as was done in Lemma 4.2:

$$w_N = v_N - \lfloor \tfrac{1}{10}(1-\rho)rN^{2/3}\rfloor e_1.$$

Break up the northeast boundary of $[\![0, v_N]\!]$ into two regions $\mathcal{L}$ and $\mathcal{D}$ as in the diagram on the right of Figure 4.3. Recall the parameter $\lambda = \rho + \frac{r}{N^{1/3}}$ defined at the beginning of the proof, and note that the $\big(-(1-\lambda)^2, -\lambda^2\big)$-directed ray started from $w_N$ still goes through

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the interval $[arN^{2/3}, brN^{2/3}]$ on the $e_1$-axis. We now require $0 < a < \frac{1}{10}(1-\rho) < 10\frac{2}{\rho^2} < b$ for $a, b$ in order to apply Lemma 4.2 directly in the later part of the proof.

First consider geodesics that hit $\mathcal{D}$. In the remainder of this erratum, we will show

$$\widetilde{\mathbb{P}}\big\{\exists z \in \mathcal{D} : 1 \le \mathbf{Z}^{0\to z} < \lfloor arN^{2/3} \rfloor\big\} \le Cr^{-3}, \tag{4.9}$$

and this replaces the estimate (4.9) in the original paper.

Let $\sigma_1^{0\to x}$ denote the exit time of the optimal path among those $0 \to x$ paths whose first step is $e_1$. Then we have

$$\begin{aligned}\widetilde{\mathbb{P}}\big\{\exists z \in \mathcal{D} : 1 \le \mathbf{Z}^{0\to z} < \lfloor arN^{2/3} \rfloor\big\} &\le \widetilde{\mathbb{P}}\big\{\exists z \in \mathcal{D} : \sigma_1^{0\to z} < \lfloor arN^{2/3} \rfloor\big\} \\ &\le \widetilde{\mathbb{P}}\big\{\sigma_1^{0\to w_N} < \lfloor arN^{2/3} \rfloor\big\}.\end{aligned} \tag{4.10}$$

The second inequality comes from the uniqueness of maximizing paths: the maximizing path to $w_N$ cannot go to the right of a maximizing path to $\mathcal{D}$.

The task is to bound $\widetilde{\mathbb{P}}\big\{\sigma_1^{0\to w_N} < \lfloor arN^{2/3} \rfloor\big\}$. Define an environment with $\mathbb{P}^\lambda$ distribution by multiplying the $\mathbb{P}^\rho$ boundary weights by $\frac{1-\rho}{1-\lambda}$ on the $e_1$-axis and by $\frac{\rho}{\lambda}$ on the $e_2$-axis. We have now three coupled weight configurations with marginal distributions $\widetilde{\mathbb{P}}, \mathbb{P}^\rho$ and $\mathbb{P}^\lambda$. Denote their joint distribution by $\mathbb{P}$. Let $\widetilde{G}$, $G^\rho$, and $G^\lambda$ denote the last-passage values under these three environments. Additionally, let $\widetilde{G}_{0,w_N}(I)$ denote the last-passage value restricted to paths that exit through the set $I$.

To obtain

$$\widetilde{\mathbb{P}}\big\{\sigma_1^{0\to w_N} < \lfloor arN^{2/3} \rfloor\big\} \le Cr^{-3}$$

we show

$$\mathbb{P}\big\{\widetilde{G}_{0,w_N}([\![e_1, \lfloor arN^{2/3} - 1\rfloor e_1]\!]) < \widetilde{G}_{0,w_N}([\![\lfloor arN^{2/3}\rfloor e_1, \lfloor brN^{2/3}\rfloor e_1]\!])\big\} \ge 1 - Cr^{-3}. \tag{4.11}$$

By Lemma 4.2 there exists an event $A_1$ with $\mathbb{P}(A_1) \ge 1 - e^{-Cr^3}$ such that on this event the geodesic of $G^\lambda_{0,w_N}$ exits inside $[\![\lfloor arN^{2/3}\rfloor e_1, \lfloor brN^{2/3}\rfloor e_1]\!]$. The following equality holds on $A_1$:

$$\widetilde{G}_{0,w_N}([\![\lfloor arN^{2/3}\rfloor e_1, \lfloor brN^{2/3}\rfloor e_1]\!]) + \sum_{k=1}^{\lfloor arN^{2/3}-1\rfloor} \Big(\frac{1-\rho}{1-\lambda} - 1\Big)\omega_{ke_1} = G^\lambda_{0,w_N}.$$

Together with the fact that

$$\widetilde{G}_{0,w_N}([\![e_1, \lfloor arN^{2/3} - 1\rfloor e_1]\!]) \le G^\rho_{0,w_N},$$

the probability in (4.11) can be lower bounded as

$$(4.11) \ge \mathbb{P}\bigg(\bigg\{G^\rho_{0,w_N} < G^\lambda_{0,w_N} - \sum_{k=1}^{\lfloor arN^{2/3}-1\rfloor} \Big(\frac{1-\rho}{1-\lambda} - 1\Big)\omega_{ke_1}\bigg\} \cap A_1\bigg). \tag{4.12}$$

Up to a $\rho$-dependent constant

$$\mathbb{E}\bigg[\sum_{k=1}^{\lfloor arN^{2/3}-1\rfloor} \Big(\frac{1-\rho}{1-\lambda} - 1\Big)\omega_{ke_1}\bigg] \sim ar^2N^{1/3}, \tag{4.13}$$

and recall that the parameter $a$ can be fixed arbitrarily small. On the other hand, a computation in eqn. (5.53) in the arXiv version of [1] with $\kappa_N^1 = -\lfloor \frac{1}{10}(1-\rho)rN^{2/3}\rfloor$ and $\kappa_N^2 = 0$ gives

$$\mathbb{E}[G^\lambda_{0,w_N}] - \mathbb{E}[G^\rho_{0,w_N}] \ge c_1r^2N^{1/3} \tag{4.14}$$

where $c_1$ is another $\rho$-dependent constant. Hence for small $a > 0$ the event inside the braces in (4.12) should occur with high probability. This we now demonstrate.

Let

$$A_2 = \{G^{\lambda}_{0,w_N} > \mathbb{E}[G^{\rho}_{0,w_N}] + \tfrac{1}{2}c_1 r^2 N^{1/3}\}.$$

We show that $\mathbb{P}(A_2) \ge 1 - Cr^{-3}$. First we estimate the variance $\mathbb{V}\mathrm{ar}[G^{\rho}_{0,w_N}]$. The first equality below is Theorem 5.6 in the arXiv version of [1]:

$$\begin{aligned}\mathbb{V}\mathrm{ar}[G^{\rho}_{0,w_N}] &= -\frac{\lfloor(1-\rho)^2 N\rfloor - \lfloor\frac{1}{10}(1-\rho)rN^{2/3}\rfloor}{(1-\rho)^2} + \frac{\lfloor\rho^2 N\rfloor}{\rho^2} + \frac{2}{1-\rho}\mathbb{E}\Bigg[\sum_{k=1}^{0\vee \mathbf{Z}^{0\to w_N}} \omega^{\rho}_{ke_1}\Bigg] \\ &\le CrN^{2/3} + \frac{2}{1-\rho}\mathbb{E}\Bigg[\sum_{k=1}^{0\vee \mathbf{Z}^{0\to v_N}} \omega^{\rho}_{ke_1}\Bigg] \le CrN^{2/3} + C'N^{2/3}. \end{aligned} \tag{4.15}$$

Shifting the endpoint from $w_N$ back to $v_N$ inside the expectations increases the expected value because $\mathbf{Z}^{0\to w_N} \le \mathbf{Z}^{0\to v_N}$ almost surely. This gives the inequality between the two expectations. The last expectation is of order $N^{2/3}$ as shown through Lemma 5.8 and Proposition 5.9 in the arXiv version of [1]. Now we can bound:

$$\begin{aligned} \mathbb{P}(A_2^c) &= \mathbb{P}\big(G^{\lambda}_{0,w_N} \le \mathbb{E}[G^{\rho}_{0,w_N}] + \frac{c_1}{2}r^2N^{1/3}\big) \\ \text{(using (4.14))}\quad &\le \mathbb{P}(G^{\lambda}_{0,w_N} \le \mathbb{E}[G^{\lambda}_{0,w_N}] - \frac{c_1}{2}r^2N^{1/3}) \\ &\le \frac{c_2}{r^4N^{2/3}}\mathbb{V}\mathrm{ar}[G^{\lambda}_{0,w_N}] \\ \text{(Lemma 5.7, arXiv version of [1])}\quad &\le \frac{c_2}{r^4N^{2/3}}(\mathbb{V}\mathrm{ar}[G^{\rho}_{0,w_N}] + c_3 rN^{-1/3}(1-\rho)^2N) \le Cr^{-3}. \end{aligned}$$

For the last inequality we take $r \ge C'$ from the last line of (4.15). We have the further lower bound

$$(4.12) \ge \mathbb{P}\Bigg(\Bigg\{G^{\rho}_{0,w_N} < \mathbb{E}[G^{\rho}_{0,w_N}] + \frac{c_1}{2}r^2N^{1/3} - \sum_{k=1}^{\lfloor arN^{2/3}-1\rfloor}\Big(\frac{1-\rho}{1-\lambda}-1\Big)\omega_{ke_1}\Bigg\} \cap A_1 \cap A_2\Bigg). \tag{4.16}$$

We handle the i.i.d. sum above using large deviation of i.i.d. exponential random variables. Let $I(\cdot)$ denote the Cramér rate function of the Exp$(1-\rho)$ distribution. Then

$$\mathbb{P}\Bigg\{\Big(\frac{1-\rho}{1-\lambda}-1\Big)\sum_{k=1}^{\lfloor arN^{2/3}-1\rfloor}\omega_{ke_1} > \frac{c_1}{4}r^2N^{1/3}\Bigg\} \le e^{-arN^{2/3}I(c_5/a)} \le e^{-c_6 rN^{2/3}}$$

where $c_5$ is a certain constant, and for small enough $a > 0$, $I(c_5/a) \ge c_6/a$. Thus the event

$$A_3 = \Bigg\{\Big(\frac{1-\rho}{1-\lambda}-1\Big)\sum_{k=1}^{\lfloor arN^{2/3}-1\rfloor}\omega_{ke_1} \le \frac{c_1}{4}r^2N^{1/3}\Bigg\}$$

satisfies $\mathbb{P}(A_3) \ge 1 - e^{-c_6 rN^{2/3}}$. Continuing the lower bound,

$$(4.16) \ge \mathbb{P}\left(\left\{G^{\rho}_{0,w_N} < \mathbb{E}[G^{\rho}_{0,w_N}] + \frac{c_1}{4}r^2N^{1/3}\right\} \cap A_1 \cap A_2 \cap A_3\right). \tag{4.17}$$

The variance bound from (4.15) gives

$$\mathbb{P}\left\{G^{\rho}_{0,w_N} - \mathbb{E}[G^{\rho}_{0,w_N}] \ge \frac{c_1}{4}r^2N^{1/3}\right\} \le \frac{c_2}{r^4N^{2/3}}\mathbb{V}\mathrm{ar}[G^{\rho}_{0,w_N}] \le Cr^{-3}.$$

All four events inside the probability in (4.17) have probability at least $1 - Cr^{-3}$, and this verifies (4.9).

## References

[1] Timo Seppäläinen, *The corner growth model with exponential weights*, Random growth models, Proc. Sympos. Appl. Math., vol. 75, Amer. Math. Soc., Providence, RI, 2018, arXiv:1709.05771, pp. 133–201. MR3838898