

Critical Ising model and spanning trees partition functions

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Abstract. We prove that the squared partition function of the two-dimensional critical Ising model defined on a finite, isoradial graph $G = (V, E)$, is equal to $2^{|V|}$ times the partition function of spanning trees of the graph \tilde{G} , where \tilde{G} is the graph G extended along the boundary; edges of G are assigned Kenyon's (*Invent. Math.* **150** (2) (2002) 409–439) critical weights, and boundary edges of \tilde{G} have specific weights. The proof is an explicit construction, providing a new relation on the level of configurations between two classical, critical models of statistical mechanics.

Résumé. Nous montrons que le carré de la fonction de partition du modèle d'Ising critique en dimension deux, défini sur un graphe isoradial $G = (V, E)$ fini, est égale à $2^{|V|}$ fois la fonction de partition des arbres couvrants du graphe \tilde{G} , où le graphe \tilde{G} est le graphe G prolongé le long du bord; les arêtes de G sont munies des poids critiques de Kenyon (*Invent. Math.* **150** (2) (2002) 409–439), et les arêtes du bord de \tilde{G} ont des poids spécifiques. La preuve consiste en une construction explicite, qui donne une nouvelle relation, au niveau des configurations, entre deux modèles classiques de mécanique statistique au point critique.

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1. Introduction

Let $G = (V, E)$ be a finite, planar graph satisfying a geometric condition called *isoradiality*, then every edge e of G is naturally assigned an angle θ_e . Consider the Ising model defined on the isoradial graph G , with Baxter's [1] critical coupling constants $J = (J_e)_{e \in E}$:

$$\forall e \in E, \quad J_e = \frac{1}{2} \log \left(\frac{1 + \sin \theta_e}{\cos \theta_e} \right).$$

The partition function of the Ising model is denoted by $Z_{\text{Ising}}(G, J)$, detailed definitions are given in Sections 2.4.1 and 2.4.2. Consider the graph \tilde{G} obtained from the graph G by extending it along the boundary as in Section 3.2. The main result of this paper can loosely be stated as follows, a precise statement is given in Theorem 3.2 of Section 3.

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Theorem 1.1. *The squared partition function of the critical Ising model defined on the isoradial graph G , is equal to $2^{|V|}$ times the spanning trees partition function of the extended graph \tilde{G} , with Kenyon's [13] critical weights on edges of G , and specific weights on boundary edges of \tilde{G} :*

$$Z_{\text{Ising}}(G, J)^2 = 2^{|V|} \left| \sum_{T \in \mathcal{T}^r(\tilde{G})} \left(\prod_{e=(x,y) \in T \cap E} \tan \theta_e \right) \underbrace{\left(\prod_{(x,y) \in T \cap (\tilde{E} \setminus E)} \tau_{(x,y)} \right) \left(\prod_{(u,v) \in T^* \cap (\tilde{E}^* \setminus E^*)} \tau_{(u,v)} \right)}_{\text{boundary contributions}} \right|.$$

The sum is over oriented spanning trees of the extended graph \tilde{G} , rooted at a vertex r , such that dual spanning trees of the dual \tilde{G}^* are oriented towards a root vertex s , see Section 3.3. Weights τ at the boundary are defined in Equation (3.1).

Remark 1. A version of Theorem 1.1 in the case where the graph G is embedded in the torus rather than the plane, is proved in the papers [2,7] and [6]. The first two papers start from Fisher's correspondence [10] between the Ising model defined on a graph G and the dimer model defined on a non-bipartite, decorated graph G^F . The squared Ising partition function is replaced by the determinant of the dimer characteristic polynomial, and spanning trees are replaced by cycle rooted spanning forests of the torus. In the paper [7], we prove an explicit mapping from cycle rooted spanning forests of the torus, to configurations counted by the determinant of the dimer characteristic polynomial. The paper [6] uses Kac–Ward matrices [19] instead of Fisher's correspondence. Working with graphs embedded in the torus has the difficulty of involving the geometry of the torus, but avoids handling problems related to the graph having a boundary.

In this paper, we address issues related to the boundary of the graph, requiring a number of technical steps. The approach used to prove Theorem 1.1 is also very different: it neither uses Fisher's correspondence nor Kac–Ward matrices. Rather, our starting point is a mapping from the double Ising model defined on a planar graph G , consisting of two independent Ising models, to a dimer model defined on a related bipartite graph G^Q . This is a combination of a result of [24] and of [28] and [8]; note that a generalization to graphs embedded in surfaces of genus g is proved in [3]. Note also, that using a different approach [11,17,18,20,28,29], Dubédat [8] relates two independent Ising models, with one living on the graph G and the second on the dual graph G^* , to the bipartite dimer model on the graph G^Q .

The proof of Theorem 1.1 provides an explicit mapping between the critical dimer model on the bipartite graph G^Q and critical spanning trees of the extended graph \tilde{G} .

Outline of the paper

- **Section 2.** Definition of the Ising model on a planar graph G . Definition of the dimer model on the bipartite graph G^Q related to the double Ising model. Statement of the result [3,8,24,28] proving that the squared Ising partition function is equal, up to a constant, to the partition function of the dimer model on the bipartite graph G^Q . Definition of the critical versions of the models on isoradial graphs.
- **Section 3.** Definition of spanning trees and related notions. Definition of the extended graph \tilde{G} constructed from G . Statement of the main result, Theorem 3.2, loosely stated in Theorem 1.1.

Sections 4, 5, 6 and 7 consist of the proof of Theorem 3.2. It provides a mapping between the critical dimer model on the bipartite graph G^Q and critical spanning trees of the extended graph \tilde{G} . The most important part of the mapping is in Section 7.

- **Section 4.** Computation of the dimer partition function of the graph G^Q using a refinement of Kasteleyn/Temperley–Fisher's method [12,26], due to Kuperberg [16], see also [13]: it is equal to the determinant of a modified Kasteleyn matrix K .
- **Section 5.** Interpretation of the Kasteleyn matrix K as the Laplacian matrix of a directed graph \tilde{G}_0^Q constructed from the bipartite graph G^Q , implying that the determinant of K counts weighted, oriented spanning trees of \tilde{G}_0^Q . Transformation of the directed graph \tilde{G}_0^Q along the boundary into a directed graph \tilde{G}_1^Q : *first step used to handle the boundary of the graph.*

- *Section 6.* Instead of considering oriented spanning trees of \tilde{G}_1^Q , we consider dual spanning trees of the dual graph G_1^{Q*} . Modification of the graph G_1^{Q*} along the boundary so that it becomes the *double graph* \tilde{G}^D of the extended graph \tilde{G} and of its dual \tilde{G}^* : *second step used to handle the boundary of the graph*. Characterization of the set of spanning trees $\mathcal{T}_{(W_2)}(\tilde{G}^D)$ of the double graph \tilde{G}^D arising as duals of oriented spanning trees of \tilde{G}_1^Q . Orientation of spanning trees of $\mathcal{T}_{(W_2)}(\tilde{G}^D)$ towards a root vertex s : definition of the set $\mathcal{T}_{(W_2)}^s(\tilde{G}^D)$. The orientation plays a key role in the next section, this is the reason why we pass to the dual: the information on the orientation of dual spanning trees is not available when looking at primal ones.
- *Section 7. Heart of the construction.* Assignment of a dimer configuration of $\tilde{G}^D(s)$ to each oriented spanning tree of $\mathcal{T}_{(W_2)}^s(\tilde{G}^D)$: *first key step of the construction*. The orientation of spanning trees of $\mathcal{T}_{(W_2)}^s(\tilde{G}^D)$ is key to constructing the dimer configuration. Partition of the set of spanning trees of $\mathcal{T}_{(W_2)}(\tilde{G}^D)$ according to their underlying dimer configuration: $\mathcal{T}_{(W_2)}(\tilde{G}^D) = \bigsqcup_{M \in \mathcal{M}(\tilde{G}^D(s))} \mathcal{T}_{(W_2),M}(\tilde{G}^D)$. Characterization of spanning trees of $\mathcal{T}_{(W_2),M}(\tilde{G}^D)$: *second key step of the construction*. This allows to compute the weight of this family of trees and prove that it is, up to a constant, the weight of the corresponding dimer configuration M . Summary and conclusion of the proof of Theorem 3.2 using the generalized form of Temperley's bijection [25] of [15].

2. Square of Ising partition function via bipartite dimers, critical case

In this section, we define the two-dimensional Ising model, the bipartite dimer model related to the double Ising model and the critical versions of the models. In the whole of the paper, we suppose that the graph $G = (V, E)$ is planar, finite, connected, simple, with vertices of degree at least two.

2.1. Two-dimensional Ising model

Consider a finite, planar graph $G = (V, E)$, together with a collection of positive *coupling constants* $J = (J_e)_{e \in E}$ indexed by edges of G . The *Ising model on G , with coupling constants J* , is defined as follows. A *spin configuration* σ is a function of the vertices of G taking values in $\{-1, 1\}$. The probability of occurrence of a spin configuration σ is given by the *Ising Boltzmann measure* $\mathbb{P}_{\text{Ising}}$, defined by:

$$\mathbb{P}_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}(G, J)} \exp\left(\sum_{e=uv \in E} J_e \sigma_u \sigma_v\right),$$

where $Z_{\text{Ising}}(G, J) = \sum_{\sigma \in \{-1, 1\}^V} \exp(\sum_{e=uv \in E} J_e \sigma_u \sigma_v)$ is the normalizing constant, known as the *Ising partition function*.

2.2. Boundary conditions

Suppose that the graph G is embedded in the plane, and also denote by \tilde{G} the embedded version of the graph. The set of *boundary vertices*, respectively *boundary edges*, of the graph \tilde{G} , consists of vertices, respectively edges, of the boundary of the outer face of \tilde{G} . Let us denote by ∂V the set of boundary vertices of \tilde{G} .

The Ising model defined in Section 2.1 is also known as the *Ising model with free-boundary conditions*. We now define the Ising model with *plus-boundary conditions*. A *spin configuration* σ is a function of the vertices of \tilde{G} taking values in $\{-1, 1\}$ with the additional constraint that the value is 1 on boundary vertices. Let us denote by $\{-1, 1\}_{(\partial V, +)}^V$ this set of spin configurations. The probability of occurrence of a spin configuration σ is defined similarly to the free-boundary case. The normalizing constant $Z_{\text{Ising}}^+(G, J)$ is the *Ising partition function with plus-boundary conditions*, it is equal to:

$$Z_{\text{Ising}}^+(G, J) = \sum_{\sigma \in \{-1, 1\}_{(\partial V, +)}^V} \exp\left(\sum_{e=uv \in E} J_e \sigma_u \sigma_v\right).$$

The Ising partition function with plus-boundary conditions is equal, up to a multiplicative constant, to the Ising partition function with free-boundary conditions of a related graph, obtained in the following way.

Let $E_{\partial V}$ denote the set of edges of G joining boundary vertices. Note that an edge of $E_{\partial V}$ is not necessarily a boundary edge of the graph G . Define $G' = (V', E')$ to be the graph obtained from G by merging all boundary vertices and edges of $E_{\partial V}$ into a single vertex u_0 . Then, the graph G' is again planar. Observing that $E \setminus E_{\partial V} = E'$, we have:

$$\begin{aligned} Z_{\text{Ising}}^+(G, J) &= \left(\prod_{e \in E_{\partial V}} e^{J_e} \right) \sum_{\sigma \in \{-1, 1\}_{(\partial V, +)}^V} \left(\prod_{e=uv \in E \setminus E_{\partial V}} e^{J_e \sigma_u \sigma_v} \right) \\ &= \left(\prod_{e \in E_{\partial V}} e^{J_e} \right) \sum_{\{\sigma \in \{-1, 1\}^{V'} : \sigma_{u_0} = 1\}} \left(\prod_{e=uv \in E'} e^{J_e \sigma_u \sigma_v} \right) \\ &= \frac{1}{2} \left(\prod_{e \in E_{\partial V}} e^{J_e} \right) \sum_{\sigma \in \{-1, 1\}^{V'}} \left(\prod_{e=uv \in E'} e^{J_e \sigma_u \sigma_v} \right) = \frac{1}{2} \left(\prod_{e \in E_{\partial V}} e^{J_e} \right) Z_{\text{Ising}}(G', J), \end{aligned}$$

where in the penultimate line, we have used the fact that the contribution $\prod_{e=uv \in E'} e^{J_e \sigma_u \sigma_v}$ is invariant under the transformation $\sigma \leftrightarrow -\sigma$.

The same kind of argument can be used to relate other boundary conditions to free ones. In the sequel, we thus restrict ourselves to the Ising model with free-boundary conditions, simply referred to as the Ising model. The result we prove holds for other boundary conditions as long as the related graphs satisfy the assumptions we shall make.

2.3. Square of Ising partition function via bipartite dimers

The goal of this section is to state Theorem 2.1, due to [8,24,28], proving equality, up to an explicit constant, between the partition function of the double Ising model on a planar graph G and the partition function of the dimer model on a related bipartite graph G^Q . The *double Ising model* consists of two independent Ising models living on the same graph G , with the same coupling constants J , implying that its partition function is the square of the partition function of the Ising model. Theorem 2.1 of this section is actually the genus 0 case of the result of [3] which holds for graphs embedded in surfaces of genus g .

In the whole of this section, we suppose that the planar graph G is embedded in the sphere, and also denote by G the embedded graph. We start by defining the bipartite graph G^Q .

2.3.1. Quadri-tiling graph

Suppose that the embedding of the dual graph G^* of G is such that primal and dual edges cross exactly once. The *quad-graph*, denoted G^\diamond , is the graph whose vertices are vertices of G and vertices of the dual graph G^* . A dual vertex is then joined to all primal vertices on the boundary of the corresponding face. The embedding of G^\diamond is chosen such that its edges do not intersect those of G and G^* . Faces of the quad-graph are quadrangles and the diagonals of the quadrangles consist of a primal edge of G and its corresponding dual edge of G^* , see Figure 1 (left, dotted lines) for an example.

Consider the graph obtained by superimposing the primal graph G , the dual graph G^* , the quad-graph G^\diamond , and by adding a vertex at the crossing of each primal and dual edge. Then, the dual of this graph, denoted by $G^Q = (V^Q, E^Q)$, is the *quadri-tiling* graph of G , see Figure 1 (right, thick full lines) for an example. The graph G^Q is bipartite, it contains:

- *Quadrangles*, each quadrangle being contained in a quadrangle of the quad-graph G^\diamond . In each quadrangle, two edges cross a primal edge of G and two edges cross a dual edge of G^* .
- *External edges*, crossing edges of the quad-graph G^\diamond .

Faces of the graph G^Q are naturally partitioned as follows: quadrangles corresponding to the crossing of primal and dual edges of G and G^* , faces corresponding to primal vertices of G , and those corresponding to dual vertices of G^* . Note that all vertices of G^Q have degree three.

2.3.2. Dimer model and square of Ising partition function

Consider an Ising model on the graph G , with coupling constants $J = (J_e)_{e \in E}$. Using the coupling constants J , define the following positive weight function $\nu = (\nu_e)_{e \in E^Q}$ on edges of the bipartite graph G^Q , see also Figure 2.

$$\nu_e = \begin{cases} 1 & \text{if } e \text{ is an external edge,} \\ \cosh^{-1}(2J_e) & \text{if } e \text{ crosses a primal edge } e \text{ of } G, \\ \tanh(2J_e) & \text{if } e \text{ crosses the dual edge } e^* \text{ of an edge } e \text{ of } G. \end{cases} \quad (2.1)$$

The *dimer model* on G^Q with weight function ν , is defined as follows. A *dimer configuration* of G^Q , also known as a *perfect matching*, is a subset of edges of G^Q such that every vertex is incident to exactly one edge of the subset. Let us denote by $\mathcal{M}(G^Q)$ the set of dimer configurations of the graph G^Q . The probability of occurrence of a dimer configuration M is given by the *dimer Boltzmann measure* $\mathbb{P}_{\text{dimer}}$, defined by:

$$\mathbb{P}_{\text{dimer}}(M) = \frac{\prod_{e \in M} \nu_e}{Z_{\text{dimer}}(G^Q, \nu)},$$

where $Z_{\text{dimer}}(G^Q, \nu) = \sum_{M \in \mathcal{M}(G^Q)} \prod_{e \in M} \nu_e$ is the normalizing constant, known as the *dimer partition function*.

The square of the Ising partition function and the dimer partition function are related by the following theorem.

Theorem 2.1 ([3,8,24,28]). *Let G be a finite, planar graph embedded in the sphere, and let G^Q be the corresponding bipartite graph. Then,*

$$(Z_{\text{Ising}}(G, J))^2 = 2^{|V|} \left(\prod_{e \in E} \cosh(2J_e) \right) \cdot Z_{\text{dimer}}(G^Q, \nu).$$

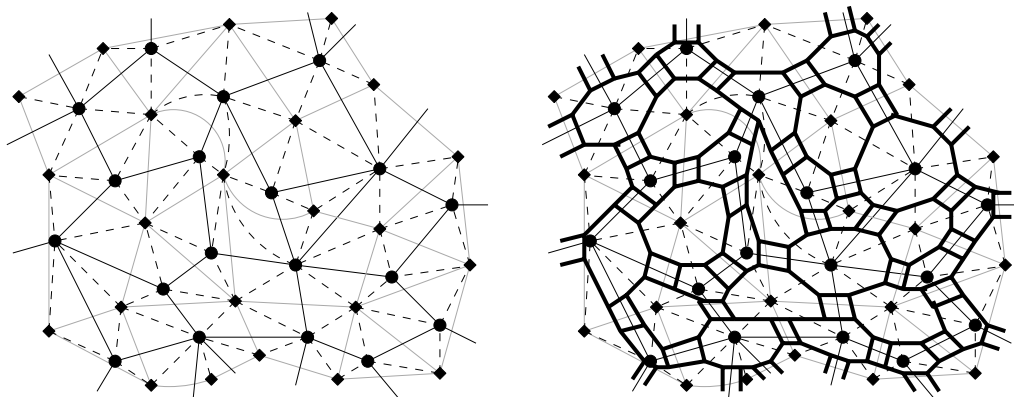


Fig. 1. Left: a piece of a finite graph G embedded in the sphere (plain full lines), of the dual graph G^* (grey full lines), of the quad-graph G^\diamond (dotted lines). Right: a piece of the quadri-tiling graph G^Q (thick full lines).

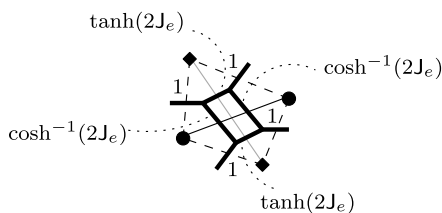


Fig. 2. Positive weight function ν on edges of the bipartite graph G^Q .

2.4. Critical versions of the model

In this section, we define the critical version of the Ising model defined on isoradial graphs, and the critical version of the corresponding bipartite dimer model. We need the following definition. If G is a planar graph, embedded in the plane, the *restricted dual graph* of G , denoted \underline{G}^* , is the dual of the graph G from which are removed: the vertex corresponding to the outer face and the dual edges corresponding to boundary edges of G .

2.4.1. Isoradial graphs

Isoradial graphs probably first appeared in the work of Duffin [9], see also Mercat [23]. The definition we present here and the name come from the paper [13] by Kenyon. A planar graph G is *isoradial*, if it can be embedded in the plane in such a way that all faces are inscribable in a circle, with all circles having the same radius. The embedding is said to be *regular* if all circumcenters are in the closure of the faces.

Let G be a finite, isoradial graph having a regular isoradial embedding. We fix such an embedding, take the common radius to be 1, and also denote by G the embedded graph. A regular isoradial embedding of the restricted dual \underline{G}^* , with radius 1, is obtained by taking as dual vertices the circumcenters of the corresponding faces. An example is provided in Figure 3 (left: plain and grey lines).

The *restricted quad-graph*, denoted \underline{G}^\diamond , is the graph whose vertices are vertices of the graph G and of the restricted dual \underline{G}^* . A dual vertex of \underline{G}^* is joined to all primal vertices on the boundary of the corresponding face, see Figure 3 (left, dotted lines). Note that this definition does not use isoradiality but, when the graphs G and \underline{G}^* are isoradially embedded, faces of the restricted quad-graph are side-length 1 rhombi, or half-rhombi along the boundary of the outer face.

Every edge e of G is the diagonal of exactly one rhombus, or half-rhombus, of the restricted quad-graph, and we let θ_e be the half-angle at the vertex it has in common with e , see Figure 3 (right).

2.4.2. Critical versions of the Ising and dimer models

Consider an Ising model defined on a finite, isoradial graph G with coupling constants J , having a regular isoradial embedding. We fix such an embedding and for every edge e of G , we let θ_e be the rhombus half-angle assigned to e . The Ising model is said to be *critical* if the coupling constants are equal to:

$$\forall e \in E, \quad J_e = \frac{1}{2} \log \left(\frac{1 + \sin \theta_e}{\cos \theta_e} \right). \quad (2.2)$$

These coupling constants were first derived by Baxter [1], using Z-invariance, a generalized form of self-duality and assumption of uniqueness of the critical point. They have been shown to be critical in the case of infinite periodic graphs by Li [21] and Cimasoni and Duminil-Copin [4]. This result has recently been extended to the case of general planar graphs by Lis [22].

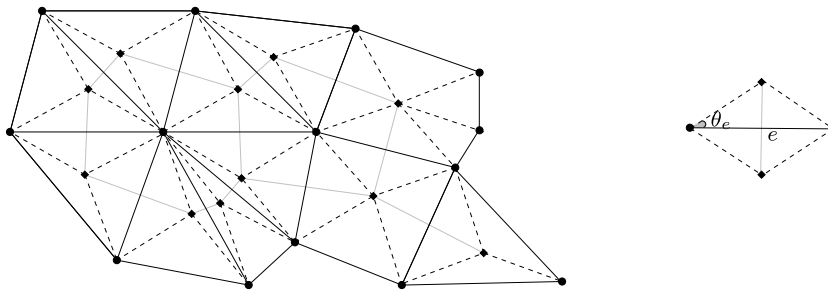


Fig. 3. Left: isoradial embedding of a finite planar graph G (plain black lines) and of the restricted dual graph \underline{G}^* (grey lines); restricted quad-graph \underline{G}^\diamond (dotted lines). Right: half-rhombus angles assigned to edges of G .

Consider a spherical embedding of the graph G , and let G^Q be the corresponding bipartite graph defined in Section 2.3.1. Using Equation (2.1), the dimer weights corresponding to the critical Ising weights, are equal to:

$$\forall e \in E^Q, \quad v_e = \begin{cases} 1 & \text{if } e \text{ is an external edge,} \\ \cos(\theta_e) & \text{if } e \text{ crosses a primal edge } e \text{ of } G, \\ \sin(\theta_e) & \text{if } e \text{ crosses the dual edge } e^* \text{ of an edge } e \text{ of } G. \end{cases} \quad (2.3)$$

3. Statement of main result

The goal of this section is to state the main result of this paper proving that the square of the critical Ising model partition function is equal, up to a multiplicative constant, to the partition function of critical spanning trees with appropriate boundary conditions. In Section 3.1, we define spanning trees, the Laplacian matrix and state the matrix-tree theorem. Then, in Section 3.2, we consider the isoradial graph G on which the critical Ising model is defined, extend the graph along its boundary into a graph \tilde{G} , and explain how the extended graph is embedded in the plane. Using this embedding we define a weight function on edges of \tilde{G} and of its dual \tilde{G}^* , which turns out to be the spanning trees' critical weight function of Kenyon [13], with specific boundary conditions. In Section 3.3, we state Theorem 3.2, the main theorem of the paper.

3.1. Spanning trees, Laplacian matrix, matrix-tree theorem

For the purpose of this section only, we let \tilde{G} be an arbitrary finite, directed graph; and we let G be an arbitrary undirected graph. Let r denote a distinguished vertex of G or \tilde{G} , referred to as the *root vertex*, or simply *root*.

An *r-rooted oriented spanning tree* (r OST) of \tilde{G} is a connected subset of oriented edges of \tilde{G} , such that each vertex of \tilde{G} has a unique outgoing edge, except the root r which has none. Equivalently, it is a subset of oriented edges, such that each vertex has a unique outgoing edge, except the root r which has none, and such that its unoriented version contains no cycle.

We denote by $\mathcal{T}^r(\tilde{G})$ the set of r OSTs of the graph \tilde{G} . Suppose that a weight function τ is assigned to oriented edges of \tilde{G} , meaning that an oriented edge (x, y) has weight $\tau_{(x, y)}$. The weighted sum of r OSTs of the graph \tilde{G} is denoted $Z_{\text{OST}}^r(\tilde{G}, \tau)$ and referred to as the *r OST partition function*.

$$Z_{\text{OST}}^r(\tilde{G}, \tau) = \sum_{T \in \mathcal{T}^r(\tilde{G})} \prod_{e=(x, y) \in T} \tau_{(x, y)}.$$

The *Laplacian matrix* of the directed graph \tilde{G} , denoted by $\Delta_{\tilde{G}}$, has lines and columns indexed by vertices of the graph. Coefficients of the matrix $\Delta_{\tilde{G}}$ are defined by:

$$(\Delta_{\tilde{G}})_{x, y} = \begin{cases} \tau_{(x, y)} & \text{if } (x, y) \text{ is an oriented edge of } \tilde{G}, \\ -\sum_{\{x': (x, x') \text{ is an oriented edge of } \tilde{G}\}} \tau_{(x, x')} & \text{if } y = x, \\ 0 & \text{else.} \end{cases}$$

The following theorem is the version for directed graphs of the classical theorem of Kirchhoff. It is due to Tutte [27], see also Chaiken [5].

Theorem 3.1 ([27]). *Let $\Delta_{\tilde{G}}$ be the Laplacian matrix of the directed graph \tilde{G} with weight function τ on the edges. Let $\Delta_{\tilde{G}}^{(r)}$ be the matrix $\Delta_{\tilde{G}}$ from which the line and the column corresponding to the root vertex r have been removed. Then,*

$$Z_{\text{OST}}^r(\tilde{G}, \tau) = \pm \det(\Delta_{\tilde{G}}^{(r)}).$$

Consider an undirected graph G . A *spanning tree* of G is a connected subset of $|V| - 1$ edges, or equivalently, a subset of $|V| - 1$ edges containing no cycle. An *r-rooted oriented spanning tree of G* is a spanning tree whose edges are oriented towards the root vertex r . With such an orientation, every vertex except the root r has exactly one outgoing

edge. Similarly to the directed case, $\mathcal{T}^r(\mathbf{G})$ denotes the set of r -OSTs of the graph \mathbf{G} . Suppose that a weight function τ is assigned to edges of \mathbf{G} , meaning that an edge xy has weight τ_{xy} . The *spanning tree partition function* of the graph \mathbf{G} is the weighted sum of spanning trees of \mathbf{G} . Since the graph is undirected, it is equal to the r -OST *partition function* $Z_{\text{OST}}^r(\mathbf{G}, \tau)$ for any choice of root vertex r .

To the graph \mathbf{G} is naturally associated a directed graph $\vec{\mathbf{G}}$, referred to as the *directed version* of \mathbf{G} : it has the same set of vertices, and two oriented edges (x, y) and (y, x) for every edge xy of \mathbf{G} . Then, an r -rooted oriented spanning tree of \mathbf{G} , is also an r -rooted oriented spanning tree of its directed version $\vec{\mathbf{G}}$. The weight function τ on edges of \mathbf{G} yields a symmetric weight function, also denoted τ , on the directed version $\vec{\mathbf{G}}$, defined by: $\tau_{(x,y)} = \tau_{(y,x)} = \tau_{xy}$. The Laplacian matrix of the graph \mathbf{G} , denoted $\Delta_{\mathbf{G}}$, is the Laplacian matrix of its directed version $\vec{\mathbf{G}}$; it is symmetric. In this case, Theorem 3.1 is the classical version of the matrix-tree theorem [14].

A classical fact about spanning trees is that if T is a spanning tree of a graph \mathbf{G} embedded in the plane and if T^* denotes the complement of the edges dual to T , then T^* is a spanning tree of the dual graph \mathbf{G}^* , it is referred to as the *dual spanning tree* of a spanning tree of \mathbf{G} . If a spanning tree is oriented, by dual spanning tree, we mean the dual of its unoriented version.

If a graph is partially directed, the above definitions are easily adapted.

3.2. Extended graphs, isoradial embeddings

Let \mathbf{G} be a finite, embedded planar graph. A planar embedding of the dual graph \mathbf{G}^* is obtained by assigning a dual vertex to the outer face of \mathbf{G} . The *extended dual graph*, denoted $\vec{\mathbf{G}}^* = (\vec{\mathbf{V}}^*, \vec{\mathbf{E}}^*)$, is obtained from the dual graph \mathbf{G}^* by splitting the vertex corresponding to the outer face of \mathbf{G} into n vertices, where n is the number of boundary edges of \mathbf{G} , and by joining these vertices in a circular way, see Figure 4 (left: grey lines) for an example. The *extended graph* $\vec{\mathbf{G}} = (\vec{\mathbf{V}}, \vec{\mathbf{E}})$ is the dual graph of the extended dual, see Figure 4 (left: plain and thick black lines). Let us denote by r the vertex of $\vec{\mathbf{G}}$ corresponding to the outer face of the extended dual $\vec{\mathbf{G}}^*$. Note that the graph \mathbf{G} , the extended dual $\vec{\mathbf{G}}^*$, and the extended graph $\vec{\mathbf{G}}$ can be embedded in such a way that they are planar, and such that primal and dual edges cross at most once.

The *extended quad-graph*, denoted $\vec{\mathbf{G}}^\diamond$ is obtained from the restricted quad-graph \mathbf{G}^\diamond by adding the missing half-quadrangles along the boundary of \mathbf{G} ; in doing so, vertices of the extended dual graph are also added.

Suppose now that the graph \mathbf{G} is finite, isoradial with a regular embedding, and fix such an embedding. Then, an embedding of the extended quad-graph $\vec{\mathbf{G}}^\diamond$ is obtained by adding, in the natural embedding of \mathbf{G}^\diamond , the missing half-rhombi along the boundary see Figure 4 (right: dotted lines).

Using this embedding, we now assign half-angles to edges of the extended graph $\vec{\mathbf{G}}$ that are incident to the vertex r . Consider an edge $e^\partial = xr$ of $\vec{\mathbf{G}}$ incident to the outer vertex r , then it is the dual of an edge (u, v) of the extended dual $\vec{\mathbf{G}}^*$, where u comes before v when going clockwise along the boundary. The half-angle θ^∂ assigned to the edge e^∂ , is half the angle $\angle(xv, xu)$ defined by the embedding of the extended quad-graph $\vec{\mathbf{G}}^\diamond$, see Figure 4 (right).

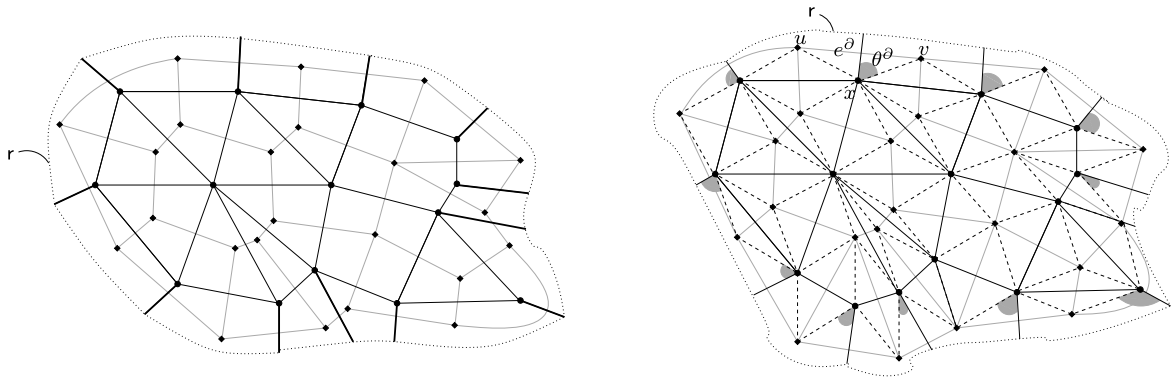


Fig. 4. Left: a planar graph \mathbf{G} (plain black lines), the extended dual $\vec{\mathbf{G}}^*$ (grey lines), and the extended graph $\vec{\mathbf{G}}$ (plain and thick black lines); the dotted line is a spread out way of representing the vertex r corresponding to the outer face of $\vec{\mathbf{G}}^*$. Right: extended quad-graph $\vec{\mathbf{G}}^\diamond$ (dotted lines) and half-angles assigned to edges of $\vec{\mathbf{G}}$ incident to the outer vertex r .

Remark 2. Note that half-rhombi added to construct the extended quad-graph from the reduced quad-graph in the isoradial embedding, may overlap. Then, half-angles assigned to edges along the boundary of the extended graph \tilde{G} may be negative; note that they may also be larger than $\frac{\pi}{2}$.

3.3. Statement of main result

Let G be a finite, isoradial graph, with a fixed isoradial embedding. Consider the extended dual graph \tilde{G}^* and the extended graph \tilde{G} defined in Section 3.2. We now define the weight function τ on edges of \tilde{G} and \tilde{G}^* . It is defined on oriented edges, but is non-symmetric only on the boundary of the graphs, implying that only boundary edges of the graphs are ‘really’ oriented. If uv is a boundary edge of the extended dual graph \tilde{G}^* , then by convention when writing (u, v) , we mean that u comes before v when traveling clockwise along the boundary. The weight function τ is defined as follows.

$$\begin{cases} \tau_{(x,y)} = \tau_{(y,x)} = \tan \theta_e & \text{if } xy \text{ is an } e \text{ edge of } G, \\ \tau_{(x,r)} = 2 \sin \frac{\theta^\partial}{2} & \text{if } xr \text{ is an edge of } \tilde{G} \text{ incident to } r, \\ \tau_{(r,x)} = 0, \\ \tau_{(u,v)} = \tau_{(v,u)} = 1 & \text{if } uv \text{ is the dual of an edge } e \text{ of } G, \\ \tau_{(u,v)} = e^{-i\theta^\partial/2} & \text{if } (u, v) \text{ is a boundary edge of } \tilde{G}^*, \\ \tau_{(v,u)} = e^{i\theta^\partial/2} & \text{if } (u, v) \text{ is a boundary edge of } \tilde{G}^*. \end{cases} \quad (3.1)$$

Note that away from the boundary, this is the spanning trees critical weight function introduced by Kenyon [13].

Fix a root vertex s on the boundary of the extended dual \tilde{G}^* . Given an oriented spanning tree T of \tilde{G} rooted at r , let us denote by T^* the dual spanning tree of \tilde{G}^* oriented towards the vertex s . Let $Z_{\text{OST}}^r(\tilde{G}, \tau)$ be the following weighted sum over spanning trees of \tilde{G} :

$$Z_{\text{OST}}^r(\tilde{G}, \tau) = \sum_{T \in \mathcal{T}^r(\tilde{G})} \left(\prod_{e=(x,y) \in T \cap E} \tau_{(x,y)} \right) \left(\prod_{(x,y) \in T \cap (\tilde{E} \setminus E)} \tau_{(x,y)} \right) \left(\prod_{(u,v) \in T^* \cap (\tilde{E}^* \setminus E^*)} \tau_{(u,v)} \right),$$

where the sum is over r OSTs of \tilde{G} . Recall that we have chosen dual spanning trees of r OSTs of \tilde{G} to be oriented towards the root vertex s .

Remark 3. Note that dual spanning trees contribute only along the boundary of the extended dual graph \tilde{G}^* . If the boundary of the extended graph \tilde{G}^* were undirected, we could factor out the contributions of dual spanning trees; but this is not possible because the boundary of the extended graph \tilde{G}^* is directed.

Theorem 3.2. Consider the critical Ising model defined on a finite isoradial graph G , having a regular embedding, with critical coupling constants J assigned to edges. Let \tilde{G} and \tilde{G}^* be the extended versions of the graph G and of its dual graph, with weight function τ of Equation (3.1) assigned to edges. Then,

$$(Z_{\text{Ising}}(G, J))^2 = 2^{|V|} |Z_{\text{OST}}^r(\tilde{G}, \tau)|.$$

The remainder of the paper consists in the proof of Theorem 3.2. It starts from Theorem 2.1 where the squared Ising partition function of the planar graph G is shown to be equal to the dimer partition function of the bipartite graph G^Q .

4. Kasteleyn theory for the critical dimer model on the graph G^Q

In this section we write an explicit formula for the partition function of the dimer model on the bipartite graph G^Q with weights ν assigned to edges, corresponding to the critical Ising weights. In a way similar to Kenyon [13], we use the approach of Kuperberg [16], which generalizes the work of Kasteleyn [12], and Temperley and Fisher [26].

4.1. Explicit formula for the dimer partition function

Kasteleyn [12] and independently Temperley and Fisher [26] proved an explicit formula for the partition function of the dimer model defined on a finite, planar graph G . The set of vertices V can naturally be split into two subsets $W \cup B$, where W denotes white vertices, B black ones, and vertices in W are only incident to vertices in B . Then, the dimer partition function is equal to the determinant of a weighted, oriented adjacency matrix of the graph G , whose lines are indexed by white vertices and columns by black ones. Edges of the graph G are oriented in such a way that all cycles surrounding inner faces of the graph are *clockwise odd*, i.e., when traveling clockwise around such a cycle, the number of co-oriented edges is odd [12].

Instead of an admissible orientation, Kuperberg [16] proves that one can use an appropriate *phasing of the edges*: every edge $e = wb$ is assigned a modulus-one, complex number $e^{i\phi_{wb}}$, where ϕ_{wb} is referred to as the *phase* of the edge e . Let F be an inner face of the graph G , whose boundary vertices are $w_1, b_1, \dots, w_k, b_k$ in clockwise order. The *Kasteleyn curvature* at the face F , denoted by $C(F)$, is:

$$C(F) = (-1)^{k-1} \frac{\prod_{j=1}^k e^{i\phi_{w_j b_j}}}{\prod_{j=1}^k e^{i\phi_{w_{j+1} b_j}}}.$$

The phasing of the edges is *flat* if, for every inner face F of the graph G , $C(F) = 1$.

Suppose that edges of the graph G are assigned positive weights $(v_e)_{e \in G}$. Define the *Kasteleyn matrix* K to be the following complex valued, adjacency matrix of the graph G :

$$\forall w \in W, b \in B, \quad K_{w,b} = \begin{cases} e^{i\phi_{wb}} v_e & \text{if } e = wb \text{ is an edge of } G, \\ 0 & \text{else.} \end{cases} \quad (4.1)$$

Kuperberg [16] proves that if the phasing of the edges is flat, the dimer partition function is equal to the modulus of the determinant of the Kasteleyn matrix:

$$Z_{\text{dimer}}(G, v) = |\det(K)|.$$

4.2. Choice of phases

Let G be a finite, planar, embedded graph. Recall that the bipartite graph G^Q is constructed from a spherical embedding of the graph G and of its dual graph G^* . A planar embedding of G^Q is obtained by taking a dual vertex of G^* as the outer face of G^Q , see Figure 5 (thick black lines).

Suppose moreover that the graph G is isoradial with a fixed regular isoradial embedding. Consider the induced embedding of the extended quad-graph \tilde{G}^\diamond of Section 3.2, see also Figure 5 (dotted lines). A near-isoradial embedding

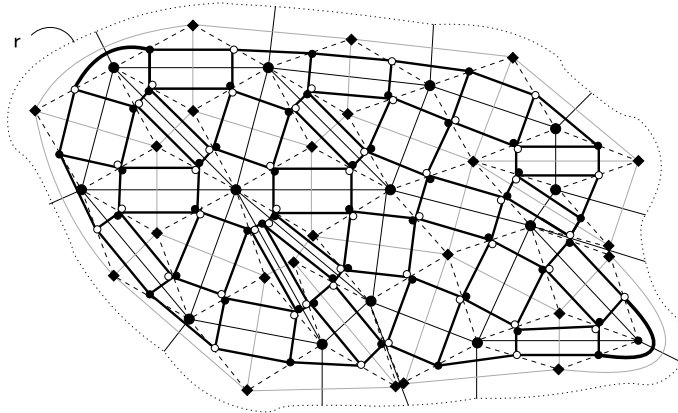


Fig. 5. Near isoradial embedding of the graph G^Q .

of the graph G^Q is constructed as follows: draw quadrangles of the bipartite graph G^Q in such a way that they are rectangles included in the rhombi of the extended quad-graph; draw vertices of the rectangles as midpoints of the sides of the rhombi. An external edge which is not on the boundary of the outer face of G^Q has length 0. An external edge on the boundary of the outer face is drawn so that the face contains the primal vertex in its interior.

Remark 4.

- Away from the boundary, this defines a regular isoradial embedding of the graph G^Q with radius $\frac{1}{2}$. Critical dimer weights on the graph G^Q induced by the critical Ising model, are the critical dimer weights defined by Kenyon [13] for bipartite isoradial graphs.
- Along the boundary, if one draws straight lines to join boundary vertices, then boundary faces are inscribed in a circle of radius $\frac{1}{2}$, but it might be that the circumcenter is outside of the face (this happens in the lower right corner of Figure 5). The embedding is thus isoradial, but not regular. What can also happen is that rhombi of the extended quad-graph \tilde{G}^\diamond overlap. In this case, the embedding is not planar. One can nevertheless define the notion of face, and show that it can be embedded in a circle of radius $\frac{1}{2}$.

The bipartite coloring of the vertices is fixed so that it is as in Figure 5. We now define phases assigned to edges of G^Q . Recall that edges of G^Q either belong to quadrangles/rectangles, or are external edges.

1. Each rectangle belongs to a rhombus of the extended quad-graph \tilde{G}^\diamond , which contains an edge e of G . An edge $e = wb$ of the rectangle either crosses the dual edge e^* of the edge e , in which case we let $\phi_{wb} = 0$; or it crosses the edge e , and we let $\phi_{wb} = \frac{\pi}{2}$.
- 2(a). An external edge $e = wb$ which is not on the boundary of G^Q crosses an edge shared by two rhombi of the extended quad-graph. The vertex w belongs to one of the two rhombi, and we let $\phi_{wb} = \frac{3\pi}{2} - \theta_{e(w)}$, where $\theta_{e(w)}$ is the rhombus half-angle of the edge $e(w)$ of G contained in the rhombus, see Figure 6 (left).
- 2(b). An external edge $e = wb$ on the boundary of G^Q crosses an edge e^∂ of the extended graph \tilde{G} . Moreover, the vertex w belongs to a quadrangle contained in a rhombus of the extended quad-graph. We let $\phi_{wb} = \frac{3\pi}{2} - (\theta_{e(w)} + \theta^\partial)$, where $\theta_{e(w)}$ is the rhombus half-angle of the edge $e(w)$ of G contained in the rhombus, and θ^∂ is the half-angle assigned in Section 3.2 to edges of the extended graph \tilde{G} , incident to the vertex r , see Figure 6 (right). When no confusion occurs, we shall omit the argument w in the notation of $\theta_{e(w)}$.

Away from the boundary, the choice of phases is, up to a gauge transformation, the complex conjugate of the choice made in [13].

Lemma 4.1. *The phasing of the edges of G^Q defined by Points 1 and 2 above is flat.*

Proof. We need to check that for every inner face F of G^Q , $C(F) = 1$. Inner faces are of three types, there are rectangles, faces corresponding to vertices of G and faces corresponding to vertices of the restricted dual \mathcal{G}^* . Using [13], we are left with proving that if F is a face of length $2k$, corresponding to a vertex of the primal graph G , and incident to the boundary, then $C(F) = 1$. The situation, as well as the labeling of the vertices is represented in Figure 7.

By Points 1, 2(a) and 2(b), the Kasteleyn curvature is:

$$C(F) = (-1)^{k-1} \frac{e^{i[\sum_{j=1}^k (3\pi/2 - \theta_{e_j}) - \theta^\partial]}}{i^k} = - \frac{i^k e^{-i[\sum_{j=1}^k \theta_{e_j} + \theta^\partial]}}{i^k}.$$

Observing that $\sum_{j=1}^k \theta_{e_j} + \theta^\partial = \pi$, yields $C(F) = 1$. □

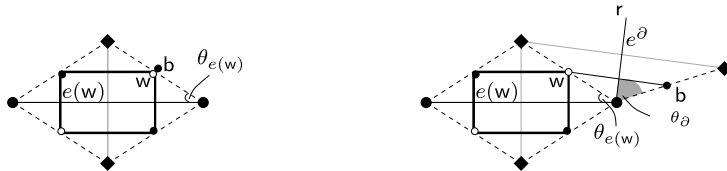


Fig. 6. Definition of the phase ϕ_{wb} of an external edge wb . Left: wb is not on the boundary of G^Q . Right: wb is on the boundary of G^Q .

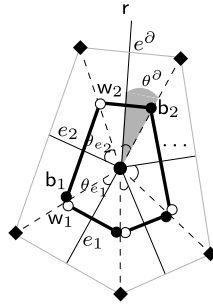


Fig. 7. Notations for a face F of length $2k$, corresponding to a vertex of the primal graph G , incident to the boundary, with $k = 4$.

Wrapping up, we obtain the following. Let G be an isoradial graph having a regular isoradial embedding. Consider an Ising model on G , with critical coupling constants J of Equation (2.2). Consider the corresponding dimer model on the bipartite graph G^Q , with the critical dimer weights ν of Equation (2.3). Let K be the Kasteleyn matrix of the graph G^Q , with the above choice of phases: $\forall w \in W^Q, b \in B^Q$,

$$K_{w,b} = \begin{cases} \sin \theta_e & \text{if } e \text{ crosses the dual edge } e^* \text{ of an edge } e \text{ of } G, \\ i \cos \theta_e & \text{if } e \text{ crosses a primal edge } e \text{ of } G, \\ e^{i(3\pi/2 - \theta_{e(w)})} & \text{if } e \text{ is an external edge, not on the boundary of } G^Q, \\ e^{i[3\pi/2 - (\theta_{e(w)} + \theta^\partial)]} & \text{if } e \text{ is an external edge on the boundary of } G^Q, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Proposition 4.2. *The dimer partition function of the graph G^Q with critical weight function ν , is equal to:*

$$Z_{\text{dimer}}(G^Q, \nu) = |\det K|.$$

Remark 5. *Note that the determinant of K is real. In general, assigning phases to edges might induce a complex phase factoring out in the determinant. The complex phase is that of any of the dimer configurations. Since the contribution of the dimer configuration of G^Q consisting of all edges crossing dual edges of G is real, we know that the determinant is real.*

5. The Kasteleyn matrix as a Laplacian matrix

In Section 5.1, we compute the sum of the columns of the Kasteleyn matrix K of the graph G^Q . Using this information and a reinterpretation of the vertices of G^Q this allows us, in Section 5.2, to interpret the Kasteleyn matrix K as the Laplacian matrix of a directed graph \tilde{G}_0^Q constructed from G^Q . Using the matrix-tree theorem for directed graphs, we deduce that the determinant of the matrix K counts oriented spanning trees of the directed graph \tilde{G}_0^Q . In Section 5.3, we modify the directed graph \tilde{G}_0^Q along the boundary. This is key to being able to handle the boundary of the graph.

5.1. Column sum of the Kasteleyn matrix

Consider the bipartite graph G^Q . Recall that all vertices of G^Q have degree 3; given a white vertex w , let us denote its three neighbors b_1, b_2, b_3 in counterclockwise order, where wb_3 denotes the external edge of G^Q , see Figure 8. Let K be the Kasteleyn matrix of the graph G^Q with critical dimer weights ν of Equation (4.2).

Lemma 5.1. *For every white vertex w of G^Q , the coefficient K_{w,b_3} of the Kasteleyn matrix is equal to:*

$$K_{w,b_3} = \begin{cases} -K_{w,b_1} - K_{w,b_2} - i e^{-i\theta_e} (e^{-i\theta^\partial} - 1) & \text{if } w \text{ is a boundary vertex of } G^Q, \\ -K_{w,b_1} - K_{w,b_2} & \text{otherwise.} \end{cases} \quad (5.1)$$



Fig. 8. Labeling of the neighbors b_1, b_2, b_3 of a white vertex w , when the vertex w is not on the boundary of G^Q (left), and when it is (right).

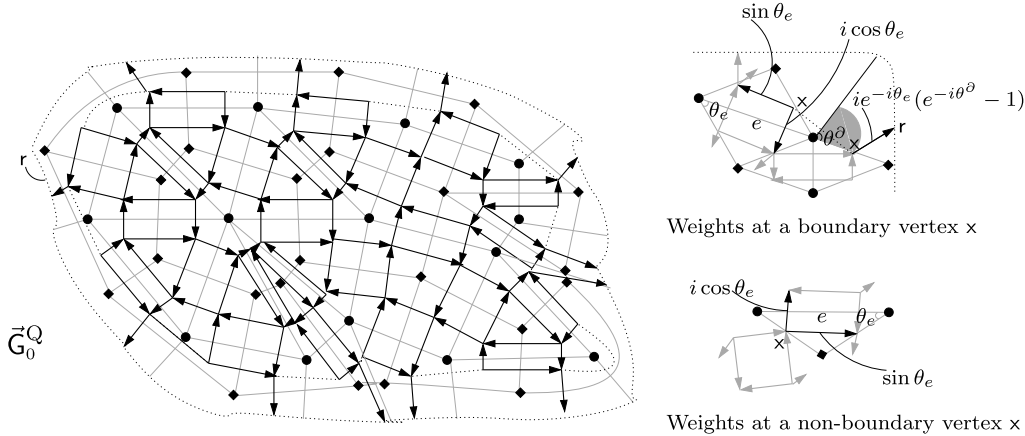


Fig. 9. Left: the directed graph \vec{G}_0^Q obtained from the bipartite graph G^Q of Figure 5; edges that are dotted are reduced to a point. Right: weights ρ_0 assigned to edges.

Proof. If the vertex w is not on the boundary of G^Q , this is a consequence of the fact that constant functions are discrete holomorphic [13]. A straightforward computation also shows that $\sum_{j=1}^3 K_{w,b_j} = 0$. If the vertex w is on the boundary of G^Q , we have:

$$\sum_{j=1}^3 K_{w,b_j} = \sin \theta_e + i \cos \theta_e + e^{i(3\pi/2 - \theta_e - \theta^\delta)} = i e^{-i\theta_e} (1 - e^{-i\theta^\delta}).$$

□

5.2. Writing the Kasteleyn matrix as a Laplacian matrix

Using Lemma 5.1, we interpret the Kasteleyn matrix of the graph G^Q as the Laplacian matrix of a related, directed graph. Let us order the lines and columns of the matrix K in such a way that if the i th line corresponds to the white vertex w , then the i th column represents the black vertex b_3 . The coefficient K_{w,b_3} becomes a diagonal element, and we write it as in Equation (5.1). Note that this transformation only changes the determinant by an overall \pm sign. Now, instead of labeling the line by w and the column by b_3 , we label it by a common vertex, denoted x . Geometrically, consider the directed graph \vec{G}_0^Q obtained from the graph G^Q as follows, see also Figure 9.

- Replace edges of quadrangles of G^Q by oriented ones, from the white vertex to the black vertex. An oriented edge (x, y) of a quadrangle is assigned weight $\rho_{0(x,y)} = \sin \theta_e$, if it crosses a dual edge e^* of an edge e of G ; and weight $\rho_{0(x,y)} = i \cos \theta_e$, if it crosses a primal edge e of G . Note that each quadrangle has two corners with two outgoing edges, and two corners with two incoming edges.
- Each external edge $w b_3$ is merged into a single vertex x .
- Each vertex x arising from the merging of an edge $w b_3$ on the boundary of the outer face of G^Q is called a *boundary vertex* of \vec{G}_0^Q . It is joined to the vertex r , where recall that the vertex r is the vertex of the extended graph \vec{G} corresponding to the outer face of the extended dual \vec{G}^* . The oriented edge (x, r) is assigned weight $\rho_{0(x,r)} = i e^{-i\theta_e} (1 - e^{-i\theta^\delta})$. Note that boundary vertices have out-degree 3.

In accordance with the notation introduced in Section 3.1, we let $\Delta_{\vec{G}_0^Q}$ be the Laplacian matrix of the directed graph \vec{G}_0^Q with weight function ρ_0 assigned to edges. Observing that the Kasteleyn matrix K is exactly the matrix $\Delta_{\vec{G}_0^Q}$ from which the line and the column corresponding to the vertex r have been removed we deduce, from Theorem 3.1, the following proposition.

Proposition 5.2. *The dimer partition function of the graph G^Q with weight function v , is equal to the absolute value of the r OST partition function of the graph \vec{G}_0^Q with weight function ρ_0 :*

$$Z_{\text{dimer}}(G^Q, v) = |Z_{\text{OST}}^r(\vec{G}_0^Q, \rho_0)|.$$

5.3. One more graph transformation

From the directed graph \vec{G}_0^Q with weight function ρ_0 , we construct a directed graph \vec{G}_1^Q with weight function ρ on the edges, such that their respective r OST partition functions are equal. The directed graph \vec{G}_1^Q only differs from the graph \vec{G}_0^Q along the boundary. The purpose of this transformation is to have a directed graph with out-degree two at every vertex, and to have half-quadrangles along the boundary, with one corner and two outgoing edges. This is key to being able to handle the boundary of the graph. The transformation is defined as follows, see also Figure 10.

- For every boundary vertex x of \vec{G}_0^Q arising from the merging of a boundary edge wb_3 of G^Q , undo the merging procedure. That is, we replace the vertex x by the two vertices w and b_3 , add the oriented edge (b_3, w) and assign it weight, $\rho_{(b_3, w)} = 1$. The edge (x, r) exiting the vertex x is drawn to exit the vertex b_3 , and the edge (b_3, r) is assigned weight, $\rho_{(b_3, r)} = e^{-i\theta^\partial} - 1$.
- All other vertices, edges and edge-weights are left unchanged: $\rho_{(x, y)} = \rho_{0(x, y)}$.

The next proposition proves that this transformation preserves r OST partition functions.

Proposition 5.3. *The r OST partition function of the graph \vec{G}_1^Q with weight function ρ is equal to the r OST partition function of the graph \vec{G}_0^Q with weight function ρ_0 :*

$$Z_{\text{OST}}^r(\vec{G}_1^Q, \rho) = Z_{\text{OST}}^r(\vec{G}_0^Q, \rho_0).$$

Proof. The proof consists in providing a weight-preserving mapping between r OSTs of \vec{G}_0^Q and r OSTs of \vec{G}_1^Q . Recall that an OST is characterized by the fact that it has one outgoing edge at every vertex except the root r , and that its unoriented version is connected. We start by exhibiting a mapping between oriented edge configurations of \vec{G}_0^Q and \vec{G}_1^Q having one outgoing edge at every vertex except the root, then show that it preserves weights and connectedness.

In the graph \vec{G}_0^Q , a boundary vertex x has out-degree three; the two corresponding vertices b_3 and w of \vec{G}_1^Q each has out-degree two. As a consequence, an oriented edge configuration of \vec{G}_0^Q , resp. \vec{G}_1^Q , having one outgoing edge at every vertex except the root, is locally one of the three configurations A1–A2–A3 of \vec{G}_0^Q , resp. B1–B2–B3a–B3b

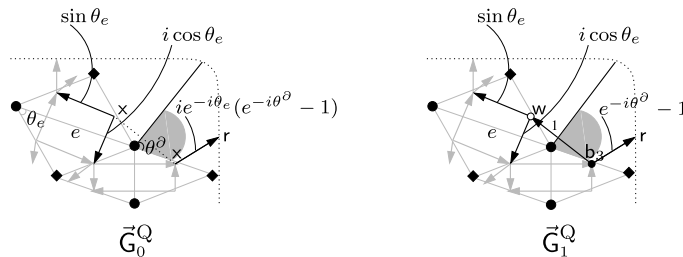


Fig. 10. From the graph \vec{G}_0^Q to the graph \vec{G}_1^Q .

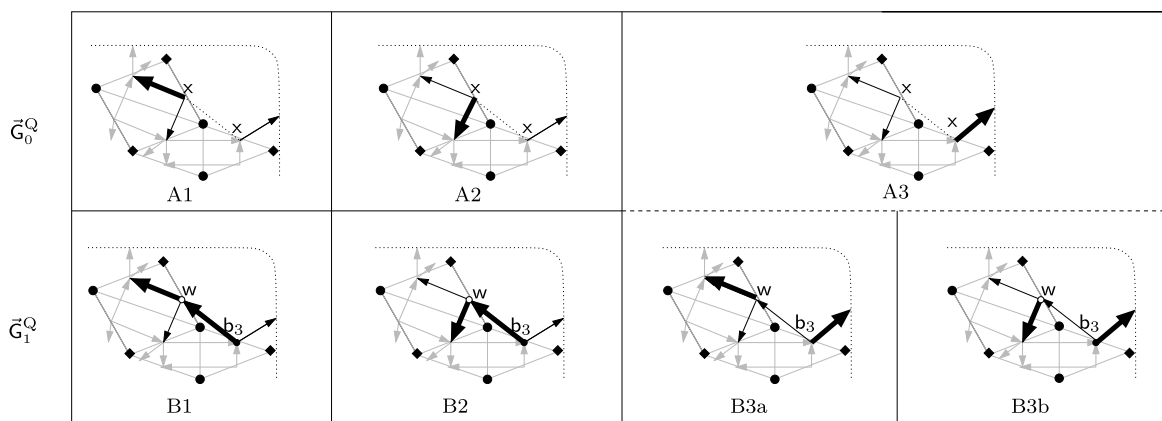


Fig. 11. From 'OSTs of \vec{G}_0^Q to 'OSTs of \vec{G}_1^Q .

of \vec{G}_1^Q . Consider the mapping between such configurations, which does not change the configuration at non-boundary vertices, and which acts as in Figure 11 (line A–line B) for boundary vertices.

Let us first prove that the mapping is weight preserving. The mapping leaves edges exiting non-boundary vertices unchanged, and by definition ρ_0 and ρ are equal on such edges, implying that the contributions are equal. It thus remains to check that the contributions of edges exiting boundary vertices are the same. Let us fix a boundary vertex x of \vec{G}_0^Q and the corresponding boundary vertices w, b_3 of \vec{G}_1^Q . By definition, the weight function ρ assigns weight 1 to the edge (b_3, w) , so that the weight is preserved for Cases 1 and 2. Let us check Case 3. The contribution of the edge exiting x is:

$$\rho_{0(x,r)} = ie^{-i\theta_e}(e^{-i\theta^\partial} - 1).$$

The contribution at w and b_3 of the corresponding configurations of \vec{G}_1^Q is:

$$\rho_{(b_3,r)}[\rho_{(w,b_1)} + \rho_{(w,b_2)}] = (e^{-i\theta^\partial} - 1)(\sin\theta_e + i\cos\theta_e) = (e^{-i\theta^\partial} - 1)ie^{-i\theta_e},$$

and the mapping is indeed weight-preserving. We are left with proving that it preserves connectedness. Consider the unoriented version T of an 'OSTs of \vec{G}_0^Q , and consider T as a subgraph of \vec{G}_1^Q . Then T is a connected subgraph of \vec{G}_1^Q spanning all vertices of \vec{G}_1^Q , except b_3 in Cases 1 and 2, resp. w in Case 3 (otherwise T would contain a cycle in \vec{G}_0^Q). The configuration of \vec{G}_1^Q constructed from T in line B consists of T and an edge connecting b_3 to T in Cases 1 and 2, resp. w to T in Case 3; it is thus connected. The reverse argument also holds. \square

From Proposition 5.2 and Proposition 5.3, we deduce the following:

Corollary 5.4. *The dimer partition function of the graph G^Q with weight function v , is equal to the absolute value of the 'OST partition function of the graph \vec{G}_1^Q with weight function ρ :*

$$Z_{\text{dimer}}(G^Q, v) = |Z_{\text{OST}}'(\vec{G}_1^Q, \rho)|.$$

6. Oriented spanning trees of \vec{G}_1^Q and of the extended double

In this section, we describe a weight-preserving mapping from 'OSTs of the graph \vec{G}_1^Q to a family of spanning trees of the extended double graph \vec{G}^D of G , which we then characterize.

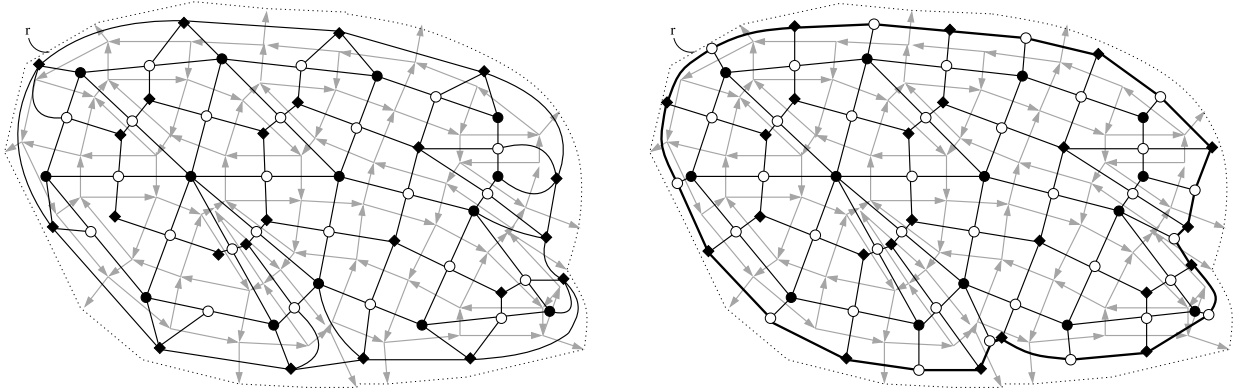


Fig. 12. Left: the dual graph G_1^{Q*} (black) of \tilde{G}_1^Q (grey). Right: the extended double graph \tilde{G}^D of G (black).

6.1. Dual spanning trees of the extended double graph

Consider the dual graph G_1^{Q*} of the undirected version of \tilde{G}_1^Q , see Figure 12 (left). Even though G_1^{Q*} is not bipartite, its vertices are naturally split into black and white vertices as follows: \bullet -black: vertices of the primal graph G ; \blacklozenge -black: vertices of the extended dual graph \tilde{G}^* ; \circ -white: dual vertices of the quadrangles of \tilde{G}_1^Q .

Recall from Section 3.1 that if T is a spanning tree of a graph, the dual configuration T^* consisting of the complement of the edges dual to T , is a spanning tree of the dual graph. As a consequence, there is a one-to-one correspondence between r -OSTs of \tilde{G}_1^Q and dual spanning trees of G_1^{Q*} .

The graph G_1^{Q*} is nearly the graph we are aiming for; we need to make one more transformation so that it becomes the *extended double graph* of G , defined as follows. Consider the extended dual graph \tilde{G}^* of G , and the dual graph \tilde{G} of \tilde{G}^* . The *extended double graph* \tilde{G}^D of G , consists of the graph \tilde{G}^* and the graph \tilde{G} , with a white vertex added at the crossing of each primal and dual edge; one then removes all edges connected to the vertex r of \tilde{G} corresponding to the outer face of \tilde{G}^* . An example of extended double graph is given in Figure 12 (right). The construction of the double graph first appears in the paper of Temperley [25] proving a bijection between spanning trees of the square grid and perfect matchings of its double graph. This bijection has been generalized to more general planar graphs and oriented spanning trees by Kenyon, Propp and Wilson [15].

Observe that the extended double graph \tilde{G}^D of G is obtained from the dual graph G_1^{Q*} by splitting every boundary \blacklozenge -black vertex of the extended dual \tilde{G}^* , into a \blacklozenge -black vertex b and a white vertex w , and by adding the edge wb . As a consequence, to every spanning tree of G_1^{Q*} corresponds a spanning tree of the extended double \tilde{G}^D , obtained by adding the edges arising from the splitting of the boundary \blacklozenge -black vertices. This mapping is weight-preserving if the new edges are assigned weight 1.

Consider an r -OST T of the directed graph \tilde{G}_1^Q and its dual spanning tree in G_1^{Q*} . Then, the *dual spanning tree* in \tilde{G}^D of T is defined to be the corresponding spanning tree of the extended double graph \tilde{G}^D , see Figure 13 for an example.

6.2. Weight preserving mapping

Let us now assign weights ρ^* to edges of the double graph \tilde{G}^D , so that the mapping from r -OSTs of \tilde{G}_1^Q with weight function ρ , to dual spanning trees in \tilde{G}^D with weight function ρ^* , preserves weights.

By construction of dual spanning trees in \tilde{G}^D , we know that edges arising from the splitting of boundary \blacklozenge -black vertices are always present and are assigned weight 1.

Recall that each vertex x of the graph \tilde{G}_1^Q , except the root r , has two outgoing edges; and that an r -OST T of \tilde{G}_1^Q takes exactly one of the two edges. Then, the dual of the present (respectively absent) edge in T , is absent (respectively present) in the dual spanning tree in \tilde{G}^D , see Figure 14. This yields a one-to-one correspondence between edges of r -OSTs of \tilde{G}_1^Q and edges of dual spanning trees in \tilde{G}^D .

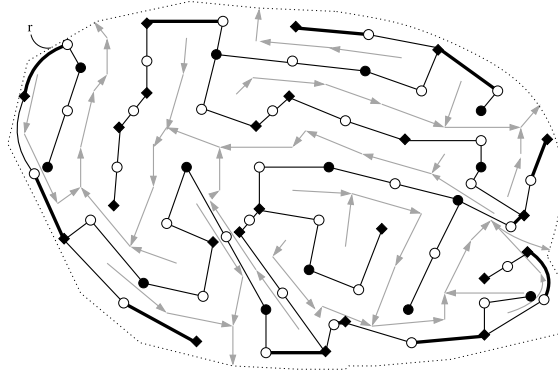
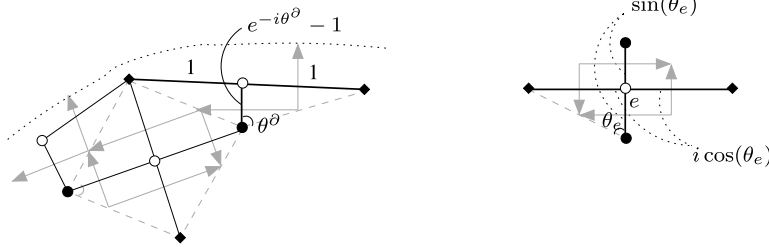


Fig. 13. An r OST of \bar{G}_1^Q (grey) and its dual spanning tree in the extended double \bar{G}^D graph (black). By definition of the *dual in \bar{G}^D* , the thicker edges are always present.



Fig. 14. Possible primal and dual configurations at a vertex x of \bar{G}_1^Q . Thick lines represent present edges, thin lines absent ones.



Weights around a boundary white vertex

Weights around a non-boundary white vertex

Fig. 15. Definition of weights ρ^* assigned to edges of the extended double graph \bar{G}^D .

Returning to the definition of the weight function ρ , see Section 5.3, this implies that the following weight function ρ^* defined on edges of the extended dual \bar{G}^D , preserves weights, see also Figure 15:

$$\rho_{wb}^* = \begin{cases} \sin \theta_e & \text{if } wb \text{ is half a primal edge } e \text{ of } G, \\ i \cos \theta & \text{if } wb \text{ is half a dual edge } e^* \text{ of an edge } e \text{ of } G, \\ 1 & \text{if } wb \text{ is half a dual edge on the boundary of the extended dual } \bar{G}^*, \\ e^{-i\theta\partial} - 1 & \text{if } wb \text{ is half an edge of } \bar{G} \setminus G. \end{cases} \quad (6.1)$$

6.3. Characterization of dual spanning trees of the double graph

The next lemma characterizes spanning trees of \bar{G}^D arising as duals of r OSTs of \bar{G}_1^Q .

Lemma 6.1. *Consider an r OST of \bar{G}_1^Q . Then, the restriction around a white vertex w of \bar{G}^D of the dual spanning tree in \bar{G}^D , is one of the following four configurations (black lines):*

1. *if the vertex w is not on the boundary of \bar{G}^D (see Figure 16);*
2. *if the vertex w is on the boundary of \bar{G}^D (see Figure 17).*

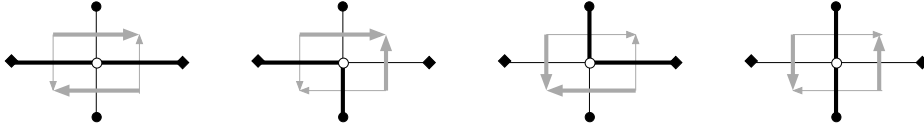


Fig. 16. The four possible configurations around a non-boundary white vertex of \tilde{G}^D .

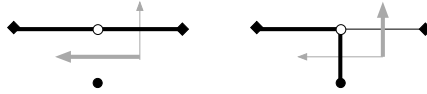


Fig. 17. The two possible configurations around a boundary white vertex of \tilde{G}^D .

Conversely, consider a spanning tree of \tilde{G}^D satisfying 1 and 2, then the oriented version of its dual spanning tree in \tilde{G}_1^Q , oriented towards the root vertex r , is an r OST of \tilde{G}_1^Q .

Remark 6. Observe that dual spanning trees of \tilde{G}^D , arising as duals of r OSTs of \tilde{G}_1^Q , have two edges incident to every white vertex w of \tilde{G}^D . Note also that the two cases of Point 2 are two of the four cases of Point 1. This is because along the boundary, there are half-quadrangles with one corner and two outgoing edges.

Proof of Lemma 6.1. Suppose that the vertex w is not on the boundary of \tilde{G}^D , the argument being similar in the boundary case. The vertex w corresponds to a quadrangle of \tilde{G}_1^Q . Two edges of \tilde{G}_1^Q exit one corner of the quadrangle and two the opposite corner. These two corners have no other exiting edges than those of the quadrangle, implying that the restriction of an r OST T of \tilde{G}_1^Q to the quadrangle around w is one of the four possible grey configurations of Point 1. Looking at dual configurations, we obtain the four black configurations of Point 1.

Conversely, suppose we are given a spanning tree of \tilde{G}^D such that the restriction around every white vertex satisfies Points 1 and 2. We need to check that the orientation of its dual spanning tree, induced by the orientation of the edges of \tilde{G}_1^Q yields an r OST of \tilde{G}_1^Q . To prove this, it suffices to show that every vertex of \tilde{G}_1^Q , except the root r , has exactly one outgoing edge. This holds as a consequence of our observation on the orientation of edges of the quadrangles. \square

Let us denote the conditions of Points 1 and 2 of Lemma 6.1 by (W2): (W2) stands for the fact that white vertices of \tilde{G}^D are restricted to having degree two. Note that *not all* configurations of degree two at white vertices are licit for (W2).

It is now convenient to orient spanning trees of the double graph \tilde{G}^D towards one of the boundary \blacklozenge -black vertex, denoted by s . Let us denote by $\mathcal{T}_{(W2)}^s(\tilde{G}^D)$ the set of s OSTs of the double \tilde{G}^D which satisfy (W2). Then, as a consequence of Section 6.1, Section 6.2 and Lemma 6.1, we have the following proposition.

Proposition 6.2. The r OST partition function of the oriented graph \tilde{G}_1^Q with weight function ρ is equal to the s OST partition function of the double graph \tilde{G}^D with weight function ρ^* , where s OSTs satisfy (W2).

$$Z_{\text{OST}}^r(\tilde{G}_1^Q, \rho) = Z_{\text{OST}, (W2)}^s(\tilde{G}^D, \rho^*).$$

7. From spanning trees of the extended double to spanning trees of \tilde{G}

We are now done with modifying the graph along the boundary to be able to handle the boundary. This section consists in the heart of the explicit construction. In Section 7.1, we assign a dimer configuration of $\tilde{G}^D(s)$ to every s OST of \tilde{G}^D satisfying (W2). Note that the orientation of spanning trees of $\mathcal{T}_{(W2)}^s(\tilde{G}^D)$ is key to constructing the dimer configuration. This yields a partition of the set of spanning trees of $\mathcal{T}_{(W2)}^s(\tilde{G}^D)$ according to their underlying dimer

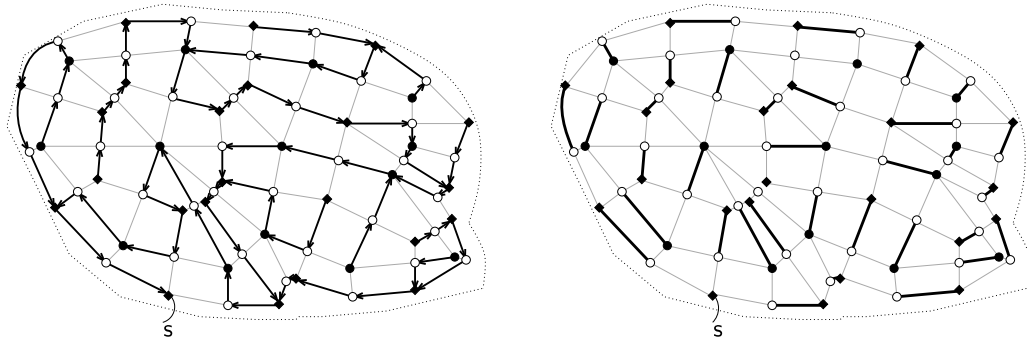


Fig. 18. Left: ${}^s\text{OST}$ of $\mathcal{T}_{(W2)}^s(\bar{G}^D)$. Right: corresponding perfect matching M_T of $\bar{G}^D(s)$.

configuration: $\mathcal{T}_{(W2)}^s(\bar{G}^D) = \bigsqcup_{M \in \mathcal{M}(\bar{G}^D(s))} \mathcal{T}_{(W2),M}^s(\bar{G}^D)$. In Section 7.2, we characterize spanning trees of $\mathcal{T}_{(W2),M}(\bar{G}^D)$ having a fixed underlying dimer configuration M . This important step allows us, in Section 7.3, to compute the weight of the family of spanning trees $\mathcal{T}_{(W2),M}(\bar{G}^D)$, and prove that it is, up to a constant, the weight of the corresponding dimer configuration M . In Section 7.4, we summarize the proof of Theorem 3.2 and conclude it using the generalized form of Temperley's bijection [25] of [15].

7.1. From spanning trees of \bar{G}^D to perfect matchings of $\bar{G}^D(s)$

Recall that a perfect matching, or dimer configuration of a graph, is a subset of edges such that each vertex of the graph is incident to exactly one edge of the subset. Let s be the boundary \blacklozenge -black vertex of \bar{G}^D chosen to be the root in the previous section. Let us denote by $\bar{G}^D(s)$ the graph \bar{G}^D from which the vertex s and all incident edges have been removed. In accordance with the notation introduced, $\mathcal{M}(\bar{G}^D(s))$ denotes the set of perfect matchings of the graph $\bar{G}^D(s)$.

The next lemma gives a natural way of obtaining a perfect matching of $\bar{G}^D(s)$ from an ${}^s\text{OST}$ of \bar{G}^D satisfying (W2). An example is provided in Figure 18.

Lemma 7.1. *Let T be an ${}^s\text{OST}$ of $\mathcal{T}_{(W2)}^s(\bar{G}^D)$. Then, the edge configuration M_T of $\bar{G}^D(s)$, consisting of edges of T exiting from black vertices, is a perfect matching of $\bar{G}^D(s)$.*

Proof. Let T be an ${}^s\text{OST}$ of $\mathcal{T}_{(W2)}^s(\bar{G}^D)$. We need to prove that every vertex of \bar{G}^D except the root s is incident to exactly one edge of M_T . By definition of ${}^s\text{OST}$ s, every vertex except the root s has exactly one outgoing edge. In particular, this is true for all black vertices, which are, by definition, part of M_T . Thus every black vertex of \bar{G}^D except the root is incident to a unique edge of M_T .

Moreover, since T satisfies (W2), every white vertex has degree exactly 2, with one incoming edge and one outgoing edge. Since the graph is bipartite, the edge of T entering the white vertex must exit a black one. By definition, it is thus part of M_T , and we conclude that every white vertex is incident to exactly one edge of M_T . \square

Given a perfect matching M of $\mathcal{M}(\bar{G}^D(s))$, an ${}^s\text{OST}$ T of $\mathcal{T}_{(W2)}^s(\bar{G}^D)$ is said to be *compatible* with M , if $M_T = M$. Let us denote by $\mathcal{T}_{(W2),M}^s(\bar{G}^D)$ the set of ${}^s\text{OST}$ s of $\mathcal{T}_{(W2)}^s(\bar{G}^D)$ compatible with a perfect matching M of $\mathcal{M}(\bar{G}^D(s))$. Then, we have:

Lemma 7.2.

$$\mathcal{T}_{(W2)}^s(\bar{G}^D) = \bigsqcup_{M \in \mathcal{M}(\bar{G}^D(s))} \mathcal{T}_{(W2),M}^s(\bar{G}^D).$$

Proof. The fact that $\mathcal{T}_{(W2)}^s(\tilde{G}^D)$ is the union of $\mathcal{T}_{(W2),M}^s(\tilde{G}^D)$ is a consequence of Lemma 7.1. Let us prove that the union is disjoint. Suppose that there are two distinct matchings M_1 and M_2 of $\mathcal{M}(\tilde{G}^D(s))$, and an s OST T which belongs to $\mathcal{T}_{(W2),M_1}^s(\tilde{G}^D) \cap \mathcal{T}_{(W2),M_2}^s(\tilde{G}^D)$. Then, by definition of being compatible with M_1 and M_2 , the s OST T must contain all edges of M_1 and all edges of M_2 . But the two matchings M_1 and M_2 being distinct, their superimposition contains a cycle. This yields a contradiction with T being an s OST. \square

7.2. Characterization of spanning trees compatible with a matching

By definition, given a perfect matching M of $\mathcal{M}(\tilde{G}^D(s))$, the unoriented version of an s OST T of $\mathcal{T}_{(W2),M}^s(\tilde{G}^D)$ satisfies the following two conditions: it contains all edges of M , and it satisfies (W2). The next proposition proves that if an edge configuration satisfies these two conditions, then it is a spanning tree. This is a remarkable fact. Indeed, being a spanning tree requires not having cycles, a non-local condition. Containing edges of a perfect matching M is a non-local condition, but it determines only half of the edges. The point of the proposition is to show that the configuration of the other half of the edges is determined locally.

Proposition 7.3. *Let M be a perfect matching of $\mathcal{M}(\tilde{G}^D(s))$. Then, an edge configuration containing all edges of M and satisfying (W2), is the unoriented version of an s OST of $\mathcal{T}_{(W2),M}^s(\tilde{G}^D)$.*

Proof. Let T be an edge configuration containing all edges of M and satisfying (W2). Since the graph \tilde{G}^D is bipartite, each edge of M joins a black and a white vertex. Orient edges of M from the black vertex to the white one.

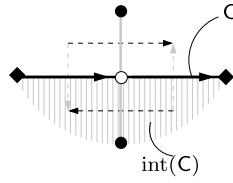
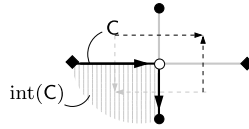
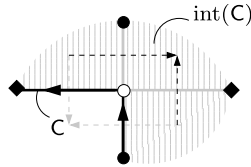
Since T satisfies (W2), every white vertex w of \tilde{G}^D is incident to exactly two edges of T . Exactly one of the two edges is an edge of M , because M is a perfect matching which only leaves the root vertex s unmatched, and the root vertex s is black. By our choice of orientation of M , this edge enters the white vertex. Let us orient the second edge of T incident to w away from w . We have thus defined an oriented version of the edge configuration T such that every vertex of \tilde{G}^D except the root s has exactly one outgoing edge, and such that edges exiting black vertices are exactly those of M . As a consequence, if T contains no cycle, its oriented version is an s OST satisfying (W2), such that $M_T = M$, i.e., it is an s OST of $\mathcal{T}_{(W2),M}^s(\tilde{G}^D)$.

It thus remains to show that T contains no cycle. Consider the oriented version of T defined above and suppose it contains a cycle, denoted by C . The graph \tilde{G}^D being bipartite, vertices of C alternate between black and white. By our choice of orientation, edges exiting black vertices belong to M , and those exiting white vertices belong to $T \setminus M$. Moreover, since each vertex of \tilde{G}^D has exactly one outgoing edge of T , edges of C must be oriented in the same direction, clockwise or counterclockwise. This implies that edges of the cycle C alternate between edges of M and edges of $T \setminus M$. Note that since the root vertex s has only incoming edges, it cannot belong to the cycle, and since it is on the boundary of the graph, it cannot be in the interior of the cycle.

As a consequence, the edge configuration M' consisting of edges of M away from the cycle C , and edges of $T \setminus M$ on the cycle C , is a dimer configuration of $\mathcal{M}(\tilde{G}^D(s))$. The superimposition of M and M' contains as single cycle, the cycle C , and the unmatched vertex s is not in the interior of C . The interior of C is thus covered by doubled edges, so that it must contain an *even* number of vertices.

We now prove that the fact that T satisfies (W2) implies that the cycle C must have an *odd* number of vertices in its interior, thus yielding a contradiction. We use Euler's formula to prove this. Without loss of generality, let us suppose that edges of C are oriented clockwise, and consider the graph $\tilde{G}^D(C) = (\tilde{V}^D(C), \tilde{E}^D(C))$ consisting of the restriction of \tilde{G}^D to C and its interior. We now prove that the number of inner vertices of the graph $\tilde{G}^D(C)$ is odd. Let us split vertices of $\tilde{G}^D(C)$ as follows.

- Type 1: white vertices of C preceded and followed by a black vertex of the same type, either \blacklozenge -black or \bullet -black. Every white vertex of Type 1 is incident to 3 edges and is on the boundary of 2 faces of $\tilde{G}^D(C)$, see Figure 19.
- Type 2: white vertices of C preceded by a \blacklozenge -black vertex and followed by a \bullet -black vertex. Since the edge configuration T satisfies (W2), only a right turn can occur at such a white vertex, see Figure 20. Every white vertex of Type 2 is thus incident to 3 edges and is on the boundary of one face of $\tilde{G}^D(C)$.

Fig. 19. White vertex preceded and followed by a \blacklozenge -black vertex.Fig. 20. White vertices preceded by a \blacklozenge -black vertex and followed by \bullet -black vertexFig. 21. White vertices preceded by a \bullet -black vertex and followed by \blacklozenge -black vertex.

- Type 3: white vertices of C preceded by a \bullet -black vertex and followed by \blacklozenge -black vertex. By (W2), only a left turn can occur at such a white vertex, see Figure 21. Every white vertex of Type 3 is thus incident to four edges and is on the boundary of three faces of $\tilde{G}^D(C)$.
- Type 4: white vertices in the interior of $\tilde{G}^D(C)$. Every white vertex of Type 4 is incident to 4 edges and is on the boundary of 4 faces of $\tilde{G}^D(C)$.
- Type 5: black vertices in the interior of $\tilde{G}^D(C)$.

For every Type i , let n_i denote the number of vertices of Type i . The total number $|\tilde{V}^D(C)|$ of vertices of the graph $\tilde{G}^D(C)$ is equal to twice the number of boundary white vertices plus the number of inner white and black vertices, that is:

$$|\tilde{V}^D(C)| = 2(n_1 + n_2 + n_3) + n_4 + n_5. \quad (7.1)$$

Since the graph $\tilde{G}^D(C)$ is bipartite, counting the number of edges $|\tilde{E}^D(C)|$ amounts to counting the number of edges incident to white vertices. By the description of the different types, we have:

$$|\tilde{E}^D(C)| = 3n_1 + 2n_2 + 4n_3 + 4n_4. \quad (7.2)$$

Faces of the graph $\tilde{G}^D(C)$ are bounded by a cycle of length 4, containing exactly two white vertices. Thus, summing the number of faces that white vertices are on the boundary of, yields twice the number of inner faces $|F_C^D|$ of $\tilde{G}^D(C)$:

$$|F_C^D| = \frac{1}{2}(2n_1 + n_2 + 3n_3 + 4n_4). \quad (7.3)$$

Euler's formula states that $|\tilde{V}^D(C)| + |F_C^D| - |\tilde{E}^D(C)| = 1$. Plugging Equations (7.1), (7.2), (7.3) yields:

$$\frac{1}{2}(n_2 - n_3) - (n_4 - n_5) = 1.$$

Moreover, since C is a cycle, $n_2 = n_3$, implying that: $n_5 - n_4 = 1$. Observing that the number of inner vertices of the graph $\tilde{G}^D(C)$ is $n_4 + n_5$, we deduce that this number is odd, thus concluding the proof. \square

7.3. Weights of spanning trees compatible with a perfect matching

Let M be a perfect matching of $\mathcal{M}(\tilde{G}^D(s))$. Using Proposition 7.3, and the definition of the weight function ρ^* , see Equation (6.1), we now compute the ρ^* -weighted sum $\rho^*(\mathcal{T}_{(W2),M}^s(\tilde{G}^D))$ of s OSTs of the extended double graph \tilde{G}^D , satisfying (W2) and compatible with M . Since the graph is bipartite, the contribution is the product of the contribution of edges incident to white vertices. Recall that they all have degree exactly 2. Given a white vertex w of \tilde{G}^D , the edge entering the white vertex is that of the matching M and is common to all s OSTs of $\mathcal{T}_{(W2),M}^s(\tilde{G}^D)$, let us denote it by e_w .

- If the white vertex w is not on the boundary of \tilde{G}^D , the s OST satisfies Point 1 of Lemma 6.1 at w . Having the edge e_w leaves only two possible configurations; and by Proposition 7.3, the two are possible. Thus the contribution is:

$$\rho_{e_w}^*(i \cos \theta_e + \sin \theta_e) = i e^{-i\theta_e} \rho_{e_w}^*. \quad (7.4)$$

- If the white vertex w is on the boundary of \tilde{G}^D , the s OST satisfies Point 2 of Lemma 6.1 at w . There are three cases to consider. If the edge e_w is half the dual edge of \tilde{G}^* arising from the splitting of a boundary \blacklozenge -black vertex, it contributes 1. By Proposition 7.3, the two configurations of Point 2 are possible. Thus the contribution is:

$$1 \cdot (1 + e^{-i\theta^\partial} - 1) = e^{-i\theta^\partial} = (-i e^{-i\theta^\partial/2})(i e^{-i\theta^\partial/2}). \quad (7.5)$$

If the edge e_w is the other half dual edge, it contributes 1. By Point 2, only the first configuration is possible, and by Proposition 7.3, it is indeed possible. Thus, the contribution is:

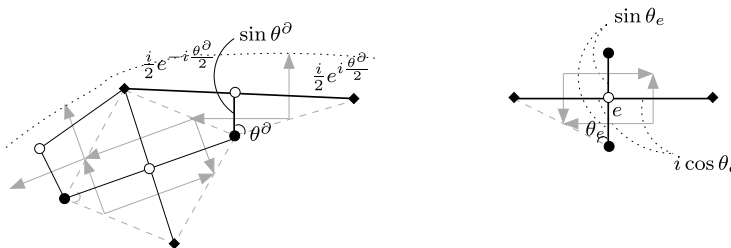
$$1 = (-i e^{-i\theta^\partial/2})(i e^{i\theta^\partial/2}). \quad (7.6)$$

If the edge e_w is half the primal edge of \tilde{G} incident to r , it contributes $e^{-i\theta^\partial} - 1$. Only the second configuration of Point 2 is possible, and by Proposition 7.3 it indeed is. Thus, the contribution is:

$$e^{-i\theta^\partial} - 1 = (-i e^{-i\theta^\partial/2}) \left(2 \sin \frac{\theta^\partial}{2} \right). \quad (7.7)$$

Let us assign the following weight function τ to edges of the extended double graph $\tilde{G}^D(s)$, see also Figure 22.

$$\tau_{wb} = \begin{cases} \sin \theta_e & \text{if } wb \text{ is half a primal edge } e \text{ of } G, \\ i \cos \theta_e & \text{if } wb \text{ is half a dual edge } e^* \text{ of an edge } e \text{ of } G, \\ i e^{-i\theta^\partial/2} & \text{if } wb \text{ is half a boundary dual edge of } \tilde{G}^*, \text{ as in Figure 22,} \\ i e^{i\theta^\partial/2} & \text{if } wb \text{ is half a boundary dual edge of } \tilde{G}^*, \text{ as in Figure 22,} \\ 2 \sin \frac{\theta^\partial}{2} & \text{if } wb \text{ is half an edge of } \tilde{G} \text{ incident to } r. \end{cases} \quad (7.8)$$



Weights τ around a boundary white vertex

Weights τ around a non-boundary white vertex

Fig. 22. Definition of the weights τ assigned to edges of the extended double graph \tilde{G}^D .

Observing that non-boundary white vertices of \bar{G}^D are in bijection with edges of G , and that boundary white vertices of \bar{G}^D are in bijection with edges of $\bar{E} \setminus E$, i.e., edges of \bar{G} incident to the boundary vertex r , we deduce from Equations (7.4), (7.5), (7.6), (7.7) and Lemma 7.2, the following proposition.

Proposition 7.4.

1. Given a perfect matching of $\mathcal{M}(\bar{G}^D(s))$, the ρ^* -weighted sum $\rho^*(\mathcal{T}_{(W2),M}^s(\bar{G}^D))$ is equal to:

$$\rho^*(\mathcal{T}_{(W2),M}^s(\bar{G}^D)) = \left(\prod_{e \in E} i e^{-i\theta_e} \right) \left(\prod_{e \in \bar{E} \setminus E} -i e^{-i\theta^\partial/2} \right) \prod_{e \in M} \tau_e.$$

2. The weighted sum of s OSTs of the extended double \bar{G}^D , satisfying (W2) is equal, up to an explicit multiplicative constant, to the dimer partition function of the graph $\bar{G}^D(s)$, with weight function τ :

$$Z_{\text{OST},(W2)}^s(\bar{G}^D, \rho^*) = \left(\prod_{e \in E} i e^{-i\theta_e} \right) \left(\prod_{e \in \bar{E} \setminus E} -i e^{-i\theta^\partial/2} \right) Z_{\text{dimer}}(\bar{G}^D(s), \tau).$$

7.4. Concluding the proof of Theorem 3.2

From Theorem 2.1 used with critical coupling constants, we have:

$$(Z_{\text{Ising}}(G, J))^2 = 2^{|V|} \left(\prod_{e \in E} \cos^{-1} \theta_e \right) Z_{\text{dimer}}(G^Q, \nu).$$

From Corollary 5.4, we have:

$$Z_{\text{dimer}}(G^Q, \nu) = |Z_{\text{OST}}^r(\bar{G}_1^Q, \rho)|.$$

From Proposition 6.2, we have:

$$Z_{\text{OST}}^r(\bar{G}_1^Q, \rho) = Z_{\text{OST},(W2)}^s(\bar{G}^D, \rho^*).$$

Combining this with Point 2 of Proposition 7.4, yields:

$$(Z_{\text{Ising}}(G, J))^2 = 2^{|V|} \left(\prod_{e \in E} \cos^{-1} \theta_e \right) |Z_{\text{dimer}}(\bar{G}^D(s), \tau)|.$$

Then by [15], we know that every perfect matching M of $\mathcal{M}(\bar{G}^D(s))$ defines a pair of primal and dual oriented spanning trees of \bar{G} and \bar{G}^* , where the primal oriented spanning tree is rooted at the vertex r , and the dual one is rooted at the vertex s . In the bijection, half-edges of \bar{G}^D correspond to oriented edges of the graphs \bar{G} and \bar{G}^* , and weights of oriented edges are obtained using the bijection. Starting from the weights of Equation (7.8), we nearly have the weights of Equation (3.1). We need two more observations.

Since edges of G and G^* are unoriented, for every spanning tree T of \bar{G} and dual spanning tree T^* of \bar{G}^* , we can write:

$$\left(\prod_{e \in E} \cos^{-1} \theta_e \right) = \left(\prod_{e=(x,y) \in T \cap E} \cos^{-1} \theta_e \right) \left(\prod_{e^*=(u,v) \in T^* \cap E^*} \cos^{-1} \theta_e \right).$$

This amounts to multiplying edge-weights τ of the graph G and of the graph G^* by $\cos^{-1} \theta_e$, yielding the 1 and $\tan \theta_e$ of (3.1) rather than $\cos \theta_e$ and $\sin \theta_e$ of (7.8).

Note that edge-weights of the extended dual graph \bar{G}^* differ by a constant i . This is because a spanning tree of a graph contains a fixed number of edges, so that the factor i on dual edges factors out, and does not change the modulus of the partition function.

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References

- [1] R. J. Baxter. Free-fermion, checkerboard and Z-invariant lattice models in statistical mechanics. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **404** (1826) (1986) 1–33. [MR0836281](#)
- [2] C. Boutillier and B. de Tilière. The critical Z-invariant Ising model via dimers: The periodic case. *Probab. Theory Related Fields* **147** (2010) 379–413. [MR2639710](#)
- [3] C. Boutillier and B. de Tilière. Height representation of XOR-Ising loops via bipartite dimers. *Electron. J. Probab.* **19** (80) (2014) 1–33. [MR3256880](#)
- [4] D. Cimaioni and H. Duminil-Copin. The critical temperature for the Ising model on planar doubly periodic graphs. *Electron. J. Probab.* **18** (44) (2013) 1–18. [MR3040554](#)
- [5] S. Chaiken. A combinatorial proof of the all minors matrix tree theorem. *SIAM J. Algebr. Discrete Methods* **3** (3) (1982) 319–329. [MR0666857](#)
- [6] D. Cimaioni. The critical Ising model via Kac–Ward matrices. *Comm. Math. Phys.* **316** (1) (2012) 99–126. [MR2989454](#)
- [7] B. de Tilière. From cycle rooted spanning forests to the critical Ising model: An explicit construction. *Comm. Math. Phys.* **319** (1) (2013) 69–110. [MR3034026](#)
- [8] J. Dubédat. Exact bosonization of the Ising model, 2011. Available at [arXiv:1112.4399](#).
- [9] R. J. Duffin. Potential theory on a rhombic lattice. *J. Combinatorial Theory* **5** (1968) 258–272. [MR0232005](#)
- [10] M. E. Fisher. On the dimer solution of planar Ising models. *J. Math. Phys.* **7** (1966) 1776–1781.
- [11] C. Fan and F. Y. Wu. General lattice model of phase transitions. *Phys. Rev. B* **2** (1970) 723–733.
- [12] P. W. Kasteleyn. Graph theory and crystal physics. In *Graph Theory and Theoretical Physics* 43–110. Academic Press, London, 1967. [MR0253689](#)
- [13] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.* **150** (2) (2002) 409–439. [MR1933589](#)
- [14] G. Kirchhoff. Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird. *Annalen der Physik* **148** (1847) 497–508.
- [15] R. W. Kenyon, J. G. Propp and D. B. Wilson. Trees and matchings. *Electron. J. Combin.* **7** (1) (2000) R25. [MR1756162](#)
- [16] G. Kuperberg. An exploration of the permanent-determinant method. *Electron. J. Combin.* **5** (1) (1998) R46. [MR1663576](#)
- [17] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. Part I. *Phys. Rev.* **60** (3) (1941) 252–262. [MR0004803](#)
- [18] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. Part II. *Phys. Rev.* **60** (3) (1941) 263–276. [MR0004804](#)
- [19] M. Kac and J. C. Ward. A combinatorial solution of the two-dimensional Ising model. *Phys. Rev. II. Ser.* **88** (1952) 1332–1337.
- [20] L. P. Kadanoff and F. J. Wegner. Some critical properties of the eight-vertex model. *Phys. Rev. B* **4** (1971) 3989–3993.
- [21] Z. Li. Critical temperature of periodic Ising models. *Comm. Math. Phys.* **315** (2012) 337–381. [MR2971729](#)
- [22] M. Lis. Phase transition free regions in the Ising model via the Kac–Ward operator. *Comm. Math. Phys.* **331** (2014) 1071–1086. [MR3248059](#)
- [23] C. Mercat. Discrete Riemann surfaces and the Ising model. *Comm. Math. Phys.* **218** (1) (2001) 177–216. [MR1824204](#)
- [24] B. Nienhuis. Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas. *J. Stat. Phys.* **34** (5–6) (1984) 731–761. [MR0751711](#)
- [25] H. N. V. Temperley. In *Combinatorics: Proceedings of the British Combinatorial Conference 1973* 202–204. London Mathematical Society Lecture Notes Series **13**. Cambridge University Press, Cambridge, 1974. Contributed talk by H. N. V. Temperley.
- [26] H. N. V. Temperley and M. E. Fisher. Dimer problem in statistical mechanics – An exact result. *Philos. Mag. (8)* **6** (1961) 1061–1063. [MR0136398](#)
- [27] W. T. Tutte. The dissection of equilateral triangles into equilateral triangles. *Proc. Cambridge Philos. Soc.* **44** (1948) 463–482. [MR0027521](#)
- [28] F. Y. Wu and K. Y. Lin. Staggered ice-rule vertex model – The Pfaffian solution. *Phys. Rev. B* **12** (1975) 419–428.
- [29] F. W. Wu. Ising model with four-spin interactions. *Phys. Rev. B* **4** (1971) 2312–2314.