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Errata for *Perturbation by non-local operators*

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There is a gap in the proof of (3.19) in [1, Theorem 3.6] in that the constant C_{14} in [1, (3.22)] depends on $r^{1/\alpha}\lambda$ rather than $\lambda > 0$ and so when applying [1, Lemma 3.4] it gives a new A_0 depending also on r. This gap affects only the proof of (1.16) of [1, Theorem 1.1(v)] (or [1, (3.23)]). The rest of [1, Theorem 3.6] including the estimates (3.20)–(3.21), (3.6) and (3.8) hold without any issue. The proof of (3.19) in [1, Theorem 3.6] works if we drop λ and replace $M_{b,\lambda}$ defined in [1, (1.13)] by $\|b\|_\infty$.

In this errata, instead of establishing [1, (3.19)], we show directly that the estimate (1.16) of [1, Theorem 1.1(v)] hold for every $\lambda > 0$. We point out that all the main results stated in the Introduction of [1] remain true.

First note that by Lemma 0.1 below, Lemmas 3.1 and 3.4, Theorems 3.6 and 3.7 of [1] hold for $\lambda = +\infty$ with (3.2), (3.11), (3.12), (3.19) and (3.23) being replaced by

$$\left|q^b\right|_n(t,x,y) \le C_{11}\left(\|b\|_\infty C_7 c\right)^n g_1(t,x,y), \quad t \in (0,T], x, y \in \mathbb{R}^d, \tag{3.2$'$}$$

$$\left|q^b_{n+1}(t,x,y)\right| \le C_{13} 2^{-n} \|b\|_\infty p_1(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d, \tag{3.11$'$}$$

$$\left|\mathcal{S}^b_x q^b_n(t,x,y)\right| \le C_{12} \|b\|_\infty 2^{-n} f_0(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d, \tag{3.12$'$}$$

$$\left|q^b_n(t,x,y)\right| \le C_{14} 2^{-n}\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{\|b\|_\infty t}{|x-y|^{d+\beta}}\right)\right) \tag{3.19$'$}$$

and

$$\left|q^b(t,x,y)\right| \le 2C_{14}\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{\|b\|_\infty t}{|x-y|^{d+\beta}}\right)\right), \tag{3.23$'$}$$

respectively, where the constant c is the one in Lemma 0.1 and that the constant A_0 in [1, Lemma 3.4] can be chosen to be smaller than $1/(2C_{12})$. This gives the existence and uniqueness of the fundamental solution $q^b(t,x,y)$ and all the stated properties in [1, Theorem 1.1] except that we need to replace $p_{M_{b,\lambda}}$ by $p_{\|b\|_\infty}$ in the estimate [1, (1.16)].

For $a \ge 0$, denote by $p_a(t,x,y)$ the fundamental solution of $\Delta^{\alpha/2} + a\Delta^{\beta/2}$. Recall that for each $\lambda > 0$ and $a \ge 0$, $f_{a,\lambda}(t,x,y)$ is defined as in [1, (2.6)], and that $f_{a,\infty}(t,x,y) = f_0(t,x,y)$, which is given by [1, (2.1)].

By a similar argument as [1, Lemma 2.5], one obtains the following inequality.

Lemma 0.1. *There exists $c = c(d,\alpha,\beta) > 0$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$\int_0^t \int_{\mathbb{R}^d} p_1(t-s,x,z) f_0(s,z,y)\,dz\,ds \le c p_1(t,x,y).$$

Note that (3.23′) in particular implies that for every $A > 0$, there is a positive constant $C_0 = C_0(d, \alpha, \beta, A) \geq 1$ so that for any b with $\|b\|_\infty \leq A$,

$$\left|q^b(t, x, y)\right| \leq C_0 p_1(t, x, y) \quad \text{on } (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{0.1}$$

The following is an immediate consequence of [1, Lemma 2.4].

Lemma 0.2. *For each $\lambda > 0$, there is a constant $C = C(d, \alpha, \beta, \lambda) > 0$ such that for every $a \in [0, 1]$,*

$$\int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \leq C t^{1-\beta/\alpha}, \quad t \in (0, 1], y \in \mathbb{R}^d. \tag{0.2}$$

Lemma 0.3. *For each $\lambda > 0$, there exists $C_1 = C_1(d, \alpha, \beta, A, \lambda) > 0$ such that for any b with $\|b\|_\infty \leq A$ and for every $a \in [0, 1]$ and for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,*

$$\int_0^t \int_{\mathbb{R}^d} \left|q^b(t-s, x, z)\right| f_{a,\lambda}(s, z, y)\, dz\, ds \leq C_1 p_a(t, x, y) + \int_0^t \int_{|x-z|>|x-y|/2} \left|q^b(t-s, x, z)\right| f_{a,\lambda}(s, z, y)\, dz\, ds.$$

Proof. Let $I = \int_0^t \int_{\mathbb{R}^d} |q^b(t-s, x, z)| f_{a,\lambda}(s, z, y)\, dz\, ds$. By (0.1) and a similar proof as that for [1, Lemma 2.5], there exists $c_1 > 0$ independent of $a \in [0, 1]$ such that $I \leq c_1 t^{-d/\alpha}$ for $|x - y| \leq t^{1/\alpha}$. Hence by [1, (1.10)], there exists $c_2 > 0$ such that $I \leq c_2 p_a(t, x, y)$ for every $a \in [0, 1]$ and $|x - y| \leq t^{1/\alpha}$.

Next assume that $|x - y| > t^{1/\alpha}$. We divide I into two parts of the integrals on $|x - z| \leq |x - y|/2$ and on $|x - z| > |x - y|/2$. By (0.1) and a similar argument as that for [1, Lemma 2.5] with p_1 in place of g_a, there exists $c_3 > 0$ independent of $a \in [0, 1]$ such that the first integral

$$\int_0^t \int_{|x-z|\leq|x-y|/2} \left|q^b(t-s, x, z)\right| f_{a,\lambda}(s, z, y)\, dz\, ds \leq c_3 \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right) \leq c_4 p_a(t, x, y).$$

This completes the proof. □

Lemma 0.4. *For each $\lambda > 0$ and $A > 0$, there exists $C_k = C_k(d, \alpha, \beta, A, \lambda) > 1, k = 2, 3$ such that for any b with $\|b\|_\infty \leq A$ and for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,*

$$\left|q^b(t, x, y)\right| \leq C_2 p_{M_{b,\lambda}/A}(t, x, y) + C_3 \int_0^t \int_{|x-z|>|x-y|/2} \left|q^b(t-s, x, z)\right| f_{M_{b,\lambda}/A,\lambda}(s, z, y)\, dz\, ds. \tag{0.3}$$

Proof. By [1, Theorem 1.1(ii)], $q^b(t, x, y)$ satisfies the following Duhamel's formula

$$q^b(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s, x, z) \mathcal{S}_z^b p_0(s, z, y)\, dz\, ds, \quad t > 0, x, y \in \mathbb{R}^d. \tag{0.4}$$

Note that since $M_{b,\lambda}/A \leq 1$, there exists $c_1 > 0$, independent of λ and A, such that $p_0(t, x, y) \leq c_1 p_{M_{b,\lambda}/A}(t, x, y)$ for $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$. Moreover, by [1, (3.1)], there exists $c_2 > 0$ such that

$$\left|\mathcal{S}_z^b p_0(s, z, y)\right| \leq c_2 f_{M_{b,\lambda}/A,\lambda}(s, z, y), \quad s \in (0, 1], z, y \in \mathbb{R}^d.$$

Then the desired conclusion follows from (0.4) and Lemma 0.3 with $a = M_{b,\lambda}/A$. □

Define $|\tilde{q}_1^b(t, x, y)| := C_2 p_{M_{b,\lambda}/A}(t, x, y)$ and

$$\left|\tilde{q}_n^b(t, x, y)\right| := C_3 \int_0^t \int_{|x-z|>|x-y|/2} \left|\tilde{q}_{n-1}^b(t-s, x, z)\right| f_{M_{b,\lambda}/A,\lambda}(s, z, y)\, dz\, ds, \quad n \geq 2.$$

Define

$$II_1(t,x,y) := C_3 \int_0^t \int_{|x-z|>|x-y|/2} \left|q^b(t-s,x,z)\right| f_{M_{b,\lambda}/A,\lambda}(s,z,y)\,dz\,ds$$

and

$$II_n(t,x,y) := C_3 \int_0^t \int_{|x-z|>|x-y|/2} II_{n-1}(t-s,x,z) f_{M_{b,\lambda}/A,\lambda}(s,z,y)\,dz\,ds, \quad n \ge 2.$$

Applying Lemma 0.4 recursively, we have for $n \ge 1$,

$$\left|q^b(t,x,y)\right| \le \sum_{k=1}^n \left|\tilde{q}_k^b(t,x,y)\right| + II_n. \tag{0.5}$$

Lemma 0.5. *For each $\lambda > 0$, there exists $C_4 = C_4(d,\alpha,\beta,\lambda) > 0$ such that for every $a \in [0,1]$ and every $t \in (0,1]$, $x, y \in \mathbb{R}^d$,*

$$\int_0^t \int_{\{|x-z|>|x-y|/2\}} p_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds \le C_4 t^{1-\beta/\alpha} p_a(t,x,y). \tag{0.6}$$

Proof. We consider the Lemma in two cases when $|x-y| \le t^{1/\alpha}$ and when $|x-y| > t^{1/\alpha}$. When $|x-y| \le t^{1/\alpha}$, we can estimate the larger item $\int_0^t \int_{\mathbb{R}^d} p_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds$. Then by an argument very similar to that for [1, Lemma 2.5] but with p_a and Lemma 0.2 in place of g_a and [1, Lemma 2.4] there, we can obtain the desired conclusion. □

By Lemma 0.5 with $a = M_{b,\lambda}/A$ and induction, we have the following result.

Lemma 0.6. *For every $\lambda > 0$ and $n \ge 1$, we have*

$$\left|\tilde{q}_n^b(t,x,y)\right| \le C_2 \left(C_3 C_4 t^{1-\beta/\alpha}\right)^{n-1} p_{M_{b,\lambda}/A}(t,x,y) \quad \textit{for } t \in (0,1] \textit{ and } x, y \in \mathbb{R}^d.$$

By (0.1), $|q^b(t,x,y)| \le C_0 p_1(t,x,y)$. On the other hand, it follows from the definition that $f_{a,\lambda}(s,z,y) \le f_{1,\lambda}(s,z,y)$ for every $a \in [0,1]$ and $\lambda > 0$. Hence, by induction, we conclude again from Lemma 0.5 with $a = 1$ the following estimate.

Lemma 0.7. *For every $\lambda > 0$ and $n \ge 1$, we have*

$$II_n(t,x,y) \le C_0 \left(C_3 C_4 t^{1-\beta/\alpha}\right)^n p_1(t,x,y) \quad \textit{for } t \in (0,1] \textit{ and } x, y \in \mathbb{R}^d.$$

Now we can show that [1, Theorem 1.1(v)] holds.

Theorem 0.8. *For each $A > 0$ and $\lambda > 0$, there exists a constant $C_5 = C_5(d,\alpha,\beta,A,\lambda) > 0$ such that for any b with $\|b\|_\infty \le A$ and for every $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$\left|q^b(t,x,y)\right| \le C_5 p_{M_{b,\lambda}}(t,x,y).$$

Proof. Let $t_0 := (2C_3C_4)^{-\alpha/(\alpha-\beta)}$. By (0.5) and Lemmas 0.6 and 0.7, for $t \in (0, t_0]$ and $x, y \in \mathbb{R}^d$,

$$\left|q^b(t,x,y)\right| \le C_2 \sum_{k=1}^n 2^{-(k-1)} p_{M_{b,\lambda}/A}(t,x,y) + C_0 2^{-n} p_1(t,x,y).$$

Passing $n \to \infty$ yields the desired estimate for $t \in (0, t_0)$. We then use Chapman–Kolmogrov equation to extend it to all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$. □

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References

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