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Rayleigh Random Flights on the Poisson line SIRSN

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Abstract

We study scale-invariant Rayleigh Random Flights ("RRF") in random environments given by planar Scale-Invariant Random Spatial Networks ("SIRSN") based on speed-marked Poisson line processes. A natural one-parameter family of such RRF (with scale-invariant dynamics) can be viewed as producing "randomly-broken local geodesics" on the SIRSN; we aim to shed some light on a conjecture that a (non-broken) geodesic on such a SIRSN will never come to a complete stop *en route*. (If true, then all such geodesics can be represented as doubly-infinite sequences of sequentially connected line segments. This would justify a natural procedure for computing geodesics.) The family of these RRF ("SIRSN-RRF"), is introduced *via* a novel axiomatic theory of abstract scattering representations for Markov chains (itself of independent interest). Palm conditioning (specifically the Mecke-Slivnyak theorem for Palm probabilities of Poisson point processes) and ideas from the ergodic theory of random walks in random environments are used to show that at a critical value of the parameter the speed of the scale-invariant SIRSN-RRF neither diverges to infinity nor tends to zero, thus supporting the conjecture.

Keywords: abstract scattering representation; critical SIRSN-RRF; Crofton cell; delineated scattering process; Dirichlet forms; dynamical detailed balance; environment viewed from particle; ergodic theorem; fibre process; Kesten-Spitzer-Whitman range theorem; Mecke-Slivnyak theorem; Metropolis-Hastings acceptance ratio; neighbourhood recurrence; Palm conditioning; Poisson line process; RRF (Rayleigh Random Flight); RWRE (Random Walk in a Random Environment); SIRSN (Scale-invariant random spatial network); SIRSN-RRF.
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1 Introduction

Aldous and Ganesan (2013) and Aldous (2014) introduced the notion of *Scale-Invariant Random Spatial Networks* (SIRSN), motivated by the now ubiquitous navigational tool of online maps (Google Maps, Bing Maps, OpenStreetMap). Informal experiments suggest that at normal scales the route-finding algorithms of these map tools exhibit scale-invariance (Aldous, 2014, Section 1.5), and the notion of a SIRSN was

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introduced to model this behaviour. A SIRSN is a random mechanism that generates networks built out of almost surely unique random routes between specified locations, required both to deliver scale-invariant statistics and to ensure considerable route-sharing between different routes.

Of course it is easy to produce random networks with translation- and isotropy-invariant statistics: the challenge is to find route-finding models which are also statistically invariant under change of scale.

In particular Aldous and Ganesan (2013) and Aldous (2014) introduced an elegant construction based on speed-marked Poisson lines (actually related to the "random pattern of streets" described by Mandelbrot, 1977, Plate 105). Significant mathematical effort (Kendall, 2017; Kahn, 2016) delivered rigorous proof that this led to a random map, the geodesics of which did indeed provide a model for the SIRSN mechanism. However one issue is still unresolved: can the geodesics of this map always be expressed as sequentially connected doubly-infinite lists of segments from the Poisson lines? Colloquially this can be expressed as the conjecture that geodesics on such a SIRSN will never come to a complete stop *en route*. If this conjecture is true, then it justifies the natural approximation of geodesics using finite-line approximations to the SIRSN.

Motivated by these considerations, this paper characterizes and describes a natural one-parameter family of random flight processes on the SIRSN. Such a process may be viewed as producing "randomly-broken local geodesics". We show that there is a critical value of the parameter at which the speed of the random process is neighbourhood-recurrent, amounting to evidence in favour of the conjecture.

To fix ideas and notation, we summarize the definition of a general SIRSN mechanism (Aldous, 2014):

Definition 1.1. *A SIRSN (based on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$) is a random mechanism that takes as input a set of nodes $x_1, \ldots, x_n$ in $\mathbb{R}^d$, and outputs a (random) network $\mathcal{N}(x_1, \ldots, x_n) = \mathcal{N}_\omega(x_1, \ldots, x_n)$ composed of continuous paths or* routes *$\mathcal{R}(x_i, x_j) = \mathcal{R}_\omega(x_i, x_j)$ connecting all pairs of distinct nodes x_i and x_j. (The explicit dependence on $\omega \in \Omega$ will typically be suppressed in the following). The connecting route $\mathcal{R}_\omega(x, y)$ between two specified endpoints x and y must be uniquely determined for almost all $\omega \in \Omega$. In addition the following* axioms *must be satisfied:*

1.1.1 Similarity-invariant statistics: *For each Euclidean similarity S (translation, rotation and scaling dilation), the networks $\mathcal{N}(Sx_1, \ldots, Sx_n)$ and $S\,\mathcal{N}(x_1, \ldots, x_n)$ have the same statistical law.*

1.1.2 Finite mean length: *Let $D_1 = \operatorname{len}(\mathcal{R}(x, y))$ be the length of the route $\mathcal{R}(x, y)$ between two nodes x and y separated by unit distance. It is required that the mean $\mathbb{E}\left[D_1\right]$ of this length be finite.*

1.1.3 *The* (Strong) SIRSN property*: Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ supports independent unit intensity Poisson processes Ξ_1, Ξ_2, … which are also independent of the SIRSN. Consider the extended network connecting all points of the* dense *Poisson point process $\widetilde{\Xi} = \bigcup\{\Xi_1, \Xi_2, \ldots\}$. Restrict attention to the "long-range" part of the network, containing those portions of connecting paths which are more than distance 1 from source or destination, with union given by $\bigcup\{\mathcal{R}(x, y) \setminus (\operatorname{ball}(x, 1) \cup \operatorname{ball}(y, 1)) : x, y \in \widetilde{\Xi}\}$. Viewing this part of the network as a* fibre process *(Chiu, Stoyan, Kendall, and Mecke, 2013, Chapter 8), it is required that this "long-range" fibre process should have finite length fraction ρ, which is to say, finite mean length per unit area / volume / hyper-volume. (Aldous uses the term "edge-intensity" for the length fraction ρ).*

Remark 1.2. Note that Aldous defines SIRSN only in the planar case of $d = 2$. Despite the complete absence of intersections between Poisson lines in spaces of dimension 3 or higher, SIRSN based on Poisson line processes in higher dimensions do in fact exist (Kendall, 2017; Kahn, 2016). Nevertheless, this paper focusses on the case $d = 2$; our questions (in particular the conjecture concerning Π-geodesics discussed below) have trivially negative answers for SIRSN based on Poisson line processes in dimensions $d \geq 3$.

Remark 1.3. The notion of a "dense" Poisson point process $\widetilde{\Xi}$ needs careful measure-theoretic interpretation (Aldous and Barlow, 1981; Kendall, 2000): it is used here as a convenient short-hand to refer to the union of a countable infinite ensemble of independent unit-intensity Poisson point processes Ξ_1, Ξ_2, ...

Remark 1.4. The assertion that Ξ_1, Ξ_2, . . . are independent of the SIRSN should be interpreted as saying that they are independent of the σ-algebra $\sigma\{\mathcal{N}(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}^d, n = 2, 3, \ldots\}$, viewing the networks $\mathcal{N}(x_1, \ldots, x_n)$ as random closed sets (the theory of random closed sets is covered for example in Chiu et al., 2013, Chapter 6).

Remark 1.5. Axioms 1.1.1, 1.1.2, 1.1.3 have strong implications. For example:

(a) The network obtained by using straight lines for routes (thus non-random) cannot be a SIRSN; almost all pairs of distinct routes have intersections which are singletons or empty, and if the network is used to connect the points of the Poisson point process Ξ then almost surely any distant point would then be connected to some other distant point of Ξ by a straight line passing within $1/2$ of the origin $\mathbf{o}$ and hence contributing length at least a positive amount ($\sqrt{3} = 2\sqrt{(1-1/4)}$ in dimension $d = 2$) within unit distance of $\mathbf{o}$. As a consequence, the intersection of the "long-range" fibre process with any bounded open set will almost surely have infinite total length, violating Axiom 1.1.3 and indeed its weaker variants $1.1.3'$ and $1.1.3''$ discussed below.

(b) Axiom 1.1.2 excludes networks generated by means of coupled Brownian bridges.

(c) The route $\mathcal{R}(x, y)$ is almost surely uniquely determined by its endpoints x and y. Nevertheless this uniqueness need not (and typically does not) hold simultaneously for all possible inputs.

(d) A related notion of a *weak SIRSN* replaces Axiom 1.1.3 by:

$1.1.3'$ The *Weak SIRSN property*: The infinite network $\mathcal{N}(\Xi) = \bigcup\{\mathcal{R}(x, y) : x, y \in \Xi\}$, (which connects all points of an independent unit intensity Poisson point process Ξ) should have finite mean length per unit area / volume / hyper-volume.

Of course Axiom 1.1.3 implies Axiom $1.1.3'$.

(e) A still weaker notion is that of a *pre-SIRSN*, further weakening Axiom $1.1.3'$:

$1.1.3''$ The *pre-SIRSN property* (Kendall, 2017): The infinite network $\mathcal{N}(\Xi) = \bigcup\{\mathcal{R}(x, y) : x, y \in \Xi\}$, connecting all points of an independent unit intensity Poisson point process Ξ, should have locally-finite random length measure (the mean length measure need not be locally finite).

Similarly Axiom $1.1.3'$ implies Axiom $1.1.3''$.

A priori the axioms in Definition 1.1 might be mutually exclusive, in which case no SIRSN could exist. Aldous (2014) proposed and rigorously justified a concrete example of a (planar) SIRSN, namely the "binary hierarchy". The routes of the network $\mathcal{N}(x_1,\ldots,x_n)$ are constructed as fastest paths lying in a dyadic cartesian network marked by varying speeds. The statistics of this network are neither stationary, isotropic, nor scale-invariant; however all these difficulties are removed by suitable randomization.

Aldous (2014) also proposed a possible SIRSN based on a speed-marked improper planar Poisson line process Π, in which the individual routes composing the network $\mathcal{N}(x_1,\ldots,x_n)$ are fastest paths using Π (we call these paths Π-*geodesics*). This mechanism is determined by choice of a parameter $\gamma > 2$: each line of Π is marked by a positive speed-limit $v \geq 0$, and Π is defined using a marked planar Poisson line process with intensity measure ν given in two equivalent forms by

$$\begin{aligned} \nu(\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\theta) &= \tfrac{\gamma-1}{2} v^{-\gamma}\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\theta\,, && (1.1)\\ &= \tfrac{\gamma-1}{2} v^{-\gamma}\;\sin\phi\;\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\phi\,. && (1.2)\end{aligned}$$

The first form (1.1) is based on parametrization of an (unsensed) line $\mathcal{L}$ using coordinates $r \in \mathbb{R}$ and $\theta \in [0,\pi)$: here r is the signed distance of $\mathcal{L}$ from a reference point often taken to be the origin $\mathbf{o}$, and θ is the angle made by $\mathcal{L}$ with a reference line often taken to be the x-axis. The second and equivalent form (1.2) is based on a parametrization which replaces r by the signed distance s *along* the reference line to the intersection with $\mathcal{L}$; the angle ϕ made by $\mathcal{L}$ with the reference line now has to be sine-weighted. We will use both kinds of parametrization below, signalled by reference to (1.1) or (1.2).

It is convenient to write $v(\mathcal{L})$ for the speed of a line $\mathcal{L} \in \Pi$.

Note that $\gamma > 1$ is required if all lines $\mathcal{L}$ of speed $v(\mathcal{L}) \geq v_0 > 0$ taken from such a speed-marked Poisson line process are to form a *proper* (non-speed-marked) Poisson line process $\Pi_{\geq v_0}$ of finite intensity. The factor $\frac{\gamma-1}{2}$ in (1.1) is a convenient normalization, chosen so that $\Pi_{\geq 1}$ (without speed-marks) forms a unit intensity Poisson line process. Routes of the SIRSN are fastest-possible Lipschitz paths whose almost-everywhere-defined velocities integrate the highly singular orientation field provided by Π and obey the speed limits given by the speed-marks v. If $\gamma > 2$ then Π can be used to define a random metric space on $\mathbb{R}^2$: the random metric is given by the time spent travelling from one point to another by the fastest route; and this is indeed a SIRSN (proof is a combination of Kendall, 2017 and Kahn, 2016).

The Poisson line process model for a SIRSN has the advantage of being intrinsically stationary and isotropic, with no need for extra randomization; this follows because the intensity measure (1.1) is invariant under Euclidean isometries of the underlying plane $\mathbb{R}^2$. Moreover the scaling transformation

$$\begin{aligned} r &\mapsto a\,\overline{r}\,, && (1.3)\\ v &\mapsto a^{\frac{1}{\gamma-1}}\;\overline{v}\,, \end{aligned}$$

also leaves both (1.1) and the equivalent (1.2) invariant. Consequently the distribution of the Poisson line SIRSN is invariant under scaling if the speed marks are adjusted as indicated in (1.3).

As noted above, the following conjecture on Poisson line SIRSN remains open.

Conjecture 1.6. *Given a SIRSN generated by a planar Poisson line process* Π*, consider a* Π-geodesic *providing the fastest route between two specified points. It is conjectured that such a* Π*-geodesic never comes to a complete halt strictly between its start and its destination.*

This unresolved conjecture is related to various observations in Aldous (2014) concerning singly and doubly infinite geodesics in general planar SIRSN: however it emphasizes the behaviour of the Π-geodesic along its entire length rather than at its start- and end-points. In complete contrast, note that in dimension $d \geq 3$ all non-trivial paths would have to halt *en route* a great deal, since paths in dimension $d \geq 3$ can change from one line to another only by using infinitely iterated infinite cascades of intervening lines.

It is fairly straightforward to use the methods of Kendall (2017, Section 4) to show that a straight-line positive-speed internal portion of a Π-geodesic in dimension $d = 2$ must connect directly to two other straight-line portions. However in principle it might still be possible (albeit implausible) for a Π-geodesic to contain a point which lies at the start and finish of two successive infinite sequences of sequentially connected straight-line portions whose speeds decay to zero near that point.

One consequence of an affirmative answer to Conjecture 1.6 would be that all Π-geodesics correspond to doubly-infinite lists of sequentially connected segments of lines from Π. Furthermore Π-geodesics between two points x and y lying on Π would be contained in the locally finite network of lines of Π of speed exceeding any small enough $v > 0$; highly relevant when simulating Π-geodesics.

We seek insight concerning Conjecture 1.6 by investigating an associated question of intrinsic interest: namely whether one can build a natural scale-invariant random process on Π which can be viewed as a "randomly-broken local Π-geodesic", and yet which is *speed-neighbourhood-recurrent* (that is to say, neighbourhood recurrence holds for the process given by the speed of the random process; so that this speed neither tends to zero nor drifts off to infinity, but returns at arbitrarily large times to a neighbourhood of the original speed). Failure to construct a natural random process of this form would reasonably count as evidence against the conjecture.

The study of random processes on SIRSN is also prompted by the widespread study of natural random processes on a random structure (compare for example the study of Liouville diffusions for Brownian maps and associated structures: Berestycki, 2015; Garban, Rhodes, and Vargas, 2016). Such random processes can be used to express a natural geometry for the structure. For example, in a different context, the Riemannian geometry expressed by a diffusion has been used to describe those smooth elliptic diffusions which admit Markovian maximal couplings (Banerjee and Kendall, 2017).

There are various options for defining random processes on a planar Poisson line SIRSN Π:

1: a conventional random walk on the plane, independent of the SIRSN, and connect successive random walk locations by Π-geodesic interpolation. However this construction is only weakly linked to the SIRSN structure of Π, and yields a random process for which speeds are trivially always revisiting zero, since almost surely each random walk location would miss all the lines of Π;

2: construction of Brownian motion on the line structure of Π, using a generalization of Walsh or "spider" Brownian motion (Barlow, Pitman, and Yor, 1989) to describe the way in which the Brownian motion switches between lines of different speeds. However such constructions are not easily related to the speed structure of Π, except indirectly by relating Brownian diffusion rate to Π speed marks.

3: we choose instead to adapt the notion of a Rayleigh Random Flight (RRF: introduced in Pearson, 1905, and associated correspondence). Implementation is scale-invariant using the following inductive construction: proceed at top speed along a chosen line, switch to intersecting lines in a manner controlled by relative speeds, (requiring switches to faster lines always to occur), and choose the new direction of movement equiprobably from the two directions along the new line.

The RRF construction is a planar version of the one-dimensional scattering processes studied by Kendall (1987), in which coupling is used to prove limits of Brownian type for an inhomogeneous random scattering process on the line. Such scattering processes also arise naturally in statistical mechanics (see for example McKean, 2014, chapter 11). In the SIRSN context, interest lies in whether it is possible to choose parameters for a scale-invariant RRF on a Poisson line SIRSN (essentially, to determine the probability of switching when encountering an intersection) such that the speed of movement of the RRF particle forms a neighbourhood-recurrent random process, neither diverging to infinite speed nor converging to zero speed. The resulting SIRSN-RRF can be viewed as a "randomly-broken local Π-geodesic", so if neighbourhood-recurrence of speed can be obtained by a natural choice of parameters then this supports the conjecture that Π-geodesics do not halt *en route*.

We ease the task of describing constructions of such SIRSN-RRF by assuming that each line of the SIRSN is additionally furnished with a random choice of direction.

The current section has explained the rationale and the mathematical content of the notion of a SIRSN, and has motivated the study of SIRSN-RRF by relating the possibility of speed-neighborhood recurrence of SIRSN-RRF to the question of whether SIRSN Π-geodesics can contain interior points at which they come to a complete stop. Section 2 then introduces concepts which are helpful in analysing RRF on SIRSN, for which possible switching points form countable dense subsets on each line of the SIRSN. The complexity of this situation is usefully addressed by taking an axiomatic approach. We consider an *abstract scattering representation* (Definition 2.1); namely an algebraic representation of non-lazy discrete-state-space Markov chains (chains that have no chance of not moving) in terms of *transmission probabilities* (intuitively, the probability of arriving at a state but not necessarily stopping and changing direction there) and *scattering probabilities* (intuitively the probability of stopping and changing direction at a state given that the process arrives there). In the context of a Poisson line SIRSN Π, the states of the chain are ordered distinct pairs of lines, corresponding to points at which the RRF switches from one line to another; so the state-space can be written as $(\Pi\times\Pi)\setminus\Delta$ where Δ is the diagonal of $\Pi\times\Pi$.

All non-lazy Markov chains admit abstract scattering representations: the presence of an involution $a\mapsto\widetilde{a}$ of the state-space (corresponding to reversal of direction of travel in our application) permits the state-space to be broken up into *scattering classes* $\mathcal{E}$ (eventually corresponding to the lines of Π), and a suitably compatible total ordering for each scattering class (see Definition 2.6; in the RRF case, a selection for each line of one of the two possible linear orderings) then permits the transition probabilities to be expressed purely in terms of the scattering probabilities and probabilities $\omega_{a,\pm}$ of initial binary choices of direction within the relevant scattering class (see Theorem 2.8; and note that it is at this stage that it pays to assume preferred directions for the lines of Π). If furthermore the involution leads to dynamical detailed balance with respect to a given invariant measure π on Π (Definition 2.13) and the scattering representation is unbiased, in the sense given in that definition, that then the scattering probabilities themselves, and π, are necessarily defined in terms of ratios of prescribed functions $\kappa(\mathcal{E})$ of equivalence classes (Theorem 2.23). This leads to a highly desirable conclusion: the stochastic dynamics of a dynamically reversible RRF on a SIRSN Π can be defined using prescribed a scattering class function $\kappa(\mathcal{E})$ (where each $\mathcal{E}$ is actually a line $\mathcal{L}$ of Π).

Section 3 continues the story by taking account of the similarity symmetries of the SIRSN Π controlled by intensity measure $\frac{\gamma-1}{2}v^{-\gamma}\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\theta$. A dynamically reversible RRF on Π is said to have similarity-equivariant dynamics if scattering probabilities and ratios of evaluations of π (considered as functions of Π) are similarity-invariant, while π itself is Euclidean-invariant. Palm distribution theory (specifically the Slivnyak-Mecke Theorem 3.4) and ergodicity of Π (Theorem 3.5) can now be used to argue that ratios of $\kappa(\mathcal{L})$ must equal the α-th power of ratios of speeds $v(\mathcal{L})$, for a fixed positive exponent

α (Theorem 3.6). If line-changes are given by a recipe of Metropolis-Hastings form then Theorem 3.6 shows that scattering probabilities and π are determined entirely by line-speeds and the exponent α: moreover $\alpha > \gamma - 1$ is required if scattering is to be non-degenerate (i.e: is not to happen immediately after time 0).

We thus obtain a natural definition of a SIRSN-RRF on the SIRSN Π, parametrized by the exponent α (Definition 3.7); moreover this SIRSN-RRF is then an irreducible Markov chain on the state-space of ordered intersections $(\Pi \times \Pi) \setminus \Delta$ when conditioned on Π (Lemma 3.10).

Section 4 considers the Markov chain given by the *relative environment* of the SIRSN-RRF, which is to say, the environment viewed from the RRF after using the group of similarities to transform the RRF state to be the intersection of a unit-speed line along the x-axis (corresponding to the current line of travel) and a further variable-speed line (corresponding to the previous line of travel) intersecting the x-axis at the origin $\mathbf{o}$. Working with Dirichlet forms related to the dynamically reversible SIRSN-RRF, and using the Slivnyak-Mecke Theorem 3.4, we establish that the Markov chain given by the relative environment has stationary *probability* distribution given (Theorem 4.3) by the independent superposition of a unit-speed line along the x-axis with a line through $\mathbf{o}$ of

(a) a random log-speed given by a possibly asymmetric Laplace density;

(b) and a random angle ϕ with the unit-speed line (with ϕ having sine-weighted density);

(c) together with a copy of Π.

The density of the log-speed is symmetric exactly at the critical value $\alpha = 2(\gamma - 1)$. The non-ergodic part of Birkhoff's ergodic theorem now allows us to rule out non-critical SIRSN-RRF, as in these cases the average log-relative speed has positive chance of converging to a non-zero limit, and thus the log-speed must have positive chance of never returning to any bounded interval around the initial log-speed.

Section 5 uses all this to show that the critical SIRSN-RRF is speed-neighbourhood-recurrent. This is done by establishing that the Markov chain given by the relative environment of the SIRSN-RRF, when started according to the stationary probability distribution, is in fact ergodic. This follows from Theorem 5.2, a variation on an argument of Kozlov (1985), using the ergodicity of Π (note that the argument works for all $\alpha > \gamma - 1$). The main result of this paper, the neighborhood-recurrence of the log-speed process in the critical case (Theorem 5.3) now follows from Theorem 5.1, an adaptation of the classic Kesten-Spitzer-Whitman range theorem to the case of continuous one-dimensional state-space.

The concluding Section 6 discusses related results and possibilities for future work.

2 Rayleigh random flights (RRF) and abstract scattering

The first task is to define a suitable family of Rayleigh random flight processes on Π with scale-invariant dynamics; a principal criterion for suitability is that the resulting process should be amenable to calculation. Moreover it is appropriate for the process to be able to change direction whenever encountering any one of the dense countable set of line-intersections along a given Poisson line. It is useful to control the complexity of this set-up by adopting an abstract approach based on general scattering processes. An additional merit of this approach is that it permits isolation of a particular one-parameter family of *discrete-time* Rayleigh random flight processes (RRF) on Π which can be naturally described as SIRSN-RRF.

We motivate the definition of abstract scattering by first making a few remarks about possible (continuous-time) RRF on Poisson line SIRSN. Let Π be an improper speed-marked planar Poisson line process, with intensity measure given by Equation (1.1) above. Our primary interest is in the SIRSN case $\gamma > 2$, although our results extend to the borderline *SIRSN candidate* (but non-SIRSN) case $\gamma = 2$. Note that, in case $\gamma = 2$, Π still possesses Euclidean- and scaling-invariance, even though it no longer possesses the SIRSN property. A reasonable if informal definition of a (continuous-time) Rayleigh random flight X on Π runs as follows: it is a continuous-time process living on the set which is the (countable) union of the lines of Π (this random dense F_σ Lebesgue null-set is called the *silhouette* in Kendall, 2017: it can be understood as the countable union of the random closed sets formed by lines of speed exceeding $1/m$ for $m = 1, 2, \ldots$). The continuous-time Rayleigh random flight process travels along the lines of Π, moving at the maximum speed permitted by the relevant speed-limits on Π, with changes of direction (switching onto different lines) occurring at a carefully defined sequence of random Markov times $0 < \tau_1 < \tau_2 < \ldots$ which will be made up of some (but by no means all) encounters with intersections of lines of Π. We will consider only cases in which the resulting sequence of random times will in fact be almost surely locally finite up to a possibly infinite "explosion time" which is the accumulation point of the times of direction-change. In fact if $\gamma \geq 2$ then the explosion time almost surely cannot correspond to the path becoming unbounded in finite time. This is because (in the terminology of Kendall, 2017) the path of the RRF is a Π-*path*, namely a locally Lipschitz path on Π with top speed almost always locally bounded above by relevant speed marks. When $\gamma \geq 2$ a comparison argument (Kendall, 2017, Theorem 2.6) bounds distances travelled by Π-paths begun in a specified compact set and travelling for specified time $T < \infty$. Note that the case $\gamma < 2$ is *not* convenient for our purposes; the results of Kendall (2017) then imply that it is possible for locally Lipschitz Π-paths to obey the Π speed limits and yet to diverge to infinity in finite time, resulting in sterile questions about failure of stochastic completeness.

As noted above, we facilitate discussion of the direction of travel along lines of Π, by making arbitrary choices of sense of direction to endow all the lines of Π with preferred directions.

The resulting continuous-time RRF X will be piecewise-linear, and so its paths can be required to be càdlàg and right-differentiable. Let Y denote the right-hand time-derivative of X:

$$Y(t) \quad = \quad \lim_{s \downarrow 0} \frac{X(t+s) - X(t)}{s} \,.$$

In particular, the speed $|Y|$ of X is the maximum permitted on the current line, which is to say that it is determined by the speed-mark of the current line:

$$|Y(t)| \quad = \quad v(\mathcal{L}) \qquad \text{for } \mathcal{L} = X(t) + Y(t) \cdot \mathbb{R}\,,$$

where $\mathcal{L} = X(t) + Y(t) \cdot \mathbb{R}$ is always a line of Π when $Y(t) \neq 0$, and $v(\mathcal{L})$ is the speed-limit of $\mathcal{L}$.

It is convenient to consider the *augmented* process

$$((X(t) + Y(t) \cdot \mathbb{R}, |Y(t)|, X(t) + Y(t-) \cdot \mathbb{R}, |Y(t-)|) \;:\; t \geq 0)$$

recording both the current unsensed line of travel $X(t) + Y(t) \cdot \mathbb{R}$ and the previous unsensed line $X(t)+Y(t-)\cdot\mathbb{R}$ as well as the corresponding absolute speeds $|Y(t)|, |Y(t-)|$. It is convenient to omit the actual sense or signed direction of travel; this augmented process is Markov conditional on Π even given the augmentation by $|Y(t)|$ and the further augmentation by $|Y(t-)|$ (which will facilitate later discussion of dynamical detailed balance), this is because knowledge of $|Y(t)|$ and $|Y(t-)|$ can be obtained from knowledge of the speed-marks of the corresponding lines of Π. We derive a discrete-time RRF by sampling the augmented process at the times τ_n when it switches from one line to another. This (discrete-time) RRF is the main subject of study for this paper. Letting $\mathcal{L}_-(\tau_n)$ be the previous line of travel and letting $\mathcal{L}_0(\tau_n)$ be the current line of travel, we know that $X(\tau_n)$ is the unique point in the intersection $\mathcal{L}_-(\tau_n) \cap \mathcal{L}_0(\tau_n)$. This corresponds to obtaining the RRF by sampling the continuous-time Rayleigh random flight process at the instants of scattering and just before the choices of direction of travel on the new line; we may then consider the RRF as the sampled process as $Z = (Z_n = (\mathcal{L}_-(\tau_n), \mathcal{L}_0(\tau_n)) \ : \ n \geq 1)$ with state-space

$$(\Pi \times \Pi) \setminus \Delta \quad = \quad \{(\mathcal{L}_-, \mathcal{L}_0) \ : \ \mathcal{L}_-, \mathcal{L}_0 \in \Pi, \ \mathcal{L}_- \neq \mathcal{L}_0\}\,, \tag{2.1}$$

given by ordered pairs of (speed-marked) lines $\mathcal{L}_-, \mathcal{L}_0 \in \Pi$, removing the diagonal set $\Delta = \{(\mathcal{L}, \mathcal{L}) : \mathcal{L} \in \Pi\}$ so that $\mathcal{L}_-, \mathcal{L}_0$ must be distinct. We repeat for emphasis that Z is a Markov chain when *quenched*, which is to say, when conditioned on the random environment given by Π.

The process $(Z_n : n \geq 1)$ is a particular instance of a *generalized scattering process*. It can be viewed as moving from the intersection $\mathcal{L}_- \cap \mathcal{L}_0$ along $\mathcal{L}_0$ past further intersections until it chooses to stop (is scattered) at a new intersection $\mathcal{L}_0 \cap \mathcal{L}_+$, where it will switch to the new line $\mathcal{L}_+$ and continue. This is a planar variation of the scattering processes discussed for example in Kendall (1987). We rise above confusing detail about SIRSNs by introducing a novel algebraic representation of general scattering for discrete state-space Markov chains, always keeping in mind the motivating example of RRF on a Poisson line SIRSN. A further benefit of this abstract approach is that it will later allow us to characterize a natural one-parameter family of RRFs respecting the symmetries of the SIRSN Π.

Definition 2.1. *Consider a non-lazy discrete-time countable state-space Markov chain Z, (non-lazy, so the transition probability matrix has zeroes on the main diagonal). An* abstract scattering representation *for Z expresses the one-step transition probabilities $p_{a,b}$ of Z in product form*

$$p_{a,b} \quad = \quad \omega_{a,b} s_b \qquad \textit{for all states } a, b\,,$$

for prescribed $s_a \in (0,1]$ and $\omega_{a,b} \in [0,1]$, where the transmission probabilities $\omega_{a,b}$ *form a matrix with zeroes on the diagonal and the* scattering probabilities s_a *are all positive.*

Remark 2.2. Because of the positivity requirement $s_a > 0$, it follows that $p_{a,b} > 0$ if and only if $\omega_{a,b} > 0$. Since the matrix $(\omega_{a,b})$ vanishes on the diagonal, the same must be true of all such matrices $(p_{a,b})$.

The content of this definition is algebraic rather than probabilistic. In particular the system of scattering and transmission probabilities is not uniquely defined by the resulting Markov kernel (note that the choice of transmission probabilities $\omega_{a,b} = p_{a,b}$ and scattering probabilities $s_a = 1$ for all states a and b always determines an abstract scattering representation, since all $p_{a,a}$ are required to vanish!) and the $\omega_{a,b}$ and s_a are described as "probabilities" only because all are required to lie in the unit interval $[0,1]$. Indeed the system of scattering and transmission probabilities need not necessarily reflect a specific stochastic mechanism of transmission and reflection (though this will eventually be the case for our particular example). In this sense a general abstract scattering representation is purely formal. Interesting examples arise by combining scattering probabilities with transmission according to a fixed stochastic dynamical system: in our case the very simple system of constant-speed movement in fixed directions. The resulting axiomatic approach may offer useful perspectives on questions of statistical survival analysis (see for example Andersen, Borgan, Gill, and Keiding, 1993). Scattering processes based on deterministic movement also arise in the *ZigZag sampler* in Markov chain Monte Carlo theory (described for example by Bierkens, Fearnhead, and Roberts, 2019; see also the notion of piecewise-deterministic Markov processes introduced by Davis, 1984).

In the context of possible RRF on a SIRSN or SIRSN candidate, as noted above, the relevant state space is the set of ordered pairs of distinct lines $(\mathcal{L}_-, \mathcal{L}_0)$ (for $\mathcal{L}_-, \mathcal{L}_0 \in \Pi$), corresponding to pre- and post-scattering lines. The quantity $\omega_{a,b}$ can be interpreted as the probability of getting at least as far as $b = (\mathcal{L}_0, \mathcal{L}_+)$ along a line $\mathcal{L}_0$ from $a = (\mathcal{L}_-, \mathcal{L}_0)$, while s_b measures the probability of the process being scattered from the current line $\mathcal{L}_0$ onto the new line $\mathcal{L}_+$.

Remark 2.3. As an aside, we indicate a partial answer to an interesting foundational question: which matrices of probabilities $(\omega_{a,b} :\ \text{states } a, b)$ (with zeroes down the diagonal) can serve as the matrix of transmission probabilities for an abstract scattering representation of some Markov chain? Consider the vectors $w^{(a)}$ given by $w^{(a)}_b = \omega_{a,b}$. Suppose these all lie in the Banach space $\ell^1(S)$, so $\|w^{(a)}\|_1 = \sum_b \omega_{a,b} < \infty$ for all states a. Let C be the ℓ^1-closure of the convex hull of the vectors $te^{(a)}$, where the $e^{(a)}$ are canonical basis vectors of $\ell^1(S)$ and $-\infty < t < 1$. Then an application of the Hahn-Banach theorem shows that the matrix $(\omega_{a,b} :\ \text{states } a, b)$ (with non-negative entries, and zeroes down the diagonal) can serve as part of an abstract scattering representation of some Markov chain exactly when the ℓ^1-closure of the affine span of the $w^{(a)}$ (for all states a) does not intersect the interior of the set C. (However ℓ^1-summability of rows $w^{(a)}$ of the matrix $(\omega_{a,b})$ will not hold in the case of our motivating example.)

Consider a Markov chain admitting a scattering representation, such that its state-space admits an involution $a \mapsto \widetilde{a}$ (with no fixed points). Suppose further (to simplify future exposition) that $p_{a,\widetilde{a}} = 0$ for all states a. Then the state-space can be partitioned into equivalence classes as follows.

Definition 2.4. *Consider a general abstract scattering representation of a Markov chain. Suppose $a \mapsto \widetilde{a}$ is an involution on the state-space with no fixed points and such that $p_{a,\widetilde{a}} = 0$ for all states a. Then the state-space supports an equivalence relation $a \sim b$ which is obtained by saturating the relation $a \leftrightsquigarrow b$, holding if either $\omega_{a,\widetilde{b}} > 0$ or $\omega_{b,\widetilde{a}} > 0$. Equivalence classes $\mathcal{E}$, $\mathcal{E}'$ (which we will refer to as* scattering classes*) are said to be* connected *when there exists a finite connecting chain of equivalence classes $\mathcal{E} = \mathcal{E}_1$, $\mathcal{E}_2, \ldots, \mathcal{E}_k = \mathcal{E}'$ and states $a_k \in \mathcal{E}_k$ in these classes such that $\widetilde{a}_k \in \mathcal{E}_{k+1}$ for $k = 1, \ldots, k-1$.*

In the case of RRFs on Π, the relevant involution is given by transposition of lines: if $a = (\mathcal{L}_-, \mathcal{L}_0)$ then $\widetilde{a} = (\mathcal{L}_0, \mathcal{L}_-)$. Subject to regularity conditions, scattering classes will then correspond to the lines of Π; the scattering class $\mathcal{E}(\mathcal{L})$ corresponding to line $\mathcal{L}$ is given by the set of all states $(\mathcal{L}_-, \mathcal{L})$ from which the scattering process moves off along line $\mathcal{L}$;

$$\mathcal{E}(\mathcal{L}) \quad = \quad \{(\mathcal{L}_-, \mathcal{L}) : \mathcal{L}_- \in \Pi \setminus \{\mathcal{L}\}\} \ . \tag{2.2}$$

Note moreover that, in the Rayleigh Random Flight application, a state a equivalent to its involution $\widetilde{a}$ would correspond to a "reverse scatterer", which would be able to reverse the direction of travel of the RRF. These are not present in the simple version studied here, but correspond to the one-dimensional symmetric scatterers considered by Kendall (1987). Their introduction might lead to "Brownian-like limits" for RRF on Π, but we do not pursue this here. In particular we would then need to adjust this exposition and definitions to account well for instances when $p_{a,\widetilde{a}}$ could be positive.

Remark 2.5. In general $\omega_{a,\widetilde{b}} > 0$, $\omega_{b,\widetilde{c}} > 0$ need *not* imply $\omega_{a,\widetilde{c}} > 0$. If $a \sim c$ then either $a = c$ or there is a finite sequence of states $a = b_0$, b_1, $\ldots b_{n-1}$, $b_n = c$ such that, for each successive pair of indices b_i and b_{i+1}, either $\omega_{b_i,\widetilde{b}_{i+1}} > 0$ or $\omega_{b_{i+1},\widetilde{b}_i} > 0$.

Given an abstract scattering representation and involution, if the scattering classes can be furnished with total orderings which are compatible with the scattering representation then then we can obtain an attractively simple representation of the transmission probabilities (see Theorem 2.8 below).

Definition 2.6. *A* delineated scattering process *is a Markov chain which admits an abstract scattering representation together with an involution $a \mapsto \widetilde{a}$, and satisfying the following compatibility property: each scattering class $\mathcal{E}$ possesses a total ordering $\prec$ such that the following holds: if $a \prec b \prec c$ in $\mathcal{E}$ then $\omega_{b,\widetilde{a}} + \omega_{b,\widetilde{c}} \leq 1$ and moreover*

$$\omega_{a,\widetilde{c}} \quad \leq \quad \omega_{a,\widetilde{b}}\left(1 - s_{\widetilde{b}}\right) \qquad \text{if } a \prec b \prec c \text{ or } c \prec b \prec a\,, \tag{2.3}$$

*If it is required to emphasize the rôles of the involution $a \mapsto \widetilde{a}$ and the family of scattering classes $\mathfrak{E} = \{\mathcal{E} : \mathcal{E} \text{ a scattering class}\}$ then we speak of an $(a \mapsto \widetilde{a}, \mathfrak{E})$-*delineated scattering process*.*

Inequality (2.3) implies that $\omega_{a,\widetilde{b}}$ is weakly increasing in b if b ranges over the scattering class of a and $b \prec a$, and weakly decreasing in b if b ranges over the scattering class of a and $a \prec b$. The requirement $\omega_{a,\widetilde{b}} + \omega_{a,\widetilde{c}} \leq 1$ if $b \prec a \prec c$ is suggestive of a scattering mechanism that chooses the direction of travel on $\mathcal{E}$ at random once it is scattered at $a \in \mathcal{E}$.

Remark 2.7. Definition 2.6 could be expressed in terms of separation rather than ordering: however the use of orderings permits easier notation.

In the case of RRF on Π, the required total ordering on a scattering class is obtained from the natural linear ordering on the corresponding line (using the arbitrary preferred direction chosen for that line).

In general the presence of an involution $a \mapsto \widetilde{a}$ as in Definition 2.6, together with compatible total orderings on scattering classes $\mathcal{E} \in \mathfrak{E}$, makes it possible to write down explicit expressions for the transmission probabilities largely in terms of scattering probabilities. Suppose that a lies in the scattering class $\mathcal{E}$ Recall that the transition probabilities $p_{a,\widetilde{b}} = \omega_{a,\widetilde{b}} s_{\widetilde{b}}$ form a stochastic matrix, moreover $\omega_{a,\widetilde{a}} = 0$ (a consequence of the requirement that $p_{a,\widetilde{a}} = 0$ together with the requirement that $s_a > 0$), while $\omega_{a,\widetilde{b}} = 0$ if $b \notin \mathcal{E}$ (following from the definition of scattering class $\mathcal{E}$ in Definition 2.4). Accordingly,

$$\sum_{z \in \mathcal{E}\,:\, z \prec a} \omega_{a,\widetilde{z}} s_{\widetilde{z}} + \sum_{z \in \mathcal{E}\,:\, a \prec z} \omega_{a,\widetilde{z}} s_{\widetilde{z}} \quad = \quad \sum_{z \in \mathcal{E}\,:\, z \neq a} \omega_{a,\widetilde{z}} s_{\widetilde{z}} \quad = \quad \sum_{z} p_{a,\widetilde{z}} \quad = \quad 1\,.$$

This permits us to represent the statistical behaviour of a delineated scattering process solely in terms of the scattering probabilities and of limiting transmission probabilities $\omega_{a,+}$ and $\omega_{a,-}$ for $a \in S$.

Theorem 2.8. *For an $(a \mapsto \widetilde{a}, \mathfrak{E})$-delineated scattering process, if states a and b lie in the same scattering class $\mathcal{E} \in \mathfrak{E}$ then*

$$p_{a,\widetilde{b}} \quad = \quad \omega_{a,\widetilde{b}} s_{\widetilde{b}} \quad = \quad \begin{cases} \omega_{a,+}\left(\prod_{z \in \mathcal{E}\,:\, a \prec z \prec b}(1 - s_{\widetilde{z}})\right) s_{\widetilde{b}} & \text{if } a \prec b\,, \\ \omega_{a,-}\left(\prod_{z \in \mathcal{E}\,:\, b \prec z \prec a}(1 - s_{\widetilde{z}})\right) s_{\widetilde{b}} & \text{if } b \prec a\,. \end{cases} \tag{2.4}$$

Here

$$\begin{aligned} \omega_{a,+} \quad &= \quad \sup\{\omega_{a,\widetilde{b}} \,:\, b \in \mathcal{E},\, a \prec b\} \quad \leq \quad 1\,, \\ \omega_{a,-} \quad &= \quad \sup\{\omega_{a,\widetilde{b}} \,:\, b \in \mathcal{E},\, b \prec a\} \quad \leq \quad 1\,. \end{aligned} \tag{2.5}$$

Moreover $\omega_{a,-} + \omega_{a,+} = 1$, so that $\omega_{a,-}$, $\omega_{a,+}$ may be interpreted as the conditional probabilities of scattering in $-$ and $+$ directions, given scattering has occurred, while $\omega_{a,+}\prod_{z \in \mathcal{E}\,:\, a \prec z}(1 - s_{\widetilde{z}}) = \omega_{a,-}\prod_{z \in \mathcal{E}\,:\, z \prec a}(1 - s_{\widetilde{z}}) = 0$, so that eventual scattering occurs with probability 1 whichever direction is taken.

Remark 2.9. In Equation (2.5) we may write $\omega_{a,+} = \lim_{b\downarrow a} \omega_{a,\widetilde{b}}$ as a monotonely increasing limit, and similarly $\omega_{a,-} = \lim_{b\uparrow a} \omega_{a,\widetilde{b}}$.

Proof. By the symmetry between $\prec$ and $\succ$ in Definition 2.6, it is sufficient to deal with the case of states $a \prec b$, with a and b lying in the same scattering class $\mathcal{E}$. Consider the following multiplicative relationship (an inductive consequence of Definition 2.6): if $a \prec u \prec v \prec \ldots \prec z \prec b$ lie in $\mathcal{E}$ then

$$\omega_{a,\widetilde{b}} \quad \leq \quad \omega_{a,\widetilde{u}} \times (1 - s_{\widetilde{u}})\,(1 - s_{\widetilde{v}}) \ldots (1 - s_{\widetilde{z}})\;. \tag{2.6}$$

As the ordered chain $a \prec u \prec v \prec \ldots \prec z \prec b$ is refined, so $(1 - s_{\widetilde{u}})\,(1 - s_{\widetilde{v}}) \ldots (1 - s_{\widetilde{z}})$ decreases. Taking the limit over the lattice set of refinements, we obtain an upper bound for $p_{a,\widetilde{b}}$ in terms of a (typically infinite) product:

$$p_{a,\widetilde{b}} \quad \leq \quad \omega_{a,+} \times \left(\prod_{z \in \mathcal{E}\,:\,a \prec z \prec b} (1 - s_{\widetilde{z}}) \right) s_{\widetilde{b}}\,. \tag{2.7}$$

Similarly, when $b \prec a$, we obtain

$$p_{a,\widetilde{b}} \quad \leq \quad \omega_{a,-} \times \left(\prod_{z \in \mathcal{E}\,:\,b \prec z \prec a} (1 - s_{\widetilde{z}}) \right) s_{\widetilde{b}}\,. \tag{2.8}$$

Using simple algebra and then taking limits over successive refinements,

$$\sum_{z \in \mathcal{E}\,:\,a \prec z} \left(\prod_{c \in \mathcal{E}\,:\,a \prec c \prec z} (1 - s_{\widetilde{c}}) \right) s_{\widetilde{z}} \quad = \quad 1 - \prod_{c \in \mathcal{E}\,:\,a \prec c} (1 - s_{\widetilde{c}}) \quad \leq \quad 1\,, \tag{2.9}$$

with equality holding if and only if $\sum_{c \in \mathcal{E}\,:\,a \prec c} s_{\widetilde{c}}$ diverges or one of the $s_{\widetilde{c}}$ is equal to 1. Likewise

$$\sum_{z \in \mathcal{E}\,:\,z \prec a} \left(\prod_{c \in \mathcal{E}\,:\,z \prec c \prec a} (1 - s_{\widetilde{c}}) \right) s_{\widetilde{z}} \quad \leq \quad 1\,, \tag{2.10}$$

with equality holding if and only if $\sum_{c \in \mathcal{E}\,:\,c \prec a} s_{\widetilde{c}}$ diverges or one of the $s_{\widetilde{c}}$ is equal to 1.

Finally it has been stipulated that $\omega_{a,\widetilde{y}} + \omega_{a,\widetilde{x}} \leq 1$ when $x \prec a \prec y$ (Definition 2.6) and therefore (using definition (2.5)) it is the case that $\omega_{a,-} + \omega_{a,+} \leq 1$. Consequently we can deduce

$$\begin{aligned}
1 \quad = \quad \sum_{z} p_{a,\widetilde{z}} \quad &= \quad \sum_{z \in \mathcal{E}\,:\,z \prec a} \omega_{a,\widetilde{z}} s_{\widetilde{z}} \;+ \sum_{z \in \mathcal{E}\,:\,a \prec z} \omega_{a,\widetilde{z}} s_{\widetilde{z}} \\
\leq\; \omega_{a,-} \sum_{z \in \mathcal{E}\,:\,z \prec a} &\left(\prod_{c \in \mathcal{E}\,:\,z \prec c \prec a} (1 - s_{\widetilde{c}}) \right) s_{\widetilde{z}} \;+\; \omega_{a,+} \sum_{z \in \mathcal{E}\,:\,a \prec z} \left(\prod_{c \in \mathcal{E}\,:\,a \prec c \prec z} (1 - s_{\widetilde{c}}) \right) s_{\widetilde{z}} \\
&\leq \quad \omega_{a,-} + \omega_{a,+} \quad \leq \quad 1\,.
\end{aligned} \tag{2.11}$$

Thus all these inequalities become equalities, also forcing equality for (2.7) and (2.8), and (2.9) and (2.10). The proof is completed by noting that convergence of the infinite sum $\sum_z p_{a,\widetilde{z}} = 1$ now forces $\omega_{a,+} \prod_{z \in \mathcal{E}\,:\,a \prec z} (1 - s_{\widetilde{z}}) = \omega_{a,-} \prod_{z \in \mathcal{E}\,:\,z \prec a} (1 - s_{\widetilde{z}}) = 0$. This is forced because (2.11) becomes a sequence of equalities: hence for example either $\omega_{a,+} = 0$ or $\sum_{z \in \mathcal{E}\,:\,a \prec z} \left(\prod_{c \in \mathcal{E}\,:\,a \prec c \prec z} (1 - s_{\widetilde{c}}) \right) s_{\widetilde{z}} = 1$: and in the second case (by the reasoning following (2.9)) either at least one of the constituent $s_{\widetilde{z}}$ is equal to 1 or the sum $\sum_{c \in \mathcal{E}\,:\,a \prec c} s_{\widetilde{c}}$ diverges – and in either case $\prod_{z \in \mathcal{E}\,:\,a \prec z} (1 - s_{\widetilde{z}}) = 0$. ☐

The following is an immediate corollary of Theorem 2.8.

Corollary 2.10. *For a delineated scattering process, if $a \in \mathcal{E}$ is a state in a scattering class then*

$$\begin{aligned}\prod_{c\in\mathcal{E}\,:\,a\prec c}(1-s_{\widetilde{c}}) &= 0 \qquad \text{if } \omega_{a,+}>0\,,\\ \prod_{c\in\mathcal{E}\,:\,c\prec a}(1-s_{\widetilde{c}}) &= 0 \qquad \text{if } \omega_{a,-}>0\,.\end{aligned} \tag{2.12}$$

Nevertheless, the sum of scattering probabilities has a local summability property.

Corollary 2.11. *Consider a scattering class $\mathcal{E}$ for a delineated scattering process. If $a \prec b \in \mathcal{E}$ then*

$$\sum_{c\in\mathcal{E}\,:\,a\preceq c\preceq b} s_{\widetilde{c}} \quad < \quad \infty\,.$$

Proof. Since $a \prec b \in \mathcal{E}$, there must be a finite chain of states $a = c_0, c_1, \ldots, c_n, c_{n+1} = b$ satisfying either $\omega_{c_{i-1}\widetilde{c}_i} > 0$ or $\omega_{c_i\widetilde{c}_{i-1}} > 0$ for each i. It follows from (2.3) and the representation provided by Theorem 2.8 that if $a \prec c \prec b$ and $s_c = 1$ then the finite chain $c_1, \ldots, c_n$ has to include c. Hence there can be at most n distinct states c with $a \prec c \prec b$ and $s_{\widetilde{c}} = 1$.

Moreover, applying (2.4) of Theorem 2.8 and the theory of infinite products, if one of $\omega_{c_{i-1}\widetilde{c}_i} > 0$ or $\omega_{c_i\widetilde{c}_{i-1}} > 0$, then $\sum_{c\in\mathcal{E}\,:\,c_{i-1}\prec c\prec c_i} s_{\widetilde{c}} < \infty$ if $c_{i-1} \prec c_i$, while if $c_i \prec c_{i-1}$ then $\sum_{c\in\mathcal{E}\,:\,c_i\prec c\prec c_{i-1}} s_{\widetilde{c}} < \infty$.

Now dissection of the range of summation then shows that

$$\sum_{c\in\mathcal{E}\,:\,a\preceq c\preceq b} s_{\widetilde{c}} \quad \leq \quad \sum_{i=1}^{n+1}\left(s_{\widetilde{c}_{i-1}} + \sum_{c\in\mathcal{E}\,:\,c \text{ lying between } c_{i-1},c_i} s_{\widetilde{c}}\right) + s_{\widetilde{b}} \quad < \quad \infty\,. \qquad \square$$

We isolate a particular situation which will be important later on.

Definition 2.12. *Consider a delineated scattering process. If $\omega_{a,\pm} = \frac{1}{2}$ for all states a (for $\omega_{a,\pm}$ defined as in equation* (2.5) *of Theorem 2.8) then we say that the delineated scattering process is* balanced.

Useful structure is added if the abstract scattering representation concerns a Markov chain satisfying dynamical detailed balance (see for example Kelly, 1979, Theorem 1.14 and preceding material). In this case equations of detailed balance relate (a) an invariant measure π defined on the state-space, (b) the transition probabilities, and (c) an involution of state-space which can be thought of as corresponding to reversal of direction of travel.

Definition 2.13. *Let $(p_{a,b})$ be the transition matrix of a Markov chain admitting an abstract scattering representation by $p_{a,b} = \omega_{a,b}s_b$ (so in particular it is necessary that $p_{a,a} = 0$). The process governed by $p_{a,b} = \omega_{a,b}s_b$ satisfies* dynamical detailed balance *(is* dynamically reversible*) with invariant measure π if there is a state-space involution $a \leftrightarrow \widetilde{a}$ with $p_{a,\widetilde{a}} = 0$, and π is a non-negative measure on the state-space, with involution and measure related to $p_{a,b}$ as follows:*

- (a) $\pi_a = \pi_{\widetilde{a}}$ *for all states a;*
- (b) $\pi_a p_{a,\widetilde{b}} = \pi_a\omega_{a,\widetilde{b}}s_{\widetilde{b}} = \pi_b\omega_{b,\widetilde{a}}s_{\widetilde{a}} = \pi_b p_{b,\widetilde{a}}$ *for all states $a \neq b$;*
- (c) $\pi_a > 0$ *for all states a.*
- (d) $\omega_{a,\widetilde{a}} = 0$ *for all states a (so $p_{a,\widetilde{a}} = 0$).*

Additionally, the abstract scattering representation is said to be unbiased dynamically reversible *if $\omega_{a,\widetilde{b}} = \omega_{b,\widetilde{a}}$ for all states a, b.*

Note that the (typically σ-finite) measure π is not normalized: therefore the π_a are defined only up to a common multiplicative constant.

The conditions (a) and (b) amount to the assertion that the chain is statistically identical to its time-reversal, so long as we also use the involution to reverse the "direction of travel" of the chain.

Unbiased dynamical detailed balance refers primarily to the scattering process representation rather than the Markov chain. Under unbiased dynamical detailed balance, condition (b) is equivalent to the following simpler condition which does not involve the transmission probabilities:

(b') $\pi_a s_{\widetilde{b}} = \pi_b s_{\widetilde{a}}$ for all states $a \neq b$ such that $\omega_{a,\widetilde{b}} = \omega_{b,\widetilde{a}} > 0$.

Remark 2.14. When we consider a Markov chain which is both an $(a \mapsto \widetilde{a}, \mathfrak{E})$-delineated scattering process and satisfies dynamical detailed balance, then we will suppose the same involution $a \mapsto \widetilde{a}$ is used in the definition of delineation and in the definition of dynamical reversibility. In this case if the delineated scattering process is balanced then Theorem 2.8 implies that it is automatically unbiased as a scattering process satisfying dynamical reversibility. We will describe such a process as a balanced delineated reversible scattering process.

Remark 2.15. It is a consequence of conditions (b) and (c) that $\omega_{a,\widetilde{b}} > 0$ if and only if $\omega_{b,\widetilde{a}} > 0$. For $\omega_{a,\widetilde{b}} > 0$ implies $p_{a,\widetilde{b}} > 0$ (since $s_{\widetilde{b}} > 0$ for all b). Since $\pi_a > 0$ and $\pi_b > 0$ by condition (c), it follows from condition (b) that $p_{b,\widetilde{a}} > 0$ and hence $\omega_{b,\widetilde{a}} > 0$.

As noted above, in the context of RRF on Π, we always consider the state-space involution supplied by $(\mathcal{L}_-, \mathcal{L}_0) \longleftrightarrow (\mathcal{L}_0, \mathcal{L}_-)$. Setting $a = (\mathcal{L}_-, \mathcal{L}_0)$ and $b = (\mathcal{L}_0, \mathcal{L}_+)$ for distinct $\mathcal{L}_-, \mathcal{L}_0, \mathcal{L}_+ \in \Pi$, unbiased dynamical reversibility means that transmission along $\mathcal{L}_0$ from $(\mathcal{L}_-, \mathcal{L}_0)$ to $(\mathcal{L}_0, \mathcal{L}_+)$ has the same probability as transmission in the reverse direction along $\mathcal{L}_0$ from $(\mathcal{L}_+, \mathcal{L}_0)$ to $(\mathcal{L}_0, \mathcal{L}_-)$.

Recall the representation of transition probabilities (2.4) for a delineated scattering process. The choices of $\omega_{a,\pm}$ are rather more constrained than might at first be supposed.

Corollary 2.16. *Consider a delineated scattering process satisfying dynamical reversibility (not necessarily balanced or unbiased), and choose states $a \prec b \prec c$ with $\omega_{a,\widetilde{c}} > 0$, equivalently $\omega_{c,\widetilde{a}} > 0$. Then $\omega_{b,+} = \omega_{b,-} = \frac{1}{2}$.*

Proof. Without loss of generality, suppose $a \prec b \prec c$. By (2.7), $\omega_{a,\widetilde{c}} = \omega_{a,+} \prod_{u\,:\,a \prec u \prec c}(1 - s_{\widetilde{u}})$, and so positivity of $\omega_{c,\widetilde{a}}$ forces positivity of $\prod_{u\,:\,a \prec u \prec c}(1 - s_{\widetilde{u}})$. Let π be the invariant measure. Using $\pi_a p_{a,\widetilde{c}} = \pi_c p_{c,\widetilde{a}}$, and representations derived from (2.7) and (2.8), we can deduce that $\pi_a \omega_{a,+} s_{\widetilde{c}} = \pi_c \omega_{c,-} s_{\widetilde{a}}$. Write this as

$$\frac{\pi_a}{s_{\widetilde{a}}}\omega_{a,+} \quad = \quad \frac{\pi_c}{s_{\widetilde{c}}}\omega_{c,-}\,.$$

Likewise it follows that

$$\frac{\pi_a}{s_{\widetilde{a}}}\omega_{a,+} \quad = \quad \frac{\pi_b}{s_{\widetilde{b}}}\omega_{b,-} \qquad \text{and} \qquad \frac{\pi_b}{s_{\widetilde{b}}}\omega_{b,+} \quad = \quad \frac{\pi_c}{s_{\widetilde{c}}}\omega_{c,-}\,.$$

We deduce

$$\frac{\pi_b}{s_{\widetilde{b}}}\omega_{b,-} \quad = \quad \frac{\pi_b}{s_{\widetilde{b}}}\omega_{b,+}\,,$$

and hence (since $\pi_b / s_{\widetilde{b}} > 0$, and Theorem 2.8 yields $\omega_{b,+} + \omega_{b,-} = 1$) it follows that $\omega_{b,+} = \omega_{b,-} = \frac{1}{2}$. $\square$

It is a consequence that, for all delineated scattering processes satisfying dynamical reversibility, the $\omega_{b,\pm}$ probabilities are all equal to $\frac{1}{2}$ except perhaps in the special case when $s_{\widetilde{b}} = 1$. This conclusion can be extended to all states in the case of a delineated scattering process, satisfying unbiased dynamical reversibility, and with sufficiently many states b with $s_{\widetilde{b}} < 1$. (This includes the SIRSN-RRF which will be defined later.)

Corollary 2.17. *Consider a delineated scattering process satisfying unbiased dynamical reversibility, such that if $\omega_{a,\widetilde{c}} > 0$ then there is a state b lying between a and c. In that case $\omega_{a,+} = \omega_{a,-} = \frac{1}{2}$ for all states a and so the delineated scattering process is balanced.*

Proof. First note for any state a there must be another state c with $\omega_{a,\widetilde{c}} > 0$, for otherwise all feasible transitions from a would actually have zero probability. According to the stated conditions there must be a state b lying between a and c: suppose without loss of generality that $a \prec b$. Now $\omega_{b,-} = \frac{1}{2}$ by Corollary 2.16: moreover the representation (2.7) applied to $\omega_{a,\widetilde{c}}$ implies that $\prod_{u\,:\,a\prec u\prec b}(1 - s_{\widetilde{u}}) < 1$.

The unbiasedness condition (b') of Definition 2.13 asserts that $\omega_{a,\widetilde{b}} = \omega_{b,\widetilde{a}}$, together with the equations $\omega_{a,\widetilde{b}} = \omega_{a,+}\prod_{u\,:\,a\prec u\prec b}(1 - s_{\widetilde{u}})$ and $\omega_{b,\widetilde{a}} = \omega_{b,-}\prod_{u\,:\,a\prec u\prec b}(1 - s_{\widetilde{u}})$, then imply that $\omega_{a,+} = \omega_{b,-}$. But $\omega_{b,-} = \frac{1}{2}$ by Corollary 2.16. The argument is concluded by using $\omega_{a,+} + \omega_{a,-} = 1$, as established in Theorem 2.8. □

Within a fixed scattering class of the form specified in Definition 2.4, an immediate algebraic consequence of general unbiased dynamic reversibility is that the equilibrium measure of a state is proportional to the scattering probability at that state. This, together with exploitation of delineated structure as expressed in Theorem 2, will allow us to describe suitable RRF efficiently in terms of a prescribed positive function on scattering classes (Theorem 2.23 below).

Lemma 2.18. *Consider an unbiased dynamically reversible scattering process. Suppose $a \sim b$; if π is the invariant measure then*

$$\pi_a/s_{\widetilde{a}} \quad = \quad \pi_b/s_{\widetilde{b}}\,. \tag{2.13}$$

For a scattering class $\mathcal{E}$ defined as in Definition 2.4, we write $\kappa(\mathcal{E})$ for the common value of $\pi_a/s_{\widetilde{a}}$ for $a \in \mathcal{E}$.

Proof. It suffices to establish (2.13) when $\omega_{a,\widetilde{b}} > 0$, equivalently $\omega_{b,\widetilde{a}} > 0$. But (2.13) holds in this case because of condition (b') of Definition 2.13. □

Since π is not normalized, the $\kappa(\mathcal{E})$ are defined only up to a common multiplicative constant.

Corollary 2.19. *Consider a unbiased dynamically reversible scattering process. Suppose $a \in \mathcal{E}_1$ and $\widetilde{a} \in \mathcal{E}_2$ for scattering classes defined as in Definition 2.4 using the involution supplied by dynamical reversibility. Then*

$$s_a/s_{\widetilde{a}} \quad = \quad \kappa(\mathcal{E}_1)/\kappa(\mathcal{E}_2) \tag{2.14}$$

In particular, $s_a = s_{\widetilde{a}}$ when a is equivalent to its involution (thus, $a \sim \widetilde{a}$).

Proof. By Lemma 2.18, $s_{\widetilde{a}} = \pi_a/\kappa(\mathcal{E}_1)$ likewise $s_a = \pi_{\widetilde{a}}/\kappa(\mathcal{E}_2)$. The result now follows from condition (a) of Definition 2.13. □

Since $\pi_a = \pi_{\widetilde{a}}$, it follows from Lemma 2.18 that $\pi_a \leq \min\{\kappa(\mathcal{E}_1), \kappa(\mathcal{E}_2)\}$ when $a \in \mathcal{E}_1$ and $\widetilde{a} \in \mathcal{E}_2$.

Definition 2.20. *Consider a unbiased dynamically reversible scattering process. Suppose $a \in \mathcal{E}_1$ and $\widetilde{a} \in \mathcal{E}_2$ for scattering classes defined as in Definition 2.4. The* deficit δ_a *is given by*

$$\pi_a \quad = \quad (1 - \delta_a)\min\{\kappa(\mathcal{E}_1), \kappa(\mathcal{E}_2)\}\,.$$

Note that the deficit $\delta_a = \delta_{\widetilde{a}}$ is left invariant by the involution, since $\pi_a = \pi_{\widetilde{a}}$, while the involution $a \mapsto \widetilde{a}$ simply exchanges the two scattering classes involved with state a.

The following interpretation of $\kappa(\mathcal{E})$ clarifies its rôle.

Corollary 2.21. *Consider a balanced delineated reversible scattering process with invariant measure π. Let $\mathcal{E}$ be a scattering class: if $a \in \mathcal{E}$ then*

$$\kappa(\mathcal{E}) \quad = \quad 2 \sum_{c \in \mathcal{E}\,:\, c \prec a} \pi_c \omega_{c,\widetilde{a}}\,. \tag{2.15}$$

Hence $\kappa(\mathcal{E})$ can be interpreted as measuring (invariantly) the in-flow of the process arriving at the state $a \in \mathcal{E}$ from the left side of $\mathcal{E} \setminus \{a\}$, (alternatively, from the right side), but not necessarily stopping there.

Proof. Apply dynamical reversibility:

$$\sum_{c \in \mathcal{E}\,:\, c \prec a} \pi_c \omega_{c,\widetilde{a}} s_{\widetilde{a}} \quad = \quad \pi_a \sum_{c \in \mathcal{E}\,:\, c \prec a} \omega_{a,\widetilde{c}} s_{\widetilde{c}} \quad = \quad \pi_a \omega_{a,-} \times \sum_{c \in \mathcal{E}\,:\, c \prec a} \left(\prod_{b \in \mathcal{E}\,:\, c \preceq b \prec a} (1 - s_{\widetilde{b}}) \right) s_{\widetilde{c}} \quad = \quad \frac{1}{2} \pi_a$$

where the last step uses balance, and also the fact established in the proof of Theorem 2.8, that Inequality (2.10) is in fact an equality. Equation (2.15) now follows by multiplying through by $2/s_{\widetilde{a}}$. □

Remark 2.22. Combining Equation (2.15) with the equation corresponding to the right side of $\mathcal{E} \setminus \{a\}$, we also obtain

$$\kappa(\mathcal{E}) \quad = \sum_{c \in \mathcal{E}\,:\, c \neq a} \pi_c \omega_{c,\widetilde{a}}\,; \tag{2.16}$$

so $\kappa(\mathcal{E})$ measures (invariantly) the in-flow of the process arriving at a from any other state in $\mathcal{E}$, but not necessarily stopping there. (This alternate interpretation holds even if the delineated reversible scattering process is not balanced).

We conclude the discussion of abstract scattering processes by noting a converse result: given an involution and a decomposition of state-space into candidate scattering classes with attached κ values, there are simply-specified assignments of scattering probabilities (motivated by the constructions of Metropolis-Hastings Markov chain Monte Carlo – see for example Gilks, Richardson, and Spiegelhalter, 1996, Chapter 1) which lead to valid delineated scattering processes satisfying unbiased dynamical reversibility. For this, we require that all deficits δ_a (Definition 2.20) are set identically equal to zero.

Theorem 2.23. *Given a state-space S supporting an involution $a \mapsto \widetilde{a}$ and a decomposition $\mathfrak{E} = \{\mathcal{E}_1, \mathcal{E}_2, \ldots\}$ into disjoint subsets $\mathcal{E}_1$, $\mathcal{E}_2$, ..., a positive function $\kappa : \mathcal{E} \mapsto \kappa(\mathcal{E}) > 0$ defined on $\mathfrak{E}$, and a total ordering on each scattering class $\mathcal{E} \in \mathfrak{E}$, consider the following "zero-deficit" assignment of scattering probabilities:*

$$s_{\widetilde{a}} \quad = \quad \min\left\{1, \frac{\kappa(\mathcal{E}_2)}{\kappa(\mathcal{E}_1)}\right\} \qquad \text{when } a \in \mathcal{E}_1 \text{ and } \widetilde{a} \in \mathcal{E}_2\,. \tag{2.17}$$

Suppose for convenience that no scattering class contains either maximal or minimal elements. Taking $\omega_{a,\pm} = \frac{1}{2}$ for all states a, we can use the scattering probabilities to define transmission probabilities as in (2.4) of Theorem 2.8. The resulting scattering representation corresponds to a (balanced) $(a \mapsto \widetilde{a}, \mathfrak{E})$-delineated scattering process precisely when

2.23.1 *for each $\mathcal{E} \in \mathfrak{E}$, and for each $a \prec b \in \mathcal{E}$,*

$$\sum_{c \in \mathcal{E}\,:\,a \prec c \prec b} s_{\widetilde{c}} \quad < \quad \infty\,.$$

2.23.2 *for each $\mathcal{E} \in \mathfrak{E}$, and for each $a \in \mathcal{E}$,*

$$\prod_{c \in \mathcal{E}\,:\,c \prec a} (1 - s_{\widetilde{c}}) \quad = \quad 0\,, \qquad \prod_{c \in \mathcal{E}\,:\,a \prec c} (1 - s_{\widetilde{c}}) \quad = \quad 0\,.$$

Finally this process satisfies unbiased dynamical reversibility, and is therefore a balanced delineated reversible scattering process, with identically zero deficits and equilibrium measure given by

$$\pi_a \quad = \quad \min\{\kappa(\mathcal{E}_1), \kappa(\mathcal{E}_2)\} \qquad \textit{when } a \in \mathcal{E}_1 \textit{ and } \widetilde{a} \in \mathcal{E}_2\,,$$

Proof. Note that forcing all deficits to be set to zero also forces $\pi_a = \min\{\kappa(\mathcal{E}_1), \kappa(\mathcal{E}_2)\}$ for $a \in \mathcal{E}_1$ and $\widetilde{a} \in \mathcal{E}_2$: (2.17) then follows from Lemma 2.18.

The necessity of 2.23.1, respectively 2.23.2, follows from Corollary 2.11, respectively Corollary 2.10.

Existence of the required balanced delineated scattering process Z can be demonstrated by noting that it is well-defined using the following recursive procedure involving a sequence of fair coin tosses and draws from a Uniform$(0, 1)$ distribution:

1. Suppose $Z_n = a \in \mathcal{E}$. Choose between the two branches of $\mathcal{E}$ with probability $\omega_{a,+} = \frac{1}{2}$ for $\{b \in \mathcal{E} \;:\; a \prec b\}$ and $\omega_{a,-} = \frac{1}{2}$ for $\{b \in \mathcal{E} \;:\; b \prec a\}$;
2. Draw U_{n+1} from a Uniform$(0, 1)$ distribution. If $\{b \in \mathcal{E} \;:\; a \prec b\}$ is chosen, then set Z_{n+1} to be the smallest $\widetilde{b}$ (with $a \prec b$, $b \in \mathcal{E}$) such that

   $$\prod_{c \in \mathcal{E}\,:\,a \prec c \preceq b} (1 - s_{\widetilde{c}}) \quad \leq \quad U_{n+1} \quad < \quad \prod_{c \in \mathcal{E}\,:\,a \prec c \prec b} (1 - s_{\widetilde{c}})$$

   (this uses the fact that $\prod_{c \in \mathcal{E}\,:\,a \prec c} (1 - s_{\widetilde{c}}) = 0$, which itself arises from (2.12)). Similarly, if $\{b \in \mathcal{E} \;:\; b \prec a\}$ is chosen, then set Z_{n+1} to be the largest $\widetilde{b}$ (with $b \prec a$, $b \in \mathcal{E}$) such that

   $$\prod_{c \in \mathcal{E}\,:\,b \preceq c \preceq a} (1 - s_{\widetilde{c}}) \quad \leq \quad U_{n+1} \quad < \quad \prod_{c \in \mathcal{E}\,:\,b \prec c \prec a} (1 - s_{\widetilde{c}})\,.$$
3. Increment n by 1 and go to step 1.

Finally, unbiased dynamical reversibility follows immediately from computations verifying the conditions in Definition 2.13 and particularly the condition (b'), which can be substituted for condition (b) in the case of unbiased dynamical reversibility. □

Part of the proof of Theorem 2.23 describes a simulation procedure which it is convenient to reference explicitly.

Corollary 2.24. *In the situation of Theorem 2.23, the (balanced) $(a \mapsto \widetilde{a}, \mathfrak{E})$-delineated scattering process Z can be simulated using*

1. *a choice Z_0 of initial position;*
2. *a sequence of independent fair coin tosses to determine direction of travel along each successive line;*

3. *and independently a sequence of independent draws from the Uniform* $(0,1)$ *distribution, to determine the distance of travel along each successive line, based only on the relevant scattering probabilities.*

The n^{th} *corresponding pair of coin toss followed by uniform draw can be thought of as the innovation for time* n *generating the evolution of* Z*.*

Remark 2.25. Condition 2.23.1 implies that the pattern of intersections of a scattering class $\mathcal{E}$ with other scattering classes of higher κ levels must be locally finite in $\mathcal{E}$. Condition 2.23.2 holds if the pattern of intersections of a scattering class $\mathcal{E}$ with other scattering classes of higher κ must be infinite in number to the right or left of any fixed state, which will certainly be the case for SIRSN; but note that condition 2.23.2 could still be satisfied even if this is not the case.

Remark 2.26. More general scattering probabilities can be considered which introduce non-zero deficits:

$$s_{\widetilde{a}} \quad = \quad \sigma(\tfrac{\kappa(\mathcal{E}_2)}{\kappa(\mathcal{E}_1)}) \min\left\{1, \frac{\kappa(\mathcal{E}_2)}{\kappa(\mathcal{E}_1)}\right\} \qquad \text{when } a \in \mathcal{E}_1 \text{ and } \widetilde{a} \in \mathcal{E}_2\,, \tag{2.18}$$

where the positive function σ is required to satisfy the inversion symmetry $\sigma(u) = \sigma(1/u)$ (in order to ensure that dynamical reversibility holds) and it is further required that $s_{\widetilde{a}}$ and s_a satisfy the contraint of lying in $(0,1)$. In the following, we will consider only the (zero-deficit) case $\sigma(u) \equiv 1$ introduced above, corresponding to an acceptance mechanism of Metropolis-Hastings type. In the next section we will see that this is the only similarity-invariant possibility once we require the scattering process to have identically zero deficits.

3 Scale-invariant RRF on SIRSN (SIRSN-RRF)

RRFs based on the Poisson line SIRSN or SIRSN candidate Π (after sampled at times of switching lines, and quenching by conditioning on the random environment Π) can be considered as $(a \mapsto \widetilde{a}, \mathfrak{E})$-delineated scattering processes, where

(a) the state-space is the set $(\Pi \times \Pi) \setminus \Delta$ of ordered pairs of distinct speed-marked lines from Π, as given by (2.1);

(b) the involution is given by $a = (\mathcal{L}_-, \mathcal{L}_0) \mapsto \widetilde{a} = (\mathcal{L}_0, \mathcal{L}_-)$,

(c) and scattering classes $\mathcal{E} \in \mathfrak{E}$ turn out to be of the form $\mathcal{E} = \{(\mathcal{L}, \mathcal{L}_1) \ : \ \mathcal{L}_1 \in \Pi \setminus \{\mathcal{L}\}\}$ for a corresponding line $\mathcal{L} \in \Pi$, furnished with the total ordering taken from $\mathcal{L}$ (using our specification of preferred direction for each $\mathcal{L} \in \Pi$).

We say that the lines of Π are scattering classes for the process.

So we will consider RRFs which under quenching are balanced delineated reversible scattering processes based on Π as above, with scattering classes corresponding to the lines of Π. The stochastic dynamics can then be specified by defining the invariant measure π_a and the scattering probability s_a for all states a. We require these to deliver processes on Π which behave well under Euclidean symmetry and changes of scale, leading in due course to the natural family of SIRSN-RRF of Theorem 3.6 and Definition 3.7. Viewing the invariant measure and scattering probabilities as depending not just on location but also on all of Π, we are led to:

Definition 3.1. *Consider a balanced delineated reversible scattering process based on the Poisson line SIRSN or SIRSN candidate* Π *(quenching by conditioning on* Π*), with the lines of* Π *as scattering classes. This is said to satisfy* similarity equivariance *if*

1. *the scattering probability, when viewed as a function* $s(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})$ *of location* $(\mathcal{L}_1, \mathcal{L}_2)$ *and reduced environment* $\Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\}$*, is invariant under the group of similarities (generated by Euclidean motions and changes of scale);*

2. *the invariant measure, viewed as a function* $\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})$ *of location* $(\mathcal{L}_1, \mathcal{L}_2)$ *and* reduced environment $\Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\}$*, is invariant under the Euclidean motion group; moreover the ratio*

$$\frac{\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{\pi(\mathcal{L}_3, \mathcal{L}_4; \Pi \setminus \{\mathcal{L}_3, \mathcal{L}_4\})}$$

is invariant under scale-change.

Remark 3.2. A simple geometric argument shows it suffices to check scale-invariance only for the ratios

$$\frac{\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{\pi(\mathcal{L}_1, \mathcal{L}_3; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_3\})},$$

since $\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})$ is symmetric in $\mathcal{L}_1$ and $\mathcal{L}_2$ as a consequence of dynamical reversibility, and almost surely all lines in Π intersect.

Also note that it follows from Lemma 2.18 that the similarity-equivariance and Euclidean invariance properties of Definition 3.7 imply Euclidean-invariance of

$$\kappa(\mathcal{L}_1) \;=\; \kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\}) \quad = \quad \frac{\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{s(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})},$$

and similarity-invariance for

$$\frac{\kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\})}{\kappa(\mathcal{L}_2; \Pi \setminus \{\mathcal{L}_2\})} \quad = \quad \frac{s(\mathcal{L}_2, \mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{s(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}.$$

Finally Euclidean-invariance for the deficit (viewed again as a function $\delta(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})$ of location $(\mathcal{L}_1, \mathcal{L}_2)$ and Π) follows from

$$1 - \delta(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\}) \quad = \quad \frac{\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{\min\{\kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\}), \kappa(\mathcal{L}_2; \Pi \setminus \{\mathcal{L}_2\})\}},$$

as does similarity-invariance for

$$\frac{1 - \delta(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{1 - \delta(\mathcal{L}_3, \mathcal{L}_4; \Pi \setminus \{\mathcal{L}_3, \mathcal{L}_4\})} \quad = \quad \frac{\pi(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{\pi(\mathcal{L}_1, \mathcal{L}_3; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_3\})}.$$

Consideration of these remarks also proves the following converse.

Lemma 3.3. *Consider a balanced delineated reversible scattering process based on the Poisson line SIRSN or SIRSN candidate* Π*, with the lines of* Π *as scattering classes. It is said to satisfy* similarity-equivariance *if* $\kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\})$ *and* $\delta(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})$ *determine Euclidean-invariant functions, and the ratios*

$$\frac{\kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\})}{\kappa(\mathcal{L}_2; \Pi \setminus \{\mathcal{L}_2\})} \qquad \text{and} \qquad \frac{1 - \delta(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})}{1 - \delta(\mathcal{L}_3, \mathcal{L}_4; \Pi \setminus \{\mathcal{L}_3, \mathcal{L}_4\})}$$

are scale-invariant. In this case the stochastic dynamics of the scattering process are determined by specifying the functions $\kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\})$ *and* $\delta(\mathcal{L}_1, \mathcal{L}_2; \Pi \setminus \{\mathcal{L}_1, \mathcal{L}_2\})$*.*

Remarkably, choice of $\kappa(\mathcal{L}_1; \Pi \setminus \{\mathcal{L}_1\})$ is heavily constrained by similarity-equivariance. This follows from an ergodic theorem for Poisson line SIRSN or SIRSN candidates Π. This in turn requires the Slivnyak-Mecke theorem for Palm conditioning of Poisson processes.

Theorem 3.4 (Slivnyak-Mecke). *Suppose Π is a Poisson point process on Euclidean space $\mathbb{R}^d$ with diffuse intensity measure ν. For any measurable non-negative function f on $\mathbb{R}^d$,*

$$\mathbb{E}\left[\sum_{x\in\Pi} f(x,\Pi\setminus\{x\})\right] \quad = \quad \int \mathbb{E}\left[f(x,\Pi)\right]\nu(\,\mathrm{d}x)\,.$$

The proof of this result is described in Example 4.3 of Chiu et al. (2013, Section 4.4.4).

We now state and prove the required ergodic theorem.

Theorem 3.5. *Let Π be a Poisson line SIRSN or SIRSN candidate, and let $\xi(\mathcal{L},\Pi)$ be a non-negative measurable function of $\mathcal{L}$ and Π which is invariant under Euclidean motion applied to the pair $(\mathcal{L},\Pi)$ of speed-marked line $\mathcal{L}$ and speed-marked line pattern Π. Consider $\xi(\mathcal{L},\Pi\setminus\{\mathcal{L}\})$ as $\mathcal{L}$ varies over the speed-marked line process Π: this is almost surely a deterministic function of the speed $v(\mathcal{L})$ alone.*

Proof. It is enough to consider non-negative bounded measurable ξ, since the general result then follows by consideration of $(n\wedge\xi)$ for increasing n. For a fixed speed-marked line $\mathcal{L}$, consider the random process $t\mapsto\xi(\mathcal{L};T_t\Pi)$, where T_t is Euclidean translation parallel to $\mathcal{L}$ for $t\in\mathbb{R}$. The law of Π is translation invariant, so this bounded random process is stationary.

Now view Π in $v-s-\phi$ coordinates corresponding to (1.2) based on the fixed speed-marked line $\mathcal{L}$. In these coordinates the action of T_t sends (v,s,ϕ) to $(v,s+t,\phi)$. Arguing as in the standard proof of the Hewitt-Savage zero-one law (for example Kallenberg, 2002, Theorem 3.15), the bounded shift-invariant function $\xi(\mathcal{L};\Pi)$ can be approximated by a non-negative bounded measurable function $\widetilde{\xi}(\mathcal{L};\Pi)$ depending only on lines of Π intersecting $\mathcal{L}$ in a bounded interval I. Choosing t such that T_tI and I are disjoint, $\xi(\mathcal{L};\Pi)$ is similarly approximated by the statistically independent $\widetilde{\xi}(\mathcal{L};T_t\Pi)$. It follows that $\xi(\mathcal{L};\Pi)$ is independent of itself and thus is almost surely a deterministic function $c(\mathcal{L})$ of $\mathcal{L}$ alone. Euclidean-invariance of ξ implies that $\xi(\mathcal{L};\Pi)=c(\mathcal{L})=c(v(\mathcal{L}))$ depends only on the speed $v(\mathcal{L})$ of $\mathcal{L}$.

Recall that the speed-marked Poisson line process Π can be viewed as a Poisson point process in $v-r-\theta$ space $((0,\infty)\times\mathrm{linespace}))$ with intensity $\nu(\,\mathrm{d}\mathcal{L})=\nu(\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\theta)$ given by (1.1). So Theorem 3.4 applies to $f(\mathcal{L},\Pi\setminus\{\mathcal{L}\})=\xi(\mathcal{L};\Pi\setminus\{\mathcal{L}\})\,\mathbb{I}\left[\mathcal{L}\in A\right]\times\mathbb{I}\left[\Pi\setminus\{\mathcal{L}\}\in B\right]$, for any compact subset $A\subset(0,\infty)\times\mathrm{linespace}$ and any measurable subset B of the space of speed-marked patterns inducing locally finite point patterns on $(0,\infty)\times\mathrm{linespace}$. Thus

$$\begin{aligned}\mathbb{E}\left[\sum_{\mathcal{L}\in\Pi\cap A}\xi(\mathcal{L};\Pi\setminus\{\mathcal{L}\})\,\mathbb{I}\left[\Pi\setminus\{\mathcal{L}\}\in B\right]\right] \quad &= \\ \int_A \mathbb{E}\left[\xi(\mathcal{L};\Pi)\,;\,\Pi\in B\right]\nu(\,\mathrm{d}\mathcal{L}) \quad &= \quad \mathbb{P}\left[\Pi\in B\right]\int_A c(v(\mathcal{L}))\nu(\,\mathrm{d}\mathcal{L})\end{aligned}$$

(finite because we required A to be a compact subset of $(0,\infty)\times\mathrm{linespace}$).

Viewing this as an equality between measures evaluated on product sets $A\times B$, a Π-system argument allows us to deduce that almost surely $\xi(\mathcal{L};\Pi\setminus\{\mathcal{L}\})=c(v(\mathcal{L}))$ for all $\mathcal{L}\in\Pi$. □

As a direct consequence we have the following result, which characterizes similarity-equivariant scattering processes based on Π as a one-parameter family when the deficit $\delta(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\{\mathcal{L}_1,\mathcal{L}_2\})$ is required to vanish identically (equivalently if the invariant measure $\pi(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\{\mathcal{L}_1,\mathcal{L}_2\})$ is maximized subject to the specification of $\kappa(\mathcal{L};\Pi\setminus\{\mathcal{L}\})$). Note that the result requires one to check non-triviality of the scattering process (namely, that scattering can not occur instantaneously, and therefore that the scattering classes do indeed correspond to entire lines).

Theorem 3.6. *Consider a balanced delineated reversible scattering process based on* Π *(quenched by conditioning on* Π*), which has the lines of* Π *as scattering classes, so that its stochastic dynamics are specified by* $\kappa(\mathcal{L};\Pi\setminus\{\mathcal{L}\})$ *and* $\delta(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\{\mathcal{L}_1,\mathcal{L}_2\})$*. Suppose further that the stochastic dynamics are similarity-equivariant. Then, for some real parameter* α*,*

$$\frac{\kappa(\mathcal{L}_1;\Pi\setminus\{\mathcal{L}_1\})}{\kappa(\mathcal{L}_2;\Pi\setminus\{\mathcal{L}_2\})} \quad = \quad \frac{v(\mathcal{L}_1)^\alpha}{v(\mathcal{L}_2)^\alpha}\,. \tag{3.1}$$

Moreover, if the deficit $\delta(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\{\mathcal{L}_1,\mathcal{L}_2\})$ *vanishes identically (so that* $\pi(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\mathcal{L}_1,\mathcal{L}_2)$ *reduces to* $\min\{v(\mathcal{L}_1)^\alpha,v(\mathcal{L}_2)^\alpha\}$*, and additionally* $s(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\mathcal{L}_1,\mathcal{L}_2)$ *reduces to* $\min\{1,(v(\mathcal{L}_2)/v(\mathcal{L}_1))^\alpha\}$*), then* $\alpha>\gamma-1$ *is necessary and sufficient for the scattering process to be non-trivial.*

Proof. Applying Theorem 3.5 to the function $\kappa(\mathcal{L};\Pi\setminus\{\mathcal{L}\})$, we deduce that we may write $\kappa(\mathcal{L};\Pi\setminus\{\mathcal{L}\})=\psi(v(\mathcal{L}))$. Now similarity-invariance for the ratio

$$\frac{\kappa(\mathcal{L}_1;\Pi\setminus\{\mathcal{L}_1\})}{\kappa(\mathcal{L}_2;\Pi\setminus\{\mathcal{L}_2\})}$$

implies, for all $\lambda>0$,

$$\frac{\psi(v)}{\psi(1)} \quad = \quad \frac{\psi(\lambda v)}{\psi(\lambda)}\,.$$

But then a multiplicative form of Cauchy's functional equation must hold,

$$\frac{\psi(\lambda v)}{\psi(1)} \quad = \quad \frac{\psi(v)}{\psi(1)}\times\frac{\psi(\lambda)}{\psi(1)}\,,$$

and thus (ψ being measurable) there must be a constant α such that

$$\psi(v) \quad = \quad \psi(1)v^\alpha\,.$$

Equation (3.1) follows.

Suppose that the deficit vanishes identically, so in particular

$$s(\mathcal{L}_1,\mathcal{L}_2;\Pi\setminus\mathcal{L}_1,\mathcal{L}_2) \quad = \quad \min\left\{1,\frac{v(\mathcal{L}_2)^\alpha}{v(\mathcal{L}_1)^\alpha}\right\}\,.$$

The resulting process is a non-trivial RRF exactly when the conditions of Corollaries 2.11 and 2.10 are satisfied. The condition for Corollary 2.10 follows immediately from the observation that, for any given line $\mathcal{L}$, there is an everywhere dense set of intersections with lines of slower speed (in case $\alpha<0$) and an infinite unbounded set of intersections with lines of faster speed (in case $\alpha\geq 0$).

Consider the condition for Corollary 2.11, which is required if the lines of Π do indeed correspond to the scattering classes of the process. We shall first show that $\alpha>\gamma-1$ implies finiteness of all sums of the form

$$\mathbb{E}\left[\sum_{\mathcal{L}\in\Pi}\xi(\mathcal{L})\sum_{\operatorname{dist}(\mathcal{L}',\mathbf{o})<A}s(\mathcal{L},\mathcal{L}';\Pi\setminus\{\mathcal{L},\mathcal{L}'\})\right]$$

whenever $\xi(\mathcal{L})$ is non-negative and measurable, and $\mathbb{E}\left[\sum_{\mathcal{L}\in\Pi}\xi(\mathcal{L})v(\mathcal{L})^{-(\gamma-1)}\right]<\infty$. Apply Theorem 3.4 (Slivnyak-Mecke) twice over, and expand using the line-space coordinates leading to (1.1):

$$\begin{aligned}
&\mathbb{E}\left[\sum_{(\mathcal{L}_-,\mathcal{L}_0)\in(\Pi\times\Pi)\setminus\Delta} \xi(\mathcal{L}_-)\,\mathbb{I}\left[|\operatorname{dist}_{\text{signed}}(\mathcal{L}_0,\mathbf{o})|<R\right]\min\left\{1,\frac{v(\mathcal{L}_0)^\alpha}{v(\mathcal{L}_-)^\alpha}\right\}\right]\\
&\qquad= \int\int_{|\operatorname{dist}_{\text{signed}}(\mathcal{L}_0,\mathbf{o})|<R} \xi(\mathcal{L}_-)\,\mathbb{E}\left[\min\left\{1,\frac{v(\mathcal{L}_0)^\alpha}{v(\mathcal{L}_-)^\alpha}\right\}\right]\nu(\,\mathrm{d}\mathcal{L}_0)\nu(\,\mathrm{d}\mathcal{L}_-)\\
&= \left(\tfrac{\gamma-1}{2}\right)^2\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{\infty}\xi(v_-,\theta_-,r_-)\int_{-R}^{R}\int_0^{\pi}\int_0^{\infty}\left(1\wedge\tfrac{v_0^\alpha}{v_-^\alpha}\right)v_0^{-\gamma}\,\mathrm{d}v_0\,\mathrm{d}\theta_0\,\mathrm{d}r_0\;v_-^{-\gamma}\,\mathrm{d}v_-\,\mathrm{d}\theta_-\,\mathrm{d}r_-\\
&= \left(\tfrac{\gamma-1}{2}\right)^2\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{\infty}\xi(v,\theta_-,r_-)\int_{-R}^{R}\int_0^{\pi}\int_0^{\infty}(1\wedge u^\alpha)\,v(uv)^{-\gamma}\,\mathrm{d}u\,\mathrm{d}\theta_0\,\mathrm{d}r_0\;v^{-\gamma}\,\mathrm{d}v\,\mathrm{d}\theta_-\,\mathrm{d}r_-\\
&\qquad\qquad\qquad\qquad\qquad\qquad(\text{using } v_-=v, v_0=vu \text{ so that } \mathrm{d}v_0\,\mathrm{d}v_-=v\,\mathrm{d}v\,\mathrm{d}u)\\
&\quad= \left(\tfrac{\gamma-1}{2}\right)^2\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{\infty}\xi(v,\theta_-,r_-)v^{-(2\gamma-1)}\,\mathrm{d}v\,\mathrm{d}\theta_-\,\mathrm{d}r_-\times 2\pi R\int_0^{\infty}\min\{1,u^\alpha\}\,u^{-\gamma}\,\mathrm{d}u\\
&\qquad= (\gamma-1)\,\pi R\times\mathbb{E}\left[\sum_{\mathcal{L}\in\Pi}\xi(\mathcal{L})v(\mathcal{L})^{-(\gamma-1)}\right]\times\left(\int_0^1 u^{\alpha-\gamma}\,\mathrm{d}u+\int_1^\infty u^{-\gamma}\,\mathrm{d}u\right)
\end{aligned}$$

(where the last step uses $\alpha>0$). This is finite only if $\alpha>\gamma-1$. Moreover in that case the theory of infinite products implies that the next scattering cannot occur instantaneously, and that there is a positive chance of the next scattering taking place at an intersection with a line faster than the current line. Since such intersections form a locally finite pattern along the current line, this implies that the scattering classes do indeed correspond to the lines, and therefore that the scattering process is non-trivial.

If $\alpha\leq 0$ then

$$\sum_{\mathcal{L}\in\Pi}\xi(\mathcal{L})\sum_{\operatorname{dist}(\mathcal{L}',\mathbf{o})<A} s(\mathcal{L},\mathcal{L}';\Pi\setminus\{\mathcal{L},\mathcal{L}'\}) \;=\; \sum_{\mathcal{L}\in\Pi}\xi(\mathcal{L})\sum_{\operatorname{dist}(\mathcal{L}',\mathbf{o})<A}\min\left\{1,\frac{v(\mathcal{L}')^\alpha}{v(\mathcal{L})^\alpha}\right\} \;=\; \infty\,,$$

because of density everywhere of the set of lines in Π of speed lower than any prescribed positive threshold. In case $0<\alpha\leq\gamma-1$, monotonicity considerations mean that it suffices to consider the boundary case $\alpha=\gamma-1$. The relevant quantity is then

$$\sum_{\mathcal{L}\in\Pi, v(\mathcal{L})\in(0,1)} v(\mathcal{L})^{\gamma-1}\,\mathbb{I}\left[\mathcal{L}\text{ hits a fixed unit interval}\right]\,.$$

This admits the stochastic lower bound $\lim_{n\to\infty}H_n$, where:

$$H_n \quad=\quad \sum_{r=1}^n N_r a_r^{\gamma-1}\,, \tag{3.2}$$

where the N_r are independent Poisson(1) random variables and $0<a_{r+1}<a_r\leq 1$ is chosen so that

$$\frac{a_{r+1}^{-\gamma-1}-a_r^{-\gamma-1}}{\gamma-1}\quad=\quad 1\,.$$

Decomposing the process H_n as the sum of a convergent L^2 martingale and a divergent harmonic sum,

$$H_n \quad=\quad \left(\sum_{r=1}^n a_r^{\gamma-1}\times(N_r-1)\right)+\sum_{r=1}^n a_r^{\gamma-1}\,,$$

we deduce that the lower bound, which is the limit of (3.2) as $n\to\infty$, almost surely diverges to $+\infty$. $\square$

We can now formally define the notion of a SIRSN-RRF.

Definition 3.7. *Consider a Poisson line SIRSN or SIRSN candidate* Π *with parameter* $\gamma \geq 2$*. A SIRSN-RRF based on* Π *(when quenched by conditioning on* Π*) is a balanced delineated reversible scattering process on state-space* $(\Pi \times \Pi) \setminus \Delta$*. with scattering classes given by the lines of* Π*, and satisfying similarity-equivariance, with zero deficit, with* $\kappa(\mathcal{L}) = v(\mathcal{L})^\alpha$ *for some* $\alpha > \gamma - 1$*.*

In summary, the (scattering) probability for the SIRSN-RRF switching from $\mathcal{L}_1$ to $\mathcal{L}_2$ is given by

$$s_a \quad = \quad s_{(\mathcal{L}_1,\mathcal{L}_2)} \quad = \quad \min\left\{1, \frac{v_2^\alpha}{v_1^\alpha}\right\} \qquad \text{for some } \alpha \in (\gamma - 1, \infty)\,. \tag{3.3}$$

Thus the RRF always switches to faster lines, but switches to slower lines with probability proportional to a power of the relative speed of the new, slower, line. (After a successful switch to a new line, the new direction of travel is chosen equiprobably.)

Remark 3.8. The work of this section implies that, if $\alpha > \gamma - 1$ and there is zero defect, then the equilibrium measure $\pi_{\mathcal{L}_1,\mathcal{L}_2} = \min\{v(\mathcal{L}_1)^\alpha, v(\mathcal{L}_2)^\alpha\}$ automatically generates scale-invariance for the SIRSN-RRF viewed as a random process in a random environment. The constraint $\alpha > \gamma - 1$ ensures that the scattering times $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \zeta$ are well-defined as the line switching times mentioned in the procedure described after Definition 3.7.

Remark 3.9. If Z is a SIRSN-RRF of parameter α on Π, and S is a similarity, then SZ is a SIRSN-RRF of parameter α on $S\Pi$. This follows from similarity-invariance of the scattering probabilities and consideration of the simulation algorithm for such a scattering process Z, given in the proof of Theorem 2.23.

Lemma 3.10. *A SIRSN-RRF based on a Poisson line SIRSN or SIRSN candidate* Π *(when quenched by conditioning on* Π*) forms an irreducible Markov chain on the state-space* $(\Pi \times \Pi) \setminus \Delta$*.*

Proof. It suffices to show that the SIRSN-RRF can move from $(\mathcal{L}_-, \mathcal{L}_0)$ to $(\mathcal{L}_0, \mathcal{L}_+)$.

The key observation is that the SIRSN-RRF is always compelled to switch onto a faster line, but may or may not choose to switch to a slower line.

Consequently, if these intersections are not separated by a line of greater speed than $v(\mathcal{L}_0)$ then the SIRSN-RRF can travel from $(\mathcal{L}_-, \mathcal{L}_0)$ to $(\mathcal{L}_0, \mathcal{L}_+)$ in a single move. If there are such lines, then consider the sequence of cells from the Crofton tessellation formed by $\Pi_{\geq v(\mathcal{L}_0)}$ which intersect the segment of $\mathcal{L}_0$ between $(\mathcal{L}_-, \mathcal{L}_0)$ and $(\mathcal{L}_0, \mathcal{L}_+)$. With positive probability, the SIRSN-RRF can move from $(\mathcal{L}_-, \mathcal{L}_0)$ along $\mathcal{L}_0$ in the direction of $(\mathcal{L}_0, \mathcal{L}_+)$, but has to switch to the Crofton tessellation when it is first encountered. The SIRSN-RRF can then use the boundaries of these cells to move to a point on $\mathcal{L}_0$ also lying on the boundary Crofton cell containing $(\mathcal{L}_0, \mathcal{L}_+)$; $\mathcal{L}_0$ can then be used to move to $(\mathcal{L}_0, \mathcal{L}_+)$. □

The continuous-time variant of the SIRSN-RRF, $(X_t : t \geq 0)$, can be recovered from the sampled process $(Z_n = X_{\tau_n} : n \geq 0)$ simply by interpolating between sampling points, requiring the RRF X to travel at top permissible speed Y_{τ_n} between scattering times τ_n and $\tau_{n+1)}$. In principle there is the possibility that the resulting continuous-time process might explode to infinity in finite time $\zeta < \infty$. We shall discuss this further in section 6.

In the next section we address the question of the long-run behaviour of the (log-)speed process $\log(Y)$ of the SIRSN-RRF.

4 Environment viewed from the SIRSN-RRF

We now focus on the (discrete-time) SIRSN-RRF Z of index $\alpha > \gamma - 1$ based on the planar Poisson line SIRSN of parameter $\gamma > 2$, or even the non-SIRSN case of $\gamma = 2$, as discussed in Section 3. This scattering process can be viewed as possessing a random and Π-dependent state-space $(\Pi \times \Pi) \setminus \Delta$. Recall that the lines of Π are speed-marked, so the state $(\mathcal{L}_0, \mathcal{L}_1)$ of Z_1 includes information on the speed $v(\mathcal{L}_0)$ previous to the switch and also the current speed $v(\mathcal{L}_1)$. Conditioned on Π, the process Z is Markovian with a discrete invariant measure $\pi_{(\mathcal{L}_0,\mathcal{L}_1)} = \min\{v(\mathcal{L}_0)^\alpha, v(\mathcal{L}_1)^\alpha\}$ for some $\alpha > \gamma - 1$, with respect to which Z satisfies dynamical reversibility. The discrete invariant measure is never summable, since any summation has to extend over all the intersection points of the stationary line process Π. Consequently a stationary version of Z cannot exist, and indeed the invariant measure is defined only up to a positive multiplicative constant. Nonetheless we will see that the environment viewed from Z can be converted into a stationary process (following the classic construction for a random walk in a random environment), so long as it is reduced by centering, rotation, *and* (most especially) rescaling.

To begin with, consider the RRF Z in its quenched environment Π. This dynamically reversible process can be related to a symmetric Dirichlet form which is quenched (conditioned on Π) and defined for the random state-space $(\Pi \times \Pi) \setminus \Delta$ as follows: suppose f and g are functions on $(\Pi \times \Pi) \setminus \Delta$ satisfying the symmetry condition $f(a) = f(\widetilde{a})$, $g(b) = g(\widetilde{b})$. Then

$$\begin{aligned} B^{\text{quenched}}(f,g) &= \sum_{a \in (\Pi\times\Pi)\setminus\Delta} \pi_a f(a)\, \mathbb{E}\left[g(Z_1) | Z_0 = a, \Pi\right] \\ &= \sum\sum_{a \neq b \in (\Pi\times\Pi)\setminus\Delta} \pi_a p_{a,\widetilde{b}}\, f(a) g(b) \\ &= \sum\sum_{a \neq b \in (\Pi\times\Pi)\setminus\Delta} \pi_a \omega_{a,\widetilde{b}} s_{\widetilde{b}}\, f(a) g(b) \quad = \sum\sum_{a \neq b \in (\Pi\times\Pi)\setminus\Delta} \pi_b p_{b,\widetilde{a}}\, f(a) g(b)\,, \end{aligned} \tag{4.1}$$

where the last step arises from dynamical reversibility (since $\pi_a p_{a,\widetilde{b}} = \pi_b p_{b,\widetilde{a}}$), and establishes the symmetry of the quenched Dirichlet form. Note that the equilibrium probabilities π_a, the transition probabilities $p_{a,\widetilde{b}}$, and the transmission probabilities $\omega_{a,\widetilde{b}}$ all depend implicitly on the random environment given by the Poisson line SIRSN or SIRSN candidate Π.

A Cauchy-Schwarz argument shows that the Dirichlet form $B^{\text{quenched}}(f,g)$ is well-defined if the functions f and g belong to the random Hilbert space $\mathfrak{H}_\Pi$ of functions h defined on $(\Pi \times \Pi) \setminus \Delta$ satisfying the symmetry requirement $h(\mathcal{L}_1, \mathcal{L}_2) = h(\mathcal{L}_2, \mathcal{L}_1)$ and

$$\sum_a h(a)^2 \pi_a \quad = \sum_{a \in (\Pi\times\Pi)\setminus\Delta} h(a)^2 \pi_a \quad < \quad \infty\,. \tag{4.2}$$

For completeness of exposition, we observe that measure-theoretic details for such symmetric $h(a) = h(\mathcal{L}_0, \mathcal{L}_1)$ can be dealt with by viewing $h(\mathcal{L}_0, \mathcal{L}_1) = h(x, \theta_0, v_0, \theta_1, v_1)$ as a measurable function of one planar and four real variables, using the $2:1$ mapping $(\Pi \times \Pi) \setminus \Delta \to \mathbb{R}^2$ determined by $\mathcal{L}_0 \cap \mathcal{L}_1 = \{x\}$ to deliver the planar variable $x \in \mathbb{R}^2$, while θ_i signifies the direction, v_i the speed of $\mathcal{L}_i$. We can thus regard f and g as functions of $\mathbb{R}^2 \times (0,\pi) \times (0,\infty) \times (0,\pi) \times (0,\infty)$. To be pedantic, we focus on functions in the subspace of $\mathfrak{H}_\Pi$ which is the L^2 closure of $h(x, \theta_0, v_0, \theta_1, v_1)$ which depend continuously on the arguments $x, \theta_0, v_0, \theta_1, v_1$ and depend only on finitely many evaluations of events involving whether specified open subsets of $\mathbb{R}^2$ are hit by Π-lines of speeds exceeding specified positive thresholds.

By Fubini-argument methods this leads us to consider an annealed Dirichlet form defined for functions $h(a;\Pi) = h(x,\theta_0,v_0,\theta_1,v_1;\Pi)$ in the deterministic Hilbert space $\mathfrak{H}$ of functions which also depend on the random pattern specified by Π, defined by

$$\mathfrak{H} \quad = \quad \left\{ h \;:\; \mathbb{E}\left[\sum_{a\in(\Pi\times\Pi)\setminus\Delta} h(a;\Pi)^2\pi_a^\Pi\right] < \infty \quad \text{with} \quad h(a;\Pi) = h(\widetilde{a},\Pi)\right\}. \tag{4.3}$$

Again we restrict to functions in the L^2 closure of $h(x,\theta_0,v_0,\theta_1,v_1)$ which depend continuously on the arguments $x,\theta_0,v_0,\theta_1,v_1$ and depend only on finitely many evaluations of events involving whether specified open subsets of $\mathbb{R}^2$ are hit by Π-lines of speeds exceeding specified positive thresholds. In particular, measurability of functions in the subspace will use the σ-algebra $\sigma\{\Pi_{\geq u} : u > 0\}$, where $\Pi_{\geq u}$ is the locally finite Poisson line process of Π-lines of speed exceeding u, viewed as a random closed set and endowed with the hitting σ-algebra (also called *Effros σ-algebra*) generated by hitting events $[\Pi_{\geq u}\cap K \neq \emptyset]$ for compact sets $K\subset\mathbb{R}^2$ (Chiu et al., 2013, §6.1.2).

The annealed Dirichlet form is given by

$$\begin{aligned} B(f,g) \quad &= \quad \mathbb{E}\left[\sum_{a\in(\Pi\times\Pi)\setminus\Delta}\pi_a^\Pi f(a;\Pi\setminus a)\,\mathbb{E}\left[g(\widetilde{Z}_1;\Pi\setminus Z_1)|Z_0 = a,\Pi\right]\right] \\ &= \quad \mathbb{E}\left[\sum\sum_{a\neq b\in(\Pi\times\Pi)\setminus\Delta}\pi_a^\Pi p_{a,\widetilde{b}}^\Pi\, f(a;\Pi\setminus a)g(b;\Pi\setminus b)\right] \\ &= \quad \mathbb{E}\left[\sum\sum_{a\neq b\in(\Pi\times\Pi)\setminus\Delta}\pi_a^\Pi\omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)}s_{\widetilde{b}}\, f(a;\Pi\setminus a)g(b;\Pi\setminus b)\right], \end{aligned} \tag{4.4}$$

where f and g are functions of the random environment Π as well as of $\mathbb{R}^2\times(0,\pi)\times(0,\infty)\times(0,\pi)\times(0,\infty)$, and both belong to $\mathfrak{H}$. The superscripts in π_a^Π, $p_{a,\widetilde{b}}^\Pi$ and $\omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)}$, in (4.4) and (4.3) emphasize dependence on the environment Π as well as a and b. We use an abbreviated notation $\Pi\setminus a = \Pi\setminus\{\mathcal{L}_-,\mathcal{L}_+\}$ when $a = (\mathcal{L}_-,\mathcal{L}_+)$, and $\Pi\setminus(a,b) = \Pi\setminus\{\mathcal{L}_-,\mathcal{L}_+,\mathcal{L}_0,\mathcal{L}_1\}$ when $a = (\mathcal{L}_-,\mathcal{L}_+)$ and $a = (\mathcal{L}_0,\mathcal{L}_1)$.

The annealed symmetric Dirichlet form (4.4) can be associated with the (rather trivial) augmentation of the Markov chain Z which is given by $((Z_n,\Pi) : n\geq 0)$. Thus the augmentation simply consists of adding the time-constant random process Π. Note that we can use $B(f,g)$ to recover the joint distribution of Z_0 and Z_1, and hence the conditional probability distribution $\mathrm{Law}(Z_1|Z_0,\Pi)$. So knowledge of the annealed symmetric Dirichlet form (4.4) identifies the annealed stochastic dynamics of the SIRSN-RRF Z.

We now introduce the notion of the *relative environment process* Ψ and the *reduced relative environment process* $\Psi^{(0)}$ for Z; $\Psi_n^{(0)}$ is the environment Π viewed from $Z_n = (\mathcal{L}_{-,n},\mathcal{L}_{0,n})$, obtained by removing the lines $\mathcal{L}_{-,n}$ and $\mathcal{L}_{0,n}$ (reduction), then translating, rotating and rescaling $\Pi\setminus\{\mathcal{L}_{-,n},\mathcal{L}_{0,n}\}$ into a standard form (relativization). In detail, for each state $a = (\mathcal{L}_-,\mathcal{L}_0)$ we introduce a proper similarity S_a whose inverse can be used to deliver the required standard form. (Recall that a similarity is simply an affine-linear transformation of Euclidean space: a *proper* similarity is one which preserves the sign of the area differential.) If $\mathcal{L}_-\cap\mathcal{L}_0 = \{z\}$ then we require that $S_a\,\mathbf{o} = z$; furthermore S_a must send the x-axis $\mathcal{L}_*^0$ (with sense given by standard direction, and unit speed) to the line $\mathcal{L}_0$ (with prescribed sense and rescaling so it has the required speed $v(\mathcal{L}_0)$); finally we require that S_a sends $\mathcal{L}_*^\theta$ to $\mathcal{L}_-$, where the line $\mathcal{L}_*^\theta$ passing through $\mathbf{o}$ makes angle $\theta = \sphericalangle(\mathcal{L}_-,\mathcal{L}_0)$ with $\mathcal{L}_*^0$. The scaling component of the similarity S_a is fixed by the requirement that $\mathcal{L}_*^0$ has unit speed: as a consequence the speed of $\mathcal{L}_*^\theta$ must be $v(\mathcal{L}_-)/v(\mathcal{L}_0)$. These requirements uniquely define the proper similarity S_a.

Definition 4.1. *The* relative environment Ψ_n *of* Z_n *is given by* $S_{Z_n}^{-1}\Pi$*. The* reduced relative environment $\Psi_n^{(0)}$ *of* $Z_n = (\mathcal{L}_-, \mathcal{L}_0)$ *is obtained by removing the lines* $S_{Z_n}^{-1}\mathcal{L}_-$ *and* $S_{Z_n}^{-1}\mathcal{L}_0$ *corresponding to* $\mathcal{L}_-$ *and* $\mathcal{L}_0$*:*

$$\Psi_n^{(0)} \quad = \quad S_{Z_n}^{-1}(\Pi \setminus Z_n) \quad = \quad S_{Z_n}^{-1}(\Pi) \setminus S_{Z_n}^{-1} Z_n \quad = \quad S_{Z_n}^{-1}(\Pi) \setminus \left\{ S_{Z_n}^{-1}\mathcal{L}_-, S_{Z_n}^{-1}\mathcal{L}_0 \right\} .$$

So the relative environment of Z_n is parametrized by the relative speed $v(\mathcal{L}_-)/v(\mathcal{L}_0)$ of the immediately preceding line $\mathcal{L}_-$ when compared with the current speed $\mathcal{L}_0$, the angle $\sphericalangle(\mathcal{L}_-, \mathcal{L}_0)$ between current and immediately preceding lines, and the point pattern $\Psi_n^{(0)}$.

The transmission probability $\omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)}$ in (4.4) must vanish unless a and b belong to the same scattering class $\mathcal{E}$. Moreover in this case the states a and b must share a line: $a = (\mathcal{L}_-, \mathcal{L}_0)$ and $b = (\mathcal{L}_+, \mathcal{L}_0)$. Applying dynamical reversibility of the quenched process, we find $\pi_a / s_{\widetilde{a}} = v(\mathcal{L}_0)^\alpha = \kappa(\mathcal{E})$ is a function of the scattering class $\mathcal{E}$ alone. Hence (4.4) can be rewritten as

$$\begin{aligned} B(f,g) \quad &= \quad \mathbb{E}\left[\sum_{\mathcal{E}\in\mathfrak{E}} \left(\sum\sum_{a\neq b\in\mathfrak{E}} \kappa(\mathcal{E}) s_{\widetilde{a}} \omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)} s_{\widetilde{b}}\, f(a;\Pi\setminus a) g(b;\Pi\setminus b) \right) \right] \\ &= \mathbb{E}\left[\sum_{\mathcal{L}_0\in\Pi} v(\mathcal{L}_0)^\alpha \left(\sum\sum_{\mathcal{L}_-\neq\mathcal{L}_+\in\Pi\setminus\{\mathcal{L}_0\}} s_{(\mathcal{L}_0,\mathcal{L}_-)} f((\mathcal{L}_-,\mathcal{L}_0);\Pi\setminus\{\mathcal{L}_-,\mathcal{L}_0\}) \right.\right. \\ &\qquad\qquad \left.\left. s_{(\mathcal{L}_0,\mathcal{L}_+)} g((\mathcal{L}_+,\mathcal{L}_0);\Pi\setminus\{\mathcal{L}_+,\mathcal{L}_0\})\, \omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)} \right) \right], \end{aligned}$$

where (compare Theorem 2.8)

$$\omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)} \quad = \quad \omega_{(\mathcal{L}_-,\mathcal{L}_0),(\mathcal{L}_0,\mathcal{L}_+)}^{\Pi\setminus\{\mathcal{L}_-,\mathcal{L}_0,\mathcal{L}_+\}} \quad = \quad \prod_{\substack{\mathcal{L}\text{ separates}\\ \mathcal{L}_0\cap\mathcal{L}_+\text{ and }\mathcal{L}_0\cap\mathcal{L}_+}} \left(1 - s_{(\mathcal{L}_0,\mathcal{L})}\right) .$$

Note that the stochastic dynamics of Z are invariant under similarity transformations, because they depend only on the scattering probabilities $s_{(\mathcal{L}_0,\mathcal{L}_-)}$ and $s_{(\mathcal{L}_0,\mathcal{L}_+)}$, and the transmission probability $\omega_{a,\widetilde{b}}^{\Pi\setminus(a,b)}$, all of which possess this invariance. It follows that the relative environment process $\Psi = (\Psi_n = S_{Z_n}^{-1}\Pi : n \geq 0)$, when quenched by conditioning on Π, is again a Markov chain.

However we can say more. The annealed Dirichlet form (4.4) can be used to establish that Ψ *when not conditioned on* Π (hence annealed) forms a stationary process for suitably distributed random initial starting points $X_0 \in (\Pi\times\Pi)\setminus\Delta$, and it can then be used to compute the ensuing stationary distribution of Ψ. The theory is closely related to that of Palm conditioning for point processes, and similarly requires careful interpretation although the underlying idea is simple enough: for some $c > 0$, the SIRSN candidate Π is conditioned to have at least one intersection within c of the origin $\mathbf{o}$, such that the current line has speed exceeding c, and the intersection with the previous line is chosen with weight based on the probability of switching. Our results will cover the evolution of the relative environment process Ψ for Z begun at one of these intersection points chosen according to the indicated weighting.

To facilitate our argument, we first establish a factorization result for suitable π-weighted sums over $(\Pi\times\Pi)\setminus\Delta$.

Lemma 4.2. *Given a Poisson line process* Π *based on the parameter* $\gamma > 1$*, consider a non-negative measurable function* $\xi((\mathcal{L}_-,\mathcal{L}_0);\Pi\setminus\{\mathcal{L}_-,\mathcal{L}_0\})$*, defined for* $(\mathcal{L}_-,\mathcal{L}_0) \in$

$(\Pi \times \Pi) \setminus \Delta$, *which admits a factorization*

$$\begin{aligned} \xi((\mathcal{L}_-, \mathcal{L}_0); \Pi \setminus \{\mathcal{L}_-, \mathcal{L}_0\}) \quad &= \quad \xi_{invar}((\mathcal{L}_-, \mathcal{L}_0); \Pi \setminus \mathcal{L}_-, \mathcal{L}_0) \times \\ \times\ \xi_{speed}(v(\mathcal{L}_0))\ \times\ \xi_{config}(&\mathrm{dist}_{signed}(\mathcal{L}_- \cap \mathcal{L}_0, \mathcal{L}_0^{\perp} \cap \mathcal{L}_0), \mathrm{dist}_{signed}(\mathcal{L}_0, \mathbf{o}), \sphericalangle(\mathcal{L}_0, \mathcal{L}_*^0)))\,, \end{aligned} \tag{4.5}$$

where $\mathcal{L}_0^{\perp}$ is the line through $\mathbf{o}$ perpendicular to $\mathcal{L}_0$. Here ξ_{invar} is a similarity-invariant function of its arguments, while ξ_{speed} is a function of the current speed and ξ_{config} is a function of three parameters describing the location and orientation of the configuration $(\mathcal{L}_-, \mathcal{L}_0)$. Suppose the intersections $a = (\mathcal{L}_-, \mathcal{L}_0) \in (\Pi \times \Pi) \setminus \Delta$ are weighted by $\pi_a = \min\{v(\mathcal{L}_-)^\alpha, (\mathcal{L}_0)^\alpha\}$ for some fixed $\alpha \in (\gamma - 1, \infty)$. Then

$$\begin{aligned} \mathbb{E}\left[\sum_{\mathcal{L}_- \neq \mathcal{L}_0 \in \Pi} \xi((\mathcal{L}_-, \mathcal{L}_0); \Pi \setminus \{\mathcal{L}_-, \mathcal{L}_0\})\, \pi_{(\mathcal{L}_-, \mathcal{L}_0)}\right] \quad &= \\ \left(\frac{\gamma-1}{2}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi} \xi_{config}(s, r, \theta)\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}r\ \times\ \int_0^\infty \xi_{speed}(v) v^{\alpha - 2\gamma + 1}\,\mathrm{d}v \times \\ \times \int_{-\infty}^{\infty}\int_0^{\pi} \mathbb{E}\left[\xi_1(e^t, \phi, \Pi)\right] \min\{e^{(\alpha - (\gamma - 1))t}, e^{-(\gamma-1)t}\} \sin\phi\,\mathrm{d}\phi\ \mathrm{d}t\,. \end{aligned} \tag{4.6}$$

Here $\xi_1(e^t, \phi, \Pi)$ is defined in terms of the similarity-invariant function ξ_{invar} by

$$\xi_1\left(\frac{v(\mathcal{L}_-)}{v(\mathcal{L}_0)}, \sphericalangle(\mathcal{L}_-, \mathcal{L}_0), S^{-1}_{(\mathcal{L}_-, \mathcal{L}_0)}(\Pi \setminus \{\mathcal{L}_-, \mathcal{L}_0\})\right) \quad = \quad \xi_{invar}((\mathcal{L}_-, \mathcal{L}_0); \Pi \setminus \{\mathcal{L}_-, \mathcal{L}_0\})\,.$$

Proof. As in the proof of Theorem 3.6, first apply the Slivnyak-Mecke theorem (Theorem 3.4) twice in succession to the left-hand side of (4.6):

$$\begin{aligned} \mathbb{E}\left[\sum_{\mathcal{L}_- \neq \mathcal{L}_0 \in \Pi} \xi((\mathcal{L}_-, \mathcal{L}_0); \Pi \setminus \{\mathcal{L}_-, \mathcal{L}_0\})\, \pi_{(\mathcal{L}_-, \mathcal{L}_0)}\right] \quad &= \\ \int\int \mathbb{E}\left[\xi((\mathcal{L}_-, \mathcal{L}_0); \Pi)\right] \pi_{(\mathcal{L}_-, \mathcal{L}_0)} \nu(\,\mathrm{d}\mathcal{L}_-)\nu(\,\mathrm{d}\mathcal{L}_0)\,. \end{aligned}$$

Using the representation corresponding to (1.1) for $\nu(\,\mathrm{d}\mathcal{L}_0)$ (based on $\mathbf{o}$ and $\mathcal{L}_*^0$ for reference point and line), and the representation corresponding to (1.2) for $\nu(\,\mathrm{d}\mathcal{L}_-)$ (based on $\mathcal{L}_0 \cap \mathcal{L}_*^0$ and $\mathcal{L}_0$ for reference point and line), we obtain

$$\begin{aligned} \mathbb{E}\left[\sum_{\mathcal{L}_- \neq \mathcal{L}_0 \in \Pi} \xi((\mathcal{L}_-, \mathcal{L}_0); \Pi \setminus \{\mathcal{L}_-, \mathcal{L}_0\})\, \pi_{(\mathcal{L}_-, \mathcal{L}_0)}\right] \quad &= \\ \left(\frac{\gamma-1}{2}\right)^2 \int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{\infty} \Bigg(\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{\infty} \mathbb{E}\left[\xi_{\mathrm{invar}}\left(\frac{v_-}{v_0}, \phi, S^{-1}_{(\mathcal{L}_-, \mathcal{L}_0)}(\Pi)\right)\right] \times \\ \times\, \xi_{\mathrm{speed}}(v_0) \times \xi_{\mathrm{config}}(s, r, \theta) \times \min\{v_-^\alpha, v_0^\alpha\} v_-^{-\gamma} \sin\phi\,\mathrm{d}v_-\,\mathrm{d}\theta\,\mathrm{d}s\Bigg) v_0^{-\gamma}\,\mathrm{d}v_0\,\mathrm{d}\theta\,\mathrm{d}r\,. \end{aligned}$$

But scale invariance implies that $\mathbb{E}\left[\xi_{\mathrm{invar}}\left(\frac{v_-}{v_0}, \phi, S^{-1}_{(\mathcal{L}_-, \mathcal{L}_0)}(\Pi)\right)\right] = \mathbb{E}\left[\xi_{\mathrm{invar}}\left(\frac{v_-}{v_0}, \phi, \Pi\right)\right]$. The result now follows by a simple change of coordinates: set $v_0 = v$ and $v_- = v_0 e^t$ so that $\mathrm{d}v_-\,\mathrm{d}v_0 = e^t v\,\mathrm{d}v\,\mathrm{d}t$. □

We can now state and prove the main theorem of this section.

Theorem 4.3. *Given a SIRSN-RRF Z on a Poisson line SIRSN or SIRSN candidate Π, parametrized by $\alpha > \gamma - 1$ where γ is the Poisson line SIRSN parameter, the relative environment process Ψ for the SIRSN-RRF Z can be made stationary if its initial distribution can be expressed as three independent components as follows:*

(1) the log-relative speed of the line before the current line has an asymmetric Laplacian density over $\mathbb{R}$: rate parameter $\gamma - 1$ for positive values, $\alpha - (\gamma - 1)$ for negative values;

(2) the angle between current and previous lines has a sine-weighted density;

(3) the ensemble $\Psi^{(0)}$ of all lines other than current and previous lines (the reduced relative environment) is distributed as the original speed-marked Poisson line process Π.

Proof. Because $B(f,g)$ is given by an expression involving the Markovian kernel $\pi_a \omega_{a,\widetilde{b}}$,

$$B(f,g) \quad = \quad \mathbb{E}\left[\sum\sum_{a \neq b \in (\Pi\times\Pi)\setminus\Delta} \pi_a \omega_{a,\widetilde{b}} s_{\widetilde{b}}\, f(a;\Pi) g(b;\Pi)\right],$$

it possesses a completion which applies to the case when g is bounded and f satisfies the L^1 condition

$$\mathbb{E}\left[\sum_{a\in(\Pi\times\Pi)\setminus\Delta} |f(a;\Pi)|\, \pi_a\right] < \infty\,.$$

Suppose f is non-negative and admits a factorization as in Lemma 4.2;

$$\begin{aligned} f((\mathcal{L}_-,\mathcal{L}_0);\Pi) \quad &= \quad f_{\text{invar}}((\mathcal{L}_-,\mathcal{L}_0);\Pi\setminus\mathcal{L}_-,\mathcal{L}_0)\times \\ &\times\ f_{\text{speed}}(v(\mathcal{L}_0))\ \times\ f_{\text{config}}(\operatorname{dist}_{\text{signed}}(\mathcal{L}_-\cap\mathcal{L}_0,\mathcal{L}_{x,0}\cap\mathcal{L}_0),\operatorname{dist}_{\text{signed}}(\mathcal{L}_0,\mathbf{o}),\sphericalangle(\mathcal{L}_0,\mathcal{L}_*^0)))\,, \end{aligned}$$

where f_{invar} is similarity-invariant. In particular, if $\phi_- = \sphericalangle(\mathcal{L}_-,\mathcal{L}_0)$ and $v(\mathcal{L}_-) = e^{t_-}v(\mathcal{L}_0)$ then we write

$$\begin{aligned} f_{\text{invar}}((\mathcal{L}_-,\mathcal{L}_0);\Pi\setminus\{\mathcal{L}_-,\mathcal{L}_0\}) \ &= \ f_{\text{invar}}\left(\frac{v(\mathcal{L}_-)}{v(\mathcal{L}_0)},\sphericalangle(\mathcal{L}_-,\mathcal{L}_0),S^{-1}_{(\mathcal{L}_-,\mathcal{L}_0)}(\Pi\setminus\{\mathcal{L}_-,\mathcal{L}_0\})\right) \\ &= \quad f_1(e^{t_-},\phi_-,\Psi_0^{(0)})\,. \end{aligned}$$

Suppose further that the bounded g is itself similarity-invariant: we write

$$g(\widetilde{b};\Pi) \quad = \quad g((\mathcal{L}_+,\mathcal{L}_0);\Pi) \quad = \quad g_{\text{invar}}((\mathcal{L}_+,\mathcal{L}_0);\Pi\setminus\{\mathcal{L}_0,\mathcal{L}_+\})\,.$$

Since the dynamics of Z are similarity-invariant, this means that

$$\mathbb{E}\left[g(\widetilde{Z}_1;\Pi)|Z_0=a,\Pi\right] \quad = \quad \mathbb{E}\left[g(S\widetilde{Z}_1;S\Pi)|SZ_0=a,S\Pi\right]$$

for any similarity S, so $\mathbb{E}\left[g(\widetilde{Z}_1;\Pi)|Z_0=a,\Pi\right]$ is similarity-invariant as a function of a and $\Pi\setminus a$. Consequently we may apply Lemma 4.2 to

$$\begin{aligned} \xi((\mathcal{L}_-,\mathcal{L}_0);\Pi) \quad &= \quad f((\mathcal{L}_-,\mathcal{L}_0);\Pi)\,\mathbb{E}\left[g(\widetilde{Z}_1;\Pi)|Z_0=(\mathcal{L}_-,\mathcal{L}_0),\Pi\right] \quad = \\ & f_{\text{invar}}((\mathcal{L}_-,\mathcal{L}_0);\Pi\setminus\{\mathcal{L}_0,\mathcal{L}_-\})\times\mathbb{E}\left[g_{\text{invar}}(\widetilde{Z}_1;\Pi\setminus\{\mathcal{L}_0,\mathcal{L}_+\})\mid Z_0=(\mathcal{L}_-,\mathcal{L}_0),\Pi\right]\times \\ &\times f_{\text{speed}}(v(\mathcal{L}_0))\ \times\ f_{\text{config}}(\operatorname{dist}_{\text{signed}}(\mathcal{L}_-\cap\mathcal{L}_0,\mathcal{L}_{x,0}\cap\mathcal{L}_0),\operatorname{dist}_{\text{signed}}(\mathcal{L}_0,\mathbf{o}),\sphericalangle(\mathcal{L}_0,\mathcal{L}_*^0)))\,. \end{aligned}$$

We deduce

$$\begin{aligned}B(f,g) &= \mathbb{E}\left[\sum_{\mathcal{L}_-\neq\mathcal{L}_0\in\Pi}\sum \pi_{(\mathcal{L}_-,\mathcal{L}_0)} f((\mathcal{L}_-,\mathcal{L}_0);\Pi)\,\mathbb{E}\left[g(\widetilde{Z}_1;\Pi)|Z_0=(\mathcal{L}_-,\mathcal{L}_0),\Pi\right]\right]\\
&= \left(\frac{\gamma-1}{2}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi} f_{\text{config}}(s,r,\theta)\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}r \;\times\; \int_0^{\infty} f_{\text{speed}}(v)v^{\alpha-2\gamma+1}\,\mathrm{d}v\;\times\\
&\qquad\int_{-\infty}^{\infty}\int_0^{\pi}\min\{e^{(\alpha-(\gamma-1))t_-},e^{-(\gamma-1)t_-}\}\times\\
&\times\mathbb{E}\left[f_{\text{invar}}((\mathcal{L}_*^{\phi_-}(e^{t_-}),\mathcal{L}_*^0);\Pi)\,\mathbb{E}\left[g_{\text{invar}}(\widetilde{Z}_1;\Pi)|Z_0=(\mathcal{L}_*^{\phi_-}(e^{t_-}),\mathcal{L}_*^0),\Pi\right]\right]\sin\phi_-\,\mathrm{d}\phi_-\;\mathrm{d}t_-\,,\end{aligned}$$

where the line $\mathcal{L}_*^{\phi_-}(e^{t_-})$ meets the unit-speed (x-axis) $\mathcal{L}_*^0$ at $\mathbf{o}$, making an angle ϕ_-, and has speed e^{t_-}.

We may deduce the following by arranging for $f_{\text{config}}(s,r,\theta)=\mathbb{I}\left[s^2+r^2\leq c^2\right]$ and $f_{\text{speed}}(v)=\mathbb{I}\left[v>c\right]$ for fixed $c>0$. Sample the speed-marked line process Π, and sample $\mathcal{L}_0$ uniformly at random from the set of lines of Π lying within c of $\mathbf{o}$ and with speed exceeding c. Then sample $\mathcal{L}_-$ from the lines of Π intersecting $\mathcal{L}_0$ and such that (i) the intersection point is within c of $\mathbf{o}$, using sampling weights $\min\{1,(v(\mathcal{L}_1)/v(\mathcal{L}_0))^\alpha\}$ If there is no such line $\mathcal{L}_0$, or there turn out to be no such intersections, then re-sample Π and repeat till successful. Use the resulting $(\mathcal{L}_-,\mathcal{L}_0)$ as the initial point of the SIRSN-RRF Z. Then the resulting relative environment process is associated with the following Dirichlet form:

$$\begin{aligned}&B^{\text{relative}}(f_{\text{invar}},g_{\text{invar}}) \;=\\
&\quad\mathbb{E}\left[\min\left\{1,\left(\frac{v(\mathcal{L}_-)}{v(\mathcal{L}_0)}\right)^\alpha\right\} f_{\text{invar}}((\mathcal{L}_-,\mathcal{L}_0);\Pi)\,\mathbb{E}\left[g_{\text{invar}}(\widetilde{Z}_1;\Pi)|Z_0=(\mathcal{L}_-,\mathcal{L}_0),\Pi\right]\right] \;=\\
&\qquad\int_{-\infty}^{\infty}\int_0^{\pi}\mathbb{E}\Bigg[f_1(e^{t_-},\phi_-,\Pi)\sum_{\mathcal{L}_+\in\Pi}\omega^{\Pi\setminus\{\mathcal{L}_+\}}_{\mathbf{o},(\mathcal{L}_*^0\cap\mathcal{L}_+)}\min\{1,v(\mathcal{L}_+)^\alpha\}\times\\
&\qquad\times g_1(v(\mathcal{L}_+),\sphericalangle(\mathcal{L}_*^0,\mathcal{L}_+),\Pi)\Bigg]\min\{e^{(\alpha-(\gamma-1))t_-},e^{-(\gamma-1)t_-}\}\sin\phi_-\,\mathrm{d}\phi_-\;\mathrm{d}t_-\,.\end{aligned}$$

Here

$$\omega^{\Pi\setminus\{\mathcal{L}_+\}}_{\mathbf{o},(\mathcal{L}_*^0\cap\mathcal{L}_+)} \quad=\quad \prod_{\substack{\mathcal{L}\in\Pi\setminus\{\mathcal{L}_+\}\text{ separating}\\ \mathbf{o}\text{ and }(\mathcal{L}_*^0\cap\mathcal{L}_+)}}\left(1-s_{\mathcal{L}_0^*,\mathcal{L}}\right).$$

One further application of the Slivnyak-Mecke theorem (Theorem 3.4), using the representation (1.2) for $\nu(\,\mathrm{d}\mathcal{L}_+)$, now yields

$$\begin{aligned}&B^{\text{relative}}(f_{\text{invar}},g_{\text{invar}}) \quad=\\
&\quad\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi}\int_{-\infty}^{\infty}\int_0^{\pi}\mathbb{E}\left[f_1(e^{t_-},\phi_-,\Pi)\omega^{\Pi}_{[\mathbf{o},s]}\min\{1,e^{\alpha t_+}\}g_1(e^{t_+},\phi_+,\Pi)\right]\times\\
&\quad\times\min\{e^{(\alpha-(\gamma-1))t_-},e^{-(\gamma-1)t_-}\}\sin\phi_-\,\mathrm{d}\phi_-\;\mathrm{d}t_-\;e^{-(\gamma-1)t_+}\sin\phi_+\,\mathrm{d}\phi_+\;\mathrm{d}t_+\,\mathrm{d}s_+\\
&\qquad=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi}\int_{-\infty}^{\infty}\int_0^{\pi}\mathbb{E}\left[f_1(e^{t_-},\phi_-,\Pi)g_1(e^{t_+},\phi_+,\Pi)\,\omega^{\Pi}_{[\mathbf{o},s]}\right]\times\\
&\qquad\times\min\{e^{(\alpha-(\gamma-1))t_-},e^{-(\gamma-1)t_-}\}\min\{e^{(\alpha-(\gamma-1))t_+},e^{-(\gamma-1)t_+}\}\times\\
&\qquad\qquad\times\sin\phi_-\,\mathrm{d}\phi_-\sin\phi_+\,\mathrm{d}\phi_+\;\mathrm{d}t_-\;\mathrm{d}t_+\,\mathrm{d}s_+\,,\end{aligned}\tag{4.7}$$

where $[\mathbf{o}, s]$ is short-hand for the interval along $\mathcal{L}^0_*$ with one endpoint given by $\mathbf{o}$ and with the signed length s.

The invariance of f_{invar}, g_{invar}, and f_{invar} collectively imply that the Dirichlet form $B^{\text{relative}}(f_{\text{invar}}, g_{\text{invar}})$ is symmetric. This in turn implies that the relative environment process Ψ is stationary, with an invariant measure which is a probability measure which makes the three coordinates of log relative speed t of previous line, angle ϕ between previous and current lines, and reduced environment $\Psi^{(0)}$ independent with joint distribution as follows

1. corresponding to the log relative speed t, a (possibly asymmetric) Laplacian density over $\mathbb{R}$ (Johnson, Kotz, and Balakrishnan, 1994, ch. 24), given by

$$f_{\alpha,\gamma}(y) \quad = \quad \begin{cases} \frac{(\gamma-1)(\alpha-(\gamma-1))}{\alpha} e^{-(\gamma-1)|y|} & \text{when } y \geq 0\,, \\ \frac{(\gamma-1)(\alpha-(\gamma-1))}{\alpha} e^{-(\alpha-(\gamma-1))|y|} & \text{when } y < 0\,; \end{cases} \tag{4.8}$$

2. corresponding to ϕ, a half-sine density over $[0, \pi)$, given by $\frac{1}{2}\sin\phi$;

3. corresponding to the reduced relative environment $\Psi^{(0)}$, a distribution which agrees with that of the underlying SIRSN candidate Π.

This completes the proof. □

We are actually interested in the log-relative speed of the current line with respect to the previous line. The distribution of this in stationary state is readily computed directly from Theorem 4.3, bearing in mind that if Z is at $(\mathcal{L}_-, \mathcal{L}_0)$ then its next state is $(\mathcal{L}_0, \mathcal{L}_+)$, where $\mathcal{L}_+$ is drawn from the reduced relative environment $\Psi^{(0)}$, such that $(\mathcal{L}_0, \mathcal{L}_+)$ is the first intersection along $\mathcal{L}_0$ is the chosen direction which is accepted by the rule summarized by the acceptance probability (3.3). We obtain

Corollary 4.4. *In the situation of Theorem 4.3, when Ψ is stationary, the distribution of the log of the speed of the current line relative to the speed of the previous line has density given by the (possibly asymmetric) Laplacian density prescribed by* (4.8)*, with mean value given by*

$$\int_{-\infty}^{\infty} y f_{\alpha,\gamma}(y)\,\mathrm{d}y \quad = \quad \frac{1}{\gamma-1} - \frac{1}{\alpha-(\gamma-1)} \quad = \quad \frac{\alpha - 2(\gamma-1)}{(\gamma-1)(\alpha-(\gamma-1))}\,. \tag{4.9}$$

Proof. Leaving $(\mathcal{L}_-, \mathcal{L}_0)$, Z encounters intersections with lines of $\Psi^{(0)}$ according to a speed-marked Poisson process with intensity measure $v^{-\gamma}\,\mathrm{d}v\,\mathrm{d}s$, where v is now the speed of the new line relative to the speed of $\mathcal{L}_0$ and s is the scaled distance along $\mathcal{L}_0$ from $(\mathcal{L}_-, \mathcal{L}_0)$. Note that the mark measure does not have finite mass. We have to thin this marked Poisson process with retention probability $\min\{1, v^\alpha\}$, according to the acceptance probability (3.3) (bearing in mind that v is the relative speed of the new line), so the mark distribution of the retained lines does have finite mass. Accordingly the mark distribution of retained lines, and thus the relative speed of the new line, is proportional to (hence equal to) (4.8). □

In particular the log-relative-speed density has zero mean in the symmetric case, when $\alpha = 2(\gamma-1)$, in which case the log-relative-speed stationary distribution is is symmetric and is given by the Laplace or double-headed exponential distribution, with rate parameter $\gamma - 1$, and density

$$f_{2(\gamma-1),\gamma}(y) \quad = \quad \frac{\gamma-1}{2} e^{-(\gamma-1)|y|}\,. \tag{4.10}$$

Finally note that the reduced relative environment process, and hence the relative environment process, is very far from being irreducible. Indeed, any particular realization of the state Ψ of the reduced environment process defines a countable set of intersection angles $A = \{\sphericalangle(\mathcal{L}_1, \mathcal{L}_2)) : (\mathcal{L}_1, \mathcal{L}_2) \in (\Psi \times \Psi) \setminus \Delta\}$ *which remains time-constant under the evolution of* Ψ. But Ψ has equilibrium distribution given by the SIRSN candidate Π, and there will be zero probability of *any* of the countably many intersections of lines from Π having an angle belonging to a fixed countable set. Indeed, by the Slivnyak-Mecke theorem (Theorem 3.4) we know that for any fixed angle ϕ we must have

$$\mathbb{E}\left[\sum_{(\mathcal{L}_1,\mathcal{L}_2)\in(\Pi\times\Pi)\setminus\Delta} \mathbb{I}\left[\sphericalangle(\mathcal{L}_1, \mathcal{L}_2) = \phi\right]\right] \quad = \quad \int\int \mathbb{I}\left[\sphericalangle(\mathcal{L}_1, \mathcal{L}_2) = \phi\right] \nu(\,\mathrm{d}\mathcal{L}_1)\nu(\,\mathrm{d}\mathcal{L}_2) \quad = \quad 0.$$

Thus conventional Markovian arguments cannot be applied. However, as we will see in the next section, ergodic theory allows us to prove the results we need.

5 Long-term behaviour of SIRSN-RRF speed

If $Z = (Z_0, Z_1, \ldots)$ is the (discrete-time) SIRSN-RRF and if $V = (V_0, V_1, \ldots)$ yields the corresponding sequence of speeds for the current line, then the relative speed change V_n/V_{n-1} can be determined using only Ψ_n the relative environment (relativized by centering, rotating, and scaling). Theorem 4.3 implies that the log-relative speed-changes $U_n = \log(V_n/V_{n-1})$ form a stationary sequence, if the initial relative environment is given the stationary distribution discussed in the previous section, and thus U_0 is given the equilibrium density specified in Equation (4.8). We may therefore apply the non-ergodic part of Birkhoff's ergodic theorem (see for example Kallenberg, 2002, Theorem 10.6) to show

$$\frac{1}{n}\log(V_n/V_0) \quad = \quad \frac{1}{n}\sum_{m=1}^{n} U_m \quad \to \quad \mathbb{E}\left[U_0|\mathcal{I}\right], \tag{5.1}$$

where $\mathcal{I}$ is the σ-algebra of shift-invariant events for the random process U, and convergence is both almost sure and in L^1.

It immediately follows that, away from the critical case $\alpha = 2(\gamma - 1)$, the speed V_n has at least a positive chance of either diverging to $+\infty$ exponentially fast as $n \to \infty$ (if $\alpha > 2(\gamma - 1)$) or converging to 0 exponentially fast as $n \to \infty$ (if $\alpha < 2(\gamma - 1)$). We may therefore rule out non-critical cases ($\alpha \neq 2(\gamma - 1)$) in our search for an example which is speed-neighbourhood-recurrent.

Suppose it can be shown that Ψ (and therefore U) is ergodic, so that we can replace $\mathbb{E}\left[U_0|\mathcal{I}\right]$ by $\mathbb{E}\left[U_0\right]$ in (5.1). In the non-critical cases discussed above, this means we can replace "has at least a positive chance" by "will almost surely end up". But in the critical case $\mathbb{E}\left[U_0\right] = 0$ is not sufficient in itself to guarantee neighborhood-recurrence for U. However in the ergodic case neighborhood-recurrence actually follows rather simply from the celebrated (but unpublished) Kesten, Spitzer and Whitman range theorem (described by Spitzer, 1976, page 38). In its original form the range theorem implies concerns recurrence for integer-valued stationary ergodic processes of zero mean. The real-valued / neighbourhood recurrent case is a simple variation on the original idea:

Theorem 5.1. *(Kesten-Spitzer-Whitman, real-valued case.) Suppose* $U_1, \ldots, U_n, \ldots$ *form a stationary ergodic sequence, with* $\mathbb{E}\left[U_1\right] = 0$*. Set* $W_n = U_1 + \ldots + U_n$*. Then for all* $\varepsilon > 0$ *it is the case that*

$$\mathbb{P}\left[|W_n - W_0| \leq \varepsilon \text{ infinitely often in } n\right] \quad = \quad 1\,.$$

Proof. Birkhoff's ergodic theorem guarantees that $W_n/n \to 0$ almost surely, hence

$$n^{-1}\sup\{|W_1|, \ldots, |W_n|\} \quad \to \quad 0 \qquad \text{almost surely.} \tag{5.2}$$

Set $A_n = [|W_m - W_n| > \varepsilon \text{ for all } m > n]$.

Applying Birkhoff's ergodic theorem again,

$$n^{-1}(\mathbb{I}[A_1] + \ldots + \mathbb{I}[A_n]) \quad \to \quad \mathbb{P}[A_0] \,. \tag{5.3}$$

If $m \neq n$ then $|W_m - W_n| > \varepsilon$ on $A_n \cap A_m$, and therefore a simple packing argument shows that

$$n^{-1}(\mathbb{I}[A_1] + \ldots + \mathbb{I}[A_n]) \quad \leq \quad (n\varepsilon)^{-1} \sup\{|W_1|, \ldots, |W_n|\} \,. \tag{5.4}$$

Letting $n \to \infty$ in (5.4) and using (5.2), we can deduce from (5.3) that $\mathbb{P}[A_0] = 0$. Consequently

$$\mathbb{P}[|W_n - W_0| \leq \varepsilon \text{ at least once for } n > 0] \quad = \quad 1 \,.$$

The same argument applies for the sub-sampled process $(W_0, W_m, W_{2m}, \ldots)$, for any sub-sampling gap $m > 0$. Therefore the event $\bigcap_m \bigcup_{n \geq m} [|W_n - W_0| \leq \varepsilon]$ happens almost surely. Consequently it is almost sure that W_n returns to within ε of W_0 for infinitely many n and the result follows. □

Accordingly speed-neighborhood-recurrence is established in the critical case $\alpha = 2(\gamma - 1)$ if we can show that the reduced environment process Ψ is ergodic. This is the main result of this paper:

Theorem 5.2. *Given a SIRSN-RRF Z on a Poisson line SIRSN or SIRSN candidate Π, parametrized by $\alpha > \gamma - 1$ where γ is the Poisson line SIRSN parameter, the relative environment process Ψ for the SIRSN-RRF Z is ergodic.*

Proof. The key part of the argument is a variation on an argument of Kozlov (1985, Lemma 1, page 82).

Firstly, consider the process Ψ of the relative environment viewed from the particle. Let h be a bounded harmonic function on the relative environment state-space (harmonic with respect to the process Ψ). It suffices to show that $h(\Psi_n)$ is almost surely constant in (discrete) time n.

Now consider

$$\mathbb{E}\left[(h(\Psi_0) - h(\Psi_1))^2\right] \quad = \quad 2\,\mathbb{E}\left[h(\Psi_0)^2\right] - 2\,\mathbb{E}\left[h(\Psi_0)h(\Psi_1)\right] \quad = \quad 0\,,$$

where the first step follows from stationarity and the second because $h(\Psi)$ is a martingale. Consequently $\mathbb{P}[h(\Psi_1) = h(\Psi_0)] = 1$.

Secondly, using Ψ to explore the network represented by Π, we see that there is a Π-measurable random variable $H = H(\Pi)$ such that $h(\Psi_n) = H(\Pi)$ for all n, for environment Π. Moreover $H(\Pi)$ inherits similarity-invariance from Ψ. It follows from the ergodicity of Π (Theorem 3.5) that $H(\Pi)$ must be non-random. □

This together with Theorem 5.1 implies speed-neighborhood-recurrence for the RRF Z, as required. It also shows that in non-critical cases the speed will either almost surely diverge to infinity or almost surely converge to zero. Accordingly a version of Conjecture 1.6 holds for the randomly-broken local Π-geodesics formed by a critical SIRSN-RRF: in the critical case $\alpha = 2(\gamma - 1)$ the RRF provides a "randomly-broken local Π-geodesic", which avoids slowing down to zero speed (or speeding up to infinite speed).

We conclude this section with a formal statement of the speed-neighborhood-recurrence result.

Theorem 5.3. *Let Π be a Poisson line SIRSN or SIRSN candidate with parameter $\gamma \geq 2$. Then there exists a (discrete-time) SIRSN-RRF Z on Π (an RRF with similarity-invariant dynamics with zero defect) such that the speed process V (given by $V_n = v(\mathcal{L}_0)$ when $Z_n = (\mathcal{L}_-, \mathcal{L}_0)$) almost surely returns infinitely often to any neighbourhood of V_0.*

Proof. Bearing in mind the characterization (Theorem 3.6) of such SIRSN-RRF by index $\alpha > \gamma - 1$, we choose the SIRSN-RRF with critical index $\alpha = 2(\gamma - 1)$. Considering the relative environment process Ψ run in stationarity, and noting that in stationarity the mean log-relative speed $U = \log(V)$ has mean zero (Theorem 4.3), and forms an ergodic process (Theorem 5.2), the desired speed-neighborhood-recurrence result follows from the adapted Kesten-Spitzer-Whitman Theorem 5.1. □

6 Conclusion

This paper defines and characterises SIRSN-RRF (similarity-equivariant discrete-time Rayleigh random flights taking place on scale invariant random spatial networks) using an axiomatic approach to scattering processes. It is shown that the relative environment viewed from the SIRSN-RRF is ergodic stationary, and that there exists a critical SIRSN-RRF whose speed process is neighborhood-recurrent. We offer this as evidence in favour of Conjecture 1.6, that Π-geodesics in a Poisson line SIRSN never come to a complete halt, and therefore can be constructed using doubly infinite sequences of segments taken from the Poisson line SIRSN.

We note that the abstract approach to scattering set out in Section 2 merits further exploration in its own right.

In conclusion, we briefly discuss some points going beyond the question of speed-neighbourhood recurrence in the critical case.

A little more can be said concerning the two non-critical cases. Bearing in mind Corollary 4.4, if $\alpha < 2(\gamma - 1)$, so that the log-relative-speed distribution of the next line relative to the current line for the SIRSN-RRF has negative mean, then ergodicity of the relative environment means that the SIRSN-RRF process itself must almost surely converge to a limiting random point as time tends to infinity. This is because its speed must tend to zero, and so almost surely it must eventually get trapped in a cell of the proper Poisson line tessellation formed by $\Pi_{\geq\varepsilon}$ (recall $\Pi_{\geq\varepsilon}$ is the part of Π for which speeds are higher than some $\varepsilon > 0$). The trapping occurs as $\varepsilon \to 0$, since $\Pi_{\geq\varepsilon}$ is a proper Poisson line tessellation, increasing monotonically as a random set as $\varepsilon \to 0$, with intensity tending to infinity and with cells shrinking down to zero size. Since the SIRSN-RRF has positive chance of not escaping from $\Pi \setminus \Pi_{\geq\varepsilon}$ onto $\Pi_{\geq\varepsilon}$ for large time, and has a positive chance of moving freely within the current connected component of $\Pi \setminus \Pi_{\geq\varepsilon}$ (by considerations akin to those of the irreducibility Corollary 3.10), it follows that the limiting point of the SIRSN-RRF must indeed be random. We call this case the *converging case*.

On the other hand, if $\alpha > 2(\gamma - 1)$, then the log-relative-speed distribution of the next line relative to the current line for the SIRSN-RRF has positive mean, and so the SIRSN-RRF process must almost surely diverge to infinity. This is because its speed must tend to infinity (using again ergodicity of the relative environment), and so almost surely the process must get trapped on $\Pi_{\geq v}$ for any $v > 0$. The divergence occurs as $v \to \infty$, since $\Pi_{\geq v}$ is a proper Poisson line tessellation, decreasing monotonically as a random set as $v \to \infty$, with intensity tending to zero, and therefore with cells blowing up to arbitrarily large size with cell boundaries almost surely being eventually arbitrarily far from the origin $\mathbf{o}$. We call this case the *diverging case*.

Whether the case is critical, diverging, or converging, the discrete-time process is defined for all time (since it will take an infinite number of jumps for the speed to exceed all bounds, or to reduce to zero). Consequently the continuous-time variant (defined by interpolating between scatterings using top-speed linear motion) is defined for all time in the critical case $\alpha = 2(\gamma - 1)$. More generally, under stationarity consider Palm-conditioning on the current line at the n^{th} scattering instant. The marginal distribution of the distance D_n travelled till next scattering must be exponentially distributed, because the pattern of speed-marked intersections on the current line is Poisson. Note that $T_n = D_n/V_n$ is the time till the next scattering. Now $D_n/V_n^{\gamma-1}$ is a function of the relative environment Ψ_n (using the scaling transformation (1.3)) and therefore forms an ergodic sequence. Considering independent thinning of all lines for which scattering fails, one calculates the (conditioned) exponential rate of $D_n/V_n^{\gamma-1} = T_n/V_n^{\gamma-2}$ to be $\frac{\alpha}{\alpha-(\gamma-1)}$. By the ergodic theorem it follows that $\frac{1}{n}\sum_{r=0}^{n} T_r/V_r^{\gamma-2}$ converges almost surely to $\frac{\alpha-(\gamma-1)}{\alpha}$. Thus in the non-SIRSN case of $\gamma=2$ it follows that scattering happens at a constant rate in time, and thus the continuous process will be defined for all time.

In the SIRSN case of $\gamma>2$, if the diverging case holds (so $\alpha > 2(\gamma-1)$) then V_n will eventually tend to ∞ at a geometric rate. Thus in that case almost sure convergence of $\frac{1}{n}\sum_{r=0}^{n} T_r/V_r^{\gamma-2}$ to a positive constant forces us to conclude that $\sum_{r=0}^{n} T_r$ diverges to ∞, and therefore again the continuous process will be defined for all time.

In contrast, in the converging case $\alpha < 2(\gamma-1)$ a similar argument shows that the continuous-time process will reach zero-speed in finite time, trading off the asymptotically linear decrease of the log-speed against the asymptotically linear increase of $\sum_{r=0}^{n} T_r/V_r^{\gamma-2}$.

In the diverging case $\alpha > 2(\gamma-1)$ it is natural to ask whether the (discrete or continuous time) scattering process achieves a limiting direction as viewed from $\mathbf{o}$. In fact it will not do so. This follows by an argument involving:

(i) an ergodic theorem for Π under scaling symmetries: (this is proved in the same manner as Theorem 3.5 but using the r-θ coordinatization used in Equation (1.1) for the intensity measure ν of Π, instead of the s-ϕ parametrization used for Equation (1.2));

(ii) a demonstration that if C_v is the zero-cell for $\Pi_{\geq v}$ (the *Crofton cell* containing the origin for the corresponding tessellation) then there is $p>0$ such that if Z_0 lies in C_v then p is a lower bound for the probability that Z makes a complete circuit of ∂C_v when first hitting ∂C_v. (This is established by noting that by scaling it suffices to consider $v=1$; then the relevant probability can be estimated by thinning $\Pi_{<1}$ such that lines are only retained if they hit ∂C_1 and additionally will not scatter Z on the two occasions when its circuit encounters the line in question.) In fact Calka (2002) gives distributional bounds on the out-radius of C_1, though here we need only use the fact that C_1 is stochastically bounded;

(iii) finally combining these two to show that

$$\mathbb{P}\left[Z \text{ makes a circuit of } C_n \text{ for infinitely many } n\right] \quad = \quad 1\,.$$

Since C_n will intersect any fixed line for large enough n, it follows that Z will eventually visit any fixed line, and therefore cannot be eventually confined within any wedge, and therefore cannot possess a limiting direction.

We conclude with some questions for further work.

In the critical case $\alpha = 2(\gamma-1)$ it is an open question whether the (discrete or continuous time) process (as opposed to its speed) is positive-recurrent on neighborhoods of the origin $\mathbf{o}$. Note that simple arguments imply that positive-recurrence on neighbourhoods would force the conclusion that the process was *point*-recurrent; if Z will always eventually return to a neighbourhood A of the origin then it may (and therefore eventually will) move on to the intersection $A\cap\Pi_{\geq v}$ between the neighbourhood and the proper Poisson line process $\Pi_{\geq v}$ (choosing the positive speed v depending on Π so that $A\cap\Pi_{\geq v}\neq\emptyset$); irreducibility (Lemma 3.10) then implies that Z has positive chance of visiting any specified point on $A \cap \Pi_{\geq v}$, and therefore will succeed in doing so eventually after repeated visits to $A \cap \Pi_{\geq v}$. However there is some evidence that in fact Z is transient: Z is caricatured by the two-dimensional integrated Brownian motion $\int V \, \mathrm{d}s$ (for V a 2-dimensional Brownian motion), which can be shown almost surely to have only finite total length of path within any bounded neighbourhood of its starting point, and thus to be transient in the sense of almost surely eventually leaving this starting point never to return.

It is natural to ask whether some kind of central limit behaviour can be established. Certainly this question makes sense for the log-speed process, as this is produced by partial sums of the stationary ergodic process of log-relative speeds. We leave this question to further work, but note that the approach to this will depend a great deal on whether or not the scattering process itself is point-recurrent.

Central limit behaviour for the scattering process itself is complicated by the fact that in the true SIRSN case $\gamma > 2$ the times between scattering have statistics depending monotonically on the current speed (see remarks earlier in this section). However it may be possible to formulate the process as being approximated by a constructed process based on a Brownian motion, using the coupling techniques of Kendall (1987).

It has been noted that the SIRSN-RRF defined here cannot exist on high-dimensional SIRSN (with $\gamma > d > 2$), for the simple reason that lines of Poisson line processes in space of dimension 3 or higher will almost surely never intersect. However it *does* make sense to ask whether this construction can be generalized to line patterns in 3-space formed by a Poisson process of planes. However it would first be necessary to extend the results of Kendall (2017) and Kahn (2016) to this situation. Finally, it would be an interesting exercise to establish similar results for Rayleigh random flights on Aldous's (2014) binary hierarchy SIRSN.

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