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TRANSIENCE AND RECURRENCE OF A BROWNIAN PATH WITH LIMITED LOCAL TIME

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In this paper, we study the behavior of Brownian motion conditioned on the event that its local time at zero stays below a given increasing function f up to time T. For a class of nonincreasing functions f, we show that the conditioned process converges, as $T \to \infty$, to a limit process and we derive necessary and sufficient conditions for the limit to be transient. In the transient case, the limit process is described explicitly, and in the recurrent case we quantify the entropic repulsion phenomenon by describing the repulsion envelope, stating how much slower than f the local time of the process grows as a result of the conditioning. The methodology is based on a fine analysis of the subordinator given by the inverse local time of the Brownian motion. We describe the probability of general subordinator to stay above a given curve up to time T via the solution of a general ordinary linear differential equation. For the specific case of the inverse local time of the Brownian motion, we explicitly and precisely compute the asymptotics of this probability for a large class of functions.

CONTENTS

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1. Introduction. Let $(B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion. In this paper, by developing a very general methodology for studying the asymptotics of the probability of increasing Lévy processes (subordinators) to stay above a given curve, we study the behavior of Brownian paths, which have a limited growth of local time at the origin. Following previous work [2] of Berestycki and Benjamini, we consider the problem of describing the measures

$$\mathbb{P}_t := \mathbb{P}\big(\cdot | L_s \leq f(s), \forall s \leq t\big) = \mathbb{P}(\cdot | \tau_{f(s)} > s, \forall s \leq t)$$

in the limit $t \to \infty$, where $(L_s)_{s\geq 0}$ denotes the local time of $(B_t)_{t\geq 0}$ at the origin, $(\tau_s)_{s\geq 0}$ its right-inverse and $f : [0,\infty) \to [0,\infty)$ is a suitable nonnegative increasing function satisfying some additional mild properties.

Let us now describe the main results of [2] in more detail. It is shown that the family of probability measures $\mathbb{P}_t$ on the canonical path space $\mathcal{C} = C([0,\infty), \mathbb{R})$ is in fact tight and thus has limit points. Furthermore, the authors manage to show that the condition

$$I(f) = \int_1^\infty \frac{f(t)}{t^{3/2}}\, dt < \infty \tag{1.1}$$

implies that every weak limit point $\mathbb{Q}$ of $\mathbb{P}_t$, as $t \to \infty$, is transient almost surely. This means in particular that restricting the local time growth to be smaller than $f(t) = \sqrt{t}(\log t)^{-1-\varepsilon}$, $\varepsilon > 0$, already results in a significant change of the original recurrent Brownian motion. Observe that this might be surprising as the typical growth of the local time coincides with $\sqrt{t}$ and thus we only require a slightly slower than average growth.

This intriguing result immediately leads to the question whether the tight family $\mathbb{P}_t$ is in fact weakly convergent and whether one can in some way interpret its limit. Exactly the answer of this question is one part of the present contribution.

In [2], it is conjectured that (1.1) is the precise dividing line between every possible weak limit of $\mathbb{P}_t$ being recurrent or transient. In our current work, we show that this integral distinguishes between recurrence and transience by elaborating a method which captures all classes of functions f such that $\lim_{t\to\infty} f(t)\ln^{4/5+\varepsilon}(t)/t^{1/2} = 0$, for some $\varepsilon > 0$, when $I(f) = \infty$ and $\lim_{t\to\infty} f(t)\ln^{1/2}(t)/t^{1/2} = 0$ when $I(f) < \infty$. Given that the functions $f(t) = t^{1/2}/\ln(t)$ and $f(t) = t^{1/2}/\ln^{1+\varepsilon}(t)$ are on the two sides of the integral test (1.1), we see that our restriction is irrelevant for the critical region. We further develop the results of [2] in several different directions:

- We show that $\mathbb{Q} = \lim_{t\to\infty} \mathbb{P}_t$ exists. We identify the limit explicitly in the case $I(f) < \infty$ and further prove that it corresponds to a recurrent process if $I(f) = \infty$. This settles two questions left open in [2]. The fact that $I(f) = \infty$ implies recurrence is their Conjecture 1.
- Motivated by Conjecture 2 of [2], we say that an increasing function w is in the repulsion envelope of f if even

$$\lim_{t\to\infty} \mathbb{Q}\big(L_t < f(t)/w(t)\big) = 1.$$

  Using our methods, we manage to describe the repulsion envelope of f analytically by providing a simple explicit criterion which furnishes a necessary and sufficient condition (NASC) for w to be in the envelope of f. This quantifies the idea of entropic repulsion which is often used in the physics literature.

Observe that the general scheme of conditioning on an unlikely event has similarities to papers on quasistationary distributions (see, e.g., [5]), penalizations (see, e.g., [9]) as well as to approaches investigated in the area of polymer models (see, e.g., [7, 10] and [8]). The questions considered in this paper (as well as our methods) still differ from the just mentioned ones in the sense that one of our main aims is to study the phenomenon of entropic repulsion in a simple but still highly nontrivial situation. This phenomenon has already been the main topic of several previous studies such as [1, 2] and [3] and usually refers to the fact that conditioning on a unlikely event often results in a process whose behavior appears to be even more unlikely than the one which the process is conditioned on. In our setting, the phenomenon of entropic repulsion is most clearly visible in Theorem 4 which proves that the repulsion envelope is not empty.

Let us describe the structure of the paper. In the next section, we set up the problem, the notation, present some basic facts that will be used later, provide a short discussion on the strategy of the proof and discuss the scope of our methodology. In Section 3, we consider the case $I(f) < \infty$ and describe the limiting process and prove that it is transient. In Section 4, under mild assumptions we discuss the case when $I(f) = \infty$ and show that the limiting process exists and is recurrent which solves Conjecture 1 in [2]. Additionally, we determine the repulsion envelope analytically showing that it is never empty thus settling Conjecture 2 in [2]. In Section 5, we provide the basic ODE which allows us to estimate in various ways the quantity $\mathbb{P}(\mathcal{O}_t)$ namely the probability of the event we condition on. The last parts are devoted to the proofs.

2. Notation and discussion.

2.1. *Basic notation.* We use throughout the paper the following conventions. First, we use $f \sim g$ to denote that $\lim_{t\to\infty} f(t)/g(t) = 1$ and $\mu_t(ds) \sim \nu_t(ds)$ to denote that the densities $m_t(s), v_t(s)$ of the measures μ_t, ν_t satisfy $\lim_{t\to\infty} m_t(s)/v_t(s) = 1$ for each finite $s > 0$ where we preclude the possibility of $v_t(s) = 0$. Similarly, we use $f \asymp g$ to denote the existence of two constants $0 < D_1 < D_2 < \infty$ such that

$$D_1 \leq \liminf_{t\to\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leq D_2$$

with the same meaning for measures indexed by t at the level of their densities, see above for $\sim$. The notation $f \lesssim g$ respectively, $f \gtrsim g$ then implies the existence of D_2 respectively, D_1 above.

Throughout the paper, we also use the convention that C will be an absolute positive and finite constant, whereas $C(A, B, \dots)$ will denote an absolute constant independent of any variables but $A, B, \dots$.

2.2. *The boundary function f and its inverse g*. Without loss of generality, we will assume that $f(1) = 1$, $1 > f(0) > 0$ and that $f : \mathbb{R} \to \mathbb{R}^+$ is an increasing function which drifts to infinity. We impose the following mild growth condition:

$$(0,\infty) \ni x \mapsto \frac{f(x)}{\sqrt{x}} \quad \text{is decreasing and} \quad \lim_{x\to\infty} \frac{f(x)}{\sqrt{x}} = 0. \tag{2.1}$$

Often we work with $g(x) := f^{-1}(x)$ for which (1.1) is with the help of (2.1) translated to

$$I(f) < \infty \quad \iff \quad J(g) := \int_1^\infty \frac{1}{\sqrt{g(s)}}\, ds < \infty. \tag{2.2}$$

Observe that we can continuously and monotonously extend the function g to the interval $(0, f(0))$. Note that since $f(0) > 0$ we have that $g(x) < 0$, for $x \in (0, f(0))$.

DEFINITION 2.1. We say that a function $f : \mathbb{R} \to \mathbb{R}_+$ satisfies the *usual conditions* if f is increasing, drifts to infinity and satisfies:

- $f(1) = 1$, $1 > f(0) > 0$;
- (2.1) holds true.

2.3. *Brownian motion, local time, inverse local time and related quantities*. In this paper, we work with a standard Brownian motion $B = (B_s)_{s\geq 0}$. Recall that for a real-valued Brownian motion the local time at zero can be defined as

$$L_t = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{B_s \in (-\varepsilon,\varepsilon)\}}\, ds.$$

The local time is a continuous, nondecreasing process which grows precisely on the set $\{s \geq 0 : B_s = 0\}$. It is well known that its right-inverse local time $\tau = (\tau_t)_{t\geq 0}$, where

$$\tau_u = \inf\{t > 0 : L_t > u\} \tag{2.3}$$

is a stable subordinator of index $1/2$, that is, a nondecreasing Lévy process without drift whose Lévy measure is given by $\Pi(ds) = K\,ds/s^{3/2}$, $s > 0$, where $K := 1/\sqrt{2\pi}$, and the Lévy–Khintchine exponent of τ_1 is given by

$$\mathbb{E}\big[e^{-\lambda\tau_1}\big] = e^{-K\int_0^\infty(1-e^{-\lambda s})\frac{ds}{s^{3/2}}} = e^{-K\sqrt{\lambda}\int_0^\infty(1-e^{-s})\frac{ds}{s^{3/2}}}. \tag{2.4}$$

In view of working with subordinators which are obtained from τ by truncating some of its jumps we do not compute explicitly $\int_0^\infty(1-e^{-\lambda s})\frac{ds}{s^{3/2}}$ as any truncation will be reflected in the region of integration.

Furthermore, the law and the density of τ_u can be computed via

$$\begin{aligned}\mathbb{P}(\tau_u > t) &= \mathbb{P}\left(\tau_1 > \frac{t}{u^2}\right) = \frac{u}{\sqrt{2\pi t}}\int_{-1}^{1} e^{-(ux)^2/(2t)}\,dx\\ &= \frac{2u}{\sqrt{\pi}}\int_0^{1/\sqrt{2t}} e^{-(ur)^2}\,dr \qquad \text{for } t > 0,\end{aligned}$$

$$\mathbb{P}(\tau_u \in dt) = \frac{ue^{-u^2/(2t)}}{\sqrt{2\pi t^3}}\,dt = f_u(t)\,dt \qquad \text{for } t > 0. \tag{2.5}$$

Then with f, g given above we see that

$$I(f) < \infty \quad\Longleftrightarrow\quad J(g) < \infty \quad\Longleftrightarrow\quad \mathbb{E}\big[f(\tau_1)\big] < \infty. \tag{2.6}$$

Note that the jumps of the subordinator τ correspond to the lengths of the excursions of the Brownian motion away from zero, which is due to (2.3). Therefore, we have that L_t is a constant on the span of each excursion away from zero. In more technical detail, the excursions are paths in $\mathcal{C}$ with the following properties: $\varepsilon \in \mathcal{C}$, $\varepsilon(0) = 0$, $\varepsilon(t) > 0$ or $\varepsilon(t) < 0$, for $\forall t < \zeta(\varepsilon)$, $\varepsilon(t) = 0$, $t \geq \zeta(\varepsilon)$, where ζ is called the length or life-time of the excursion and determines a jump of size ζ in the subordinator τ. We refer to [2] for a very good exposition of excursions for this setting and to [4], Chapter IV, for more general Lévy processes.

Finally, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of the inverse local time τ which is via a standard random time change generated by the natural filtration of the Brownian motion.

2.4. *The event on which the process is conditioned.* Throughout the paper, it will be convenient to work with the inverse local time τ. We use that the following sets are equal $\forall t > 0$:

$$\begin{aligned}\mathcal{O}_t &= \big\{\tau_s > g(s); s \leq t\big\} = \{L_{g(s)} \leq s; s \leq t\}\\ &= \big\{L_u \leq f(u); u \leq g(t)\big\}.\end{aligned} \tag{2.7}$$

This definition slightly differs from the sets $\mathcal{E}_t = \mathcal{O}_{f(t)}$ used in [2]. This difference is irrelevant for the limit.

Important functions in our study will be $\phi(t) = \Phi'(t) = \mathbb{P}(\mathcal{O}_t)$, where

$$\Phi(t) = \int_0^t \mathbb{P}(\mathcal{O}_s)\,ds. \tag{2.8}$$

In Section 5, we provide the explicit asymptotic behavior of $\phi(t)$ and $\Phi(t)$ via an ordinary linear differential equation of first order which links ϕ and Φ. These are the results at the heart of our main theorems. One might find it surprising that such precise estimates can be given for such highly dependent events. In fact, $\mathcal{O}_t$ depends on the whole path of the process τ up to time t.

2.5. *Discussion and strategy for the proof.* Since we condition on $\mathcal{O}_t$ the results naturally depend on the knowledge about the asymptotics of $\phi(t) = \mathbb{P}(\mathcal{O}_t)$. The functions $\phi(t)$ and $\Phi(t)$ are linked by a linear differential equation of the type

$$\phi(t) - \frac{2K}{\sqrt{\varphi(t)}}\Phi(t) = H(t),$$

where φ is a parameter that can be chosen to fit f and H is a functional determined by φ. This equation can be solved and the function $H(t)$ can be estimated rather precisely. This allows us to provide very sharp results on the asymptotics of $\phi(t)$ and $\Phi(t)$. The differential equation itself arises by simply conditioning on the time of a first jump of τ that will take τ above $g(t) = f^{-1}(t)$ which removes the dependence on the future. The fact that we can estimate H comes from the one-large jump principle which roughly states that one large jump determines the large deviation behavior of τ. Since $\lim_{t\to\infty} g(t)/t^2 = \infty$ and by scaling $\mathbb{P}(\tau_t > ct^2) = \mathbb{P}(\tau_1 > c)$ we see that we are in the regime of large deviations for $\mathbb{P}(\tau_t > g(t))$ and the one-large jump principle is expected to hold true for $\mathbb{P}(\mathcal{O}_t)$. However, this is harder to verify in our scenario as $\mathcal{O}_t$ depends on the entire past of the process.

Due to the heavy space–time dependence revealed for example, by

$$\mathbb{P}(\tau_h \in dy; \mathcal{O}_t) = \mathbb{P}\big(\tau_s > g(s+h) - y; \forall 0 \le s \le t-h\big)\mathbb{P}(\tau_h \in dy; \mathcal{O}_h),$$

information on $\phi(t)$ and $\Phi(t)$ does not suffice. Using the same differential equation (5.1) for each point (h, y) and function $g_{y,h}(s) := g(s+h) - y$ we are able to prove some uniform bounds for $\phi_y^h(t)$ and $\Phi_y^h(t)$. The functions $\phi_y^h(t)$ and $\Phi_y^h(t)$ are defined similar to ϕ and Φ by replacing g with $g_{y,h}$.

When $I(f) < \infty$, these bounds do not require heavy calculations. In the situation $I(f) = \infty$, these are much harder. Precisely in this case, we need the condition $\lim_{t\to\infty} f(t)\ln^{4/5+\varepsilon}(t)/t^{1/2} = 0$ but we have no doubt that the exponent $4/5$ can be made much smaller. However, this would unnecessary burden the exposition of the paper and our condition in any case captures the transition from the scenario $I(f) < \infty$ to $I(f) = \infty$.

Once suitable bounds on $\phi_y^h(t)$ and $\Phi_y^h(t)$ are settled, it is a matter of dominated convergence theorem and the tightness of $\mathbb{P}(\tau_h \in \cdot|\mathcal{O}_t)$ to show that in the two scenarios $I(f) < \infty$ and $I(f) = \infty$ the limiting process exists and it is correspondingly transient and recurrent. However, the estimates can be used even further. An estimate of the quantity $\mathbb{Q}(\tau_h \in (g(h), g(h)w(h)))$ can be made very precise and analytical which allows us to prove a NASC for $\lim_{h\to\infty} \mathbb{Q}(\tau_h \in (g(h), g(h)w(h))) = 0$ which in other words distinguishes the functions in the repulsion envelope.

2.6. *Brownian motion conditioned on the growth of its local time at its maximum.* The inverse local time τ at zero for the reflected Brownian motion $\sup_{s\le t} B_s - B_t$ has the same law as the inverse local time at zero. Since all our results rely first on the distribution of the inverse local time under the limit measure $\mathbb{Q} = \lim_{t\to\infty} \mathbb{P}_t$ and then on splicing of excursions of the Brownian motion we see that all results are of the same type. The differences between the transient and recurrent regime are as follows: in contrast to the recurrent regime the all time maximum is obtained in the transient case in a finite time and then the negative of a Bessel three process is started as in Theorem 1. The three-dimensional Bessel process in this setting occurs in the form of an excursion with infinite life time.

3. Transient case. Recall that $\mathcal{C} = C([0,\infty), \mathbb{R})$ is the space of continuous functions indexed by the time t and denote by $\mathcal{W}$ the Wiener measure on $\mathcal{C}$.

Next, define the family of random variables $\mathfrak{C}_t$, called clocks, with $\operatorname{supp} \mathfrak{C}_t = [0, t]$ via their densities as follows:

$$\mathbb{P}(\mathfrak{C}_t \in ds) = \frac{\mathbb{P}(\mathcal{O}_s)}{\int_0^t \mathbb{P}(\mathcal{O}_v)\,dv}\,ds = \frac{\phi(s)}{\Phi(t)}\,ds \qquad \text{for } 0 < s \le t. \tag{3.1}$$

Recall that $\mathcal{O}_t = \{\tau_s > g(s), 0 \le s \le t\}$. Denote by $\Delta_1^{g(t)} = \{s > 0 : \tau_s - \tau_{s-} > g(t)\}$. The clocks approximate very precisely the underlying structure namely the fact that the conditions represented by $\mathcal{O}_t$ are satisfied with dominating probability by the arrival of one jump larger than $g(t)$, that is,

$$\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}\bigl(\mathcal{O}_t \cap \{\Delta_1^{g(t)} \le t\}\bigr).$$

Conditioned upon arrival on $[0, t]$ the jump has uniform distribution which subsequently is reweighted in (3.1) to reflect the additional assumption that $\mathcal{O}_s$ must hold until time $\Delta_1^{g(t)}$. This size of the large jump for τ is in fact the length of an excursion of the Brownian motion away from zero. In the limit this excursion conditioned to last more than $g(t)$ converges to the three-dimensional Bessel process; see, for example, [2], pages 10–11, which is a standard result in the probability folklore.

When $\mathfrak{C} = \lim_{t\to\infty} \mathfrak{C}_t$ exists in a weak sense, namely iff $\Phi(\infty) < \infty$, then it has a density function

$$\mathbb{P}(\mathfrak{C} \in ds) = \frac{\mathbb{P}(\mathcal{O}_s)}{\Phi(\infty)}\,ds, \qquad s \ge 0.$$

Define the process $(Y_t)_{t\geq 0}$ in the following way: choose independent copies of the clock $\mathfrak{C}$; of $B=(B_s)_{s\geq 0}$; of $\varpi\in\{-1,1\}$ with $\mathbb{P}(\varpi=1)=1/2$; and of $B^{(3)}=(B^{(3)}_s)_{s\geq 0}$, where $B^{(3)}$ is a three-dimensional Bessel process; then:

1. Conditionally on $\{\mathfrak{C}=x\}$ run B conditioned on $\{L_s\leq f(s); s\leq \tau_x\}$ (note that $L_{\tau_x}=x$) and put Y_s to coincide with this conditioned process for $s\leq\tau_x$.
2. Choose 1 or -1 according to ϖ.
3. For $t>\tau_x$ put $(Y_t)_{t\geq\tau_x}=(\varpi B^{(3)}_{t-\tau_x})_{t\geq\tau_x}$.

The next result shows that under $\mathbb{Q}$, B equals precisely Y whenever $I(f)<\infty$. Recall from subsection 2.3 that $(\mathcal{F}_h)_{h\geq 0}$ denotes the natural filtration of the inverse local time.

THEOREM 1. *For all f satisfying the usual conditions given in Definition 2.1,*

$$\liminf_{t\to\infty}\frac{g(t)}{t^2\ln(t)}=\infty \quad \text{and} \quad I(f)<\infty,$$

the limit $\mathfrak{C}=\lim_{t\to\infty}\mathfrak{C}_t$ exists in a weak sense and furthermore $\mathbb{Q}(\cdot)=\lim_{t\to\infty}\mathbb{P}_t(\cdot)$ exists in the sense of the weak topology on the space $\mathcal{C}$. The process B under the measure $\mathbb{Q}$ equals the process Y. Moreover, for any fixed $h>0$ and any $y>g(h)\vee 0$ we have the formula for the density of τ under $\mathbb{Q}$

$$\mathbb{Q}(\tau_h\in dy):=\frac{\Phi^h_y(\infty)}{\Phi(\infty)}\mathbb{P}(\tau_h\in dy;\mathcal{O}_h), \tag{3.2}$$

where $\Phi^h_y(\infty)=\int_0^\infty\mathbb{P}(\tau_s>g(s+h)-y, s\leq v)\,dv<\infty$ is part of the claim. Therefore, $\mathbb{Q}(\tau_h<\infty)=\mathbb{P}(\mathfrak{C}>h)$. Finally, for any measurable $\mathcal{B}\subset\mathcal{O}_h$ and $\mathcal{B}\in\mathcal{F}_h$ we have that

$$\mathbb{Q}(\mathcal{B})=\mathbb{E}\left[\frac{\Phi^h_{\tau_h}(\infty)}{\Phi(\infty)};\mathcal{B}\right]. \tag{3.3}$$

REMARK 2. Note that this result is consistent with [1], Theorem 2, where $f(s)\equiv 1$ is studied despite the fact that the inverse function g is undefined. The clock there is a uniform random variable on $(0,1)$ and the local time is accumulated until this random variable is attained. Then a Bessel process with random sign is issued forth. The Bessel process is a result of the limit of longer and longer excursions away from zero which in turn are a consequence of the one-large jump principle.

COROLLARY 1. *We have that under $\mathbb{Q}$ the process B is transient, namely $\mathbb{Q}(\lim_{t\to\infty}|B_t|=\infty)=1$ and even $|B_s|_{s\geq\tau_{\mathfrak{C}-}}=B^{(3)}_{s-\tau_{\mathfrak{C}-}}$. Therefore after time $\tau_{\mathfrak{C}}$ the process is explicit and its density and rate of growth, which determines the speed of transience are computed as those of the three-dimensional Bessel process.*

We proceed with the recurrence case.

4. Recurrent case.

4.1. *Weak limit and recurrence.* The recurrent case is much more demanding. We will impose the following condition as it suffices to capture the transition region:

$$\liminf_{t\to\infty} \frac{g(t)}{t^2 \ln^{8/5+\varepsilon}(t)} = \infty \qquad \text{for some } \varepsilon > 0. \tag{4.1}$$

Undoubtedly, condition (4.1) can be further relaxed but this would require more precision in the heavy computations below and will add less value as we have already captured the transitions region with (4.1).

Under the weaker condition $\liminf_{t\to\infty} g(t)/t^2 \ln(t) = \infty$, we can see from (5.6), Lemma 1 that the limit clock $\mathfrak{C}$ defined in Section 3 does not exist since $\Phi(\infty) = \infty$. This in turn is a good indicator as to why the recurrence holds: still

$$\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}\big(\mathcal{O}_t \cap \Delta_1^{g(t)} \le t\big),$$

but, for any $a < \infty$,

$$\lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_t \cap \Delta_1^{g(t)} \le a)}{\mathbb{P}(\mathcal{O}_t)} = 0,$$

see (3.1), when $\Phi(\infty) = \infty$, and the long excursion, which is the cause of the Bessel process to appear in the transient scenario, is pushed away to infinity with probability one.

We have the following statement.

THEOREM 3. *Let f satisfy the usual conditions given in Definition 2.1. Additionally, assume that $I(f) = \infty$ and that (4.1) holds. Then the limit $\lim_{t\to\infty} \mathbb{P}(\cdot | \mathcal{O}_t) = \lim_{t\to\infty} \mathbb{P}_t(\cdot) = \mathbb{Q}(\cdot)$ exists and under $\mathbb{Q}$ the process is recurrent, namely*

$$\mathbb{Q}(\exists t > T : B_t = 0) = 1 \qquad \forall T > 0.$$

Under $\mathbb{P}_t$ the inverse local time converges, as $t \to \infty$, to an increasing pure-jump process under $\mathbb{Q}$ which we call the inverse local time under $\mathbb{Q}$.

The increasing pure-jump process referred to in the above theorem is studied in more detail in Proposition 1.

We proceed to utilize this information and discuss the phenomenon of repulsion.

4.2. *Repulsion envelope.* Let us define the set of functions $\mathcal{D} = \{w : [1,\infty) \mapsto [1,\infty) : w \text{ is increasing to } \infty\}$ and

$$R_g = \mathcal{D} \cap \Big\{w : \lim_{h\to\infty} \mathbb{Q}\big(\tau_h \ge w(h)g(h)\big) = 1\Big\}.$$

We call R_g the envelope of repulsion which means that in fact under $\mathbb{Q}$ the inverse local time stays with increasing to one probability not only above g but above $g_w := gw$. Note that if $f_w = g_w^{-1}$ then solving for $u = f/f_w$ we see that $u \downarrow 0$ is such that $\lim_{t\to\infty} \mathbb{Q}(L_t \le f(t)u(t)) = 1$. It is conjectured in [2], Conjecture 2, that $R_g \neq \varnothing$ with some further quite insightful comments as to the form of functions that comprise R_g. Our next result shows that one in fact can in a simple analytical way specify R_g. We are able to do this thanks to (8.1). We have the following statement.

THEOREM 4. *Let the conditions upon f of Theorem 3 hold. Let $w \in \mathcal{D}$ then we have*

$$w \in R_g \quad \Longleftrightarrow \quad \lim_{h\to\infty} \int_h^{f(g(h)w(h))} \frac{1}{\sqrt{g(s)}}\,ds = 0. \tag{4.2}$$

REMARK 5. Take a function $f(t) = \sqrt{t}/\ln^\gamma(t)$ with $1 > \gamma > 4/5$ then we have that $g(t) \sim t^2 \ln^{2\gamma}(t)$. Define $w_\gamma(t) = e^{\ln^\gamma(t)}$ and $g_{w_\gamma} = gw_\gamma$ and then easily $g_{w_\gamma}^{-1}(t) =: f_{w_\gamma}(t) \sim e^{-\kappa \ln^\gamma(t)}\sqrt{t}/\ln^\gamma(t)$, as $t \to \infty$ and some $\kappa > 0$. Then using (4.2) we can see the conjectured function $w_\gamma(t)$ is indeed the separating line of R_g since for any w such that $\ln w = o(\ln w_\gamma)$ then $w \in R_g$ but in fact $w_\gamma \notin R_g$. Computing (4.2) explicitly we can even have the simplified criterion $w \in R_g$ iff $\ln(w(h)) = o(\ln^\gamma(h))$.

REMARK 6. The case $\gamma = 1$ is the most interesting as it correspond to the case $g(t) \sim t^2 \ln^2(t)$ at the boundary of our transition region. Then an easy computation yields that

$$\begin{aligned}
&\lim_{h\to\infty} \int_h^{f(g(h)w(h))} \frac{1}{\sqrt{g(s)}}\,ds \\
&\quad = \lim_{h\to\infty} \ln\biggl(\frac{\ln(h) + \ln(w(h)) + \ln\ln(h)}{\ln(h)}\biggr) = 0 \quad \Longleftrightarrow \quad \ln\bigl(w(h)\bigr) = o\bigl(\ln(h)\bigr).
\end{aligned}$$

REMARK 7. We would like to point out that due to the fact that we estimate many quantities with constants bounded away from zero it will be difficult to study other probabilities like $\mathbb{Q}(\tau_h \in (g(h), g(h)w(h))) \to 1$ unless we have a zero-one law, something we do not anticipate to be true.

5. Precise asymptotic estimates for $\mathbb{P}(\mathcal{O}_t)$ and $\int_0^t \mathbb{P}(\mathcal{O}_s)\,ds$. The fact that τ is a stable subordinator and thus enjoys the so-called one large jump principle allows for the very precise study of the events $\mathcal{O}_t = \{\tau_s > g(s), s \le t\}$ at least to a first order asymptotics. We recall that the one-large jump principle postulates that the probability of the subordinator to cross larger and larger barrier in an also expanding time horizon is asymptotically equivalent to the probability that the subordinator makes one jump of size exceeding the level of this barrier. It is clear that if this principle applies in this setting then the long-term dependency in the definition of $\mathcal{O}_t$ will be destroyed at the moment we make a jump bigger than $g(t)$. This is the main observation behind the ensuing estimates. However, (5.1) holds in any situation, for any subordinator, and offers the opportunity for more general studies.

Recall that $\mathbb{P}(\mathcal{O}_t) = \phi(t)$ and $\Phi(t) = \int_0^t \mathbb{P}(\mathcal{O}_s)\,ds$. Then the following general result holds.

THEOREM 8. *For any function $\varphi(t) \geq g(t) \vee 1$, for $t > 0$, we have that*

$$\phi(t) - \frac{2K}{\sqrt{\varphi(t)}}\Phi(t) = H(t), \tag{5.1}$$

where with $\Delta_1^{\varphi(t)} = \inf\{s \geq 0 : \tau_s - \tau_{s-} > \varphi(t)\}$ we have that $H(t)$ is defined as follows:

$$\begin{aligned} H(t) &:= \mathbb{P}\big(\tau_s > g(s), s \leq t;\, \Delta_1^{\varphi(t)} > t\big) \\ &\quad - \frac{4K^2}{\varphi(t)} \int_0^t \int_0^s \mathbb{P}\big(\tau_u > g(u), u \leq v \,|\, \Delta_1^{\varphi(t)} = v\big) e^{-2Kv/\sqrt{\varphi(t)}}\,dv\,ds. \end{aligned} \tag{5.2}$$

Denote by

$$\rho(t) := \frac{H(t)}{\Phi(t)}. \tag{5.3}$$

For any $t_0 \geq 0$ such that $\varphi(t) = g(t) = g(t) \vee 1$, for all $t \geq t_0$, we have that

$$\Phi(t) = \Phi(t_0) e^{\int_{t_0}^t (2K\sqrt{g(s)})\,ds + \int_{t_0}^t \rho(s)\,ds}. \tag{5.4}$$

REMARK 9. We have little doubt that a similar approach can be used to control the probability of events $\mathcal{O}_t$ arising from more general subordinators whose Lévy measure tail $\overline{\Pi}(x) = \int_x^\infty \Pi(ds)$ behaves as $x^{-\alpha}L(x)$, as $x \to \infty$. Here $0 < \alpha < 2$ and L denotes a slowly varying function. For further information on subordinators we refer to [4], Chapter III. Therefore, the main results are likely to be extended to a larger class of Lévy processes. The conditions for a Lévy process to possess a local time at zero and the form of the Lévy–Khintchine exponent of the inverse local time can be found in [4], Chapter V.

REMARK 10. It is even more interesting to understand whether these equations are applicable only for nondecreasing processes like τ or a suitable modification can be developed for, say Lévy processes. Then the problem of general Lévy process $\mathbb{P}(X_s > g(s), s \leq t)$ could be attacked with such a simple approach as ODE.

REMARK 11. It is important to note that despite that (5.1) is valid with any $\varphi(t) \geq g(t) \vee 1$ it is most beneficial to work with $g(t)$ itself since then the error term represented by $H(t)$ will be minimal.

REMARK 12. We note the striking semblance of the derivation of (5.1) to the classical renewal theory. Perusing the proof, it is apparent that the second term can be decomposed ad infinitum in terms of more and more repeated integrals involving $\Phi(s)$ and further error terms thus obtaining a differential equation involving infinitely many derivatives.

Assume the following mild technical condition:

$$\liminf_{t\to\infty} \frac{g(t)}{t^2 \ln(t)} = \infty. \tag{5.5}$$

From now on, we work with $\varphi(t) = g(t) \vee 1$. The next result shows that the finiteness of $\Phi(\infty)$ depends on $I(f)$. We recall the conditions (2.1) on f:

$$(0, \infty) \ni x \mapsto \frac{f(x)}{\sqrt{x}} \quad \text{is decreasing and} \quad \lim_{x\to\infty} \frac{f(x)}{\sqrt{x}} = 0.$$

LEMMA 1. *Let f satisfy the usual conditions without the assumption $f(1) = 1$ given in Definition 2.1 and (5.5). Then $H(t) = o(\Phi(t)/\sqrt{g(t)})$ and hence $\rho(t) = o(1/\sqrt{g(t)})$. Therefore,*

$$\Phi(\infty) < \infty \quad \Longleftrightarrow \quad \mathbb{E}\bigl[f(\tau_1)\bigr] < \infty \quad \Longleftrightarrow \quad \int_1^\infty \frac{ds}{\sqrt{g(s)}} < \infty. \tag{5.6}$$

Then equation (5.1) leads to the following essential result.

THEOREM 13. *For any f satisfying the usual conditions given in Definition 2.1, $I(f) < \infty$ and (5.5), we have that*

$$\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}\bigl(\mathcal{O}_t; \Delta_1^{g(t)} \leq t\bigr) \sim \frac{2K\Phi(\infty)}{\sqrt{g(t)}} \qquad \text{as } t \to \infty. \tag{5.7}$$

REMARK 14. Condition (5.5) is expected to hold when $I(f) < \infty$ unless the function is exceptionally bad.

The next result considers the case when $\Phi(\infty) = \infty$. We then have the following theorem.

THEOREM 15. *For any f satisfying the usual conditions given in Definition 2.1, $I(f) = \infty$ and (5.5), we have that, as $t \to \infty$,*

$$\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}\bigl(\mathcal{O}_t \cap \Delta_1^{g(t)} \leq t\bigr) \sim \frac{2K\Phi(t)}{\sqrt{g(t)}}; \qquad \ln\bigl(\Phi(t)\bigr) \sim \int_1^t \frac{2K}{\sqrt{g(s)}}\, ds, \tag{5.8}$$

where we recall that $\Delta_1^{g(t)} = \inf\{t > 0 : \tau_t - \tau_{t-} > g(t)\}$. *Furthermore, if for some* $t \geq t_0 \geq 1$, $\int_{t_0}^{\infty} |\rho(s)|\, ds < \infty$, *then* (5.8) *is augmented to*

$$\Phi(t) \sim \Phi(t_0) e^{\int_{t_0}^{\infty} \rho(s)\, ds} e^{\int_{t_0}^{t} (2K/\sqrt{g(s)})\, ds}. \tag{5.9}$$

In particular, this holds when $\liminf_{t\to\infty} g(t)/t^2 \ln^{8/5+\varepsilon}(t) = \infty$, *that is*, (4.1) *holds*.

REMARK 16. Note the strong form of the asymptotics (5.9) is essential in the proof of recurrence. As mentioned in Section 2.5, we need to study in a uniform way a family of equations for a generalized form of Φ.

We start by proving the results of this section as they are instrumental in our further analysis.

6. Proof of the results in Section 5. In this section and later, we will use the following notation. We shall attach a superscript to $\mathcal{O}_t$, τ, etc. to denote that jumps until given time above certain level are conditioned not to have occurred. For example, $\mathcal{O}_s^{g(t)} = \{\tau_v^{g(t)} > g(v), v \leq s\}$ means that $\mathcal{O}_s$ holds for the subordinator $\tau^{g(t)}$ which is constructed from τ by conditioning that jumps larger than $g(t)$ do not occur. The Lévy–Khintchine exponent of $\tau_1^{g(t)}$ can be represented by

$$\begin{aligned} \Psi_{g(t)}(\lambda) &= \ln\bigl(\mathbb{E}\bigl[e^{\lambda \tau_1^{g(t)}}\bigr]\bigr) \\ &= K \int_0^{g(t)} \bigl(e^{\lambda s} - 1\bigr) \frac{ds}{s^{3/2}} \\ &= K\sqrt{\lambda} \int_0^{\lambda g(t)} \bigl(e^{s} - 1\bigr) \frac{ds}{s^{3/2}} \qquad \forall \lambda > 0, \end{aligned} \tag{6.1}$$

where we note that only the Lévy measure $\Pi(ds) = K\, ds/s^{3/2}$ has been truncated; see (2.4). The fact that $\tau_1^{g(t)}$ has all exponential moments yields that $\Psi_{g(t)}(\cdot)$ is analytic on the complex plane. Indeed, the analyticity of $\Psi_{g(t)}(\cdot)$ can be directly read off from the first integral formula in (6.1) by a power series expansion of the exponential.

We also use the notation $\Delta_k^a = \inf\{s > \Delta_{k-1}^a : \tau_s - \tau_{s-} > a\}$, $\Delta_0^a = 0$, to denote the time of the kth jump of τ larger than a. Note that Δ_1^a is exponentially distributed with parameter $2K/\sqrt{a}$, where we recall that $\overline{\Pi}(x) = \int_x^{\infty} \Pi(ds) = 2K/\sqrt{x}$, for all $x > 0$, is the intensity measure of the jumps larger than x, see [4] for more information on Lévy processes.

We are now ready to start with our proof.

PROOF OF THEOREM 8. Note that since $\varphi(t) \geq g(t) \vee 1$ we have upon disintegration the values of the exponentially distributed random variable $\Delta_1^{\varphi(t)}$ with parameter $2K/\sqrt{\varphi(t)}$ that

$$
\begin{aligned}
\mathbb{P}(\mathcal{O}_t) &= \int_0^t \mathbb{P}(\mathcal{O}_t; \Delta_1^{\varphi(t)} \in ds) + \mathbb{P}(\Delta_1^{\varphi(t)} > t; \mathcal{O}_t) \\
&= \frac{2K}{\sqrt{\varphi(t)}} \int_0^t \mathbb{P}(\mathcal{O}_s^{\varphi(t)}) e^{-2Ks/\sqrt{\varphi(t)}} \, ds \qquad (6.2)\\
&\quad + \mathbb{P}(\tau_s > g(s), s \le t; \Delta_1^{\varphi(t)} > t).
\end{aligned}
$$

Indeed, we have that

$$
\mathbb{P}(\mathcal{O}_t; \Delta_1^{\varphi(t)} \in ds) = \mathbb{P}(\mathcal{O}_{\Delta_1^{\varphi(t)}}; \Delta_1^{\varphi(t)} \in ds),
$$

which upon conditioning on $\{\Delta_1^{\varphi(t)} = s\}$ confirms our equation. Next, note that since $\mathbb{P}(\Delta_1^{\varphi(t)} > s) = e^{-2Ks/\sqrt{\varphi(t)}}$ we obtain that

$$
\begin{aligned}
\mathbb{P}(\mathcal{O}_s) &= \mathbb{P}(\Delta_1^{\varphi(t)} \le s; \mathcal{O}_s) + \mathbb{P}(\Delta_1^{\varphi(t)} > s; \mathcal{O}_s) \\
&= \mathbb{P}(\Delta_1^{\varphi(t)} \le s; \mathcal{O}_s) + \mathbb{P}(\mathcal{O}_s^{\varphi(t)}) e^{-2Ks/\sqrt{\varphi(t)}}.
\end{aligned}
$$

Substituting back in (6.2) for $\mathbb{P}(\mathcal{O}_s^{\varphi(t)}) e^{-2Ks/\sqrt{\varphi(t)}}$ we get that

$$
\begin{aligned}
\phi(t) &= \mathbb{P}(\mathcal{O}_t) \\
&= \frac{2K}{\sqrt{\varphi(t)}} \int_0^t \mathbb{P}(\mathcal{O}_s) \, ds + \mathbb{P}(\tau_s > g(s), s \le t; \Delta_1^{\varphi(t)} > t) \\
&\quad - \frac{2K}{\sqrt{\varphi(t)}} \int_0^t \mathbb{P}(\Delta_1^{\varphi(t)} \le s; \mathcal{O}_s) \, ds \\
&= \frac{2K}{\sqrt{\varphi(t)}} \Phi(t) + \mathbb{P}(\tau_s > g(s), s \le t; \Delta_1^{\varphi(t)} > t) \\
&\quad - \frac{4K^2}{\varphi(t)} \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{\varphi(t)}) e^{-2Kv/\sqrt{\varphi(t)}} \, dv \, ds.
\end{aligned}
$$

Recalling the definition $H(t)$ [see (5.2)] we conclude (5.1). Finally, (5.4) comes as the solution of a classical first order linear ODE. □

PROOF OF LEMMA 1. We estimate the terms in $H(t)$, see (5.2). Since $\varphi(t) = g(t) \vee 1 = g(t)$ when $t > f(1)$ we can rewrite $H(t)$ as follows:

$$
\begin{aligned}
H(t) &= e^{-2Kt/\sqrt{g(t)}} \mathbb{P}(\mathcal{O}_t^{g(t)}) \\
&\quad - \frac{4K^2}{g(t)} \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{g(t)}) e^{-2Kv/\sqrt{g(t)}} \, dv \, ds.
\end{aligned}
$$

Estimating $\mathbb{P}(\mathcal{O}_v^{g(t)}) \le \mathbb{P}(\mathcal{O}_v)$, $e^{-2Kv/\sqrt{g(t)}} \le 1$ and using the fact that $\Phi(t)$ is nondecreasing we arrive at the bound

$$\begin{aligned} &\frac{4K^2}{g(t)} \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{g(t)}) e^{-2Kv/\sqrt{g(t)}} \, dv \, ds \\ &\quad \le \frac{4K^2}{g(t)} t \Phi(t) = \frac{4K^2 t}{\sqrt{g(t)}} \frac{\Phi(t)}{\sqrt{g(t)}} \\ &\quad \overset{(5.5)}{=} o\biggl(\frac{\Phi(t)}{\sqrt{g(t)}}\biggr). \end{aligned} \tag{6.3}$$

Therefore, we need to discuss the first term of $H(t)$ only.

Denote by $g_1(t) := g(t)/\ln(t)$, for $t > 2$. Distinguishing upon the times of $\Delta_1^{g_1(t)}$, $\Delta_1^{\theta g(t)}$, for some $0 < \theta < 1$, we get

$$\begin{aligned} \mathbb{P}(\mathcal{O}_t^{g(t)}) &\le \mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{\theta g(t)} \le t) \\ &\quad + \mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{g_1(t)} \le t; \Delta_1^{\theta g(t)} > t) \\ &\quad + \mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{g_1(t)} > t). \end{aligned}$$

Note that we work with the truncated subordinator $\tau^{g(t)}$ and the corresponding event $\mathcal{O}_t^{g(t)}$. Therefore, in this case, Δ_1^a is exponentially distributed with parameter $2K/\sqrt{a} - 2K/\sqrt{g(t)}$, for any $a < g(t)$. One can always in a crude manner relate the measures

$$\mathbb{P}(\Delta_1^a \in ds) \le \biggl(\frac{2K}{\sqrt{a}} - \frac{2K}{\sqrt{g(t)}}\biggr) ds.$$

This bound will be used extensively but implicitly below.

Choose any $c > 0$ and any $n \in \mathbb{N}^+ = \{1, 2, 3, \ldots\}$. With those c, n we apply Lemma 3 with $\delta = 1$, see below for its statement and proof. Thus, we get the following inequality and estimate

$$\mathbb{P}(\tau_t^{g_1(t)} > cg(t)) \le e^{K\sqrt{nc^{-1}} \frac{t \ln^{1/2}(t)}{\sqrt{g(t)}} \int_0^{n/c} (e^s - 1) \frac{ds}{s^{3/2}}} t^{-n} \lesssim t^{-n},$$

since $\liminf_{t\to\infty} g(t)/t^2 \ln(t) = \infty$, that is, (5.5) holds. Therefore, since $\Phi(t) = \int_0^t \mathbb{P}(\mathcal{O}_s) \, ds$ is nondecreasing,

$$\begin{aligned} \mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{g_1(t)} > t) &= \mathbb{P}(\mathcal{O}_t^{g_1(t)}) \mathbb{P}(\Delta_1^{g_1(t)} > t) \le \mathbb{P}(\mathcal{O}_t^{g_1(t)}) \\ &\le \mathbb{P}(\tau_t^{g_1(t)} > g(t)) \lesssim \frac{1}{t^n} \\ &\lesssim \frac{\int_0^t \mathbb{P}(\mathcal{O}_s) \, ds}{t^{n-1}} = \frac{\Phi(t)}{t^{n-1}}. \end{aligned} \tag{6.4}$$

Similarly, disintegrating the time of arrival of $\Delta_1^{g_1(t)}$ and using that the maximal jump does not exceed $\theta g(t)$ we derive that

$$\begin{aligned}
&\mathbb{P}\big(\mathcal{O}_t^{g(t)}; \Delta_1^{g_1(t)} \leq t; \Delta_1^{\theta g(t)} > t\big) \\
&\quad = \int_{s=0}^{t} \mathbb{P}\big(\mathcal{O}_t^{g(t)}; \Delta_1^{\theta g(t)} > t; \Delta_1^{g_1(t)} \in ds\big) \\
&\quad = \int_{s=0}^{t} \mathbb{P}\big(\mathcal{O}_t^{g(t)}; \Delta_1^{\theta g(t)} > t | \Delta_1^{g_1(t)} = s\big)\mathbb{P}\big(\Delta_1^{g_1(t)} \in ds\big) \\
&\quad \leq \int_{s=0}^{t} \mathbb{P}\big(\mathcal{O}_s^{g_1(t)}; \tau_t^{g(t)} > g(t); \Delta_1^{\theta g(t)} > t | \Delta_1^{g_1(t)} = s\big)\mathbb{P}\big(\Delta_1^{g_1(t)} \in ds\big).
\end{aligned}$$

We proceed to estimate the last integral above. First, we bound the measure $\mathbb{P}(\Delta_1^{g_1(t)} \in ds) \leq 2K/\sqrt{g_1(t)}\,ds$. Second, conditionally on $\{\Delta_1^{g_1(t)} = s\}$ we have the representation

$$\tau_t^{g(t)} = \tau_{s-}^{g(t)} + \tau_s^{g(t)} - \tau_{s-}^{g(t)} + \tau_t^{g(t)} - \tau_s^{g(t)} \stackrel{d}{=} \tau_{s-}^{g_1(t)} + \tau_s^{g(t)} - \tau_{s-}^{g(t)} + \tilde{\tau}_{t-s}^{g(t)},$$

because the process $\tau^{g(t)}$ runs as $\tau^{g_1(t)}$ until time s at which time it makes a jump. Note that $\tilde{\tau}^{g(t)}$ is a copy of $\tau^{g(t)}$ independent of $\tau_{s-}^{g_1(t)}$. Third, on $\{\Delta_1^{\theta g(t)} > t\}$ we have that $\tau_s^{g(t)} - \tau_{s-}^{g(t)} \in (g_1(t), \theta g(t))$. Using these three points, we continue the estimates above to get that

$$\begin{aligned}
&\mathbb{P}\big(\mathcal{O}_t^{g(t)}; \Delta_1^{g_1(t)} \leq t; \Delta_1^{\theta g(t)} > t\big) \\
&\quad \leq \frac{2K}{\sqrt{g_1(t)}} \int_0^t \mathbb{P}\big(\mathcal{O}_s^{g_1(t)}; \tilde{\tau}_{t-s}^{g(t)} + \tau_{s-}^{g_1(t)} > (1-\theta)g(t)\big)\,ds \\
&\quad = \frac{2K}{\sqrt{g_1(t)}} \int_0^t \mathbb{P}\bigg(\mathcal{O}_s^{g_1(t)}; \tau_{s-}^{g_1(t)} > \frac{(1-\theta)g(t)}{2}; \\
&\qquad \tilde{\tau}_{t-s}^{g(t)} > (1-\theta)g(t) - \tau_{s-}^{g_1(t)}\bigg)\,ds \\
&\qquad + \frac{2K}{\sqrt{g_1(t)}} \int_0^t \mathbb{P}\bigg(\mathcal{O}_s^{g_1(t)}; \tau_{s-}^{g_1(t)} \leq \frac{(1-\theta)g(t)}{2}; \\
&\qquad \tilde{\tau}_{t-s}^{g(t)} > (1-\theta)g(t) - \tau_{s-}^{g_1(t)}\bigg)\,ds \\
&\quad \leq \frac{2K}{\sqrt{g_1(t)}} \int_0^t \mathbb{P}\bigg(\mathcal{O}_s^{g_1(t)}; \tau_{s-}^{g_1(t)} > \frac{(1-\theta)g(t)}{2}\bigg)\,ds \\
&\qquad + \frac{2K}{\sqrt{g_1(t)}} \int_0^t \mathbb{P}\bigg(\mathcal{O}_s^{g_1(t)}; \tilde{\tau}_{t-s}^{g(t)} > \frac{(1-\theta)g(t)}{2}\bigg)\,ds \\
&\quad = J_1 + J_2.
\end{aligned}$$

We estimate J_1 and J_2. For J_2 we recall the facts that $\tilde{\tau}_{t-s}^{g(t)}$ is independent of $\mathcal{O}_s^{g_1(t)}$; $\tilde{\tau}^{g(t)}$ is a copy of $\tau^{g(t)}$ and τ is a stable subordinator with index $1/2$. The latter readily yields from (2.5), for all $v > 0, t > 0, a > 0$,

$$\mathbb{P}(\tau_t^v > a) \le \mathbb{P}(\tau_t > a) \le \sqrt{\frac{2}{\pi}}\frac{t}{\sqrt{a}}. \tag{6.5}$$

Therefore, recalling that $g_1(t) = g(t)/\ln(t)$, we get

$$\begin{aligned}
J_2 &= \frac{2K}{\sqrt{g_1(t)}}\int_0^t \mathbb{P}\left(\mathcal{O}_s^{g_1(t)}; \tilde{\tau}_{t-s}^{g(t)} > \frac{(1-\theta)g(t)}{2}\right) ds \\
&\le \frac{2K}{\sqrt{g_1(t)}}\mathbb{P}\left(\tau_t \ge \frac{(1-\theta)g(t)}{2}\right)\int_0^t \mathbb{P}(\mathcal{O}_s^{g_1(t)})\, ds \\
&\le \frac{Ct\ln^{1/2}(t)}{\sqrt{((1-\theta)/2)g(t)}}\frac{\Phi(t)}{\sqrt{g(t)}},
\end{aligned}$$

for some absolute constant $C > 0$. J_1 we estimate following the steps leading to (6.4) to get

$$\begin{aligned}
J_1 &= \frac{2K}{\sqrt{g_1(t)}}\int_0^t \mathbb{P}\left(\mathcal{O}_s^{g_1(t)}; \tau_{s-}^{g_1(t)} > \frac{(1-\theta)g(t)}{2}\right) ds \\
&\le \frac{2Kt}{\sqrt{g_1(t)}}\mathbb{P}\left(\tau_t^{g_1(t)} > \frac{(1-\theta)g(t)}{2}\right) \lesssim \frac{t\ln^{1/2}(t)}{\sqrt{g(t)}}\frac{\Phi(t)}{t^{n-1}}.
\end{aligned}$$

Therefore, collecting the terms above we get

$$\mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{g_1(t)} \le t; \Delta_1^{\theta g(t)} > t) \le J_1 + J_2 \le o(1)\frac{\Phi(t)}{\sqrt{g(t)}}, \tag{6.6}$$

since (5.5) holds and n can be chosen as big as we wish.

Finally, consider the case $\{\Delta_1^{\theta g(t)} \le t\}$. Then estimating

$$\begin{aligned}
\mathbb{P}(\Delta_1^{\theta g(t)} \in ds) &= \frac{2K}{\sqrt{g(t)}}\left(\frac{1}{\sqrt{\theta}} - 1\right)e^{-(\frac{2K}{\sqrt{g(t)}}(\frac{1}{\sqrt{\theta}}-1)s)}\, ds \\
&\le \frac{2K}{\sqrt{g(t)}}\left(\frac{1}{\sqrt{\theta}} - 1\right) ds,
\end{aligned}$$

we get that

$$\begin{aligned}
\mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{\theta g(t)} \le t) &= \int_0^t \mathbb{P}(\mathcal{O}_t^{g(t)}; \Delta_1^{\theta g(t)} \in ds) \\
&\le \frac{2K}{\sqrt{g(t)}}\left(\frac{1}{\sqrt{\theta}} - 1\right)\int_0^t \mathbb{P}(\mathcal{O}_s^{g(t)})\, ds \\
&\le \frac{2K}{\sqrt{g(t)}}\left(\frac{1}{\sqrt{\theta}} - 1\right)\Phi(t).
\end{aligned}$$

Collecting this term, (6.6), (6.4) we get that

$$\limsup_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_t^{g(t)})\sqrt{g(t)}}{2K\Phi(t)} \le \left(\frac{1}{\sqrt{\theta}} - 1\right).$$

Setting $\theta \to 1$, we conclude the statement that $H(t) = o(\frac{\Phi(t)}{\sqrt{g(t)}})$ and $\rho(t) = o(\frac{1}{\sqrt{g(t)}})$. Then this allows us, together with

$$\Phi(t) = \Phi\bigl(f(1)\bigr)e^{\int_{f(1)}^{t} \frac{2K}{\sqrt{g(s)}}\,ds + \int_{f(1)}^{t} \rho(s)\,ds},$$

to deduce (5.6). $\square$

PROOF OF THEOREM 13 AND (5.8) OF THEOREM 15. The assertion that $\mathbb{P}(\mathcal{O}_t) \sim \frac{2K\Phi(t)}{\sqrt{g(t)}}$ in both theorems follows from the differential equation (5.1) with $\varphi(t) = g(t) \vee 1 = g(t), t > 1$, and using therein that $H(t) = o(\frac{\Phi(t)}{\sqrt{g(t)}})$, see Lemma 1. The behavior of $\ln(\Phi(t))$ in (5.8) follows from (5.4) and the fact that Lemma 1 shows that $\rho(t) = o(\frac{1}{\sqrt{g(t)}})$.

The claim $\mathbb{P}(\mathcal{O}_t) \sim \mathbb{P}(\Delta_1^{g(t)} \le t; \mathcal{O}_t)$ follows from (6.2) as the second term there is proved to be $o(\frac{\Phi(t)}{\sqrt{g(t)}})$ and therefore $o(\mathbb{P}(\mathcal{O}_t))$. $\square$

PROOF OF (5.9) OF THEOREM 15. The proof is immediate from Lemma 2 below with $h = y = 0$ which is the classical case. Indeed, with $h = y = 0$, (6.10) is applicable for all t_0 big enough, some $\varepsilon > 0$ and $A > 3$ as follows:

$$\int_{t_0}^{\infty} \rho(s)\,ds \le C(A) \int_{t_0}^{\infty} \frac{1}{s \ln^{1+\varepsilon/2}(s)}\,ds < \infty. \qquad \square$$

The next lemma proves a stronger claim than (5.4) of Theorem 15 as it provides some form of uniformity. For any $y \ge 0$ and $h \ge 0$, define $g_y^h(s) := (g(s+h) - y) \vee 1$ and $g_{y,h}(s) := g(s+h) - y$ and define $\mathcal{O}_t(h,y) = \{\tau_s > g_{y,h}(s), s \le t\}$ and

$$\mathcal{O}_t^{g_y^h} := \mathcal{O}_t^{g_y^h}(h,y) = \bigl\{\tau_s^{g_y^h(t)} > g_{y,h}(s), s \le t\bigr\}.$$

We note that

$$\mathbb{P}\bigl(\mathcal{O}_t(h,y); \Delta_1^{g_y^h(t)} > t\bigr) \le \mathbb{P}\bigl(\mathcal{O}_t^{g_y^h}\bigr) \le \mathbb{P}\bigl(\mathcal{O}_t(h,y)\bigr). \tag{6.7}$$

We denote as well

$$\Phi_y^h(t) = \int_0^t \mathbb{P}\bigl(\mathcal{O}_t(h,y)\bigr)\,ds; \qquad \phi_y^h(t) = \bigl(\Phi_y^h(t)\bigr)'.$$

Then consider the more general differential equation with H_y^h defined as in (5.2):

$$\phi_y^h(t) - \frac{2K}{\sqrt{g_y^h(t)}}\Phi_y^h(t) = H_y^h(t). \tag{6.8}$$

We note that the functions g_y^h and $g_{y,h}$ reflect the new boundaries above which τ has to remain. Therefore, the meaning of (6.8) is that of (5.1). Finally, denote by

$$\rho_y^h(t) := \frac{H_y^h(t)}{\Phi_y^h(t)}. \tag{6.9}$$

We have the following claim.

LEMMA 2. *Let f satisfy the usual conditions that can be found in Definition 2.1 and $\liminf_{t\to\infty} g(t)/(t^2 \ln^{8/5+\varepsilon}(t)) = \infty$, for some $\varepsilon > 0$, that is, (4.1) holds. Then, for any $A > 3$, $h > 0$, $y > g(h) \vee 0$ and any $t > f(Ay) \vee t(A)$, where $t(A) = t^*(A) \vee e^2$ with $t^*(A)$ satisfying the equation $g(t^*(A)) = 1 + \frac{2}{A}$ we have the following bounds*:

$$\rho_y^h(t) \le C(A) \frac{1}{t \ln^{1+\varepsilon/2}(t)} \left(1 + \frac{1}{(f(y) - h)}\right). \tag{6.10}$$

There exists $u(t) \to 0$, as $t \to \infty$, such that for all h, y constrained as above we have that

$$\rho_y^h(t) \le \frac{u(t)}{\sqrt{g(t)}} \left(1 + \frac{1}{(f(y) - h)}\right). \tag{6.11}$$

The last estimate (6.11) holds with $\sqrt{g_y^h(t)}$ instead of $\sqrt{g(t)}$. The estimates (6.10) and (6.11) also hold for $h = y = 0$ without the factor $1/(f(y) - h)$.

REMARK 17. The fact that $y > g(h)$ is to ensure that $g_{y,h}(0) = g(h) - y < 0$. This is needed since for small times s the subordinator cannot cross immediately a positive boundary which will be the case if $g_{y,h}(0) = g(h) - y > 0$. We thus avoid trivialities like $\mathbb{P}(\mathcal{O}_s(y, h)) = 0$.

PROOF OF LEMMA 2. The case when $y = h = 0$ can be dealt with as below with the only simplification that since $\Phi(t) > \Phi(t(A)) > 0, t > t(A)$, we do not need (6.14) to introduce the function $\Phi(t)$ in inequalities (6.15), (6.16) and (6.23). So we deal only with the uniform estimates in h, y under the conditions of the lemma. Applying (6.7) to the first term of (5.2) taken with $\varphi(t) = g_y^h(t)$, we get that

$$\begin{aligned} H_y^h(t) &= \rho_y^h(t)\Phi_y^h(t) \le \mathbb{P}(\mathcal{O}_t^{g_y^h}) \\ &\quad - \frac{4K^2}{g_y^h(t)} \int_0^t \int_0^s \mathbb{P}(\tau_u > g_{y,h}(u), u \le v | \Delta_1^{g_y^h(t)} = v) e^{-2Kv/\sqrt{g_y^h(t)}} \, dv \, ds \\ &= \mathbb{P}(\mathcal{O}_t^{g_y^h}) - \frac{4K^2}{g_y^h(t)} \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{g_y^h}) e^{-2Kv/\sqrt{g_y^h(t)}} \, dv \, ds \\ &= \mathbb{P}(\mathcal{O}_t^{g_y^h}) - I(t). \end{aligned} \tag{6.12}$$

We work with $t > f(Ay) \vee t(A)$. To bound $I(t)$, we need the following important inequalities. For any $y > g(h) \vee 0$ and $t > f(Ay) \vee t(A)$, we have that $g(t) = g(t) \vee 1$ since $g(t) > g(t^*(A)) = 1 + \frac{2}{A} > 1$. Then with $B(A) = 1 - \frac{1}{A}$

$$
\begin{aligned}
g_y^h(t) &\ge g(t)\left(\frac{g(t+h)}{g(t)} - \frac{y}{g(t)}\right) \ge g(t)\left(1 - \frac{y}{g(f(Ay))}\right) \ge B(A)g(t), \\
g_{y,h}(t) &\ge g(t)\left(\frac{g(t+h)}{g(t)} - \frac{y}{g(t)}\right)g(t) \ge \left(1 - \frac{y}{g(f(Ay))}\right) \ge B(A)g(t).
\end{aligned} \tag{6.13}
$$

Then

$$
\begin{aligned}
I(t) &= \frac{4K^2}{g_y^h(t)} \int_0^t \int_0^s \mathbb{P}(\mathcal{O}_v^{g_y^h}) e^{-2Kv/\sqrt{g_y^h(t)}}\, dv\, ds \\
&\le \frac{4K^2 t}{g_y^h(t)} \Phi_y^h(t) \overset{(6.13)}{\le} \tilde{B}(A) \frac{t}{g(t)} \Phi_y^h(t),
\end{aligned}
$$

where $\tilde{B}(A) = \frac{A}{A-1}$. However, since $\liminf_{t\to\infty} g(t)/t^2 \ln^{8/5+\varepsilon}(t) = \infty$ then $t/g(t) = o(\frac{1}{t \ln^{8/5+\varepsilon}(t)})$ for $t > f(Ay) \vee t(A)$. Thus, since $\tilde{B}(A)\frac{t}{g(t)}$ does not depend on h and $y > g(h) \vee 0$, we need to consider only $\mathbb{P}(\mathcal{O}_t^{g_y^h})$ in (6.12) for the proof of both (6.10) and (6.11) of our Lemma 2.

For brevity we put $w := g_y^h$, $\tilde{w} := g_{y,h}$. Additionally, set $w_\delta(t) := w(t)/\ln^\delta(t)$. Then, we denote by $\Delta_k^{w_1(t)} = \inf\{s > \Delta_{k-1}^{w_1(t)} : \tau_s^{w(t)} - \tau_{s-}^{w(t)} > w_1(t)\}$, $\Delta_0^{w_1(t)} = 0$. Put for the duration of this proof $\mathcal{O}_t^{g_y^h} := \mathcal{O}_t^{w(t)}$. With the choice of $t > f(Ay) \vee t(A)$, we get that $w(t) = \tilde{w}(t)$, because when $y > \frac{1}{A-1}$ we have that

$$
g(t+h) - y \ge g\big(f(Ay)\big) - y = (A-1)y > 1
$$

and otherwise

$$
g(t+h) - y \ge g\big(t(A)\big) - y \ge 1 + \frac{2}{A} - \frac{1}{A-1} > 1
$$

holds for $A > 2$.

To estimate $\mathbb{P}(\mathcal{O}_t^{w(t)})$ precisely, we consider gradually several cases which correspond to different scenarios. Collecting all the estimates from each case will lead to our result.

Case 1: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} > t)$.

We note that from Lemma 3 with $\delta = 1$ and $c = 1$ we get that, for any $n \in \mathbb{N}^+$,

$$
\begin{aligned}
\mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} > t\big) &\le \mathbb{P}\big(\tau_t^{w_1(t)} > w(t)\big) \\
&\le e^{tK\sqrt{n}\frac{\ln^{1/2}(w(t))}{\sqrt{w(t)}} \int_0^n (e^s - 1)\frac{ds}{s^{3/2}}} e^{-n\ln(w(t))}.
\end{aligned}
$$

Therefore, using (6.13), we are able to deduce that for $h > 0$, $y > g(h) \vee 0$, for any $t > f(Ay) \vee t(A)$ and for each $n \in \mathbb{N}^+$,

$$\mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} > t\big) \le e^{2\tilde{B}(A)K\sqrt{n}\frac{t\ln^{1/2}(w(t))}{\sqrt{g(t)}}\int_0^n (e^s-1)\frac{ds}{s^{3/2}}} e^{-n\ln(w(t))}.$$

However, since $\liminf_{t\to\infty} g(t)/t^2 \ln^{8/5+\varepsilon}(t) = \infty$, we see that

$$2\tilde{B}(A)K\sqrt{n}\frac{t\ln^{1/2}(w(t))}{\sqrt{g(t)}}\int_0^n (e^s - 1)\frac{ds}{s^{3/2}} \le C_n(A)\ln^{1/2}\big(w(t)\big),$$

where $C_n(A) > 0$ depends solely on $n \in \mathbb{N}^+$, $A > 3$. Henceforth, we get that

$$\begin{aligned}\mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} > t\big) &\le e^{C_n(A)\ln^{1/2}(w(t)) - \ln(w(t))} e^{-(n-1)\ln(w(t))} \\ &\le \frac{C(n, A)}{w^{n-1}(t)} \le \frac{C(n, A)}{g^{n-1}(t)},\end{aligned}$$

where the last inequality follows from (6.13). $C(n, A) > 0$ from now on is a generic constant depending on n, A.

We note that in general for $s \le f(y) - h$, $y > g(h) \vee 0$, we have that $\mathbb{P}(\mathcal{O}_s^{w(t)}) = 1$ as $\tilde{w}(v) = g(v+h) - y \le 0$, $v \le s$. Hence, we obtain that for $t > t(A) \vee f(Ay) > f(y) - h$

$$(6.14)\qquad \Phi_y^h(t) = \int_0^t \mathbb{P}\big(\mathcal{O}_s^{w(t)}\big)\,ds \ge \int_0^{f(y)-h} \mathbb{P}\big(\mathcal{O}_s^{w(t)}\big)\,ds = f(y) - h.$$

From the latter and the estimate above, we get that

$$(6.15)\qquad \mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} > t\big) \le C(n, A)\frac{\Phi_y^h(t)}{(f(y) - h)g^{n-1}(t)}.$$

Case 2: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} \le t)$.

First, choose $\varepsilon < 1/4$ so that $1 - 4\varepsilon > 0$ and define

$$\Delta_k^{\varepsilon w(t)} = \inf\big\{s > \Delta_{k-1}^{\varepsilon w(t)} : \tau_s^{w(t)} - \tau_{s-}^{w(t)} > \varepsilon w(t)\big\}, \qquad \Delta_0^{\varepsilon w(t)} = 0.$$

Note that, since the jumps are defined for the truncated subordinator $\tau^{w(t)}$, each difference $\Delta_k^{\varepsilon w(t)} - \Delta_{k-1}^{\varepsilon w(t)}$, $k \ge 1$, is an exponentially distributed random variable with parameter $2K(1/\sqrt{\varepsilon w(t)} - 1/\sqrt{w(t)})$. Moreover, they form an independent sequence of random variables.

Case 2A: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} \le t; \Delta_4^{w_1(t)} > t; \Delta_1^{\varepsilon w(t)} > t)$.

We observe that putting at most 3 jumps of at most size $\varepsilon w(t)$ and conditioning on $\{\Delta_1^{\varepsilon w(t)} > t; \Delta_4^{w_1(t)} > t\}$ we get

$$\begin{aligned}&\mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} \le t; \Delta_4^{w_1(t)} > t; \Delta_1^{\varepsilon w(t)} > t\big) \\ (6.16)\qquad &= \sum_{k=1}^{3}\int_0^t \mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_k^{w_1(t)} \in ds; \Delta_{k+1}^{w_1(t)} > t; \Delta_1^{\varepsilon w(t)} > t\big)\end{aligned}$$

$$\le \sum_{k=1}^{3} \int_0^t \mathbb{P}\big(\tau_t^{w_1(t)} > (1-k\varepsilon)w(t)\big)\mathbb{P}\big(\Delta_k^{w_1(t)} \in ds; \Delta_{k+1}^{w_1(t)} > t; \Delta_1^{\varepsilon w(t)} > t\big)$$

$$\le 3\mathbb{P}\big(\tau_t^{w_1(t)} > (1-4\varepsilon)w(t)\big) \le C(\varepsilon, A, n)\frac{\Phi_y^h(t)}{(f(y)-h)g^{n-1}(t)},$$

where for the very last inequality we have used the procedure leading to (6.15) and $C(\varepsilon, A, n) > 0$ is a generic constant. Also, we have used that subtracting k jumps of size between $w_1(t)$ and $\varepsilon w(t)$ then conditionally on $\{\Delta_{k+1}^{w_1(t)} > t\}$ we have that $\tau^{w(t)} = \tau^{w_1(t)}$.

Case 2B: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_4^{w_1(t)} \le t; \Delta_1^{\varepsilon w(t)} > t)$.

Conditioning on $\Delta_1^{w_1(t)}$, we get

$$\begin{aligned}
&\mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_4^{w_1(t)} \le t; \Delta_1^{\varepsilon w(t)} > t\big)\\
&\quad= \int_0^t \mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{w_1(t)} \in ds; \Delta_4^{w_1(t)} \le t; \Delta_1^{\varepsilon w(t)} > t\big)\\
&\quad\le \int_0^t \mathbb{P}\big(\mathcal{O}_s^{w_1(t)}\big)\mathbb{P}\big(\Delta_1^{w_1(t)} \in ds; \Delta_4^{w_1(t)} \le t\big)\\
&\quad\le \mathbb{P}\big(\Delta_3^{w_1(t)} \le t\big) \int_0^t \mathbb{P}\big(\mathcal{O}_s^{w_1(t)}\big)\mathbb{P}\big(\Delta_1^{w_1(t)} \in ds\big) \qquad (6.17)\\
&\quad\le \big(\mathbb{P}\big(\Delta_1^{w_1(t)} \le t\big)\big)^3 \frac{2K\ln^{1/2}(t)}{\sqrt{w(t)}} \int_0^t \mathbb{P}\big(\mathcal{O}_s^{w_1(t)}\big)\,ds\\
&\quad\le C(A)\frac{t^3\ln^2(t)}{w^2(t)}\Phi_y^h(t) \le \tilde{C}(A)\frac{t^3\ln^2(t)}{g^2(t)}\Phi_y^h(t)\\
&\quad= \tilde{C}(A)\frac{t^3\ln^2(t)}{g^{3/2}(t)}\frac{\Phi_y^h(t)}{\sqrt{g(t)}} \le \tilde{C}'(A)\frac{\Phi_y^h(t)}{t\ln^{6/5+2\varepsilon}(t)}.
\end{aligned}$$

Indeed, (6.17) is obtained as a result of the following steps. In the first inequality, we excluded $\{\Delta_1^{\varepsilon w(t)} > t\}$ and estimated

$$\mathbb{P}\big(\mathcal{O}_t^{w(t)} | \Delta_1^{w_1(t)} = s, \Delta_4^{w_1(t)} \le t\big) \le \mathbb{P}\big(\mathcal{O}_s^{w_1(t)}\big).$$

Next, for the second inequality we enlarged the time for possible arrivals of jumps $2, 3, 4$. For the third inequality, we further allowed each jump $2, 3, 4$ to take t amount of time to occur and estimated

$$\mathbb{P}\big(\Delta_1^{w_1(t)} \in ds\big) \le \frac{2K}{\sqrt{w_1(t)}}\,ds$$

since $\Delta_1^{w_1(t)}$ is an exponentially distributed random variable with parameter $2K/\sqrt{w_1(t)} - 2K/\sqrt{w(t)}$. For the fourth inequality, we note that similarly

$$\mathbb{P}\big(\Delta_1^{w_1(t)} \le t\big) = 1 - e^{-2K(t/\sqrt{w_1(t)} - t/\sqrt{w(t)})} \le \frac{2Kt}{\sqrt{w_1(t)}}.$$

For the fifth inequality, we use (6.13) to bound the expressions with $w(t)$ uniformly with $g(t)$. Finally, we employ that $\liminf_{t\to\infty} g(t)/t^2 \ln^{8/5+\varepsilon}(t) = \infty$, that is, (4.1) holds.

Case 2C: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{\varepsilon w(t)} \le t)$.

Define $p(t) = \ln^{-\gamma}(t)$, $p^*(t) = 1 - p(t)$ and $0 < \gamma < 3/5 + \varepsilon$ to be chosen later. As before, we define the sequence of jumps exceeding the level $p^*(t)w(t)$ via $\Delta_0^{p^*(t)w(t)} = 0$ and

$$\Delta_k^{p^*(t)w(t)} = \inf\{s > \Delta_{k-1}^{p^*(t)w(t)} : \tau_s^{w(t)} - \tau_{s-}^{w(t)} \in (p^*(t)w(t), w(t))\}, \qquad k \ge 1,$$

where we recall that we already work with a subordinator whose jumps larger than $w(t)$ have been truncated. We have again that $\Delta_k^{p^*(t)w(t)} - \Delta_{k-1}^{p^*(t)w(t)}, k \ge 1$, is exponentially distributed with parameter $2K(\frac{1}{\sqrt{p^*(t)w(t)}} - \frac{1}{\sqrt{w(t)}})$. Hence, we get easily from (6.13), $\liminf_{t\to\infty} g(t)/t^2 \ln^{8/5+\varepsilon}(t) = \infty$ and $1 - \sqrt{1-x} \le x$ that

$$\begin{aligned}\mathbb{P}(\Delta_1^{p^*(t)w(t)} \in ds) &\le \frac{2Kp(t)}{\sqrt{p^*(t)w(t)}}\,ds \\ &\le \frac{C(A)}{\sqrt{g(t)}\ln^{\gamma}(t)}\,ds \\ &\le \frac{C(A)}{t\ln^{4/5+\varepsilon/2+\gamma}(t)}\,ds.\end{aligned} \tag{6.18}$$

We clarify that $t > t(A) \ge e^2$ implies $p^*(t) \ge p^*(t(A)) > 1 - \ln^{-\gamma}(e^2) > 0$.

Case 2Ca: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{\varepsilon w(t)} \le t; \Delta_1^{p^*(t)w(t)} \le t)$.

We ignore the event $\{\Delta_1^{\varepsilon w(t)} \le t\}$. Then disintegrating on the possible position of $\Delta_1^{p^*(t)w(t)}$ and estimating

$$\mathbb{P}(\mathcal{O}_t^{w(t)} | \Delta_1^{p^*(t)w(t)} = s) \le \mathbb{P}(\mathcal{O}_s^{w(t)})$$

we get the following chain of inequalities:

$$\begin{aligned}&\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{\varepsilon w(t)} \le t; \Delta_1^{p^*(t)w(t)} \le t) \\ &\quad\le \int_0^t \mathbb{P}(\mathcal{O}_s^{w(t)})\mathbb{P}(\Delta_1^{p^*(t)w(t)} \in ds) \\ &\quad\overset{(6.18)}{\le} \frac{C(A)}{\sqrt{g(t)}\ln^{\gamma}(t)}\Phi_y^h(t) \\ &\quad\le \frac{C(A)}{t\ln^{4/5+\varepsilon/2+\gamma}(t)}\Phi_y^h(t).\end{aligned} \tag{6.19}$$

Case 2Cb: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_2^{\varepsilon w(t)} \le t; \Delta_1^{p^*(t)w(t)} > t)$.

We estimate this case the same way as (6.17). To do so, we start by disintegrating the time of the first jump $\Delta_1^{\varepsilon w(t)}$. Then we invoke

$$\mathbb{P}(\Delta_1^{\varepsilon w(t)} \in ds) \le \frac{C(A,\varepsilon)}{\sqrt{g(t)}}\,ds; \qquad \mathbb{P}(\Delta_1^{\varepsilon w(t)} \le t) \le \frac{C(A,\varepsilon)t}{\sqrt{g(t)}},$$

which follow from the steps leading to (6.18) and employing the inequalities (6.13). Finally, using $\liminf_{t\to\infty} g(t)/t^2\ln^{8/5+\varepsilon}(t) = \infty$, we get

$$\begin{aligned}
&\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_2^{\varepsilon w(t)} \le t; \Delta_1^{p^*(t)w(t)} > t) \\
&\quad \le \frac{C^2(A,\varepsilon)t}{g(t)} \int_0^t \mathbb{P}(\mathcal{O}_s^{w(t)})\,ds \\
&\quad \le C^2(A,\varepsilon)\frac{t}{\sqrt{g(t)}}\frac{\Phi_y^h(t)}{\sqrt{g(t)}} \le \frac{C^2(A,\varepsilon)}{t\ln^{8/5+\varepsilon}(t)}\Phi_y^h(t).
\end{aligned} \tag{6.20}$$

Case 2Cc: $\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{\varepsilon w(t)} \le t; \Delta_2^{\varepsilon w(t)} > t; \Delta_1^{p^*(t)w(t)} > t)$.

We obtain a preliminary estimate as follows. First, we disintegrate the position of $\Delta_1^{\varepsilon w(t)}$ by conditioning upon $\{\Delta_1^{\varepsilon w(t)} = s; \Delta_2^{\varepsilon w(t)} > t; \Delta_1^{p^*(t)w(t)} > t\}$. Then upon this conditioning we have that

$$(\tau_v^{w(t)})_{v\le t} \stackrel{d}{=} (\tau_v^{\varepsilon w(t)})_{v\le t} + 1_{\{v>s\}}(\tau_s^{w(t)} - \tau_{s-}^{w(t)})$$

with the highest possible value of the jump $(\tau_s^{w(t)} - \tau_{s-}^{w(t)})$ being $p^*(t)w(t)$. Therefore, we get

$$\begin{aligned}
&\mathbb{P}(\mathcal{O}_t^{w(t)}; \Delta_1^{\varepsilon w(t)} \le t; \Delta_2^{\varepsilon w(t)} > t; \Delta_1^{p^*(t)w(t)} > t) \\
&\quad \le \int_0^t \mathbb{P}(\mathcal{O}_s^{\varepsilon w(t)}; \tau_t^{\varepsilon w(t)} + p^*(t)w(t) > w(t))\mathbb{P}(\Delta_1^{\varepsilon w(t)} \in ds) \\
&\quad \le \frac{C(A,\varepsilon)}{\sqrt{g(t)}} \int_0^t \mathbb{P}(\mathcal{O}_s^{\varepsilon w(t)}; \tau_t^{\varepsilon w(t)} > p(t)w(t))\,ds \\
&\quad \le \frac{C(A,\varepsilon)}{\sqrt{g(t)}} \int_0^t \mathbb{P}\biggl(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} \le \frac{p(t)w(t)}{2}; \tau_t^{\varepsilon w(t)} > p(t)w(t)\biggr)\,ds \\
&\quad\quad + \frac{C(A,\varepsilon)}{\sqrt{g(t)}} \int_0^t \mathbb{P}\biggl(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}\biggr)\,ds \\
&\quad = S(t) + S^*(t).
\end{aligned}$$

Let us first estimate $S(t)$ to see that its implicit dependence on y is irrelevant. We note that

$$\begin{aligned}
&\biggl\{\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} \le \frac{p(t)w(t)}{2}; \tau_t^{\varepsilon w(t)} > p(t)w(t)\biggr\} \\
&\qquad \subseteq \biggl\{\mathcal{O}_s^{\varepsilon w(t)}; \tau_t^{\varepsilon w(t)} - \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}\biggr\}.
\end{aligned}$$

Clearly, from the fact that $\tau_t^{\varepsilon w(t)} - \tau_s^{\varepsilon w(t)}$ is independent of $\mathcal{O}_s^{\varepsilon w(t)}$, $\tau_t^{\varepsilon w(t)} \le \tau_t$ and $\tau, \tau^{\varepsilon w(t)}$ are a.s. increasing, we are able to imply that

$$S(t) \le C(A,\varepsilon)\frac{\mathbb{P}(\tau_t^{\varepsilon w(t)} > p(t)w(t)/2)}{\sqrt{g(t)}}\int_0^t \mathbb{P}(\mathcal{O}_s^{w(t)})\,ds$$
$$\le C(A,\varepsilon)\frac{\mathbb{P}(\tau_t > p(t)w(t)/2)}{\sqrt{g(t)}}\int_0^t \mathbb{P}(\mathcal{O}_s^{w(t)})\,ds.$$

Since τ_t is stable with index $1/2$ then (6.5) applies and yields

$$\mathbb{P}\left(\tau_t > \frac{p(t)w(t)}{2}\right) = \mathbb{P}\left(\tau_1 > \frac{p(t)w(t)}{2t^2}\right) \le \sqrt{\frac{2}{\pi}}\frac{t}{\sqrt{p(t)w(t)}}.$$

This, together with the definition of $p(t) = \ln^{-\gamma}(t)$, the employment of (6.13) to compare uniformly $w(t)$ to $g(t)$ from below and the recurring $\liminf_{t\to\infty} g(t)/t^2\ln^{8/5+\varepsilon}(t) = \infty$, give

$$\begin{aligned}S(t) &\le \frac{D(A,\varepsilon)t}{g(t)\sqrt{p(t)}}\int_0^t \mathbb{P}(\mathcal{O}_s^{w(t)})\,ds\\ &\le D(A,\varepsilon)\frac{t\ln^{\gamma/2}(t)}{\sqrt{g(t)}}\frac{\Phi_y^h(t)}{\sqrt{g(t)}} \le \frac{D(A,\varepsilon)}{t\ln^{8/5+\varepsilon-\gamma/2}(t)}\Phi_y^h(t).\end{aligned} \tag{6.21}$$

Let us next estimate $S^*(t)$. Denote $\delta = 1+\gamma \in (1, \frac{8}{5}+\varepsilon)$ and recall that by definition $w_\delta(t) = w(t)/\ln^\delta(t)$. Define as always $\Delta_1^{w_\delta(t)}$ the time of the first jump exceeding $w_\delta(t)$ and note that its density can be estimated as in similar cases before with the help of (6.13) by

$$\mathbb{P}(\Delta_1^{w_\delta(t)} \in ds) \le \frac{C(A)\ln^{\delta/2}(t)}{\sqrt{g(t)}}\,ds.$$

We write the integrand of $S^*(t)$ as follows:

$$\begin{aligned}\mathbb{P}\left(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}\right) &= \mathbb{P}\left(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} \le s\right)\\ &\quad + \mathbb{P}\left(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} > s\right).\end{aligned}$$

Then for the first we get

$$\begin{aligned}S_1^*(t) &:= \frac{C(A,\varepsilon)}{\sqrt{g(t)}}\int_0^t \mathbb{P}\left(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} \le s\right) ds\\ &= \frac{C(A,\varepsilon)}{\sqrt{g(t)}}\int_0^t\int_0^s \mathbb{P}\left(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} \in dv\right) ds\\ &\le \frac{C(A,\varepsilon)}{\sqrt{g(t)}}\int_0^t\int_0^s \mathbb{P}(\mathcal{O}_v^{\varepsilon w(t)})\mathbb{P}(\Delta_1^{w_\delta(t)} \in dv)\,ds\end{aligned} \tag{6.22}$$

$$\leq \frac{C(A,\varepsilon)t\ln^{\delta/2}(t)}{g(t)}\Phi_y^h(t) \leq \frac{C(A,\varepsilon)t\ln^{\delta/2}(t)}{\sqrt{g(t)}}\frac{\Phi_y^h(t)}{\sqrt{g(t)}}$$
$$\leq \frac{C'(A,\varepsilon)}{t\ln^{11/10+\varepsilon-\gamma/2}(t)}\Phi_y^h(t),$$

where in the first inequality we have estimated as measures

$$\mathbb{P}\biggl(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} \in dv\biggr)$$
$$\leq \mathbb{P}\bigl(\mathcal{O}_v^{\varepsilon w(t)}; \Delta_1^{w_\delta(t)} \in dv\bigr) \leq \mathbb{P}\bigl(\mathcal{O}_v^{\varepsilon w(t)}\bigr)\mathbb{P}\bigl(\Delta_1^{w_\delta(t)} \in dv\bigr).$$

For the second integrand we simply estimate in the following generous manner truncating all events and putting the largest values at the point t, that is, $\tau_t^{w_\delta(t)}$:

$$\mathbb{P}\biggl(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} > s\biggr)$$
$$= \mathbb{P}\biggl(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}\bigg|\Delta_1^{w_\delta(t)} > s\biggr)\mathbb{P}\bigl(\Delta_1^{w_\delta(t)} > s\bigr)$$
$$\leq \mathbb{P}\biggl(\tau_t^{w_\delta(t)} > \frac{p(t)w(t)}{2}\biggr).$$

Using the exponential Markov inequality with $\lambda = 2nw_\delta^{-1}(t)$, the last expression for the Lévy–Khintchine exponent of $\tau^{w_\delta(t)}$ in (6.1), $p(t) = \ln^{-\gamma}(t)$ and $\delta = 1+\gamma$ we get that

$$\mathbb{P}\biggl(\tau_t^{w_\delta(t)} > \frac{p(t)w(t)}{2}\biggr) \leq e^{-\lambda\frac{p(t)w(t)}{2}}\mathbb{E}\bigl[e^{\lambda\tau_t^{w_\delta(t)}}\bigr]$$
$$= e^{tK\sqrt{2n}\frac{\ln^{\delta/2}(t)}{\sqrt{w(t)}}\int_0^{2n}(e^{s\frac{2n}{w_\delta(t)}}-1)\frac{ds}{s^{3/2}}}e^{-2n\frac{p(t)w(t)}{2w_\delta(t)}}$$
$$= e^{tK\sqrt{2n}\frac{\ln^{\delta/2}(t)}{\sqrt{w(t)}}\int_0^{2n}(e^{s\frac{2n}{w_\delta(t)}}-1)\frac{ds}{s^{3/2}}}e^{-n\ln(t)}$$
$$\leq e^{C(A)K\sqrt{n}\frac{1}{\ln^{3/10-\gamma/2+\varepsilon/2}(t)}\int_0^{2n}(e^{s\frac{2n}{w_\delta(t)}}-1)\frac{ds}{s^{3/2}}}e^{-n\ln(t)},$$

where for the exponent of the first factor in the last inequality we have used (6.13) and $\liminf_{t\to\infty} g(t)/t^2\ln^{8/5+\varepsilon}(t) = \infty$. Therefore,

$$S_2^*(t) = \frac{C(A,\varepsilon)}{\sqrt{g(t)}}\int_0^t \mathbb{P}\biggl(\mathcal{O}_s^{\varepsilon w(t)}; \tau_s^{\varepsilon w(t)} > \frac{p(t)w(t)}{2}; \Delta_1^{w_\delta(t)} > s\biggr)ds$$
$$\leq t\frac{C(A,\varepsilon)}{\sqrt{g(t)}}e^{C(A)K\sqrt{n}\frac{1}{\ln^{3/10-\gamma/2+\varepsilon/2}(t)}}e^{-n\ln(t)}\frac{\Phi_y^h(t)}{f(y)-h} \tag{6.23}$$
$$\leq \frac{C(A,\varepsilon)}{\sqrt{t^{n-1}g(t)}}\frac{\Phi_y^h(t)}{f(y)-h},$$

provided $\gamma = 1/5$ as then the positive exponent is bounded since the inequality $\frac{3}{10} - \frac{\gamma}{2} + \frac{\varepsilon}{2} > 0$ holds and $\frac{n}{w_\delta(t)}$ is bounded for $t > t(A)$. The appearance of the factor $\frac{\Phi_y^h(t)}{f(y)-h}$ follows from inequality (6.14).

We collect all terms in (6.22), (6.23), (6.21), (6.20), (6.19) updating for $\gamma = \frac{1}{5}$ and choosing $n = 7$ to get

$$\mathbb{P}\big(\mathcal{O}_t^{w(t)}; \Delta_1^{\varepsilon w(t)} \le t\big) \le \frac{C(A, \varepsilon, 7, 1/5)}{t \ln^{1+\varepsilon/2}(t)} \Phi_y^h(t) + \frac{1}{t^4} \frac{\Phi_y^h(t)}{f(y) - h}. \tag{6.24}$$

We note the worst logarithmic bound comes from (6.19).

We are ready now to conclude the proof. Indeed, note that with $n = 7$ all bounds in (6.16), (6.17) and (6.15) are of at most the same or faster decay as (6.24) thanks to $g(t)/t \to \infty, t \to \infty$. Therefore, we conclude that uniformly, for $t > f(Ay) \vee t(A)$, $y > g(h) \vee 0$, we have that

$$\mathbb{P}\big(\mathcal{O}_t^{w(t)}\big) \le \frac{C(A, \varepsilon, 7, 1/5)}{t \ln^{1+\varepsilon/2}(t)} \Phi_y^h(t) + \frac{1}{t^4} \frac{\Phi_y^h(t)}{f(y) - h}. \tag{6.25}$$

Using (6.12), we conclude that

$$H_y^h(t) \le C\left(A, \varepsilon, 7, \frac{1}{5}\right)\left(\frac{1}{t \ln^{1+\varepsilon/2}(t)} + \frac{1}{t^4(f(y) - h)}\right)\Phi_y^h(t).$$

We thus conclude our proof of (6.10). To prove (6.11), we note that all estimates above, which contain $g^{n-1}(t) \gtrsim t^{n-1}$ with $n = 7$, that is (6.15) and (6.16), can be uniformly majorized by $u_1(t)/(f(y) - h)\sqrt{g(t)}$ with $u_1(t) \to 0, t \to \infty$. For the other estimates (6.17), (6.19), (6.20), (6.22) and (6.21) choosing the worst estimates we get that they do not exceed with $\gamma = \frac{1}{5}$

$$\begin{aligned}
&\left(\left(\frac{t \ln^{3/5}(t)}{\sqrt{g(t)}}\right) \vee \left(\frac{1}{\ln^{1/5}(t)}\right) \vee \left(\frac{t}{\sqrt{g(t)}}\right) \vee \left(\frac{t^3 \ln^2(t)}{g^{3/2}(t)}\right)\right) \frac{\Phi_y^h(t)}{\sqrt{g(t)}} \\
&\quad = u_2(t) \frac{\Phi_y^h(t)}{\sqrt{g(t)}}.
\end{aligned}$$

Therefore, from (4.1) we get $u_2(t) \to 0, t \to \infty$. Henceforth

$$\rho_y^h(t) \le C(A)\left(\frac{u_1(t)}{(f(y) - h)\sqrt{g(t)}} + \frac{u_2(t)}{\sqrt{g(t)}}\right),$$

which settles the last claim. We could easily observe that in each bound we obtained along the way we estimated $w(t) \ge B(A)g(t)$ in the denominator and then it easily follows that (6.11) holds with $w = g_{y,h}$ for g. □

The next lemma is auxiliary and is used throughout the proof above.

LEMMA 3. *Let $a > 0$, then we have that with $a_\delta = a/\ln^\delta(a)$ and $\delta > 0$ for any $t > 0$, $c > 0$ and $n \in \mathbb{N}^+$*

$$\mathbb{P}\big(\tau_t^{a_\delta} > ca\big) \le e^{\big(\frac{tK\sqrt{n}\ln^{\delta/2}(a)}{\sqrt{ca}}\int_0^{n/c}(e^s-1)\frac{ds}{s^{3/2}}\big)} e^{-n\ln^\delta(a)}. \tag{6.26}$$

PROOF. This is a simple proof using the Markov inequality together with $\Pi(ds) = K\frac{ds}{s^{3/2}}$, (6.1) and a choice of $\lambda = \frac{n}{c}a_\delta^{-1}$. $\square$

7. Proofs for Section 3.

PROOF OF THEOREM 1. Since $\mathbb{E}[f(\tau_1)] < \infty$, we have thanks to (2.6) that $J(g) < \infty$ and hence according to Lemma 1 that $\Phi(\infty) < \infty$. Therefore, the clocks $\mathfrak{C}_t$ defined in (3.1) converge in distribution to $\mathfrak{C}$.

Next, we show that under any possible limit of $\mathbb{P}_t$, say $\mathbb{Q}$, the inverse local time at zero $\tau = \{\tau_s\}_{s\ge 0}$ satisfies the relation $\mathbb{Q}(\tau_x < \infty) = \mathbb{P}(\mathfrak{C} > x)$ and $\mathbb{Q}(\tau_x \in dy; \mathcal{B}) = \frac{\Phi_y^x(\infty)}{\Phi(\infty)}\mathbb{P}(\tau_x \in dy; \mathcal{B})$, for any $\mathcal{B} \subseteq \mathcal{O}_x, \mathcal{B} \in \mathcal{F}_x$. Thus, the possible limit of τ under $\mathbb{P}_t$ is unique. Note that, for any $x > 0$, $t > x$, $y > g(x) \vee 0$ and $\mathcal{B} \subseteq \mathcal{O}_x, \mathcal{B} \in \mathcal{F}_x$,

$$\mathbb{P}(\tau_x \in dy; \mathcal{B}|\mathcal{O}_t) = \frac{\mathbb{P}(\mathcal{O}_{t-x}(x, y))}{\mathbb{P}(\mathcal{O}_t)}\mathbb{P}(\tau_x \in dy; \mathcal{B}),$$

where $\mathcal{O}_t(x, y) = \{\tau_s > g(s+x) - y, \forall s \le t\}$. Clearly, from Theorem 13, we have that with $\Phi_y^x(t) = \int_0^t \mathbb{P}(\mathcal{O}_s(x, y))\, ds$ and fixed $x > 0$, $y > g(x) \vee 0$,

$$\lim_{t\to\infty}\frac{\mathbb{P}(\mathcal{O}_{t-x}(x, y))}{\mathbb{P}(\mathcal{O}_t)} = \frac{\Phi_y^x(\infty)}{\Phi(\infty)}\lim_{t\to\infty}\frac{\sqrt{g(t)}}{\sqrt{g(t-x+x)-y}} = \frac{\Phi_y^x(\infty)}{\Phi(\infty)}.$$

The fact that $\Phi_y^x(\infty) < \infty$ follows from Theorem 13 since $g_{y,x}(s) = g(s+x) - y$ satisfies (5.5) and it is integrable at infinity as function of s since $g(s)$ has this property. This shows the convergence in law of τ under $\mathbb{P}_t$ to a unique limit and demonstrates that

$$\mathbb{Q}(\tau_x \in dy) = \frac{\Phi_y^x(\infty)}{\Phi(\infty)}\mathbb{P}(\tau_x \in dy; \mathcal{O}_x),$$

which proves (3.2). Then (3.3) follows immediately. Since $\mathbb{P}(\mathcal{O}_t(x, y))$ is non-decreasing in y, for any $g(x) \vee 0 < y < B$ with $B > 0$ some constant, we can use the dominated convergence theorem to get, invoking the definition of Φ_y^x,

$$\begin{aligned}\mathbb{Q}\big(\tau_x \in \big(g(x), B\big)\big) &= \lim_{t\to\infty}\int_{y=g(x)}^{B}\mathbb{P}(\tau_x \in dy|\mathcal{O}_t)\\ &= \lim_{t\to\infty}\int_{y=g(x)}^{B}\frac{\mathbb{P}(\mathcal{O}_{t-x}(x, y))}{\mathbb{P}(\mathcal{O}_t)}\mathbb{P}(\tau_x \in dy; \mathcal{O}_x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Phi(\infty)} \int_{y=g(x)}^{B} \Phi_y^x(\infty)\mathbb{P}(\tau_x \in dy; \mathcal{O}_x) \\
&= \frac{1}{\Phi(\infty)} \int_{y=g(x)}^{B} \int_0^\infty \mathbb{P}\big(\mathcal{O}_v(x,y)\big)\, dv \mathbb{P}(\tau_x \in dy; \mathcal{O}_x) \\
&= \frac{\int_x^\infty \mathbb{P}(\mathcal{O}_u; \tau_x \in (g(x), B))\, du}{\Phi(\infty)}.
\end{aligned}
$$

Using the monotone convergence theorem, we get that

$$
\begin{aligned}
\mathbb{Q}(\tau_x < \infty) &= \lim_{B\to\infty} \frac{\int_x^\infty \mathbb{P}(\mathcal{O}_u; \tau_x \in (g(x), B))\, du}{\Phi(\infty)} \\
&= \frac{\int_x^\infty \mathbb{P}(\mathcal{O}_u)\, du}{\Phi(\infty)} = \mathbb{P}(\mathfrak{C} > x).
\end{aligned}
$$

Since τ is a.s. nondecreasing we conclude that under $\mathbb{Q}$ there is a random explosion time $\mathcal{T}$ for τ such that $\tau_s = \infty$, $s \geq \mathcal{T}$ and $\mathcal{T} \stackrel{d}{=} \mathfrak{C}$.

Recall that $\Delta_1^{g(t)} = \{s > 0 : \tau_s - \tau_{s-} > g(t)\}$ and that $\Delta_1^{g(t)}$ is exponentially distributed with parameter $2K/\sqrt{g(t)}$. We prove that the random elements $Z^t = \{\tau, \Delta_1^{g(t)}\} \in \mathcal{D}(0,\infty) \times \mathbb{R}^+$ converge under $\mathbb{P}_t$ to a random element Z and we specify the structure of Z under $\mathbb{Q}$. Let $x > 0$, $y > g(x)$, $t > b > a > x$ with a, b, x, y fixed. Let also $\mathcal{B} \subseteq \mathcal{O}_x$, $\mathcal{B} \in \mathcal{F}_x$. Then

$$
\begin{aligned}
J_t :&= \mathbb{P}\big(\tau_x \in dy; \mathcal{B}; \Delta_1^{g(t)} \in (a,b); \mathcal{O}_t\big) \\
&= \int_a^b \mathbb{P}\big(\Delta_1^{g(t)} \in ds; \tau_x \in dy; \mathcal{B}; \mathcal{O}_x\big) \\
&= \int_a^b \mathbb{P}\big(\Delta_1^{g(t)} \in ds; \tau_x \in dy; \mathcal{B}; \mathcal{O}_{\Delta_1^{g(t)}}\big) \\
&= \int_a^b \mathbb{P}\big(\tau_x^{g(t)} \in dy; \mathcal{B}; \mathcal{O}_s^{g(t)}\big)\mathbb{P}\big(\Delta_1^{g(t)} \in ds\big).
\end{aligned}
$$

Setting $t \to \infty$, we then get

$$
\begin{aligned}
J_t &\sim \frac{2K}{\sqrt{g(t)}} \int_a^b \mathbb{P}\big(\tau_x^{g(t)} \in dy; \mathcal{B}; \mathcal{O}_s^{g(t)}\big)\, ds \\
&\sim \frac{\mathbb{P}(\mathcal{O}_t)}{\Phi(\infty)} \int_a^b \mathbb{P}\big(\tau_x^{g(t)} \in dy; \mathcal{B}; \mathcal{O}_s^{g(t)}\big)\, ds \\
&= \frac{\mathbb{P}(\mathcal{O}_t)}{\Phi(\infty)} \int_a^b \mathbb{P}\big(\tau_x^{g(t)} \in dy; \mathcal{B} | \mathcal{O}_s^{g(t)}\big)\mathbb{P}\big(\mathcal{O}_s^{g(t)}\big)\, ds.
\end{aligned}
$$

For the preceding chain of relations, we have used that since $\Delta_1^{g(t)}$ is exponentially distributed with parameter $\frac{2K}{\sqrt{g(t)}}$ then, for $s \in [a,b]$, we have uniformly

$$\mathbb{P}\big(\Delta_1^{g(t)} \in ds\big) = \frac{2K}{\sqrt{g(t)}} e^{-2Ks/\sqrt{g(t)}}\, ds \sim \frac{2K}{\sqrt{g(t)}}\, ds$$

and then thanks to (5.7) of Theorem 13 that $\frac{2K}{\sqrt{g(t)}} \sim \frac{\mathbb{P}(\mathcal{O}_t)}{\Phi(\infty)}$. Furthermore, since by definition $\frac{\mathbb{P}(\mathcal{O}_s)}{\Phi(\infty)}\, ds = \mathbb{P}(\mathfrak{C} \in ds)$, see (3.1) in the limit, we continue the relations

$$\begin{aligned} J_t &= \mathbb{P}\big(\tau_x \in dy; \mathcal{B}; \Delta_1^{g(t)} \in (a,b); \mathcal{O}_t\big) \\ &\sim \mathbb{P}(\mathcal{O}_t) \int_a^b \mathbb{P}\big(\tau_x^{g(t)} \in dy; \mathcal{B} | \mathcal{O}_s^{g(t)}\big) \frac{\mathbb{P}(\mathcal{O}_s)}{\Phi(\infty)}\, ds \\ &= \mathbb{P}(\mathcal{O}_t) \int_a^b \mathbb{P}\big(\tau_x^{g(t)} \in dy; \mathcal{B} | \mathcal{O}_s^{g(t)}\big) \mathbb{P}(\mathfrak{C} \in ds) \\ &\sim \mathbb{P}(\mathcal{O}_t) \int_a^b \mathbb{P}(\tau_x \in dy; \mathcal{B} | \mathcal{O}_s) \mathbb{P}(\mathfrak{C} \in ds). \end{aligned}$$

Conditioning on $\mathcal{O}_t$, we then get that $Z^t = \{\tau, \Delta_1^{g(t)}\}$ converges under $\mathbb{P}_t$ to $\{\{\tau_s\}_{s \le \mathcal{T}}, \mathcal{T}\}$, which under $\mathbb{Q}$, has the law

$$\begin{aligned} \mathbb{Q}\big(\tau \in \mathcal{B}; \mathcal{T} \in (a,b)\big) &= \int_a^b \mathbb{Q}(\tau \in \mathcal{B} | \mathcal{T} = s) \mathbb{Q}(\mathcal{T} \in ds) \\ &= \int_a^b \mathbb{P}(\tau \in \mathcal{B} | \mathcal{O}_s) \mathbb{P}(\mathfrak{C} \in ds), \end{aligned} \tag{7.1}$$

for all $\mathcal{B} \in \mathcal{F}_a$ and $b > a > 0$. Clearly, since $\tau_{\Delta_1^{g(t)}} \ge \tau_{\Delta_1^{g(t)}} - \tau_{\Delta_1^{g(t)}-} \ge g(t)$, for the limit process we have that $\tau_s = \infty$, $s \ge \mathcal{T}$, and weakly on $\mathbb{R}^+$,

$$\lim_{t \to \infty} \mathbb{P}_t\big(\Delta_1^{g(t)} \in ds\big) = \mathbb{P}(\mathfrak{C} \in ds) = \mathbb{P}(\mathcal{T} \in ds).$$

Next, we consider the original Brownian motion B. To show that B converges under $\mathbb{P}_t$ to the process specified in the theorem, we shall rely on the so-called Ito's representation of the Brownian motion via its excursions away from zero. This representation is well developed and explained in the proof of Theorem 2 in [2] and we shall be brief on some details. Let us introduce the space of excursions of the Brownian motion away from zero, which we denote by E. Put $\mathcal{C} := \mathcal{C}(0,\infty)$. Let $x \ge 0$ and consider the set of functions

$$\mathrm{E}_x = \big\{\varepsilon \in \mathcal{C}; \varepsilon(0) = \varepsilon(s) = 0, \forall s \ge x\big\} \cap \big\{\varepsilon \in \mathcal{C}; \big|\varepsilon(s)\big| > 0, \forall s \in (0,x)\big\}.$$

Then $\mathrm{E} = \bigcup_{x>0} \mathrm{E}_x$ and $\mathrm{E}_x \cap \mathrm{E}_y = \varnothing$, for all $x > 0$, $y > 0$, such that $x \ne y$. The functional $\zeta : \mathrm{E} \to \mathbb{R}^+$ defined by $\zeta(\varepsilon) = x \iff \varepsilon \in \mathrm{E}_x$ is called the life-time of the excursion ε. Next, recall that $\tau = \{\tau_s\}_{s\geq 0}$ is the inverse local time of the Brownian motion and note that $\tau_t = \sum_{s\leq t}(\tau_s - \tau_{s-}), \forall t \geq 0$. We define the process $U := \{(\Delta_s, \varepsilon_s)\}_{s\geq 0}$, where $\Delta_s = \tau_s - \tau_{s-}$ encodes the jump of τ at time s and conditionally on $\{\Delta_s = x\}, x \geq 0$, ε_s is sampled in two different ways from E_x depending solely on whether $x > 0$ or not. When $x > 0$ this is done according to the measure of the Brownian meander of length x which is identified with the Brownian bridge of length x conditioned not to cross zero; see [6]. The latter is either positive or negative with equal probability. When $x = 0$, we sample ε_s from E_0 which consists only of the function that is identically zero. The definition and the construction of U is a mere reflection of the classical fact that $\{(s, \Delta_s, \varepsilon_s)\}_{s\geq 0}$ is a Poisson point process on $[0,\infty) \times \mathbb{R}^+ \times E$. The first passage time process of τ across all levels $t > 0$ coincides with the local time at zero of the original Brownian motion B, namely $\{L_t\}_{t\geq 0}$. Then $\mathrm{V} = \{(\tau_s, \varepsilon_s)\}_{s\geq 0}$ induces a standard Brownian motion via the definition $B'_u = \varepsilon_{\tau_{L(u-)}}(u - \tau_{L(u-)}), u \geq 0$, that is, $B' \stackrel{d}{=} B$. Conversely, decomposing the path of B into excursions away from zero via the Ito's excursion representation we can directly obtain V. Recall that $\Delta_1^{g(t)} = \inf\{s > 0 : \tau_s - \tau_{s-} > g(t)\}$. Consider the stopped process $\mathrm{V}^t = \{(\tau_s, \varepsilon_s)\}_{s<\Delta_1^{g(t)}}$ and the extended process $\mathfrak{V}^t = (\mathrm{V}^t, \Delta_1^{g(t)}, \varepsilon_{\Delta_1^{g(t)}})$. Note that from $\mathfrak{V}^t$ we can construct the Brownian motion until and including the first excursion away from zero of life-time longer than $g(t)$ and vice versa. We shall show that under $\mathbb{P}_t$ both V and $\mathfrak{V}_t$ have the same limit which coincides with the explicit process of the theorem. We start by considering $\mathfrak{V}_t$ which takes values in $\mathcal{D}(0,\infty) \times \mathrm{E}^\infty \times \mathbb{R} \times \mathrm{E}$. For each $h > 0$, we introduce the time truncation operator which applied on any process $X = \{X\}_{s\geq 0}$ yields $\pi_h(X) = \{X_s\}_{s\leq h}$. Fix the numbers $h > 0, b > a > h$ and the set $\mathcal{B} \subset \mathcal{F}_h$. Furthermore, we fix the bounded continuous functional $F_1 : \mathrm{E}^\infty \to \mathbb{R}$ which depends only on excursions up to time h, that is $F_1(\varepsilon) = F_1(\pi_h(\varepsilon)), \forall \varepsilon \in \mathrm{E}$, and the bounded continuous functional $F_2 : \mathrm{E} \to \mathbb{R}$. Since $1_{\mathcal{O}_t}$ is a functional of $\pi_t(\tau)$ only and conditionally on $\{\pi_h(\tau) = \vartheta, \Delta_1^{g(t)} = u > h\}$, where $\vartheta \in \mathcal{D}(0,h)$, the excursion process $\pi_h(\varepsilon) = \{\varepsilon_s\}_{s\leq h}$ forms an independent sampling of Brownian meanders with given lengths $(\vartheta(s) - \vartheta(s-))_{s\leq h}$we evaluate

$$
\begin{aligned}
&\mathbb{E}^t\big[1_{\pi_h(\tau)\in\mathcal{B}} 1_{\Delta_1^{g(t)}\in(a,b)} F_1\big(\pi_h(\varepsilon)\big) F_2(\varepsilon_{\Delta_1^{g(t)}})\big] \\
&\quad = \int_{\mathcal{B}} \int_a^b \mathbb{E}\big[F_1\big(\pi_h(\varepsilon)\big) F_2(\varepsilon_u 1_{\zeta(\varepsilon_u)>g(t)}) | \pi_h(\tau) = \vartheta; \Delta_1^{g(t)} = u\big] \\
&\qquad \times \frac{\mathbb{P}(\pi_h(\tau) \in d\vartheta, \Delta_1^{g(t)} \in du; \mathcal{O}_t)}{\mathbb{P}(\mathcal{O}_t)}.
\end{aligned}
$$

However, we have also that conditionally on $\{\pi_h(\tau) = \vartheta; \Delta_1^{g(t)} = u\}$ the law of ε_u is independent of $\pi_h(\varepsilon)$ and equals in law the law of a Brownian excursion conditioned on its life-time being longer than $g(t)$, say $\varepsilon_u^{g(t)}$, and $\pi_h(\varepsilon)$

equals in law $\pi_h(\varepsilon^t)$, where ε^t is an excursion process consisting of excursions whose individual life-times do not exceed $g(t)$. Therefore, we further compute that

$$\begin{aligned}
&\mathbb{E}^t\big[1_{\pi_h(\tau)\in\mathcal{B}}1_{\Delta_1^{g(t)}\in(a,b)}F_1\big(\pi_h(\varepsilon)\big)F_2(\varepsilon_{\Delta_1^{g(t)}})\big]\\
&\qquad=\int_{\mathcal{B}}\int_a^b\mathbb{E}_{\vartheta,u}\big[F_1\big(\pi_h(\varepsilon^t)\big)\big]\mathbb{E}\big[F_2\big(\varepsilon_u^{g(t)}\big)\big]\frac{\mathbb{P}(\pi_h(\tau)\in d\vartheta,\Delta_1^{g(t)}\in du;\mathcal{O}_t)}{\mathbb{P}(\mathcal{O}_t)}.
\end{aligned}$$

By $\mathbb{E}_{\vartheta,u}$ we understand the expectation under sampling of Brownian meanders given the position of their arrival (the start of the excursion of the Brownian motion away form zero) and their length. The arrivals are encoded in the points of jumps of ϑ, that is $\{s<u:\vartheta(s)-\vartheta(s-)>0\}$, whereas the lengths are represented by the size of the jumps themselves. It is proved in [2], proof of Theorem 2, that $\varepsilon^{g(t)}$ converges, as $t\to\infty$, to a three-dimensional Bessel process with random sign, denoted here by ε^∞ and hence

$$\lim_{t\to\infty}\mathbb{E}\big[F_2\big(\varepsilon_u^{g(t)}\big)\big]=\mathbb{E}\big[F_2\big(\varepsilon^\infty\big)\big].$$

Also, clearly, since $h>0$ is fixed

$$\lim_{t\to\infty}\mathbb{E}_{\vartheta,u}\big[F_1\big(\pi_h(\varepsilon^t)\big)\big]=\mathbb{E}_{\vartheta,u}\big[F_1\big(\pi_h(\varepsilon)\big)\big].$$

Therefore, since the measure of integration converges, as $t\to\infty$, we conclude that

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}^t\big[1_{\pi_h(\tau)\in\mathcal{B}}1_{\Delta_1^{g(t)}\in(a,b)}F\big(\pi_h(\varepsilon);\varepsilon_{\Delta_1^{g(t)}}\big)\big]\\
&\qquad=\int_{\mathcal{B}}\int_a^b\mathbb{E}_{\vartheta,u}\big[F_1\big(\pi_h(\varepsilon)\big)\big]\mathbb{E}\big[F_2\big(\varepsilon_u^\infty\big)\big]\mathbb{Q}\big(\pi_h(\tau)\in d\vartheta,\mathcal{T}\in du\big).
\end{aligned}$$

As a consequence one obtains that, conditionally on $\{\pi_h(\tau)=\vartheta;\Delta_1^{g(t)}=u\}$, ε^∞ is independent of $\pi_h(\varepsilon)$. Given the description of the joint law $(\pi_{\mathcal{T}}(\tau),\mathcal{T})$ under $\mathbb{Q}$, derived in equation (7.1) above, we conclude that the construction $B'_u=\varepsilon_{\tau_{L(u-)}}(u-\tau_{L(u-)}),\tau_{\mathcal{T}-}\geq u\geq 0$, under $\mathbb{Q}$ is the Brownian motion with its inverse local time running up to the time of the clock $\mathfrak{C}$ conditioned on $\{\tau_s>g(s),s\leq\mathfrak{C}\}$. The process ε^∞ is an independent Bessel three process with a random sign. Splicing ε^∞ at time $\tau_{\mathfrak{C}-}$ gives the process of the theorem. The uniqueness follows from the uniqueness of the law of $(\pi_{\mathcal{T}}(\tau),\mathcal{T})$ and the independence of the limit, as $t\to\infty$, of $\mathbb{E}_{\vartheta,u}(\cdot)$ above. Thus under $\mathbb{P}_t$, $\mathfrak{V}_t$ converges to the process defined in the theorem. Let us show that under $\mathbb{P}_t$, V converges to the same process. This follows easily from above since

$$\lim_{b\to\infty}\lim_{t\to\infty}\mathbb{P}\big(\Delta_1^{g(t)}>b|\mathcal{O}_t\big)=\lim_{b\to\infty}\frac{\int_b^\infty\mathbb{P}(\mathcal{O}_s)\,du}{\Phi(\infty)}=0$$

and since $\tau_{\Delta_1^{g(t)}} \geq g(t) \uparrow \infty, t \to \infty$. This completes the proof of the theorem as the construction of the Brownian motion from V converges thus to the process of the theorem. □

The proof of Corollary 1 is immediate from Theorem 1.

8. Proofs for Section 4. We prove Theorem 3 in several steps. First, we show that $\mathbb{P}(\tau \in \cdot|\mathcal{O}_t) \to \mathbb{Q}(\cdot)$ and we describe the law of τ under $\mathbb{Q}$. Recall that $\Phi_y^h(t) = \int_0^t \mathbb{P}(\tau_s > g(s+h) - y, s \leq v)\, dv$, see the introduced notation around (6.7). Then the following claim holds.

PROPOSITION 1. *Let f satisfy the usual conditions given in Definition 2.1, $I(f) = \infty$ and (4.1). Then, under $\mathbb{P}_t$ the inverse local time converges, as $t \to \infty$, to an increasing pure-jump process with law $\mathbb{Q}$ which we call the inverse local time under $\mathbb{Q}$. Fix $h > 0$. The measure $\mathbb{Q}(\tau_h \in dy)$ is absolutely continuous with respect to the measure $\mathbb{P}(\tau_h \in dy, \mathcal{O}_h)$ with density denoted by $q_h(y)$. Let $A > 3$, $t^*(A) > 0$ such that $g(t^*(A)) = 1 + \frac{2}{A}$ and $t(A) = \max\{t^*(A), e^2\}$. Then*

$$q_h(y) := \frac{\mathbb{Q}(\tau_h \in dy)}{\mathbb{P}(\tau_h \in dy, \mathcal{O}_h)}$$

$$(8.1) \qquad = \frac{\Phi_y^h(f(Ay) \vee t(A))}{\Phi(1)} e^{-\int_1^{f(Ay)\vee t(A)} \frac{2K}{\sqrt{g(s)}}\, ds} \times e^{\int_{f(Ay)\vee t(A)}^\infty (\frac{2K}{\sqrt{g(s+h)-y}} - \frac{2K}{\sqrt{g(s)}})\, ds} e^{-\int_1^\infty \rho(s)\, ds + \int_{f(Ay)\vee t(A)}^\infty \rho_y^h(s)\, ds},$$

where $\rho_y^h(s)$, $\rho(s)$ are defined in (5.3) and (6.9). Furthermore, for any $\mathcal{B} \subset \mathcal{O}_h$ and $\mathcal{B} \in \mathcal{F}_h$, we have that

$$(8.2) \qquad \mathbb{Q}(\mathcal{B}) = \mathbb{E}[q_h(\tau_h); \mathcal{B}].$$

Finally, the function $q_h : (g(h) \vee 0, \infty) \to (0, \infty)$ is nondecreasing for every $h > 0$.

PROOF. Fix $h > 0$ and $\mathcal{B} \subset \mathcal{O}_h, \mathcal{B} \in \mathcal{F}_h$. We write using the Markov property at $t > h$

$$\mathbb{P}(\tau_h \in dy; \mathcal{B}|\mathcal{O}_t) = \frac{\mathbb{P}(\mathcal{O}_t; \tau_h \in dy; \mathcal{B})}{\mathbb{P}(\mathcal{O}_t)} = \frac{\mathbb{P}(\mathcal{O}_{t-h}(h, y))}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(\tau_h \in dy; \mathcal{B}; \mathcal{O}_h),$$

where we recall that $\mathcal{O}_{t-h}(h, y) = \{\tau_s > g_{y,h}(s), s \leq t - h\}$ and $g_{y,h}(t) = g(t + h) - y$. Clearly, for every $t > h$, conditionally on $\mathcal{O}_t$, $y > g(h) \vee 0$. Fix $y \in (g(h) \vee 0, \infty)$. It remains to show the following limit:

$$(8.3) \qquad \lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_{t-h}(h, y))}{\mathbb{P}(\mathcal{O}_t)} = q_h(y).$$

However, using equation (5.8) of Theorem 15 we get that

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \sim \frac{\Phi_y^h(t-h)\sqrt{g(t)}}{\Phi(t)\sqrt{g(t-h+h)-y}} \sim \frac{\Phi_y^h(t-h)}{\Phi(t)}.$$

Next, we employ, for $t > (t(A)+h) \vee f(Ay) > 1$, the expressions of the modified solutions to (6.8):

$$\Phi_y^h(t-h) = \Phi_y^h\big(f(Ay) \vee t(A)\big)e^{\int_{f(Ay)\vee t(A)}^{t-h} \frac{2K}{\sqrt{g(s+h)-y}}\,ds + \int_{f(Ay)\vee t(A)}^{t-h} \rho_y^h(s)\,ds},$$

$$\Phi(t) = \Phi(1)e^{\int_1^{f(Ay)\vee t(A)} \frac{2K}{\sqrt{g(s)}}\,ds + \int_{f(Ay)\vee t(A)}^{t} \frac{2K}{\sqrt{g(s)}}\,ds + \int_1^t \rho(s)\,ds}.$$

Clearly then, on $y > g(h) \vee 0$,

$$\begin{aligned} q_h(y) &= \lim_{t\to\infty} \frac{\Phi_y^h(t-h)}{\Phi(t)} \\ &= \frac{\Phi_y^h(f(Ay)\vee t(A))}{\Phi(1)} e^{-\int_1^{f(Ay)\vee t(A)} \frac{2K}{\sqrt{g(s)}}\,ds} \\ &\quad \times e^{\int_{f(Ay)\vee t(A)}^{\infty}(\frac{2K}{\sqrt{g(s+h)-y}} - \frac{2K}{\sqrt{g(s)}})\,ds} e^{-\int_1^\infty \rho(s)\,ds + \int_{f(Ay)\vee t(A)}^{\infty} \rho_y^h(s)\,ds}, \end{aligned}$$

which suffices to prove the existence of density $q_h(y)$ with respect to the measure $\mathbb{P}(\tau_h \in dy, \mathcal{O}_h)$. Indeed,

$$\int_{f(Ay)\vee t(A)}^{\infty} \left(\frac{2K}{\sqrt{g(s+h)-y}} - \frac{2K}{\sqrt{g(s)}}\right) ds < \infty$$

follows from the axillary Lemma 4 below, whereas the finiteness of the quantity $\int_{f(Ay)\vee t(A)}^{\infty} \rho_y^h(s)\,ds$ follows from the bound (6.10) of Lemma 2 which holds under the assumptions for f and equivalently for $g = f^{-1}$ of Proposition 1. Finally, the fact that $q_h(y)$ is nondecreasing in $y > g(h) \vee 0$ follows from the observation that for $y_2 > y_1 > g(h) \vee 0$ we have that for any $t > h$, $\mathcal{O}_t(h, y_1) \subseteq \mathcal{O}_t(h, y_2)$ since $g_{y_1,h}(t) = g(t+h) - y_1 \ge g_{y_2,h}(t) = g(t+h) - y_2$. $\square$

LEMMA 4. *Let f satisfy condition (2.1), that is, $f(x)/\sqrt{x} \downarrow 0$, as $x \to \infty$. Then we have that, for any $h > 0$, $y > g(h) \vee 0$, $A > 3$,*

$$\begin{aligned} &\int_{f(Ay)\vee t(A)}^{\infty} \left(\frac{1}{\sqrt{g(s+h)-y}} - \frac{1}{\sqrt{g(s)}}\right) ds \\ &\quad \le \int_{f(Ay)\vee t(A)}^{\infty} \left(\frac{1}{\sqrt{g(s)-y}} - \frac{1}{\sqrt{g(s)}}\right) ds < \frac{f(y)}{2\sqrt{y(1-1/A)}}. \end{aligned} \tag{8.4}$$

PROOF. Fix $h > 0$, $y > g(h) \vee 0$. Note that, for $s > f(Ay)$, we have that $g(s) > g(f(Ay)) > Ay > 3y$. Also recall the general inequality $1 - \sqrt{1-x} \le x$, which holds for $x \in (0,1)$. Then we estimate

$$\begin{aligned}
&\int_{f(Ay)\vee t(A)}^{\infty}\biggl(\frac{1}{\sqrt{g(s+h)-y}}-\frac{1}{\sqrt{g(s)}}\biggr)\,ds \\
&\quad\le \int_{f(Ay)\vee t(A)}^{\infty}\biggl(\frac{1}{\sqrt{g(s)-y}}-\frac{1}{\sqrt{g(s)}}\biggr)\,ds \\
&\quad= \int_{f(Ay)\vee t(A)}^{\infty}\frac{1-\sqrt{1-y/g(s)}}{\sqrt{g(s)-y}}\,ds \\
&\quad\le y\int_{f(Ay)\vee t(A)}^{\infty}\frac{1}{g^{3/2}(s)\sqrt{1-y/g(s)}}\,ds \\
&\quad\le \frac{y}{\sqrt{1-A^{-1}}}\int_{f(y)}^{\infty}\frac{ds}{g^{3/2}(s)}=\frac{y}{\sqrt{1-A^{-1}}}\int_{y}^{\infty}\frac{f'(s)}{s^{3/2}}\,ds \\
&\quad= \frac{y}{\sqrt{1-A^{-1}}}\frac{f(s)}{s^{3/2}}\Big|_y^{\infty}+\frac{3y}{2\sqrt{1-A^{-1}}}\int_y^{\infty}\frac{f(s)}{s^{5/2}}\,ds \\
&\quad\le -\frac{f(y)}{\sqrt{y(1-A^{-1})}}+\frac{3f(y)}{2\sqrt{y(1-A^{-1})}},
\end{aligned}$$

where for the last line we have also used that $f(s)/\sqrt{s}$ is decreasing, that is, condition (2.1). $\square$

The proof of Theorem 3 follows several steps. Fix $h>0$ and we will first prove the following result.

LEMMA 5. *Assume that f satisfies the usual conditions given in Definition 2.1, $I(f)=\infty$ and that (4.1) for $g=f^{-1}$ holds, that is, we have that $\liminf_{t\to\infty} g(t)/t^2\ln^{8/5+\varepsilon}(t)=\infty$. Then, for all $h>0$,*

$$\mathbb{Q}\bigl(\tau_h\in\bigl(g(h),\infty\bigr)\bigr)=\int_{g(h)\vee 0}^{\infty}q_h(y)\mathbb{P}(\tau_h\in dy;\mathcal{O}_h)=1. \tag{8.5}$$

PROOF. We have trivially that

$$\begin{aligned}
1&=\lim_{t\to\infty}\mathbb{P}\bigl(\tau_h\in\bigl(g(h),\infty\bigr)\,|\,\mathcal{O}_t\bigr)\\
&=\lim_{t\to\infty}\int_{g(h)\vee 0}^{\infty}\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)}\mathbb{P}(\tau_h\in dy,\mathcal{O}_h).
\end{aligned} \tag{8.6}$$

We aim to show that we can interchange the limit and the integral to obtain (8.5). For this purpose,we choose $A>3$ and split the range of integration in (8.6) in three possibly overlapping sets, that is,

$$\begin{aligned}
&\{y>g(h)\vee 0\}\\
&\quad=\bigl\{y\in\bigl(g(h)\vee 0,g(h+1)\bigr)\bigr\}\cup\biggl\{y\ge\frac{g(t)}{A}-h\biggr\}\cup\biggl\{y<\frac{g(t)}{A}-h\biggr\}.
\end{aligned} \tag{8.7}$$

We put for brevity $I_1 := I_1(A,t) = \{y \geq \frac{g(t)}{A} - h\} = \{t \leq f(Ay+h)\}$ and $I_2 = \{y \in (g(h) \vee 0, g(h+1))\}$. We recall that g is an increasing function with $g(1) = 1$. Therefore, I_2 is not empty. We start by considering the limit in (8.6) on I_2. We know from the proof of Proposition 1 [see (8.3)] that, for $y > g(h) \vee 0$,

$$\lim_{t\to\infty} \frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} = q_h(y).$$

Moreover, we know from there that $\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} : (g(h) \vee 0, \infty) \mapsto (0,\infty)$ and $q_h(y)$ are increasing in y. Henceforth, for $y \in I_2$,

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \leq \frac{\mathbb{P}(\mathcal{O}_{t-h}(h,g(h+1)))}{\mathbb{P}(\mathcal{O}_t)}.$$

The fact that the right-hand side converges to $q_h(g(h+1))$ independently of $y \in I_2$ shows that

$$\lim_{t\to\infty} \int_{I_2} \frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(\tau_h \in dy, \mathcal{O}_h) = \int_{I_2} q_h(y) \mathbb{P}(\tau_h \in dy, \mathcal{O}_h). \tag{8.8}$$

Next, we consider (8.6) on $I_1 = \{t \leq f(Ay+h)\} = \{y \geq \frac{g(t)}{A} - h\}$. Then the trivial asymptotic estimate

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \leq \frac{1}{\mathbb{P}(\mathcal{O}_t)} \sim \frac{\sqrt{g(t)}}{2K\Phi(t)}, \tag{8.9}$$

follows from Theorem 15 and (5.8). However, (8.9) and (6.5) immediately lead to the asymptotic estimate

$$\begin{aligned} &\int_{I_1} \frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(\tau_h \in dy; \mathcal{O}_h) \\ &\quad \lesssim \frac{\sqrt{g(t)}}{\Phi(t)} \mathbb{P}\left(\tau_h \geq \frac{g(t)}{A} - h\right) = \frac{\sqrt{g(t)}}{\Phi(t)} \mathbb{P}\left(\tau_1 \geq \frac{g(t)}{Ah^2} - \frac{1}{h}\right) \\ &\quad \leq \sqrt{\frac{2}{\pi}} \frac{\sqrt{g(t)}}{\Phi(t)\sqrt{g(t)/(Ah^2) - 1/h}} = \frac{\sqrt{\frac{2}{\pi}}}{\sqrt{1/(Ah^2) - 1/(hg(t))}} \frac{1}{\Phi(t)}. \end{aligned} \tag{8.10}$$

However, (4.1) implies (5.5) and since $I(f) = \infty$ by assumption, we get from (5.6) of Lemma 1 that $\Phi(t) \to \Phi(\infty) = \infty$, as $t \to \infty$. Hence, we conclude from (8.10) that

$$\lim_{t\to\infty} \int_{I_1} \frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \mathbb{P}(\tau_h \in dy, \mathcal{O}_h) = 0. \tag{8.11}$$

We turn our attention to the region

$$\begin{aligned} I_1^c \cap I_2^c &= \{t \geq f(Ay+h)\} \cap \{f(y) > h+1\} \\ &= \left\{y \leq \frac{g(t)}{A} - h\right\} \cap \{y > g(h+1)\}. \end{aligned}$$

Next, (6.8) allows us to express

$$\mathbb{P}\big(\mathcal{O}_{t-h}(h,y)\big) = \frac{2K\Phi_y^h(t-h)}{\sqrt{(g(t)-y)\vee 1}} + \Phi_y^h(t-h)\rho_y^h(t-h)$$
$$= \Phi_y^h(t-h)\bigg(\frac{2K}{\sqrt{(g(t)-y)}} + \rho_y^h(t-h)\bigg).$$

Indeed, once $t > t(A)$, on $I_1^c \cap I_2^c$, we have since $g(h+1) > g(1) = 1$ that

$$g(t) - y > g\big(f(Ay+h)\big) - y = (A-1)y + h > (A-1)g(h+1) + h > A-1 > 1.$$

Substituting the expression for $\mathbb{P}(\mathcal{O}_{t-h}(h,y))$ above and using again (5.8) that is, $\mathbb{P}(\mathcal{O}_t) \sim 2K\Phi(t)/\sqrt{g(t)}$, we obtain the inequality for all t big enough

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \le C(A,h)\frac{\Phi_y^h(t-h)}{\Phi(t)}\bigg(\frac{\sqrt{g(t)}}{\sqrt{g(t)-y}} + \frac{\sqrt{g(t)}\rho_y^h(t-h)}{2K}\bigg)$$
$$= C(A,h)\bigg(\frac{\Phi_y^h(t-h)}{\Phi(t)\sqrt{1-y/g(t)}} + \frac{\sqrt{g(t)-y}}{\sqrt{1-y/g(t)}}\frac{H_y^h(t-h)}{2K\Phi(t)}\bigg).$$

Next, $\forall y \in I_1^c \cap I_2^c$ we have that $1 - y/g(t) > 1 - 1/A$. Thus, finally, we arrive with some generic constant $C(A,h) > 0$ at

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \le C(A,h)\frac{\Phi_y^h(t-h)}{\Phi(t)} + C(A,h)\sqrt{g(t)-y}\frac{H_y^h(t-h)}{\Phi(t)}. \tag{8.12}$$

Next, since on $I_1^c \cap I_2^c$, inequality (6.11) holds with $g_{y,h}(t) = g(t+h) - y$ too, we estimate that

$$\frac{\sqrt{g(t)-y}H_y^h(t-h)}{\Phi(t)} = \frac{\sqrt{g(t)-y}\rho_y^h(t-h)\Phi_y^h(t-h)}{\Phi(t)}$$
$$\le \frac{u(t-h)\sqrt{g(t)-y}}{\sqrt{g_{y,h}(t-h)}}\frac{\Phi_y^h(t-h)}{\Phi(t)}\bigg(1 + \frac{1}{f(y)-h}\bigg)$$
$$= u(t-h)\frac{\Phi_y^h(t-h)}{\Phi(t)}\bigg(1 + \frac{1}{f(y)-h}\bigg)$$
$$\le u(t-h)\frac{\Phi_y^h(t-h)}{\Phi(t)}\bigg(1 + \frac{1}{f(g(h+1))-h}\bigg)$$
$$= o(1)\frac{\Phi_y^h(t-h)}{\Phi(t)}.$$

Therefore, (8.12) further reduces for t big enough to

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \le C(A,h)\frac{\Phi_y^h(t-h)}{\Phi(t)}\big(1+o(1)\big). \tag{8.13}$$

Thus, we need to consider only the ratio $\frac{\Phi_y^h(t-h)}{\Phi(t)}$. We start by providing estimates for $\Phi_y^h(t-h)$ using that it solves (6.8), for $t \ge f(Ay) \vee t(A)$. From (6.10), we get on $I_1^c \cap I_2^c$

$$\begin{aligned}\int_{f(Ay+h)}^{t} \rho_y^h(s)\,ds &\le C(A)\left(1+\frac{1}{f(y)-h}\right)\int_e^\infty \frac{ds}{s\ln^{1+\varepsilon/2}(s)}\\ &< C(A,\varepsilon)\left(1+\frac{1}{f(g(h+1))-h}\right) \le 2C(A,\varepsilon).\end{aligned}$$

Therefore, the solution of (6.8) for $t \ge f(Ay)\vee t(A)$ is bounded in the following way uniformly in y and t

$$\begin{aligned}\Phi_y^h(t) &= \Phi_y^h\big(f(Ay+h)\big)e^{\int_{f(Ay+h)}^t \frac{2K}{\sqrt{(g(s+h)-y)\vee 1}}\,ds+\int_{f(Ay+h)}^t \rho_y^h(s)\,ds}\\ &\le C_1(A,h)\Phi_y^h\big(f(Ay+h)\big)e^{\int_{f(Ay+h)}^t \frac{2K}{\sqrt{(g(s+h)-y)}}\,ds}.\end{aligned}$$

Furthermore, since the elementary $\Phi_y^h(f(Ay+h)) \le f(Ay+h)$ holds, we get the following inequality:

$$\Phi_y^h(t) \le C(A,h)f(Ay+h)e^{\int_{f(Ay+h)}^t \frac{2K}{\sqrt{(g(s+h)-y)}}\,ds}. \tag{8.14}$$

Also (5.4) with $t_0 = 1$ and (6.10) of Lemma 2, for the case $h = y = 0$, yield that

$$\Phi(t) = \Phi(1)e^{\int_1^t \frac{2K}{\sqrt{g(s)}}\,ds+\int_1^t \rho(s)\,ds} \ge Ce^{\int_1^t \frac{2K}{\sqrt{g(s)}}\,ds}.$$

Thus this and (8.14), on $I_1^c \cap I_2^c$, give that with $C = C(A,h) > 0$,

$$\begin{aligned}\frac{\Phi_y^h(t-h)}{\Phi(t)} &\le Cf(Ay+h)e^{\int_{f(Ay+h)}^{t-h} \frac{2K}{\sqrt{(g(s+h)-y)}}\,ds-\int_1^t \frac{2K}{\sqrt{g(s)}}\,ds}\\ &\le Cf(Ay+h)e^{-\int_1^{f(Ay+h)+h} \frac{2K}{\sqrt{g(s)}}\,ds}e^{\int_{f(Ay+h)+h}^t (\frac{2K}{\sqrt{(g(s)-y)}}-\frac{2K}{\sqrt{g(s)}})\,ds}\\ &\le Cf(Ay+h)e^{-\int_1^{f(Ay+h)+h} \frac{2K}{\sqrt{g(s)}}\,ds},\end{aligned}$$

where for the last exponent we have used (8.4) of Lemma 4. We further continue the estimate as follows:

$$\begin{aligned}\frac{\Phi_y^h(t-h)}{\Phi(t)} &\le Cf(Ay+h)e^{-\int_1^{f(Ay+h)+h} \frac{2K}{\sqrt{g(s)}}\,ds}\\ &\le Cf(Ay+h)e^{-\int_1^{f(Ay+h)} \frac{2K}{\sqrt{g(s)}}\,ds} = f(Ay+h)e^{-\int_{g(1)}^{Ay+h} \frac{2Kf'(s)}{\sqrt{s}}\,ds}\end{aligned}$$

$$= Cf(Ay+h)e^{-K\int_{g(1)}^{Ay+h}\frac{f(s)}{s^{3/2}}\,ds-\frac{2Kf(s)}{\sqrt{s}}|_{g(1)}^{Ay+h}}$$
$$\leq Cf(Ay+h)e^{-K\int_{g(1)}^{Ay+h}\frac{f(s)}{s^{3/2}}\,ds}.$$

Using this last bound, we get that (8.13) reduces on $I_1^c \cap I_2^c$ to

$$\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)} \leq Cf(Ay+h)e^{-K\int_{g(1)}^{Ay+h}\frac{f(s)}{s^{3/2}}\,ds} := r_h(y). \tag{8.15}$$

Note that $r_h(y)$ can be extended to $y \in (0,\infty)$. We proceed to show that $r_h(y)$ is even integrable on $(0,\infty)$ with respect to $\mathbb{P}(\tau_h \in dy, \mathcal{O}_h)$. However, since f is bounded in any neighbourhood of zero it suffices to show that $r_h(y)$ is integrable on $(1,\infty)$. To achieve this, we first observe that

$$\mathbb{P}(\tau_h \in dy, \mathcal{O}_h) \leq \mathbb{P}(\tau_h \in dy) = \mathbb{P}(h^2\tau_1 \in dy) \leq C\frac{dy}{y^{3/2}},$$

see (2.5). Using this, we therefore get

$$\begin{aligned} J &:= \int_1^\infty r_h(y)\mathbb{P}(\tau_h \in dy, \mathcal{O}_h) \leq C\int_1^\infty f(Ay+h)e^{-K\int_{g(1)}^{Ay+h}\frac{f(s)}{s^{3/2}}\,ds}\frac{dy}{y^{3/2}} \\ &\leq C\int_1^\infty f(Ay+h)e^{-K\int_{g(1)}^{Ay+h}\frac{f(s)}{s^{3/2}}\,ds}\frac{d(Ay+h)}{(Ay+h)^{3/2}} \\ &\leq C\int_1^\infty f(u)e^{-\int_{g(1)}^{u}\frac{Kf(s)}{s^{3/2}}\,ds}\frac{du}{u^{3/2}}, \end{aligned}$$

where $C > 0$ is a constant depending at most on h, A. However, with $\alpha(u) = Kf(u)/u^{3/2}$ the integral above can be represented as

$$J \leq C\int_1^\infty \alpha(u)e^{-\int_{g(1)}^{u}\alpha(s)\,ds}\,du = -e^{-\int_{g(1)}^{u}\alpha(s)\,ds}|_1^\infty < \infty.$$

The integrability of $r_h(y)$ and (8.15) imply that the dominated convergence theorem applies and yields

$$\lim_{t\to\infty}\int_{I_1^c\cap I_2^c}\frac{\mathbb{P}(\mathcal{O}_{t-h}(h,y))}{\mathbb{P}(\mathcal{O}_t)}\mathbb{P}(\tau_h \in dy; \mathcal{O}_h) = \int_{I_2^c} q_h(y)\mathbb{P}(\tau_h \in dy; \mathcal{O}_h), \tag{8.16}$$

since the set $I_1^c \cap I_2^c$ increases as $t \to \infty$ to I_2^c. Gathering (8.8), (8.11) and (8.16), we show that the limits and the integral in (8.6) are interchangeable and hence (8.5) follows. □

PROOF OF THEOREM 3. The fact that (8.5) holds implies the statement that any possible weak limit is recurrent as there is no loss of mass at infinity. Moreover, we see that the limit for inverse local time always has the same law. Given that conditional on the size of the jump, we fill a Brownian excursion conditioned to have the same length we see that we can in fact pathwise construct the same process so the limit exists and it is unique. These considerations with the splicing of excursions are even simpler here compared to the transient case as we do not have appearance of explosion time and an infinite excursion. This concludes the proof. □

PROOF OF THEOREM 4. We observe that $w \in R_g \iff$

$$\begin{aligned}
&\lim_{h\to\infty} \mathbb{Q}\big(\tau_h \in \big(g(h), w(h)g(h)\big)\big) \\
&\quad = \lim_{h\to\infty} \int_{g(h)}^{g(h)w(h)} q_h(y)\mathbb{P}(\tau_h \in dy; \mathcal{O}_h) = 0.
\end{aligned} \tag{8.17}$$

Given the expression for $q_h(y)$, see (8.1), we note that thanks to Lemma 4 we have that

$$\int_{f(Ay)\vee t(A)}^{\infty} \left(\frac{1}{\sqrt{g(s+h)-y}} - \frac{1}{\sqrt{g(s)}}\right) ds < \infty$$

and thanks to Lemma 2 with $h = y = 0$, $\int_1^\infty \rho(s)\,ds < \infty$. Thus, choosing h big enough that $f(Ay) > f(Ag(h)) > t(A)$ we see that

$$q_h(y) \asymp \Phi_y^h\big(f(Ay)\big)e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds} e^{\int_{f(Ay)}^\infty \rho_y^h(s)\,ds}. \tag{8.18}$$

However, thanks to (6.10) of Lemma 2, (8.18) is augmented to the asymptotic relation

$$q_h(y) \asymp \Phi_y^h\big(f(Ay)\big)e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds} \tag{8.19}$$

once $y \ge g(h+1)$ since as in the proof of Lemma 5

$$\int_1^\infty \rho_y^h(s)\,ds \le C(A) < \infty.$$

To simplify (8.17) and be able to use (8.19), we first consider the integration on the region $y \in (g(h), g(h+1))$. Then

$$\begin{aligned}
&\int_{g(h)}^{g(h+1)} q_h(y)\mathbb{P}(\tau_h \in dy; \mathcal{O}_h) \\
&\quad \le q_h\big(g(h+1)\big)\mathbb{P}(\mathcal{O}_h) \stackrel{h\to\infty}{\sim} q_h\big(g(h+1)\big)\frac{2K\Phi(h)}{\sqrt{g(h)}},
\end{aligned} \tag{8.20}$$

where we have employed the monotonicity in y of $q_h(y)$; see Proposition 1. Observe next that (2.1) yields the trivial estimate for $y \in (g(h), g(h+1))$

$$\Phi_y^h\big(f(Ay)\big) \le \Phi_y^h\big(f\big(Ag(h+1)\big)\big) \le f\big(Ag(h+1)\big) \le \sqrt{A}(h+1).$$

This together with $f(Ag(h+1)) > h$ gives that (8.19) is further simplified for $y = g(h+1)$ to

$$q_h\big(g(h+1)\big) \lesssim he^{-\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}.$$

The relation $\Phi(h) \asymp e^{\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}$ coming from (5.9) easily implies that

$$q_h\big(g(h+1)\big)\frac{2K\Phi(h)}{\sqrt{g(h)}} \lesssim \frac{h}{\sqrt{g(h)}} \overset{(4.1)}{=} o(1).$$

Thus the portion of (8.17) contained in (8.20) never contributes to the limit in (8.17). We are then free to use the asymptotic relation (8.19) for the interval $I := y \in (g(h+1), w(h)g(h))$ which we split into $I_1 = (g(h+1), 20g(h))$ and $I_2 = I \setminus I_1$. We then get that (8.17) can be checked on I_1, I_2 separately. Let us start with I_1. We get using $\Phi_y^h(f(Ay)) \le f(Ay) \le \sqrt{A}f(y)$, since (2.1) holds, that

$$\begin{aligned}\int_{I_1} q_h(y)\mathbb{P}(\tau_h \in dy; \mathcal{O}_h) &\lesssim \int_{I_1} f(y)e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds}\mathbb{P}(\tau_h \in dy; \mathcal{O}_h)\\ &\le f\big(20g(h)\big)e^{-\int_1^{f(g(h))} \frac{2K}{\sqrt{g(s)}}\,ds}\mathbb{P}(\mathcal{O}_h)\\ &\lesssim \frac{h\Phi(h)}{\sqrt{g(h)}}e^{-\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds} \asymp \frac{h}{\sqrt{g(h)}} \overset{(4.1)}{=} o(1).\end{aligned}$$

For the last estimates above we used the asymptotic relations (5.8), (5.9). Therefore, the portion on I_1 is always negligible and we obtain thanks to Lemma 6 and (8.19) that

$$\begin{aligned} w \in R_g \iff & \lim_{h\to\infty} \mathbb{Q}\big(\tau_h \in \big(20g(h), w(h)g(h)\big)\big)\\ &\overset{(8.17)}{=} \lim_{h\to\infty} \int_{20}^{w(h)} q_h\big(yg(h)\big)\mathbb{P}\big(\tau_h \in g(h)\,dy; \mathcal{O}_h\big)\\ &\overset{(8.23)}{=} \lim_{h\to\infty} \mathbb{P}(\mathcal{O}_h)\int_{20}^{w(h)} q_h\big(yg(h)\big)\frac{dy}{y^{3/2}}\\ &\overset{(5.8)}{=} \lim_{h\to\infty} \Phi(h)\int_{20g(h)}^{w(h)g(h)} \Phi_y^h\big(f(Ay)\big)e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds}\frac{dy}{y^{3/2}}\\ &= 0.\end{aligned} \tag{8.21}$$

Let us first show that the convergence of the integral in (4.2) is sufficient for $w \in R_g$ by relating it to the last limit in (8.21). Using the inequality $\Phi_y^h(f(Ay)) \le f(Ay)$ and the change of variables $Ay \to u$, we get easily that

$$\begin{aligned}&\lim_{h\to\infty} \Phi(h)\int_{20g(h)}^{w(h)g(h)} \Phi_y^h\big(f(Ay)\big)e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds}\frac{dy}{y^{3/2}}\\ &\quad\le \sqrt{A}\lim_{h\to\infty} \Phi(h)\int_{g(h)}^{w(h)g(h)} f(u)e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds}\frac{du}{u^{3/2}}\\ &\quad\quad + \sqrt{A}\lim_{h\to\infty} \Phi(h)\int_{w(h)g(h)}^{Aw(h)g(h)} f(u)e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds}\frac{du}{u^{3/2}}\\ &\quad= \sqrt{A}\Big(\lim_{h\to\infty} J_1(h) + \lim_{h\to\infty} J_2(h)\Big).\end{aligned} \tag{8.22}$$

Let us investigate $J_2(h)$. We use $f(x)/\sqrt{x} \downarrow 0$, see (2.1), $f(x)$ is an increasing function and $\Phi(h) \asymp e^{\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}$ coming from (5.9) to get

$$\begin{aligned}
\lim_{h\to\infty} J_2(h) &\le 2\sqrt{A}\limsup_{h\to\infty}\biggl(\Phi(h)e^{-\int_1^{f(w(h)g(h))}\frac{2K}{\sqrt{g(s)}}\,ds}\frac{f(Aw(h)g(h))}{\sqrt{Aw(h)g(h)}}\biggr)\\
&\le 2\sqrt{A}\limsup_{h\to\infty}\biggl(\Phi(h)e^{-\int_1^{h}\frac{2K}{\sqrt{g(s)}}\,ds}\frac{f(Aw(h)g(h))}{\sqrt{Aw(h)g(h)}}\biggr)\\
&\le C(A)\limsup_{h\to\infty}\frac{f(Aw(h)g(h))}{\sqrt{Aw(h)g(h)}} = 0,
\end{aligned}$$

where $C(A) > 0$ is a generic constant. Thus, only $\lim_{h\to\infty} J_1(h)$ can contribute to the limit in (8.21). An integration by parts gives us that

$$\int_1^{f(y)}\frac{1}{\sqrt{g(s)}}\,ds = \frac{f(y)}{\sqrt{y}} - 1 + \frac{1}{2}\int_1^y \frac{f(s)}{s^{3/2}}\,ds \sim \frac{1}{2}\int_1^y\frac{f(s)}{s^{3/2}}\,ds,$$

since $f(s)/\sqrt{s} \downarrow 0$, see (2.1), and $I(f) = \int_1^\infty \frac{f(s)}{s^{3/2}}\,ds = \infty$. Then, clearly,

$$\begin{aligned}
J_1(h) &= \Phi(h)\int_{g(h)}^{w(h)g(h)} f(u)e^{-\int_1^{f(u)}\frac{2K}{\sqrt{g(s)}}\,ds}\frac{du}{u^{3/2}}\\
&\asymp \Phi(h)\int_{g(h)}^{w(h)g(h)} f(y)e^{-(1/2)\int_1^y\frac{2Kf(s)}{s^{3/2}}\,ds}\frac{dy}{y^{3/2}}\\
&= \Phi(h)\biggl(-\frac{1}{K}e^{-(1/2)\int_1^y\frac{2Kf(s)}{s^{3/2}}\,ds}\Big|_{g(h)}^{g(h)w(h)}\biggr).
\end{aligned}$$

Expressing back $\frac{1}{2}\int_1^y \frac{f(s)}{s^{3/2}}\,ds$ in terms of $\int_1^{f(y)}\frac{1}{\sqrt{g(s)}}\,ds$ and noting that $\Phi(h) \asymp e^{\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}$ we get that

$$J_1(h) = \Phi(h)\int_{y=g(h)}^{w(h)g(h)} f(y)e^{-\int_1^{f(y)}\frac{2K}{\sqrt{g(s)}}\,ds}\frac{dy}{y^{3/2}} \asymp \bigl(1 - e^{-\int_h^{f(g(h)w(h))}\frac{K}{\sqrt{g(s)}}\,ds}\bigr).$$

Therefore, since the limit of the right-hand side of (8.22) equals $\lim_{h\to\infty} J_1(h)$, we prove the sufficiency of the right-hand side of (4.2) for $w \in R_g$.

The necessity part of (4.2) is trickier. Choose $A \in (3, 20)$. Assume that $w \in R_g$ and hence the right-hand side of (8.21) holds. We recall that inequality (6.14) is valid with $t = f(Ay) > f(Ag(h)) > h > t(A)$, whenever $h > t(A)$. In this case, it takes the form $\Phi_h^y(f(Ay)) \ge f(y) - h, \forall y > g(h), h > t(A)$. Let us feed this inequality in the right-hand side of (8.21) to get

$$\begin{aligned}
0 &= \lim_{h\to\infty}\Phi(h)\int_{20g(h)}^{w(h)g(h)}\Phi_y^h\bigl(f(Ay)\bigr)e^{-\int_1^{f(Ay)}\frac{2K}{\sqrt{g(s)}}\,ds}\frac{dy}{y^{3/2}}\\
&\ge \lim_{h\to\infty}\Phi(h)\int_{20g(h)}^{w(h)g(h)} f(y)e^{-\int_1^{f(Ay)}\frac{2K}{\sqrt{g(s)}}\,ds}\frac{dy}{y^{3/2}}
\end{aligned}$$

$$
\begin{aligned}
&- \lim_{h\to\infty} h\Phi(h) \int_{20g(h)}^{w(h)g(h)} e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{dy}{y^{3/2}} \\
&= I_1 + I_2.
\end{aligned}
$$

Recall that $\Phi(h) \asymp e^{\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}$. Then

$$
\begin{aligned}
|I_2| &= \lim_{h\to\infty} h\Phi(h) \int_{20g(h)}^{w(h)g(h)} e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{dy}{y^{3/2}} \\
&\le \sqrt{A} \lim_{h\to\infty} h\Phi(h) \int_{u=20Ag(h)}^{Aw(h)g(h)} e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}} \\
&\le \sqrt{A} \lim_{h\to\infty} h\Phi(h) e^{-\int_1^{h} \frac{2K}{\sqrt{g(s)}}\,ds} \int_{y=20g(h)}^{\infty} \frac{du}{u^{3/2}} \\
&\le C \lim_{h\to\infty} \frac{h}{\sqrt{g(h)}} \overset{(4.1)}{=} 0.
\end{aligned}
$$

Therefore, I_2 never contributes to (8.21). Then, we get that

$$
\begin{aligned}
0 = I_1 &= \lim_{h\to\infty} \Phi(h) \int_{20g(h)}^{w(h)g(h)} f(y) e^{-\int_1^{f(Ay)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{dy}{y^{3/2}} \\
&= \lim_{h\to\infty} \sqrt{A}\Phi(h) \int_{u=20Ag(h)}^{Aw(h)g(h)} f\left(\frac{u}{A}\right) e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}} \\
&\ge C(A) \lim_{h\to\infty} \Phi(h) \int_{u=20Ag(h)}^{Aw(h)g(h)} f(u) e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}},
\end{aligned}
$$

where we have used the fact that $f(s)/s^{1/2} \downarrow 0$ and thus $f(u/A) \ge \sqrt{\frac{1}{A}} f(u)$. We evaluate the very last expression, ignoring the constant $C(A)$, to get

$$
\begin{aligned}
0 = I_1 \ge{} & \lim_{h\to\infty} -\frac{\Phi(h)}{K} e^{-(1/2)\int_1^y \frac{2Kf(s)}{s^{3/2}}\,ds} \Big|_{g(h)}^{g(h)w(h)} \\
& - \lim_{h\to\infty} \Phi(h) \int_{g(h)}^{20Ag(h)} f(u) e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}} \\
& + \lim_{h\to\infty} \Phi(h) \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}} \\
={} & K_1 + K_2 + K_3.
\end{aligned}
$$

We note that once we employ $\Phi(h) \asymp e^{\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}$, K_1 is precisely $\lim_{h\to\infty} J_1(h)$ discussed in the case of sufficiency. Thus, K_1 will then be zero if always $K_2 = K_3 = 0$. Hence, the necessity of (4.2) will follow. We trivially estimate using $\Phi(h) \asymp e^{\int_1^h \frac{2K}{\sqrt{g(s)}}\,ds}$ and $f(20Ag(h)) \le \sqrt{20A} f(g(h)) = \sqrt{20A}h$, since

$f(s)/s^{1/2} \downarrow 0$, that

$$|K_2| = \lim_{h\to\infty} \Phi(h) \int_{g(h)}^{20Ag(h)} f(u) e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}}$$
$$\lesssim \lim_{h\to\infty} f\big(20Ag(h)\big) \int_{g(h)}^{20Ag(h)} \frac{du}{u^{3/2}} \lesssim \lim_{h\to\infty} \frac{h}{\sqrt{g(h)}} \overset{(4.1)}{=} 0.$$

K_3 is also computed using identical calculations as above. Indeed

$$\Phi(h) \int_{w(h)g(h)}^{Aw(h)g(h)} f(u) e^{-\int_1^{f(u)} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{du}{u^{3/2}}$$
$$\le 2\Phi(h) f\big(Aw(h)g(h)\big) e^{-\int_1^{f(w(h)g(h))} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{1}{\sqrt{w(h)g(h)}} \left(1 - \frac{1}{\sqrt{A}}\right)$$
$$\le 2(\sqrt{A}-1)\Phi(h) e^{-\int_1^{h} \frac{2K}{\sqrt{g(s)}}\,ds} \frac{f(Aw(h)g(h))}{\sqrt{Aw(h)g(h)}}$$
$$\asymp \frac{f(Aw(h)g(h))}{\sqrt{Aw(h)g(h)}} = o(1).$$

Therefore, $K_3 = 0$ in any case and we conclude that K_1 must be zero if $w \in R_g$. However, as mentioned above $K_1 = 0$ triggers the validity of the limit of the right-hand side of (4.2). This finally concludes the proof of Theorem 4. □

The strong repulsion depends on the following lemma which studies the measures $\mathbb{P}(\tau_h \in g(h)\,dy, \mathcal{O}_h)$, $h > 0$.

LEMMA 6. *Let $\sigma_h(dy) = o_h(y)\,dy = \mathbb{P}(\tau_h \in g(h)\,dy; \mathcal{O}_h)$ be a measure on $(1,\infty)$. Then, for any $h \ge h_0$ big enough, there are absolute constants $0 < c < 1 < C < \infty$ such that for $y > 20$ we have that*

$$\frac{c}{y^{3/2}} \mathbb{P}(\mathcal{O}_h) \le o_h(y) \le \frac{C}{y^{3/2}} \mathbb{P}(\mathcal{O}_h). \tag{8.23}$$

PROOF. The absolute continuity of $\sigma_h(dy)$ follows immediately from $\sigma_h(dy) \ll \mathbb{P}(\tau_h \in dy) \ll dy$. For the proof, we introduce the quantities $T := T_h = \inf\{t > 0 : \tau_t > g(h)\}$, in the usual sense $\Delta = \Delta_1 = \inf\{t > 0 : \tau_t - \tau_{t-} > g(h)\}$ and $S_\Delta = \tau_\Delta - \tau_{\Delta-}$. We then have that

$$\mathbb{P}\big(S_\Delta \in g(h)\,dy\big) = \frac{\Pi(g(h)\,dy)}{\overline{\Pi}(g(h))} = \frac{dy}{y^{3/2}}, \tag{8.24}$$

which is a standard property for any Lévy process, namely conditionally that a jump exceeds a level $g(h) > 0$ than its size is independent of the time of the jump and the past of the process and its distribution is given by the first ratio in (8.24).

The second ratio in (8.24) holds only in this special instance of a stable subordinator of index $1/2$. Furthermore, denote by τ_{T-} the position prior to the passage time. Finally, note that we have the immediate identity from (2.5)

$$\mathbb{P}(\tau_v \in a\,du) = \frac{1}{u\sqrt{2\pi a v^{-2} u}} e^{-1/(av^{-2}u)}\,du. \tag{8.25}$$

We consider the measure $\sigma_h(dy)$ on three possibly overlapping regions. We start with $\sigma_h^1(dy) := \sigma_h(dy, \tau_T \le 2g(h))$. We use the following steps. We disintegrate on $\tau_T \in (g(h), 2g(h))$. Since the event $\mathcal{O}_h$ implies $\{T \le h\}$, we get that $\tau_h = \tau_T + \tau_h - \tau_T \overset{d}{=} \tau_T + \tau'_{h-T}$ where τ' an independent copy of τ. Therefore,

$$\begin{aligned}
\sigma_h^1(dy) &= o_h^1(y)\,dy \\
&= \int_{s=0}^{h}\int_{u=1}^{2} \mathbb{P}\big(\tau_{h-s} \in g(h)(dy-u)\big)\mathbb{P}\big(T \in ds; \tau_T \in g(h)\,du; \mathcal{O}_h\big) \\
&\le C\int_{s=0}^{h}\int_{u=1}^{2} \frac{h-s}{\sqrt{g(h)}(y-u)^{3/2}}\mathbb{P}\big(T \in ds; \tau_T \in g(h)\,du; \mathcal{O}_h\big)\,dy \\
&\le C_1\frac{h}{\sqrt{g(h)}}\frac{dy}{y^{3/2}}\mathbb{P}(\mathcal{O}_h) \overset{(4.1)}{=} o(1)\frac{dy}{y^{3/2}}\mathbb{P}(\mathcal{O}_h),
\end{aligned} \tag{8.26}$$

where we have used (8.25) and assumed that $y > 4$ so that $y - u > y/2$. We note that this density in fact decays faster with factor $\frac{h}{\sqrt{g(h)}} \overset{(4.1)}{=} o(1)$ than the required (8.23).

In the remaining two scenarios, we employ that

$$\big\{\tau_T > 2g(h)\big\} \cap \mathcal{O}_h \subset \{\Delta_1 \le h; T = \Delta_1\} \cap \mathcal{O}_h = \big\{\Delta_1 \le h; \tau_{\Delta_1-} < g(h)\big\} \cap \mathcal{O}_h,$$

which follows from the definitions of $T, \Delta_1, \mathcal{O}_h = \{\tau_s > g(s), s \le h\}$ and the fact that τ is a subordinator, that is, an increasing Lévy process.

First, we consider

$$\sigma_h^2(dy) := \sigma_h\bigg(dy; \Delta_1 \le h; \tau_{\Delta_1-} < g(h); S_\Delta < \frac{g(h)y}{2}\bigg),$$

which majorizes in terms of measures the measure

$$\sigma_h\bigg(dy; \tau_T > 2g(h); S_\Delta < \frac{g(h)y}{2}\bigg).$$

Disintegrate with respect to Δ_1 and the position prior to the jump to get

$$\begin{aligned}
\sigma_h^2(dy) &= o_h^2(y)\,dy \\
&= \int_{s=0}^{h}\int_{w=0}^{1}\int_{v=1}^{y/2} \mathbb{P}\big(\tau_{h-s} \in g(h)(dy-v-w)\big) \\
&\quad \times \mathbb{P}\big(\Delta_1 \in ds, \tau_{\Delta_1-} \in g(h)\,dw, S_\Delta \in g(h)\,dv; \mathcal{O}_h\big).
\end{aligned}$$

Using the definition of $\mathcal{O}_h = \{\tau(s) > g(s), s \le h\}$, the fact that g is an increasing function and τ is a subordinator, we have the identity

$$\begin{aligned}
&\mathbb{P}\big(\Delta_1 \in ds, \tau_{\Delta_1-} \in g(h)\,dw, S_\Delta \in g(h)\,dv; \mathcal{O}_h\big) \\
&\quad = \mathbb{P}\big(\Delta_1 \in ds, \tau_{\Delta_1-} \in g(h)\,dw, S_\Delta \in g(h)\,dv; \mathcal{O}_{\Delta_1-}\big).
\end{aligned}$$

Since conditionally on $\{\Delta_1 = s\}$ the jump S_Δ is independent of the past, we get that

$$\begin{aligned}
&\mathbb{P}\big(\Delta_1 \in ds; \tau_{s-} \in g(h)\,dw; S_\Delta \in g(h)\,dv; \mathcal{O}_{\Delta_1-}\big) \\
&\quad = \mathbb{P}\big(S_\Delta \in g(h)\,dv\big)\mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1-} \in g(h)\,dw; \mathcal{O}_{\Delta_1-}\big) \\
&\quad = \mathbb{P}\big(S_\Delta \in g(h)\,dv\big)\mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1-} \in g(h)\,dw; \mathcal{O}_h\big).
\end{aligned}$$

Substituting this back above and using (8.24) for the law of S_Δ we get that

$$\begin{aligned}
\sigma_h^2(dy) &= o_h^2(y)\,dy \\
&\le C \int_{s=0}^h \int_{w=0}^1 \int_{v=1}^{y/2} \frac{h-s}{\sqrt{g(h)}(y-w-v)^{3/2}} \\
&\qquad \times \mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1-} \in g(h)\,dw; \mathcal{O}_h\big)\frac{dv}{v^{3/2}}\,dy.
\end{aligned}$$

Using that $y - w - v \ge y/2 - 1 \ge y/3$, once $y > 10$, we have that

$$\sigma_h^2(dy) = o_h^2(y)\,dy \le \frac{h}{\sqrt{g(h)}}\frac{dy}{y^{3/2}}\mathbb{P}(\mathcal{O}_h) \overset{(4.1)}{=} o(1)\frac{dy}{y^{3/2}}\mathbb{P}(\mathcal{O}_h). \tag{8.27}$$

Second, we study the measure

$$\sigma_h^3(dy) := \sigma_h\bigg(dy; \Delta_1 \le h; \tau_{\Delta_1-} < g(h); S_\Delta > \frac{g(h)y}{2}\bigg),$$

which majorizes in terms of measures the measure

$$\sigma_h\bigg(dy; \tau_T > 2g(h); S_\Delta > \frac{g(h)y}{2}\bigg).$$

We similarly disintegrate the measure $\sigma_h^3(dy)$ to get

$$\begin{aligned}
\sigma_h^3(dy) &= o_h^3(y)\,dy \\
&= \int_{s=0}^h \int_{w=0}^1 \int_{v=y/2}^{\infty} \mathbb{P}\big(\tau_{h-s} \in g(h)(dy - v - w)\big) \\
&\qquad \times \mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1-} \in g(h)\,dw; S_\Delta \in g(h)\,dv; \mathcal{O}_h\big) \\
&\le \int_{s=0}^h \int_{w=0}^1 \int_{v=y/3-w}^{y-w} \mathbb{P}\big(\tau_{h-s} \in g(h)(dy - v - w)\big) \\
&\qquad \times \mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1-} \in g(h)\,dw; \mathcal{O}_h\big)\frac{dv}{v^{3/2}},
\end{aligned}$$

where we have used that $y-w-v>0\Rightarrow v<y-w$ and for $y>10$, $w\in(0,1)$, $\frac{y}{2}>\frac{y}{3}-w$. We note from (8.25) that for $y>v+w$

$$\begin{aligned}\mathbb{P}\bigl(\tau_{h-s}\in g(h)(dy-v-w)\bigr)\\ &=\frac{1}{\sqrt{2\pi(g(h)(y-v-w)/(h-s)^2)}(y-v-w)}\\ &\quad\times e^{-1/(2(g(h)(y-v-w)/(h-s)^2))}\,dy.\end{aligned}$$

Put $a(h,s)=g(h)/(h-s)^2$. Change variables in v such that $v\to z-w$ and then $z\to y\rho$ in the last integral to get that

$$\begin{aligned}\sigma_h^3(dy)&=o_h^3(y)\,dy\\ &\le\int_{s=0}^{h}\int_{w=0}^{1}\int_{\rho=1/3}^{1}\frac{1}{\sqrt{2\pi a(h,s)(y-y\rho)}(y-y\rho)}e^{-1/(2a(h,s)(y-y\rho))}\\ &\quad\times\mathbb{P}\bigl(\Delta_1\in ds;\tau_{\Delta_1-}\in g(h)\,dw;\mathcal{O}_h\bigr)\frac{y\,d\rho}{(y\rho-w)^{3/2}}\,dy.\end{aligned}$$

Furthermore, using the inequalities $y\rho-w>y\rho-1>y\rho/4$ once $y>20$, since $\rho\in(1/3,1)$, we obtain that

$$\begin{aligned}\sigma_h^3(dy)&=o_h^3(y)\,dy\\ &\le\int_{s=0}^{h}\int_{w=0}^{1}\int_{\rho=1/3}^{1}\frac{1}{\sqrt{2\pi a(h,s)y(1-\rho)}(y(1-\rho))}e^{-1/(2a(h,s)y(1-\rho))}\\ &\quad\times\frac{4^{3/2}\,d\rho}{\sqrt{y}\rho^{3/2}}\mathbb{P}\bigl(\Delta_1\in ds;\tau_{\Delta_1-}\in g(h)\,dw;\mathcal{O}_h\bigr)\,dy\\ &\le 12^{3/2}\int_{s=0}^{h}\int_{w=0}^{1}\int_{\sigma=0}^{2/3}\frac{1}{\sqrt{2\pi a(h,s)y\sigma}\sigma}e^{-1/(2a(h,s)y\sigma)}\\ &\quad\times d\sigma\,\mathbb{P}\bigl(\Delta_1\in ds;\tau_{\Delta_1-}\in g(h)\,dw;\mathcal{O}_h\bigr)\frac{dy}{y^{3/2}}\\ &=12^{3/2}\int_{s=0}^{h}\int_{w=0}^{1}\int_{\chi=0}^{(4/3)ya(h,s)}\frac{1}{\sqrt{\pi\chi}\chi}e^{-1/\chi}\\ &\quad\times d\chi\,\mathbb{P}\bigl(\Delta_1\in ds;\tau_{\Delta_1-}\in g(h)\,dw;\mathcal{O}_h\bigr)\frac{dy}{y^{3/2}}.\end{aligned}$$

Since $\int_{\chi=0}^{\infty}\chi^{-3/2}e^{-1/\chi}\,d\chi<\infty$ we get that, for $y>20$,

$$\sigma_h^3(dy)=o_h^3(y)\,dy\le C\frac{dy}{y^{3/2}}\mathbb{P}\bigl(\Delta_1\le h,\tau_{\Delta_1-}\le g(h);\mathcal{O}_h\bigr).\tag{8.28}$$

By trivial estimates using (8.26), (8.27) and (8.28) we conclude the upper bound in (8.23). For the lower bound, consider that

$$\sigma_h^3(dy) = \sigma_h\left(dy; \Delta_1 \le h; \tau_{\Delta_1 -} < g(h); S_\Delta > \frac{g(h)y}{2}\right) \le \sigma_h(dy).$$

Then the lower bound for (8.23) follows by observing that

$$\begin{aligned}\sigma_h^3(dy) &= o_h^3(y)\,dy \\ &= \int_{s=0}^{h}\int_{w=0}^{1}\int_{v=y/2}^{\infty} \mathbb{P}\big(\tau_{h-s} \in g(h)(dy - v - w)\big) \\ &\quad \times \mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1 -} \in g(h)\,dw; S_\Delta \in g(h)\,dv; \mathcal{O}_h\big) \\ &\ge \int_{s=0}^{h}\int_{w=0}^{1}\int_{v=2y/3-w}^{y-w} \mathbb{P}\big(\tau_{h-s} \in g(h)(dy - v - w)\big) \\ &\quad \times \mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1 -} \in g(h)\,dw; \mathcal{O}_h\big)\frac{dv}{v^{3/2}},\end{aligned}$$

once $2/3y - w \ge y/2$ which holds for $y > 20$ and $w \in (0, 1)$. Then precisely as in the derivation of inequality (8.28) we feed in the expression for $\mathbb{P}(\tau_{h-s} \in g(h)(dy - v - w))$, change in the same way the variables, estimate first $y\rho - w < y\rho$ and then instead of estimating from above $\rho^{-3/2}$ we estimate it from below with 1 since $\rho \in (2/3, 1)$, to get similarly that

$$\begin{aligned}\sigma_h^3(dy) &= o_h^3(y)\,dy \\ &\ge \int_{s=0}^{h}\int_{w=0}^{1}\int_{\chi=0}^{(2/3)ya(h,s)} \frac{1}{\sqrt{\pi\chi}\chi}e^{-1/\chi} \\ &\quad \times d\chi\,\mathbb{P}\big(\Delta_1 \in ds; \tau_{\Delta_1 -} \in g(h)\,dw; \mathcal{O}_h\big)\frac{dy}{y^{3/2}}.\end{aligned}$$

However, since $a(h, s) = \frac{g(h)}{(h-s)^2} \ge \frac{g(h)}{h^2} \overset{(4.1)}{\to} \infty$, as $h \to \infty$, we get with some $C > 0$ that

$$\sigma_h^3(dy) = o_h^3(y)\,dy \ge C\frac{dy}{y^{3/2}}\mathbb{P}\big(\Delta_1 \le h; \tau_{\Delta_1 -} \le g(h); \mathcal{O}_h\big). \tag{8.29}$$

Note that in the sense of measures so far we have that

$$\sigma_h^3(dy) \le \sigma_h(dy) \le \sigma_h^1(dy) + \sigma_h^2(dy) + \sigma_h^3(dy), \qquad y > 20.$$

Therefore, if we assume that over a subsequence $h_i \uparrow \infty$,

$$\mathbb{P}\big(\Delta_1 \le h_i; \tau_{\Delta_1 -} \le g(h_i); \mathcal{O}_{h_i}\big) = o(1)\mathbb{P}(\mathcal{O}_{h_i}),$$

as $i \to \infty$, then (8.26), (8.27), (8.28) and (8.29) yield that in sense of measures $\sigma_{h_i}(dy) = o(1)\mathbb{P}(\mathcal{O}_{h_i})\,dy/y^{3/2}$, $y > 20$, as $i \to \infty$. Therefore, upon this assumption for i big enough and $y > 20$, we have that

$$\mathbb{P}\big(\tau_{h_i} > 20g(h_i); \mathcal{O}_{h_i}\big) = \int_{20}^{\infty} \sigma_{h_i}(dy) = o(1)\mathbb{P}(\mathcal{O}_{h_i})\int_{20}^{\infty}\frac{dy}{y^{3/2}} = o(1)\mathbb{P}(\mathcal{O}_{h_i}).$$

Next, we will provide a contradiction by showing that

$$\liminf_{h\to\infty} \frac{\mathbb{P}(\tau_h > 20g(h); \mathcal{O}_h)}{\mathbb{P}(\mathcal{O}_h)} > 0.$$

However, recalling the usual notation $\Delta_1^a = \inf\{t > 0 : \tau_t - \tau_{t-} > a\}$, $a > 0$, we see that since τ is a subordinator further it suffices to show that

$$\begin{aligned}
&\liminf_{h\to\infty} \frac{\mathbb{P}(\tau_h > 20g(h); \mathcal{O}_h)}{\mathbb{P}(\mathcal{O}_h)} \\
&\quad \geq \liminf_{h\to\infty} \frac{\mathbb{P}(\Delta_1^{g(h)} = \Delta_1^{20g(h)} \leq h; \mathcal{O}_h)}{\mathbb{P}(\mathcal{O}_h)} > 0.
\end{aligned} \tag{8.30}$$

To demonstrate this, we compute that

$$\begin{aligned}
&\mathbb{P}\big(\Delta_1^{g(h)} = \Delta_1^{20g(h)} \leq h; \mathcal{O}_h\big) \\
&\quad = \int_0^h \mathbb{P}\big(\mathcal{O}_s^{g(h)}\big)\mathbb{P}\big(\Delta_1^{g(h)} \in ds; \Delta_1^{g(h)} = \Delta_1^{20g(h)}\big).
\end{aligned}$$

However,

$$\begin{aligned}
\mathbb{P}\big(\Delta_1^{g(h)} \in ds; \Delta_1^{g(h)} = \Delta_1^{20g(h)}\big) &= \mathbb{P}\big(\Delta_1^{g(h)} \in ds; S_{\Delta_1^{g(h)}} > 20g(h)\big) \\
&= \mathbb{P}\big(\Delta_1^{g(h)} \in ds\big)\mathbb{P}\big(S_{\Delta_1^{g(h)}} > 20g(h)\big) \\
&= \frac{\overline{\Pi}(20g(h))}{\overline{\Pi}(g(h))}\mathbb{P}\big(\Delta_1^{g(h)} \in ds\big) \\
&= C\mathbb{P}\big(\Delta_1^{g(h)} \in ds\big).
\end{aligned}$$

Feeding the last expression back above, we get

$$\mathbb{P}\big(\Delta_1^{g(h)} = \Delta_1^{20g(h)} \leq h; \mathcal{O}_h\big) = C\mathbb{P}\big(\Delta_1^{g(h)} \leq h; \mathcal{O}_h\big) \sim C\mathbb{P}(\mathcal{O}_h),$$

where the last follows from the first relation of (5.8) of Theorem 15 and $C \in (0, 1)$ is an absolute constant. Therefore, we conclude from (8.30) that

$$\liminf_{h\to\infty} \frac{\mathbb{P}(\tau_h > 20g(h); \mathcal{O}_h)}{\mathbb{P}(\mathcal{O}_h)} \geq C > 0$$

and thus a contradiction is furnished. □

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