

**ADDENDUM TO
“OPTIMAL STOPPING UNDER MODEL UNCERTAINTY:
RANDOMIZED STOPPING TIMES APPROACH”**

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1. Introduction. In the authors’ paper [2], the optimal stopping problems of the form

$$(1.1) \quad \sup_{\tau \in \mathcal{T}} \rho_0^\Phi(-Y_\tau)$$

were studied. Here, $(Y_t)_{t \in [0, T]}$ denotes a nonnegative stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with $T \in]0, \infty[$, \mathcal{T} stands for the set of (\mathcal{F}_t) stopping times $\tau \leq T$ and the functional ρ_0^Φ is a divergence risk measure w.r.t. a lower semicontinuous convex mapping $\Phi : [0, \infty[\rightarrow [0, \infty]$ (see [2] for a precise definition and further details). Meanwhile, we have realized that some assumptions in [2] are unnecessarily strong. More precisely, we required \mathcal{F}_t to be countably generated for any $t > 0$. However, this excludes many widely used filtered probability spaces like standard augmentations of the filtered probability spaces generated by general multidimensional diffusions. A key point in [2] is the so-called derandomization result (see Proposition 6.3 in [2]), which shows that we obtain the same optimal value for the stopping problem (1.1) if we enlarge the set of stopping times to randomized stopping times. A crucial step in the proof of Proposition 6.3 is Lemma 7.4 which in turn uses an argument from the theory of angelic spaces (see Proposition B.1). This argument relies on the assumption that (\mathcal{F}_t) is countably generated for $t > 0$. In the meantime, we have realized that this line of argumentation continues to hold true if we only require that for any $t > 0$, the L^1 -space $L^1(\Omega, \mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ is weakly separable, that is, the weak topology on $L^1(\Omega, \mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ is separable. The latter assumption turns out to be much weaker than the original one.

The aim of this addendum is to reformulate the main results of [2] under weakened assumptions on the filtration (\mathcal{F}_t) . In particular, we show that the case of augmented filtration generated by a right-continuous stochastic process is now encompassed.

Received January 2016; revised June 2016.

MSC2010 subject classifications. Primary 60G40; secondary 91G80.

Key words and phrases. Optimized certainty equivalents, optimal stopping, primal representation, additive dual representation, randomized stopping times, thin sets.

2. New versions of the main results. Let $\Phi : [0, \infty[\rightarrow [0, \infty]$ be some lower semicontinuous convex mapping with Legendre transform

$$\Phi^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}, \quad y \mapsto \sup_{x \geq 0} (xy - \Phi(x)).$$

Furthermore, let $\text{int}(\text{dom}(\Phi))$ denote the topological interior of the effective domain of Φ . As in [2] Φ satisfies

$$(2.1) \quad 1 \in \text{int}(\text{dom}(\Phi)), \quad \inf_{x \geq 0} \Phi(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = \infty.$$

In addition, let H^{Φ^*} be the set of all random variables Z on (Ω, \mathcal{F}, P) such that $\Phi^*(\lambda|Z|)$ is P -integrable for every $\lambda > 0$.

Let us formulate our main primal representation result for the stopping problem (1.1) which can be viewed as a new version of Theorem 3.1 in [2].

THEOREM 2.1. *Let $(\Omega, \mathcal{F}_t, P|_{\mathcal{F}_t})$ be atomless with $L^1(\Omega, \mathcal{F}_t, P|_{\mathcal{F}_t})$ being weakly separable for every $t > 0$. Furthermore, let (2.1) be fulfilled, and let $\sup_{t \in [0, T]} Y_t \in H^{\Phi^*}$, then*

$$\begin{aligned} \sup_{\tau \in \mathcal{T}} \rho_0^\Phi(-Y_\tau) &= \sup_{\tau \in \mathcal{T}} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_\tau) - x] \\ &= \inf_{x \in \mathbb{R}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[\Phi^*(x + Y_\tau) - x] < \infty. \end{aligned}$$

REMARK 2.2. In a similar way as in Theorem 2.1, a dual representation for the stopping problem (1.1) (see Theorem 3.9 in [2]) can be formulated under the same weaker assumptions on the filtration (\mathcal{F}_t) .

REMARK 2.3. In many applications, the filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ is assumed to be the standard augmentation of the natural filtration induced by some d -dimensional right-continuous stochastic process $S = (S_t)_{t \in [0, T]}$ on some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ such that the marginals S_t have absolute continuous distributions for any $t > 0$, and S_0 has constant value \bar{P} -a.s. By construction, the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is right-continuous with \mathcal{F}_0 consisting of sets with probability 0 or 1, and containing all null sets of \mathcal{F} . Moreover, for any $t > 0$, the probability space $(\Omega, \mathcal{F}_t, P|_{\mathcal{F}_t})$ supports at least one continuously distributed random variable, as $S_{t'}$ is assumed to be continuously distributed for $t' > 0$. In particular, the probability space $(\Omega, \mathcal{F}_t, P|_{\mathcal{F}_t})$ is atomless for any $t > 0$. Next, by right-continuity of S , the σ -algebra $\widehat{\mathcal{F}} := \sigma(\{S_t \mid t \in [0, T]\})$ generated by S is countably generated so that the space of $\bar{P}|_{\widehat{\mathcal{F}}}$ -integrable random variables is separable w.r.t. the L^1 -norm (see e.g. [3], Proposition 3.4.5). Since by construction $\mathcal{F} = \mathcal{F}_T$ is the completion of $\widehat{\mathcal{F}}$ we may conclude from Lemma A.2 (cf. Appendix 3) that for any $t > 0$, the space $L^1(\Omega, \mathcal{F}_t, P|_{\mathcal{F}_t})$ is separable w.r.t. the L^1 -norm, in particular it is weakly separable. Summing up, the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ meets the requirements of Theorem 2.1.

3. Proof of Theorem 2.1. First, by (2.1) we may apply [2], Lemma A.1, to observe

$$(3.1) \quad \sup_{\tau \in \mathcal{T}} \rho_0^\Phi(-Y_\tau) = \sup_{\tau \in \mathcal{T}} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_\tau) - x].$$

Consider the stochastic process $(Y_t^r)_{t \geq 0}$, defined as

$$Y_t^r : \Omega \times [0, 1] \rightarrow \mathbb{R}, \quad (\omega, u) \mapsto Y_t(\omega),$$

which is adapted w.r.t. the enlarged filtered probability space

$$(\Omega \times [0, 1], \mathcal{F} \otimes \mathcal{B}([0, 1]), (\mathcal{F}_t \otimes \mathcal{B}([0, 1]))_{t \in [0, T]}, \mathbb{P} \otimes \mathbb{P}^U).$$

Here, \mathbb{P}^U stands for the uniform distribution on $[0, 1]$, defined on $\mathcal{B}([0, 1])$, the usual Borel σ -algebra on $[0, 1]$. Moreover, let \mathcal{T}^r denote the set of all randomized stopping times $\tau^r \leq T$. Then due to (2.1) under assumption $\sup_{t \in [0, T]} Y_t \in H^{\Phi^*}$, we may conclude from [2], Corollary 7.3 and Proposition 6.2,

$$(3.2) \quad \sup_{\tau^r \in \mathcal{T}_f^r} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau^r}^r) - x] = \sup_{\tau^r \in \mathcal{T}^r} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau^r}^r) - x]$$

$$(3.3) \quad = \inf_{x \in \mathbb{R}} \sup_{\tau^r \in \mathcal{T}^r} \mathbb{E}[\Phi^*(x + Y_{\tau^r}^r) - x].$$

Here, \mathcal{T}_f^r denotes the set of randomized stopping times from \mathcal{T}^r with finite range. Now the following auxiliary result gives a missing link to prove Theorem 2.1.

LEMMA 3.1. *Let (2.1) be fulfilled. Furthermore, let $\tau^r \in \mathcal{T}_f^r$, and let us denote by \mathcal{T}_f the set containing all nonrandomized stopping times from \mathcal{T} with finite range. If $(\Omega, \mathcal{F}_t, \mathbb{P}|_{\mathcal{F}_t})$ is atomless with $L^1(\Omega, \mathcal{F}_t, \mathbb{P}|_{\mathcal{F}_t})$ being weakly separable for every $t > 0$, and if $Y_t \in H^{\Phi^*}$ for $t > 0$, then*

$$(3.4) \quad \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau^r}^r) - x] \leq \sup_{\tau \in \mathcal{T}_f} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_\tau) - x].$$

PROOF. In order to simplify notation, let us assume $T \in \mathbb{N}$ and that τ^r has values in $\{0, \dots, T\}$ only. By Fubini’s theorem, we obtain

$$\mathbb{P} \otimes \mathbb{P}^U(\{\tau^r = 0\}) = \int_0^1 \mathbb{P}(\{\tau^r(\cdot, u) = 0\}) du.$$

Since $\{\tau^r(\cdot, u) = 0\} \in \mathcal{F}_0$ holds for every $u \in [0, 1]$, we may conclude by assumption on \mathcal{F}_0 that $\mathbb{P} \otimes \mathbb{P}^U(\{\tau^r = 0\}) \in \{0, 1\}$ holds. Hence, if $\mathbb{P} \otimes \mathbb{P}^U(\{\tau^r = 0\}) > 0$, then $\tau^r = 0$ $\mathbb{P} \otimes \mathbb{P}^U$ -a.s. In this case, the statement of Lemma 3.1 follows immediately because $\tau \equiv 0$ belongs to \mathcal{T}_f . So without loss of generality, we may assume that τ^r has values in $\{1, \dots, T\}$ only.

Next, let $F_{Y_{\tau^r}^r}$ denote the distribution function of $Y_{\tau^r}^r$. Then by Fubini’s theorem

$$(3.5) \quad F_{Y_{\tau^r}^r}(x) = \sum_{t=1}^T \mathbb{E}[\mathbb{1}_{]-\infty, x]}(Y_t) \cdot Z_t] \quad \text{for } x \in \mathbb{R},$$

where

$$Z_t \doteq \int_0^1 \mathbb{1}_{\{t\}}(\tau^r(\cdot, u)) du \quad \text{for } t \in \{1, \dots, T\}.$$

Note that $\mathbb{1}_{\{t\}}(\tau^r)$ is $\mathbb{P}_{|\mathcal{F}_t} \otimes \mathbb{P}^U$ -integrable so that by Fubini theorem, Z_t is a random variable on $(\Omega, \mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ which satisfies $0 \leq Z_t \leq 1$ P-a.s. for every $t \in \{1, \dots, T\}$. In addition, we may observe that $\sum_{t=1}^T Z_t = 1$ holds P-a.s. Since the probability spaces $(\Omega, \mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ ($t = 1, \dots, T$) are assumed to be atomless with $L^1(\Omega, \mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ being weakly separable, we may draw on Corollary C.4 along with Lemma C.1 and Proposition B.1, all from [2], to find a sequence $((B_{1n}, \dots, B_{Tn}))_{n \in \mathbb{N}}$ in $\times_{t=1}^T \mathcal{F}_t$ such that B_{1n}, \dots, B_{Tn} is a partition of Ω for $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \mathbb{E}[\mathbb{1}_{B_{tn}} \cdot g] = \mathbb{E}[Z_t \cdot g]$$

holds for $g \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ and $t \in \{1, \dots, T\}$. In particular, we have by (3.5)

$$F_{Y_{\tau^r}}(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^T \mathbb{E}[\mathbb{1}_{]-\infty, x]}(Y_t) \cdot \mathbb{1}_{B_{kn}}] \quad \text{for } x \in \mathbb{R}.$$

We can define a sequence $(\tau_n)_{n \in \mathbb{N}}$ of nonrandomized stopping times from \mathcal{T}_f via

$$\tau_n := \sum_{t=1}^T t \mathbb{1}_{B_{tn}},$$

so that

$$F_{Y_{\tau^r}}(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^T \mathbb{E}[\mathbb{1}_{]-\infty, x]}(Y_t) \cdot \mathbb{1}_{B_{kn}}] = \lim_{n \rightarrow \infty} F_{Y_{\tau_n}}(x) \quad \text{for } x \in \mathbb{R}.$$

Now, one may show just in the same way as in the proof of Lemma 7.4 from [2] that

$$\lim_{n \rightarrow \infty} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau_n}) - x] = \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau^r}) - x]$$

holds. This completes the proof. \square

Now we are ready to show Theorem 2.1. By (3.2) along with Lemma 3.1 and (3.3), we obtain

$$\begin{aligned} \sup_{\tau \in \mathcal{T}} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_\tau) - x] &\geq \sup_{\tau \in \mathcal{T}_f} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_\tau) - x] \\ &\geq \sup_{\tau^r \in \mathcal{T}_f^r} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau^r}) - x] \\ &= \sup_{\tau^r \in \mathcal{T}^r} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau^r}) - x] \end{aligned}$$

$$\begin{aligned}
&= \inf_{x \in \mathbb{R}} \sup_{\tau^r \in \mathcal{T}^r} \mathbb{E}[\Phi^*(x + Y_{\tau^r}^r) - x] \\
&\geq \inf_{x \in \mathbb{R}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[\Phi^*(x + Y_{\tau}) - x] \\
&\geq \sup_{\tau \in \mathcal{T}} \inf_{x \in \mathbb{R}} \mathbb{E}[\Phi^*(x + Y_{\tau}) - x].
\end{aligned}$$

Then Theorem 2.1 follows immediately from (3.1).

APPENDIX

In the following, we shall denote by $L^1(\Omega, \mathcal{F}, \mathbb{P})$ the space of \mathbb{P} -integrable random variables.

LEMMA A.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space with completion $(\Omega, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, and let \mathcal{A} be any sub σ -algebra of $\overline{\mathcal{F}}$. If $L^1(\Omega, \mathcal{F}, \mathbb{P})$ is separable w.r.t. the L^1 -norm, then $L^1(\Omega, \mathcal{A}, \mathbb{P}_{|\mathcal{A}})$ is separable w.r.t. the L^1 -norm.*

PROOF. By assumption, there is some sequence $(f_n)_{n \in \mathbb{N}}$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ which is dense w.r.t. the L^1 -norm. Let us fix $g \in L^1(\Omega, \mathcal{A}, \mathbb{P}_{|\mathcal{A}})$ and $\varepsilon > 0$. Since $(\Omega, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ is the completion of $(\Omega, \mathcal{F}, \mathbb{P})$, we may find some random variable f on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $g = f$ $\overline{\mathbb{P}}$ -a.s. (cf. [1], Theorem 10.35). Obviously, f is \mathbb{P} -integrable because g is $\overline{\mathbb{P}}$ -integrable, and thus $g = \mathbb{E}_{\overline{\mathbb{P}}}[f|\mathcal{A}]$ $\overline{\mathbb{P}}$ -a.s. Then there exists some $n \in \mathbb{N}$ such that $\mathbb{E}_{\overline{\mathbb{P}}}[\|g - f_n\|] = \mathbb{E}_{\mathbb{P}}[\|f - f_n\|] < \varepsilon$. This implies

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}}[\|g - \mathbb{E}_{\overline{\mathbb{P}}}[f_n|\mathcal{A}]\|] &= \mathbb{E}_{\overline{\mathbb{P}}}[\|\mathbb{E}_{\overline{\mathbb{P}}}[f|\mathcal{A}] - \mathbb{E}_{\overline{\mathbb{P}}}[f_n|\mathcal{A}]\|] \\
&\leq \mathbb{E}_{\overline{\mathbb{P}}}[\mathbb{E}_{\overline{\mathbb{P}}}[\|f - f_n\||\mathcal{A}]] \\
&= \mathbb{E}_{\overline{\mathbb{P}}}[\|f - f_n\|] < \varepsilon.
\end{aligned}$$

Hence, we have shown that $(\mathbb{E}_{\overline{\mathbb{P}}}[f_n|\mathcal{A}])_{n \in \mathbb{N}}$ is dense in $L^1(\Omega, \mathcal{A}, \mathbb{P}_{|\mathcal{A}})$ which completes the proof. \square

Acknowledgments. This study has been funded by the Russian Academic Excellence Project “5-100.”

REFERENCES

- [1] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*, 3rd ed. Springer, Berlin. MR2378491
- [2] BELOMESTNY, D. and KRÄTSCHMER, V. (2016). Optimal stopping under model uncertainty: Randomized stopping times approach. *Ann. Appl. Probab.* **26** 1260–1295.
- [3] COHN, D. L. (1980). *Measure Theory*. Birkhäuser, Boston, MA. MR0578344

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