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EXACT FORMULAS FOR RANDOM GROWTH WITH HALF-FLAT INITIAL DATA

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We obtain exact formulas for moments and generating functions of the height function of the asymmetric simple exclusion process at one spatial point, starting from special initial data in which every positive even site is initially occupied. These complement earlier formulas of E. Lee [*J. Stat. Phys.* **140** (2010) 635–647] but, unlike those formulas, ours are suitable in principle for asymptotics. We also explain how our formulas are related to divergent series formulas for half-flat KPZ of Le Doussal and Calabrese [*J. Stat. Mech.* **2012** (2012) P06001], which we also recover using the methods of this paper. These generating functions are given as a series without any apparent Fredholm determinant or Pfaffian structure. In the long time limit, formal asymptotics show that the fluctuations are given by the $\text{Airy}_{2\to 1}$ marginals.

1. Introduction. The one-dimensional asymmetric simple exclusion process (ASEP) is a continuous time Markov process with state space $\{0,1\}^{\mathbb{Z}}$, the 1's being thought of as particles and the 0's as holes. Each particle has an independent exponential clock which rings at rate one. When it rings, the particle chooses to attempt to jump one site to the right with probability $p \in [0,1]$, or one site to the left with probability $q = 1 - p$. However, the jump is only executed if the target site is empty; otherwise, the jump is suppressed and the particle must wait for the alarm to ring again. If $q = 1$, $p = 0$ (or $q = 0$, $p = 1$, but we will assume for convenience that $q \geq p$), it is called the totally asymmetric simple exclusion process (TASEP); if $0 < q \neq p$ it is the (partially) asymmetric simple exclusion process (ASEP); if $q = p = 1/2$ it is the symmetric simple exclusion process (SSEP). We denote by $\eta_t(x) = 1$ or 0 the presence or absence of a particle at $x \in \mathbb{Z}$ at time t. The state of the system is completely determined at time $t > 0$ by the initial data $\eta_x(0)$, $x \in \mathbb{Z}$, together with the family of exponential clocks; for more details on the construction of the process, we refer the reader to [26]. Given $\eta \in \{0,1\}^{\mathbb{Z}}$, we

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define $\hat{\eta} \in \{-1, 1\}^{\mathbb{Z}}$ by $\hat{\eta}(x) = 2\eta(x) - 1$. The *height function* of ASEP is defined in terms of $\hat{\eta}_t$ by

$$h(t,x) = \begin{cases} 2N_0^{\text{flux}}(t) + \sum_{0<y\le x} \hat{\eta}_t(y), & x > 0, \\ 2N_0^{\text{flux}}(t), & x = 0, \\ 2N_0^{\text{flux}}(t) - \sum_{x<y\le 0} \hat{\eta}_t(y), & x < 0, \end{cases} \tag{1.1}$$

where $N_0^{\text{flux}}(t)$ is the net number of particles which crossed from site 1 to 0 up to time t, meaning that particle jumps $1 \to 0$ are counted as $+1$ and jumps $0 \to 1$ are counted as -1.

ASEP is an important member of the one-dimensional Kardar–Parisi–Zhang (KPZ) universality class. This is a broad class of one-dimensional driven diffusive systems, or stochastic growth models, characterized by unusual, but universal asymptotic fluctuations. These should be of size $t^{1/3}$, and decorrelate on spatial scales of $t^{2/3}$, with special distributions in the long time limit, usually given in terms of Fredholm determinants, which only depend on the initial data class. There are a few special classes of initial data characterized by scale invariance: *curved* (or *step*), corresponding to starting with particles at every nonnegative site; *flat* (or *periodic*), corresponding to starting with particles at all even sites; and *stationary*, corresponding to starting with a product Bernoulli measure. In addition, there are three crossover classes: *curved* $\to$ *flat*, *curved*$\to$ *stationary* and *flat* $\to$ *stationary*; corresponding to putting two different initial conditions on either side of the origin. Based on exact computations for TASEP and a few other models with special determinantal (Schur) structure, the asymptotic spatial fluctuations in all six cases are known to be given by the Airy processes, a family of processes defined through their finite dimensional distributions which are given by specific Fredholm determinants. The full space–time limit in this KPZ scaling $\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x)$ is believed to be a Markov process known as the *KPZ fixed point*. For more details, see the reviews [11, 31, 32].

Within the universality class, the KPZ equation

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{\gamma}{2}(\partial_x h)^2 + \xi,$$

where ξ is space–time white noise, plays a special role as a heteroclinic orbit connecting the Edwards–Wilkinson (linear) fixed point $\partial_t h = \frac{1}{2}\partial_x^2 h + \xi$ to the (nonlinear and poorly understood) KPZ fixed point. It can be obtained from other models with adjustable nonlinearity or noise in the diffusive ($t = \varepsilon^{-2} T, x = \varepsilon^{-1} X$) weakly asymmetric, or weak noise limit, with rigorous proofs available in a few cases [1, 2, 4, 5, 13, 17, 18, 29].

The importance of ASEP in this context is that it has an adjustable nonlinearity

$$\gamma = q - p.$$

Although in the case $\gamma > 0$ it does not have a determinantal structure, somewhat surprisingly exact formulas have been discovered for the distribution of the height function of ASEP at a fixed $t > 0$ and $x \in \mathbb{Z}$ for certain initial data, starting with the work of Tracy and Widom in 2008 [34, 35]. The first formula was for the *step* case $\eta_0^{\text{step}}(x) = \mathbf{1}_{x \in \mathbb{Z}_{>0}}$. In the weakly asymmetric limit exact formulas were obtained for the one-point distribution of the KPZ equation with so called *narrow wedge* initial data (corresponding to the curved class); see [2] and also [33]. In the $t \to \infty$ limit, one obtains the Tracy–Widom GUE distribution. An analogous procedure was then performed on the step Bernoulli, or curved $\to$ stationary case for ASEP, corresponding to half-Brownian initial data for KPZ; the $t \to \infty$ limit in this case gives the $\text{Airy}_{2\to\text{BM}}$ marginals, or BBP transitional distributions [12]. Parallel computations were performed on the physics side using the nonrigorous replica method. The case of Brownian initial data for KPZ (corresponding to stationary ASEP) has also recently been completed in the physics [21] and mathematics [7] literatures. It should be emphasized that these are formulas for one point distributions only, and for very special initial data. So far, multipoint distributions have resisted rigorous analysis, though some nonrigorous attempts have been made [14, 15, 30].

Among the primary scaling invariant initial data at the KPZ level, this left the flat and half-flat cases. In [23, 24], Le Doussal and Calabrese gave formulas for the one point height distribution of KPZ for the half-flat and flat initial data via the replica method. Their half-flat formula is an uncontrolled divergent series, with no apparent Fredholm structure. As such, it is a pure formalism, and is mainly used as an intermediate step in order to obtain a Fredholm Pfaffian formula for the flat initial condition, by scaling the wedge to infinity, that is, looking farther and farther into the flat region.

Here, we will work directly with ASEP, which in particular can be regarded as a microscopic version of KPZ [5], and where one can avoid the problems associated with the nonsummable moments. Later, in Section 5, we will discuss how the methods we will use can be applied in the case of KPZ, yielding some of the formulas appearing in [23, 24].

We will be primarily concerned with the *half-flat* initial condition,

$$\eta_0^{\text{h-fl}}(x) = \mathbf{1}_{x \in 2\mathbb{Z}_{>0}}. \tag{1.2}$$

The superscript h-fl will be used for probabilities and expectations computed with respect to this initial condition. The limit to the flat initial condition $\eta_0^{\text{flat}}(x) = \mathbf{1}_{x \in 2\mathbb{Z}}$ will be pursued in an upcoming paper.

E. Lee's thesis already contains exact formulas for the quantities we are interested in. Here, and in the rest of the paper, we set

$$\tau = \frac{p}{q} \in (0, 1).$$

THEOREM 1.1 ([22]).

$$\begin{aligned}
&\mathbb{P}^{\text{h-fl}}\big(h(t,0) \geq 2m - x\big) \\
&\quad = (-1)^m \sum_{k \geq m} \frac{\tau^{(k-m)(k-m+1)/2}}{(1+\tau)^{k(k-1)} k!} \binom{k-1}{k-m}_\tau \\
&\quad\quad \times \int_{C_R^k} \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{\xi_i^x e^{t\varepsilon(\xi_i)}}{(1-\xi_i)(\xi_i^2 - \tau)} \\
&\quad\quad\quad \times \prod_{i<j} \frac{1 + \tau - (\xi_i + \xi_j)}{\tau - \xi_i\xi_j} \prod_i d\xi_i,
\end{aligned} \tag{1.3}$$

where

$$\varepsilon(\xi_i) := p\xi_i^{-1} + q\xi_i - 1. \tag{1.4}$$

C_R is a contour large enough to contain all the poles of the integrand, and $\binom{n}{k}_\tau = \frac{n_\tau!}{k_\tau!(n-k)_\tau!}$ with the τ-factorial $n_\tau!$ defined in (1.10).

These formulas are similar in structure to earlier formulas of [35]. However, such formulas turn out not to be conducive to asymptotics analysis. They need considerable "postproduction" before the asymptotic behaviour can be extracted [34, 36], and no one has been able to figure out how to do this for (1.3), nor to extract the relevant asymptotics (even formally).

Our main result is an explicit formula for the one-point distribution in the half-flat case, expressed as a certain series which has a structure reminiscent of a Fredholm determinant. In an upcoming paper, we will use these formulas to obtain analogous moment formulas in the flat case and, furthermore, a Fredholm Pfaffian formula for a certain transform of the height function. Formal asymptotics lead to the expected results in the $t \to \infty$ and weakly asymmetric limits, but turning them into rigorous proofs presents some considerable technical challenges and is left for future work (see the Appendix for a discussion of the large time case).

Formulas for the half-flat case can be obtained by the method of [9], together with an ansatz coming from a study of the mechanics of (1.3). Let

$$N_x(t) = \sum_{y=-\infty}^{x} \eta_t(y) \tag{1.5}$$

be the total number of particles to the left of x at time t. It is not hard to check that when all particles start to the right of the origin, $N_0^{\text{flux}}(t) = N_0(t)$, and thus by (1.1)

$$h(t,x) = 2N_x(t) - x \tag{1.6}$$

in the half-flat case. Define

$$\widetilde{Q}_x(t) = \frac{\tau^{N_x(t)} - \tau^{N_{x-1}(t)}}{\tau - 1} = \tau^{N_{x-1}(t)} \eta_x(t).$$

THEOREM 1.2. *Consider ASEP with half-flat initial condition as in* (1.2). *Then for any* $\vec{x} \in \mathbb{Z}^k$ *we have*

$$\begin{aligned} &\mathbb{E}^{\text{h-fl}}\big[\widetilde{Q}_{x_1}(t) \cdots \widetilde{Q}_{x_k}(t)\big] \\ &\quad = \frac{\tau^{(1/2)k(k-1)}}{(2\pi \mathrm{i})^k} \\ &\qquad \times \int_{C_{1,\rho}^k} d\vec{z} \prod_{1 \le a < b \le k} \frac{z_a - z_b}{z_a - \tau z_b} \frac{1 - z_a z_b}{1 - \tau z_a z_b} \prod_{a=1}^{k} \frac{1}{\tau z_a^2 - 1} f_{x_a,t}(z_a), \end{aligned} \tag{1.7}$$

where $C_{1,\rho}$ *is a circle around* 1 *with radius* $0 < \rho < \min\{\tau^{-1/2} - 1, (1+\tau)^{-1}\}$, $C_{1,\rho}^k$ *denotes the product of* k *copies of* $C_{1,\rho}$,

$$f_{x,t}(z) = \left(\frac{1 - \tau z}{1 - z}\right)^{x-1} e^{\widetilde{\varepsilon}(z)t},$$

and $\widetilde{\varepsilon}$ *is defined in terms of the function* ε *given in* (1.4) *by*

$$\widetilde{\varepsilon}(z) = \varepsilon\left(\frac{1 - \tau z}{1 - z}\right) = p \frac{1 - z}{1 - \tau z} + q \frac{1 - \tau z}{1 - z} - 1. \tag{1.8}$$

For simplicity, throughout the rest of the paper we will omit the bound on the indices in products such as $\prod_{1 \le a \le k}$ and $\prod_{1 \le a < b \le k}$ when no confusion can arise and the factors involved in the products are defined in terms of a collection of k variables. A similar convention will sometimes be used for sums. Additionally, we will continue using the notation C^k for the product of k copies of a given contour C in the complex plane.

An analogous formula holds for the stochastic heat equation/KPZ/delta Bose gas; see Section 5 for details. On the other hand, the analogous ansatz does not work in the case of q-TASEP and the O'Connell–Yor semi-discrete polymer, at least with the most straightforward candidates for half-flat initial conditions.

The interesting new term here over earlier formulas [6, 9] is $\prod_{a<b} \frac{1 - z_a z_b}{1 - \tau z_a z_b}$, which together with the factor $\prod_a \frac{1}{\tau z_a^2 - 1}$ allows us to recover the periodic initial data. This term leads to the double product $\prod_{a<b} \mathfrak{h}(w_a, w_b; s_a, s_b)$ appearing in (1.13) below, which is the obstacle to making long time limit fluctuations rigorous (see the Appendix). Factors of this form were fortuitously absent from earlier formulas for step and step Bernoulli initial data, which only contained the double product $\prod_{a<b} \frac{z_a - z_b}{z_a - \tau z_b}$; this last factor turns into the determinant in (1.13) and this makes it much easier to deal with. Similar expressions have also proved to be an

obstacle in the replica formulas [23, 24] for half-flat and flat initial data, as well as for expressions for multipoint distributions [14–16].

The formula for $\mathbb{E}^{\text{h-fl}}[\widetilde{Q}_{x_1}(t)\cdots\widetilde{Q}_{x_k}(t)]$ can be used to write a formula for the moments of $\tau^{N_x(t)}$ by using ideas of [9, 20]. The result is given in Section 3 as Proposition 3.2. The formula for $\mathbb{E}[\tau^{kN_x(t)}]$ is given as a nested contour integral (see Figure 1). As given, such a formula is suitable neither for asymptotic analysis (not even at a formal level) nor for our later goal of deriving a formula for the full flat case. In order to obtain a formula where all the contours coincide we will expand the nested contours so that they all coincide with largest one. The resulting formula amounts to computing the residue expansion associated to the poles that we cross as we perform this deformation. It is given in Proposition 3.3 as a sum of multiple contour integrals indexed by partitions. After some rewriting, this formula leads to our main result for ASEP with half-flat initial data. Define the following functions:

$$\mathfrak{f}(w;n) = (1-\tau)^n e^{(q-p)t[1/(1+w)-1/(1+\tau^n w)]}\left(\frac{1+\tau^n w}{1+w}\right)^{x-1},$$

$$\mathfrak{g}(w;n) = \frac{(-w;\tau)_\infty}{(-\tau^n w;\tau)_\infty}\frac{(\tau^{2n}w^2;\tau)_\infty}{(\tau^n w^2;\tau)_\infty}, \tag{1.9}$$

$$\mathfrak{h}(w_1,w_2;n_1,n_2) = \frac{(w_1w_2;\tau)_\infty(\tau^{n_1+n_2}w_1w_2;\tau)_\infty}{(\tau^{n_1}w_1w_2;\tau)_\infty(\tau^{n_2}w_1w_2;\tau)_\infty},$$

where the infinite *q-Pochhammer symbols* are defined as

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-q^n a).$$

Note that $\mathfrak{g}$ and $\mathfrak{h}$ can be written in terms of ratios of finite q-Pochhammer symbols, but it will more convenient for us to write them in this form. The formulas for $\mathfrak{g}$ and $\mathfrak{h}$ can alternatively be written as ratios of *q-Gamma functions*,

$$\Gamma_q(x) = \frac{(1-q)^{1-x}(q;q)_\infty}{(q^x;q)_\infty},$$

which converge (uniformly on compact sets) to the usual Gamma function as $q \to 1$. We also define the *q-factorial*

$$m_q! = \frac{\prod_{a=1}^{k}(1-q^a)}{(1-q)^k}. \tag{1.10}$$

For later use, we further introduce the *q-exponential function*

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty}\frac{x^k}{k_q!}, \tag{1.11}$$

where the second equality only holds for $|x| < 1$ and amounts to the q-Binomial theorem (see, e.g., Theorem 10.2.1 in [3]). As $q \to 1$, this function converges to the usual exponential function, uniformly on $(-\infty, A]$ for any A. In keeping with the standard usage we have used the parameter q in the definition of these q-deformed functions, but in all that follows the parameter τ will appear in place of q.

THEOREM 1.3. *Consider ASEP with half-flat initial condition as in* (1.2) *and let* $m \in \mathbb{Z}_{\geq 0}$. *Then*

$$\mathbb{E}^{\text{h-fl}}\big[\tau^{mN_x(t)}\big] = m_\tau! \sum_{k=0}^{m} \nu_{k,m}^{\text{h-fl}}(t,x) \tag{1.12}$$

with

$$\begin{aligned}\nu_{k,m}^{\text{h-fl}}(t,x) = \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1 \\ n_1+\cdots+n_k=m}} \frac{1}{(2\pi \mathrm{i})^k} \int_{\gamma_{-1,0}^k} d\vec{w}\, \det\Big[\frac{-1}{w_a \tau^{n_a} - w_b}\Big]_{a,b=1}^{k} \\ \times \prod_a \mathfrak{f}(w_a; n_a)\mathfrak{g}(w_a; n_a) \prod_{a<b} \mathfrak{h}(w_a, w_b; n_a, n_b),\end{aligned} \tag{1.13}$$

where $\gamma_{-1,0}$ *is a* (*positively oriented*) *contour around* -1 *and* 0, *strictly contained inside the circle of radius* $\tau^{-1/2}$, *which does not include any other singularities of the integrand.*

The contour $\gamma_{-1,0}$ in the theorem can for example be chosen to be a circle around the origin with radius in $(1, \tau^{-1/2})$. In fact, the determinant clearly never vanishes for this choice, and one can check that all the other singularities of the integrand, except for $w_a = 0$ and $w_a = -1$, are outside this contour.

With a formula for the moments of $\tau^{N_x(t)}$ at our disposal we are ready to form a generating function, namely the τ-*Laplace transform* of $\tau^{N_x(t)}$. The formula involves a Mellin–Barnes integral representation of the infinite sums in $n_1, \dots, n_k$ appearing in (1.13) after summing over $m \geq 0$.

THEOREM 1.4. *Let* $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$. *Then, for* e_τ *as in* (1.11),

$$\begin{aligned}&\mathbb{E}^{\text{h-fl}}\big[e_\tau\big(\zeta \tau^{N_x(t)}\big)\big] \\ &\quad = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(2\pi \mathrm{i})^{2k}} \int_{(1/2+\mathrm{i}\mathbb{R})^k} d\vec{s} \int_{\gamma_{-1,0}^k} d\vec{w}\, \det\Big[\frac{-1}{w_a \tau^{s_a} - w_b}\Big]_{a,b=1}^{k} \\ &\qquad \times \prod_a (-\zeta)^{s_a} \mathfrak{f}(w_a; s_a)\mathfrak{g}(w_a; s_a) \prod_{a<b} \mathfrak{h}(w_a, w_b; s_a, s_b).\end{aligned} \tag{1.14}$$

Now set $\zeta = -\tau^{-t/4-t^{2/3}x/2+t^{1/3}r/2}$. Since $e_\tau(z) \longrightarrow 0$ as $z \to -\infty$ and $e_\tau(z) \longrightarrow 1$ as $z \to 0$ for fixed τ, uniformly in $z \in (-\infty, 0]$ we have (see [6], Lemma 4.1.39)

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}^{\text{h-fl}}\big[e_\tau\big(-\tau^{N_{t^{2/3}x}(t/\gamma)-(1/4)t-(1/2)t^{2/3}x+(1/2)t^{1/3}r-(1/4)t^{1/3}x^2\mathbf{1}_{x\le 0}}\big)\big] \\ = \lim_{t\to\infty} \mathbb{P}^{\text{h-fl}}\bigg(\frac{h(t/\gamma, t^{2/3}x) - (1/2)t - (1/2)t^{1/3}x^2\mathbf{1}_{x\le 0}}{t^{1/3}} \ge -r\bigg),\end{aligned} \tag{1.15}$$

where we recall $\gamma = q - p$. In the Appendix, we show that a formal steepest descent analysis of the right-hand side of (1.14) gives (a scaled version of) the one-point marginals of the $\text{Airy}_{2\to 1}$ process $\mathcal{A}_{2\to 1}(x)$.

Outline. The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.2. In Section 3 we will use the formula obtained in Theorem 1.2 to derive the moment formula given in Theorem 1.3, while in Section 4 we will derive the formula for the τ-Laplace transform of $\tau^{N_x(t)}$ (Theorem 1.4). Section 5 explains how the methods used for ASEP can be applied to the case of the SHE/KPZ equation (or, more precisely, the delta Bose gas) and discusses the relation with the work of Le Doussal and Calabrese. Finally, the Appendix contains the formal derivation of the limiting fluctuations for ASEP with half-flat initial condition.

2. Contour integral ansatz. To prove Theorem 1.2, we will use Proposition 4.10 of [9], which shows that $\mathbb{E}^{\text{h-fl}}[\widetilde{Q}_{x_1}(t)\cdots\widetilde{Q}_{x_k}(t)]$ can be represented as the solution of a certain evolution equation with boundary conditions. We describe this result next.

Let η_0 be an ASEP configuration with a leftmost particle and consider ASEP started with η_0 as initial condition. Let $\tilde{u}_0(\vec{x}) = \prod_{a=1}^k \tau^{N_{x_a-1}(0)}\eta_{x_a}(0)$ (where, of course, $N_x(0)$ is computed with respect to the initial condition η_0). Consider the following system of differential equations:

(1) For all $\vec{x} \in \mathbb{Z}^k$ and $t \ge 0$, writing $\vec{x}_\ell^{\pm} = (x_1, \dots, x_\ell \pm 1, \dots, x_k)$,

$$\frac{d}{dt}\tilde{u}(t,\vec{x}) = \sum_{j=1}^k \big[p\tilde{u}\big(t,\vec{x}_j^-\big) + q\tilde{u}\big(t,\vec{x}_j^+\big) - \tilde{u}(t,\vec{x})\big].$$

(2) For all $\vec{x} \in \mathbb{Z}^k$ such that there exists $\ell < k$ with $x_{\ell+1} = x_\ell + 1$,

$$p\tilde{u}\big(t,\vec{x}_{\ell+1}^-\big) + q\tilde{u}\big(t,\vec{x}_\ell^+\big) = \tilde{u}(t,\vec{x}).$$

(3) There exist constants $c, C, \delta > 0$ such that for all $\vec{x} \in \mathbb{Z}^k$ with $x_1 < x_2 < \cdots < x_k$ and $t \in [0, \delta]$,

$$\big|\tilde{u}(t,\vec{x})\big| \le Ce^{c\sum_j |x_a|}.$$

(4) For all $\vec{x} \in \mathbb{Z}^k$ such that $x_1 < x_2 < \cdots < x_k$ we have

$$\tilde{u}(0, \vec{x}) = \tilde{u}_0(\vec{x}).$$

PROPOSITION 2.1 ([9]). *Suppose that $\tilde{u}(t, \vec{x})$ solves* (1)–(4). *Then for all $\vec{x} \in \mathbb{Z}^k$ such that $x_1 < x_2 < \cdots < x_k$ we have*

$$\mathbb{E}^{\eta_0}\big[\widetilde{Q}_{x_1}(t) \cdots \widetilde{Q}_{x_k}(t)\big] = \tilde{u}(t, \vec{x}),$$

where the superscript on the left-hand side means that ASEP is started with initial condition η_0.

We proceed now to the proof of our formula for $\mathbb{E}^{\text{h-fl}}[\widetilde{Q}_{x_1}(t) \cdots \widetilde{Q}_{x_k}(t)]$.

PROOF OF THEOREM 1.2. In view of Proposition 2.1 and (1.7), we need to check that

$$\begin{aligned}\tilde{u}(t; \vec{x}) := \frac{\tau^{(1/2)k(k-1)}}{(2\pi \mathrm{i})^k} \int_{C^k_{1,\rho}} d\vec{z} \prod_{1 \le a < b \le k} \frac{z_a - z_b}{z_a - \tau z_b} \frac{1 - z_a z_b}{1 - \tau z_a z_b} \\ \times \prod_{a=1}^{k} \frac{1}{\tau z_a^2 - 1} f_{x_a,t}(z_a)\end{aligned} \tag{2.1}$$

satisfies (1)–(4) with $\tilde{u}_0$ defined in terms of the half-flat initial condition $\eta_0(x) = \mathbf{1}_{x \in 2\mathbb{Z}_{>0}}$. A straightforward computation shows that in this case

$$\tilde{u}_0(\vec{x}) = \prod_{a=1}^{k} \mathbf{1}_{x_a \in 2\mathbb{Z}_{>0}} \tau^{\sum_{y=-\infty}^{x_a - 1} \eta_y(0)} = \tau^{-k} \prod_{a=1}^{k} \mathbf{1}_{x_a \in 2\mathbb{Z}_{>0}} \tau^{(1/2)x_a}. \tag{2.2}$$

We will denote the integrand in (2.1) by $I_{k,t}(\vec{x}; \vec{z})$, that is,

$$I_{k,t}(\vec{x}; \vec{z}) = \prod_{a<b} \frac{z_a - z_b}{z_a - \tau z_b} \frac{1 - z_a z_b}{1 - \tau z_a z_b} \prod_a \frac{1}{\tau z_a^2 - 1} f_{x_a,t}(z_a). \tag{2.3}$$

Additionally, we will write $\vec{x}^{(i_1,\ldots,i_\ell)}$ and $\vec{z}^{(i_1,\ldots,i_\ell)}$ to denote, respectively, the vectors $\vec{x}$ and $\vec{z}$ with the components $i_1, \ldots, i_\ell$ removed.

Computing $\frac{d}{dt}\tilde{u}(t, \vec{x})$ introduces a factor $\sum_{\ell=1}^{k} \widetilde{\varepsilon}(z_\ell)$ in front of the integrand. Similarly, computing $\tilde{u}(t, \vec{x}^{\pm}_{\ell})$ introduces a factor $(\frac{1-\tau z_\ell}{1-z_\ell})^{\pm 1}$ in front of the integrand. Hence, (1) is satisfied if we can show that

$$\sum_{\ell=1}^{k} \widetilde{\varepsilon}(z_\ell) = \sum_{\ell=1}^{k} \Big[p \frac{1 - z_\ell}{1 - \tau z_\ell} + q \frac{1 - \tau z_\ell}{1 - z_\ell} - 1 \Big].$$

But this follows immediately from the definition of $\widetilde{\varepsilon}$; see (1.8)

For (2), let $\vec{x} \in \mathbb{Z}^k$ and suppose that there exists ℓ such that $x_{\ell+1} = x_\ell + 1$. Then using the above computation of $\tilde{u}(t, \vec{x}_\ell^{\pm})$, we have

$$\begin{aligned} &p\tilde{u}\big(t, \vec{x}_{\ell+1}^{-}\big) + q\tilde{u}\big(t, \vec{x}_\ell^{+}\big) - \tilde{u}(t, \vec{x}) \\ &\quad = \frac{\tau^{(1/2)k(k-1)}}{(2\pi \mathrm{i})^k} \int_{C_{1,\rho}^k} d\vec{z}\, I_{k,t}\big(\vec{x}_{\ell+1}^{-}; \vec{z}\big) \\ &\qquad \times \left[p + q\frac{1-\tau z_\ell}{1-z_\ell}\frac{1-\tau z_{\ell+1}}{1-z_{\ell+1}} - \frac{1-\tau z_{\ell+1}}{1-z_{\ell+1}} \right]. \end{aligned} \tag{2.4}$$

We need to show that the integral vanishes. The expression inside the brackets equals $\frac{(q-p)(z_\ell - \tau z_{\ell+1})}{(1-z_\ell)(1-z_{\ell+1})}$. Note that the factor $z_\ell - \tau z_{\ell+1}$ cancels a like factor in the denominator of the product $\prod_{a<b} \frac{z_a - z_b}{z_a - \tau z_b}$ coming from $I_{k,t}(\vec{x}_{\ell+1}^{-}; \vec{z})$, and thus (using the fact that $x_{\ell+1} = x_\ell + 1$) the integrand in (2.4) can be rewritten as

$$\frac{(q-p)(z_\ell - z_{\ell+1})(1 - z_\ell z_{\ell+1})}{(1-z_\ell)(1-z_{\ell+1})(1-\tau z_\ell z_{\ell+1})} f_{x_\ell,t}(z_\ell) f_{x_\ell,t}(z_{\ell+1}) G\big(\vec{x}^{(\ell,\ell+1)}, \vec{z}^{(\ell,\ell+1)}\big),$$

where, as suggested by the notation, the factor $G(\vec{x}^{(\ell,\ell+1)}, \vec{z}^{(\ell,\ell+1)})$ does not depend on x_ℓ, $x_{\ell+1}$, z_ℓ and $z_{\ell+1}$. This expression is antisymmetric in $z_\ell, z_{\ell+1}$, and thus its integral over $(z_\ell, z_{\ell+1}) \in C_{1,\rho}^2$ must vanish. This shows that the integral in (2.4) is zero, proving (2).

(3) follows directly from the form of $f_{x,t}$ and the facts that $C_{1,\rho}$ is compact and that the integrand is continuous in $\vec{z} \in C_{1,\rho}^k$.

We turn now to (4). Note that when $t = 0$ the essential singularity in the exponent of $f_{x,t}$ in $I_{k,t}$ disappears [see (2.3)], and thus we can evaluate the integral by computing residues.

First, if $x_1 \le 1$ then $f_{x_1,0}(z_1)$ has no pole at $z_1 = 1$. Hence, the integrand is analytic in z_1 inside $C_{1,\rho}$, and thus the integral is 0. Since $x_1 < \cdots < x_k$, this accounts for the condition that all x_a's be at least 2. So let us assume now that $2 \le x_1 < \cdots < x_k$. We will evaluate the z_k integral first, by expanding the contour to infinity. Note that, thanks to the decay coming from the factor $(\tau z_k^2 - 1)^{-1}$ there is no pole at infinity, and thus the integral equals minus the sum of the residues of the poles of the integrand outside $C_{1,\rho}$.

In z_k, the poles are $\pm\tau^{-1/2}$, $\tau^{-1} z_\ell$ and $\tau^{-1} z_\ell^{-1}$ for $\ell < k$. The condition imposed on ρ implies that all these poles lie outside the contour. Consider first the poles at $z_k = \tau^{-1} z_\ell$, $\ell < k$. The residue of $I_{k,0}$ at this point is given by

$$\begin{aligned} I_{k-1,0}\big(\vec{x}^{(k)}; \vec{z}^{(k)}\big) \prod_{\substack{a<k \\ a\neq\ell}} \left(\frac{z_a - \tau^{-1} z_\ell}{z_a - z_\ell} \frac{1 - \tau^{-1} z_a z_\ell}{1 - z_a z_\ell} \right) \frac{(1-\tau) z_\ell (1 - \tau^{-1} z_\ell^2)}{1 - z_\ell^2} \\ \times \frac{((1-z_\ell)/(1-\tau^{-1} z_\ell))^{x_k - 1}}{\tau^{-1} z_\ell^2 - 1} \end{aligned}$$

$$= I_{k-2,0}\big(\vec{x}^{(\ell,k)};\vec{z}^{(\ell,k)}\big)\prod_{a=1}^{\ell-1}\left(\frac{z_a-\tau^{-1}z_\ell}{z_a-\tau z_\ell}\frac{1-\tau^{-1}z_az_\ell}{1-\tau z_az_\ell}\right)$$
$$\times\prod_{b=\ell+1}^{k-1}\left(\frac{\tau^{-1}z_\ell-z_b}{z_\ell-\tau z_b}\frac{1-\tau^{-1}z_bz_\ell}{1-\tau z_bz_\ell}\right)$$
$$\times\frac{(1-\tau)z_\ell}{1+z_\ell}\frac{((1-\tau z_\ell)^{x_\ell-1}/(1-\tau^{-1}z_\ell)^{x_k-1})}{(1-\tau z_\ell^2)}(1-z_\ell)^{x_k-x_\ell-1}.$$

Observe that the factors $z_a - z_\ell$ and $1 - z_az_\ell$ appearing in the denominator of the first line are canceled by matching factors coming out of $I_{k-1,0}(\vec{x}^{(k)};\vec{z}^{(k)})$. This is crucial, because it implies that the resulting integrand has no singularities in z_ℓ inside $C_{1,\rho}$ except possibly at $z_\ell = 1$. On the other hand, since $x_k \geq x_\ell + 1$, the simplification leading to the second line above implies again that there is no pole at $z_\ell = 1$. We deduce that the integrand is analytic in z_ℓ inside $C_{1,\rho}$, and hence the integral vanishes. An analogous argument shows that the residues at $z_k = \tau^{-1}z_\ell^{-1}$ also vanish.

Thus, the only important poles are those at $\pm\tau^{-1/2}$. We have

$$\operatorname*{Res}_{z_k=\tau^{-1/2}} I_{k,0}(\vec{x};\vec{z}) = I_{k-1,0}\big(\vec{x}^{(k)};\vec{z}^{(k)}\big)\prod_{a=1}^{k-1}\left(\frac{z_a-\tau^{-1/2}}{z_a-\tau^{1/2}}\frac{1-\tau^{-1/2}z_a}{1-\tau^{1/2}z_a}\right)$$
$$\times\frac{((1-\tau^{1/2})/(1-\tau^{-1/2}))^{x_k-1}}{2\tau^{1/2}}$$
$$= I_{k-1,0}\big(\vec{x}^{(k)};\vec{z}^{(k)}\big)(-1)^{x_k-1}\frac{1}{2}\tau^{(1/2)x_k-k}.$$

Similarly,

$$\operatorname*{Res}_{z_k=-\tau^{-1/2}} I_{k,0}(\vec{x};\vec{z}) = I_{k-1,0}\big(\vec{x}^{(k)};\vec{z}^{(k)}\big)\prod_{a=1}^{k-1}\left(\frac{z_a+\tau^{-1/2}}{z_a+\tau^{1/2}}\frac{1+\tau^{-1/2}z_a}{1+\tau^{1/2}z_a}\right)$$
$$\times\frac{((1+\tau^{1/2})/(1+\tau^{-1/2}))^{x_k-1}}{-2\tau^{1/2}}$$
$$= -I_{k-1,0}\big(\vec{x}^{(k)};\vec{z}^{(k)}\big)\frac{1}{2}\tau^{(1/2)x_k-k}.$$

If x_k is odd then the two residues cancel each other out. Therefore,

$$\operatorname*{Res}_{z_k=\tau^{-1/2}} I_{k,0}(\vec{x};\vec{z}) + \operatorname*{Res}_{z_k=-\tau^{-1/2}} I_{k,0}(\vec{x};\vec{z})$$
$$= -I_{k-1,0}\big(\vec{x}^{(k)};\vec{z}^{(k)}\big)\mathbf{1}_{x_k\in 2\mathbb{Z}_{\geq 0}}\tau^{(1/2)x_k-k}.$$

Recalling that we have computed the residues on the outside of $C_{1,\rho}$, which introduces a minus sign, we get

$$\begin{aligned}\tilde{u}(0,\vec{x}) &= \frac{\tau^{(1/2)k(k-1)}}{(2\pi\mathrm{i})^k}\int_{C_{1,\rho}^k} d\vec{z}\, I_{k,0}(\vec{z}) \\ &= \tau^{-1}\mathbf{1}_{x_k\in 2\mathbb{Z}_{>0}}\tau^{(1/2)x_k}\frac{\tau^{(1/2)(k-1)(k-2)}}{(2\pi\mathrm{i})^{k-1}}\int_{C_{1,\rho}^{k-1}} d\vec{z}\, I_{k-1,0}\big(\vec{x}^{(k)};\vec{z}\big).\end{aligned}$$

Equation (2.2) follows by induction, and this proves (4). □

3. Moment formulas. Recall that Theorem 1.2 provides a formula for the expectation of $\widetilde{Q}_{x_1}(t)\cdots\widetilde{Q}_{x_\ell}(t)$, where $\widetilde{Q}_x(t)=\eta_x(t)\tau^{N_{x-1}(t)}$ and the x_a's have to be different. To turn this into a formula for the moments of $\tau^{N_x(t)}$, we will use the following identity, first proved as Proposition 3 of [20] (in [20] the identity was stated only for the expected value of both sides, the more general form stated here appears as Lemma 4.17 in [9]).

LEMMA 3.1. *Let $\eta\in\{0,1\}^{\mathbb{Z}}$ and write $N_x(\eta)=\sum_{y\le x}\eta_y$. Then*

$$\tau^{kN_x(\eta)} = \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}_\tau(\tau;\tau)_\ell \sum_{x_1<\cdots<x_\ell\le x}\eta_{x_1}\tau^{N_{x_1}(\eta)}\cdots\eta_{x_\ell}\tau^{N_{x_\ell}(\eta)}, \tag{3.1}$$

where the summand for $\ell=0$ should be interpreted as 1.

Note that this result is not specific to ASEP, which is why we have introduced the notation $N_x(\eta)$. For the case of ASEP, and in view of (1.5), we are writing $N_x(t)=N_x(\eta_t)$. The expected value of the right-hand side of (3.1) is explicit in this case (i.e., when we take η to be the ASEP configuration at time t, η_t) thanks to (1.7), and we will turn it into a single multiple integral it using arguments similar to those in Section 4 of [9].

PROPOSITION 3.2. *For any $k\in\mathbb{Z}_{\ge 0}$, we have*

$$\begin{aligned}&\mathbb{E}\big[\tau^{kN_x(t)}\big] \\ &\quad= \frac{\tau^{(1/2)k(k-1)}}{(2\pi\mathrm{i})^k}\int d\vec{y}\prod_{a<b}\Big(\frac{y_a-y_b}{y_a-\tau y_b}\frac{1-\tau^{-2}y_ay_b}{1-\tau^{-1}y_ay_b}\Big)\prod_a\frac{F_{x,t}(y_a)}{y_a},\end{aligned} \tag{3.2}$$

where

$$F_{x,t}(y)=\frac{\tau+y}{\tau-y^2}\Big(\frac{1+y}{1+\tau^{-1}y}\Big)^{x-1}e^{t\hat{\varepsilon}(y)},$$

$\hat{\varepsilon}(y)=\widetilde{\varepsilon}(-\tau^{-1}y)$, and the integration contours are given as follows. For each $a=1,\dots,k$, the y_a contour is composed of two disconnected pieces: a circle around $-\tau$ with radius small enough so that $-\tau^{1/2}$ is on its exterior, and a circle around

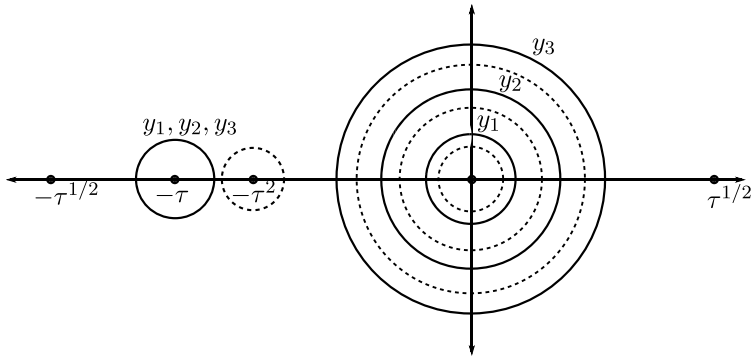

FIG. 1. *Contours appearing in Proposition 3.2 in the case $k = 3$. The dashed contours correspond to multiplying each of the contours by τ and illustrate the nesting condition described in the proposition.*

0 *with radius small enough so that $\tau^{1/2}$ is on its exterior. The radii of these circles are chosen so that, in addition, for all $a < b$ the y_a contour does not include the image under multiplication by τ of the y_b contour* (*see Figure 1*).

PROOF. By (1.7) and Lemma 3.1, we have

$$\mathbb{E}\big[\tau^{kN_x(t)}\big] = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell}_\tau (\tau;\tau)_\ell G_\ell$$

with

$$\begin{aligned} G_\ell = {} & \frac{\tau^{(1/2)\ell(\ell-1)}}{(2\pi \mathrm{i})^\ell} \int_{C^\ell_{1,\rho}} d\vec{z} \prod_{a<b} \frac{z_a - z_b}{z_a - \tau z_b} \frac{1 - z_a z_b}{1 - \tau z_a z_b} \\ & \times \sum_{x_1 < \cdots < x_\ell \le x} \prod_a \frac{e^{\tilde{\varepsilon}(z_a)t}}{\tau {z_a}^2 - 1} \left(\frac{1 - \tau z_a}{1 - z_a} \right)^{x_a - 1}. \end{aligned} \tag{3.3}$$

For ease of notation, let $\tilde{\xi}_a = \frac{1-\tau z_a}{1-z_a}$. A computation shows that

$$\sum_{x_1 < \cdots < x_\ell \le x} \prod_{a=1}^{\ell} \tilde{\xi}_a^{x_a - 1} = \prod_{a=1}^{\ell} \tilde{\xi}_a^{x-1} \prod_{a=1}^{\ell} \frac{1}{\tilde{\xi}_1 \cdots \tilde{\xi}_a - 1}.$$

Using this in (3.3), changing variables $z_a = -\tau^{-1} y_a$ and writing $\xi_a = \frac{1+y_a}{1+\tau^{-1} y_a}$ we get

$$\begin{aligned} G_\ell = {} & \frac{\tau^{(1/2)\ell(\ell-1)}}{(2\pi \mathrm{i})^\ell} \int_{C^\ell_{-\tau,\tau\rho}} d\vec{y} \prod_{a<b} \frac{y_a - y_b}{y_a - \tau y_b} \prod_{a=1}^{\ell} \frac{1}{\xi_1 \cdots \xi_a - 1} \\ & \times \prod_{a<b} \frac{1 - \tau^{-2} y_a y_b}{1 - \tau^{-1} y_a y_b} \prod_{a=1}^{\ell} \frac{\xi_a^{x-1}}{\tau - y_a^2} e^{t\hat{\varepsilon}(y_a)}, \end{aligned} \tag{3.4}$$

where the new contour $C_{-\tau,\tau\rho}$ is a circle around $-\tau$ with radius $\tau\rho$ (note that this implies that $-\tau^{1/2}$ lies on its exterior). Now the symmetrization identities appearing in Lemma 7.2 of [9] imply straightforwardly that

$$\sum_{\sigma\in S_\ell}\prod_{a<b}\frac{y_{\sigma(a)}-y_{\sigma(b)}}{y_{\sigma(a)}-\tau y_{\sigma(b)}}\prod_a\frac{1}{\xi_{\sigma(1)}\cdots\xi_{\sigma(a)}-1}$$

$$=\frac{(-1)^\ell}{(\tau;\tau)_\ell}\prod_a\frac{\tau+y_a}{y_a}\sum_{\sigma\in S_\ell}\prod_{a<b}\frac{y_{\sigma(a)}-y_{\sigma(b)}}{y_{\sigma(a)}-\tau y_{\sigma(b)}}.$$

Note that, crucially, the last two factors on the right-hand side of (3.4) are already symmetric, so the above identity can be used to symmetrize the whole integral, yielding

$$G_\ell=(-1)^\ell\tau^{(1/2)\ell(\ell-1)-(1/2)k(k-1)}\frac{1}{(\tau;\tau)_\ell}\tilde{\nu}_\ell$$

with

$$\tilde{\nu}_\ell=\frac{\tau^{(1/2)k(k-1)}}{(2\pi\mathrm{i})^\ell}\int_{C^\ell_{-\tau,\tau\rho}}d\vec{y}\prod_{a<b}\frac{y_a-y_b}{y_a-\tau y_b}\frac{1-\tau^{-2}y_ay_b}{1-\tau^{-1}y_ay_b}$$

$$\times\prod_{a=1}^{\ell}\frac{e^{t\hat{\varepsilon}(y_a)}}{\tau-y_a^2}\Big(\frac{1+y_a}{1+\tau^{-1}y_a}\Big)^{x-1}\frac{\tau+y_a}{y_a}.$$

Therefore, we have

$$\mathbb{E}\big[\tau^{kN_x(t)}\big]=\sum_{\ell=0}^{k}\binom{k}{\ell}_\tau\tau^{(1/2)\ell(\ell-1)-(1/2)k(k-1)}\tilde{\nu}_\ell.$$

We have written things so that we may easily compare with Lemma 4.20 in [9]. Note that $\tilde{\nu}_\ell$ may be rewritten as

$$\tilde{\nu}_\ell=\frac{1}{(2\pi\mathrm{i})^\ell}\int_{C^\ell_{-\tau,\tau\rho}}\prod_{a<b}\frac{y_a-y_b}{y_a-\tau y_b}s(y_a,y_b)\prod_a f(y_a)\frac{1}{y_a},$$

where $s(y,y')=(1-\tau^{-2}yy')/(1-\tau^{-1}yy')$ has no poles in y and y' in a suitable contour encircling 0 and $-\tau$, while f is a function with no poles in a ball around 0 and such that $f(0)=1$. This is exactly the structure of $\tilde{\nu}_\ell$ in Lemma 4.20 of [9], and it is easy to see the extra factor $\prod_{a<b}s(y_a,y_b)$ in our formula makes no difference in the argument. Hence, using their result, we deduce that $\mathbb{E}[\tau^{kN_x(t)}]$ has the form claimed in (3.2). □

As we explained in the Introduction, we would like to manipulate the formula (3.2) given in the last result into one where all contours coincide. Doing this involves expanding the nested contours one by one so that they all end up coinciding with the largest one. As this multiple contour deformation is performed, many poles are crossed. The associated residues group into clusters, and this leads

to a formula which is a sum of contour integrals naturally indexed by *partitions* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$. We will write $\lambda \vdash k$ if $\sum_a \lambda_a = k$ and we will denote by $\ell(\lambda)$ the number of nonzero elements of λ. Additionally, we will write $\lambda = 1^{m_1} 2^{m_2} \cdots$ if a appears m_a times in λ, so in this case $\ell(\lambda) = \sum_a m_a$ and $\lambda \vdash \sum_a a m_a$.

The contour shift argument referred to above was used in the setting of Macdonald processes in [6] and later for q-TASEP and ASEP in [9]. In the setting of the delta Bose gas (or Yang's system) with general type root systems, it goes back to the work of [19]. Section 7 of [8] contains a detailed presentation of this argument, and in fact the proposition that follows is a particular case of a result proved there.

PROPOSITION 3.3.

$$\begin{aligned}
&\mathbb{E}\big[\tau^{kN_x(t)}\big] \\
&\quad = k_\tau! \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{(1-\tau)^k}{m_1! m_2! \cdots} \frac{1}{(2\pi \mathrm{i})^{\ell(\lambda)}} \\
&\qquad \times \int_{\gamma_{-\tau,0}^{\ell(\lambda)}} d\vec{w} \det\left[\frac{-1}{w_a \tau^{\lambda_a} - w_b}\right]_{a,b=1}^{\ell(\lambda)} \\
&\qquad \times H\big(w_1, \tau w_1, \ldots, \tau^{\lambda_1 - 1} w_1, \ldots, w_{\ell(\lambda)}, \ldots, \tau^{\lambda_{\ell(\lambda)} - 1} w_{\ell(\lambda)}\big),
\end{aligned} \tag{3.5}$$

where $\gamma_{-\tau,0}$ is a (positively oriented) contour around $-\tau$ and 0, strictly contained inside the disk of radius $\tau^{1/2}$ and which does not include any other singularities of the integrand, and

$$H(y_1, \ldots, y_k) = \prod_{a<b} \frac{1 - \tau^{-2} y_a y_b}{1 - \tau^{-1} y_a y_b} \prod_a F_{x,t}(y_a).$$

PROOF. It is not hard to check that the contours and the integrand which appear on the right-hand side of (3.2) satisfy the hypotheses of Proposition 7.4 of [8], and thus

$$\begin{aligned}
&\mathbb{E}\big[\tau^{kN_x(t)}\big] \\
&\quad = \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{(-1)^k (1-\tau)^k}{m_1! m_2! \cdots} \frac{1}{(2\pi \mathrm{i})^{\ell(\lambda)}} \int_{\gamma_{-\tau,0}^{\ell(\lambda)}} d\vec{w} \det\left[\frac{1}{w_a \tau^{\lambda_a} - w(b)}\right]_{a,b=1}^{\ell(\lambda)} \\
&\qquad \times \prod_{a=1}^{\ell(\lambda)} w_a^{\lambda_a} \tau^{(1/2)\lambda_a(\lambda_a - 1)} \\
&\qquad\quad \times E\big(w_1, \tau w_1, \ldots, \tau^{\lambda_1 - 1} w_1, \ldots, w_{\ell(\lambda)}, \ldots, \tau^{\lambda_{\ell(\lambda)} - 1} w_{\ell(\lambda)}\big)
\end{aligned}$$

with

$$E(y_1,\dots,y_k) = \sum_{\sigma\in S_k}\prod_{1\le b\le a\le k}\frac{y_{\sigma(a)}-\tau y_{\sigma(b)}}{y_{\sigma(a)}-y_{\sigma(b)}}\prod_{a<b}\frac{1-\tau^{-2}y_{\sigma(a)}y_{\sigma(b)}}{1-\tau^{-1}y_{\sigma(a)}y_{\sigma(b)}}\prod_a\frac{F_{x,t}(y_{\sigma(a)})}{y_{\sigma(a)}}.$$

Note now that the second and third products in the definition of E are symmetric under permutation of the indices in $\vec{y}$. On the other hand, by III.(1.4) in [27] the first double product in the same identity can be symmetrized as

$$\sum_{\sigma\in S_k}\prod_{a>b}\frac{y_{\sigma(a)}-\tau y_{\sigma(b)}}{y_{\sigma(a)}-y_{\sigma(b)}} = (1-\tau)^{-k}(\tau;\tau)_k = k_\tau!. \tag{3.6}$$

Hence, $E(y_1,\dots,y_k) = k_\tau!\prod_{a<b}\frac{1-\tau^{-2}y_ay_b}{1-\tau^{-1}y_ay_b}\prod_{a=1}^k\frac{F_{x,t}(y_a)}{y_a}$. Evaluating E at the point $(y_1,\dots,y_k) = (w_1,\tau w_1,\dots,\tau^{\lambda_1-1}w_1,\dots,w_{\ell(\lambda)},\dots,\tau^{\lambda_{\ell(\lambda)}-1}w_{\ell(\lambda)})$ leads, after some simplifications, to (3.5). □

As we will see below, the strings of geometric progressions appearing in (3.5) account for the ratios of q-Pochhammer symbols in (1.9) [see (3.12)], which in this case can be thought of as ratios of q-Gamma functions. This is analogous to the strings of arithmetic progressions which appear in the case of the delta Bose gas, which give rise to ratios of Gamma functions (see Section 5).

We are now ready for the proof of our main moment formula for $\tau^{N_x(t)}$ in the half-flat case.

PROOF OF THEOREM 1.3. The formula given in Proposition 3.3 can be rewritten as

$$\mathbb{E}\big[\tau^{kN_x(t)}\big] = k_\tau!\sum_{\ell=0}^k\sum_{\substack{m_1,m_2,\dots\\ \sum_a m_a=\ell,\sum_a am_a=k}}\frac{1}{\ell!}\frac{\ell!}{m_1!m_2!\cdots}\frac{1}{(2\pi\mathrm{i})^\ell} \times\int_{\gamma^\ell_{-\tau,0}}d\vec{w}\,I_\ell(\lambda_{m_1,m_2,\dots};\vec{w}), \tag{3.7}$$

where $\lambda_{m_1,m_2,\dots}$ is specified by $\lambda_{m_1,m_2,\dots} = 1^{m_1}2^{m_2}\cdots$ and

$$I_\ell(\lambda;\vec{w}) = \det\Big[\frac{-1}{w_a\tau^{\lambda_a}-w_b}\Big]_{a,b=1}^{\ell(\lambda)}H\big(w_1,\dots,w_1^{\lambda_1-1},\dots,w_{\ell(\lambda)},\dots,w_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}-1}\big) \times\prod_a(1-\tau)^{\lambda_a}. \tag{3.8}$$

In the above sum, $m_1,m_2,\dots$ encodes the partition $\lambda_{m_1,m_2,\dots}$ of k of length ℓ. Observe on the other hand that, by the symmetry of the integrand, the right-hand

side of (3.7) is unchanged if we permute the λ_a's. Thus, we can get rid of the multinomial coefficient $\frac{\ell!}{m_1!m_2!\cdots}$ by replacing the sum over the m_a's by a sum over (unordered) $n_1, \dots, n_\ell$ with the following correspondence: for each a, exactly m_a out of the $n_1, n_2, \dots, n_\ell$ equal a. This gives

$$\mathbb{E}\big[\tau^{kN_x(t)}\big] = k_\tau! \sum_{\ell=0}^{k} \frac{1}{\ell!} \sum_{\substack{n_1,\dots,n_\ell \geq 1 \\ \sum n_a = k}} \frac{1}{(2\pi \mathrm{i})^\ell} \int_{\gamma^\ell_{-\tau,0}} d\vec{w}\, I_\ell\big((n_1,\dots,n_\ell);\vec{w}\big), \tag{3.9}$$

where the notation (3.8) has been extended trivially to unordered ℓ-tuples $(n_1, \dots, n_\ell)$.

What remains is to simplify the integrand. Define

$$g_1(w) = \frac{(-\tau^{-1}w;\tau)_\infty}{(\tau^{-1}w^2;\tau^2)_\infty}\left(\frac{\tau}{\tau+w}\right)^{x-1} e^{(q-p)t(\tau/(\tau+w))},$$

$$g_2(w_1,w_2) = \frac{(\tau^{-1}w_1^2;\tau^2)_\infty}{(\tau^{-3}w_2^2;\tau^2)_\infty}\frac{(\tau^{-3}w_2^2;\tau)_\infty}{(\tau^{-2}w_1w_2;\tau)_\infty},$$

and write $\vec{w}\circ\vec{n} = (w_1, \dots, w_1^{n_1-1}, \dots, w_\ell, \dots, w_\ell^{n_\ell-1})$. We have

$$H(\vec{w}\circ\vec{n}) = \widetilde{H}(\vec{w}\circ\vec{n}) \prod_{a=1}^{k}\prod_{b=0}^{n_a-1} F_{x,t}(\tau^b w_a) \tag{3.10}$$

$$\text{with } \widetilde{H}(y_1,\dots,y_k) = \prod_{a<b}\frac{1-\tau^{-2}y_ay_b}{1-\tau^{-1}y_ay_b}.$$

One checks directly that $F_{x,t}(y) = g_1(y)/g_1(\tau y)$, whence

$$\prod_{a=1}^{k}\frac{g_1(w_a)}{g_1(\tau^{n_a}w_a)} = \prod_{a=1}^{k}\prod_{b=0}^{n_a-1} F_{x,t}(\tau^b w_a). \tag{3.11}$$

On the other hand, we have

$$\widetilde{H}(\vec{w}\circ\vec{n}) = \widetilde{H}\big(\vec{w}^{(1)}\circ\vec{n}^{(1)}\big) \prod_{0\leq a_1<a_2<n_1}\frac{1-\tau^{a_1+a_2-2}w_1^2}{1-\tau^{a_1+a_2-1}w_1^2}$$

$$\times \prod_{b=2}^{k}\prod_{a_1=0}^{n_1-1}\prod_{a_2=0}^{n_b-1}\frac{1-\tau^{a_1+a_2-2}w_1w_b}{1-\tau^{a_1+a_2-1}w_1w_b}.$$

The first product on the right-hand side equals

$$\prod_{a_1=0}^{n_1-2}\frac{1-\tau^{2a_1-1}w_1^2}{1-\tau^{a_1+n_1-2}w_1^2} = \prod_{a_1=0}^{n_1-2}\frac{(\tau^{2a_1-1}w_1^2;\tau^2)_\infty}{(\tau^{2a_1+1}w_1^2;\tau^2)_\infty}\frac{(\tau^{a_1+n_1-1}w_1^2;\tau)_\infty}{(\tau^{a_1+n_1-2}w_1^2;\tau)_\infty}$$

$$= g_2\big(w_1,\tau^{n_1}w_1\big).$$

One checks similarly that, for fixed b, the second product equals $\mathfrak{h}(\tau^{-1}w_1, \tau^{-1}w_b; n_1, n_b)$. We deduce that $\widetilde{H}(\vec{w} \circ \vec{n}) = \widetilde{H}(\vec{w}^{(1)} \circ \vec{n}^{(1)}) g_2(w_1, \tau^{n_1} w_1) \prod_{b=2}^{k} \mathfrak{h}(\tau^{-1}w_1, \tau^{-1}w_b; n_1, n_b)$. Proceeding inductively to rewrite the right-hand side yields and using (3.10) and (3.11) yields

$$H(\vec{w} \circ \vec{n}) = \prod_a \frac{g_1(w_a)}{g_1(\tau^{n_a} w_a)} g_2(w_a, \tau^{n_a} w_a) \prod_{a<b} \mathfrak{h}(\tau^{-1}w_a, \tau^{-1}w_b; n_a, n_b). \tag{3.12}$$

To finish, we note that there is a simplification in the τ-Pochhammer symbols coming from the factors $g_1(w_a)/g_1(\tau^{n_a}w_a)$ and $g_2(w_a, \tau^{n_a}w_a)$:

$$\begin{aligned}
&\frac{(-\tau^{-1}w;\tau)_\infty}{(\tau^{-1}w^2;\tau^2)_\infty} \frac{(\tau^{-1+2n}w^2;\tau^2)_\infty}{(-\tau^{-1+n}w;\tau)_\infty} \frac{(\tau^{-1}w^2;\tau^2)_\infty}{(\tau^{-3+2n}w^2;\tau^2)_\infty} \frac{(\tau^{-3+2n}w^2;\tau)_\infty}{(\tau^{-2+n}w^2;\tau)_\infty} \\
&\quad = \frac{(-\tau^{-1}w;\tau)_\infty}{(-\tau^{-1+n}w;\tau)_\infty} \frac{(\tau^{-1+2n}w^2;\tau^2)_\infty}{(\tau^{-2+n}w^2;\tau)_\infty} \frac{(\tau^{-3+2n}w^2;\tau)_\infty}{(\tau^{-3+2n}w^2;\tau^2)_\infty} \\
&\quad = \frac{(-\tau^{-1}w;\tau)_\infty}{(-\tau^{-1+n}w;\tau)_\infty} \frac{(\tau^{-2+2n}w^2;\tau)_\infty}{(\tau^{-2+n}w^2;\tau)_\infty}.
\end{aligned}$$

The right-hand side is exactly $\mathfrak{g}(w, n)$. Using this in (3.12) and (3.8), we deduce that

$$\begin{aligned}
I_\ell\big((n_1, \dots, n_\ell); \vec{w}\big) = \det\Big[\frac{-1}{w_a \tau^{\lambda_a} - w_b}\Big]_{a,b=1}^{\ell} \prod_a \mathfrak{f}(\tau^{-1}w_a, n_a) \mathfrak{g}(\tau^{-1}w_a, n_a) \\
\times \prod_{a<b} \mathfrak{h}(\tau^{-1}w_a, \tau^{-1}w_b; n_a, n_b).
\end{aligned}$$

Comparing with (3.9) and (1.12) yields the result after the change of variables $w_a \mapsto \tau w_a$ (absorbing the Jacobian from the change of variables into the determinant). □

4. Generating function. Since, by definition, $N_x(t) \geq 0$, we have $\tau^{N_x(t)} \leq 1$ and thus by (1.11) we have for $|\zeta| < 1$ that

$$\mathbb{E}\big[e_\tau\big(\zeta \tau^{N_x(t)}\big)\big] = \sum_{m \geq 0} \frac{\zeta^m}{m_\tau!} \mathbb{E}^{\text{h-fl}}\big[\tau^{m N_x(t)}\big]. \tag{4.1}$$

Using (1.12) to write the expectation on the right-hand side explicitly and interchanging the sums in m and k formally leads to

$$\begin{aligned}
\mathbb{E}\big[e_\tau\big(\zeta \tau^{N_x(t)}\big)\big] = \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \dots, n_k \geq 1} \frac{1}{(2\pi \mathrm{i})^k} \int_{\gamma_{-1,0}^k} d\vec{w} \det\Big[\frac{-1}{w_a \tau^{n_a} - w_b}\Big]_{a,b=1}^{k} \\
\times \prod_a \zeta^{n_a} \mathfrak{f}(w_a, n_a) \mathfrak{g}(w_a, n_a) \prod_{a<b} \mathfrak{h}(w_a, w_b; n_a, n_b).
\end{aligned} \tag{4.2}$$

As we will see in the proof of Theorem 1.4, the application of Fubini's theorem here can be justified, which implies that the above formula holds as long as $|\zeta| < 1$. In order to analytically extend this identity beyond this region, we proceed as in [6] and use a Mellin–Barnes representation for the sums in n_a. The precise result we will use is the following.

LEMMA 4.1. *Let g be a meromorphic function and $C_{1,2,\dots}$ a negatively oriented contour enclosing all positive integers (e.g., $C_{1,2,\dots} = \frac{1}{2} + \mathrm{i}\mathbb{R}$ oriented with increasing imaginary part) but no other singularities of $g(\tau^s)$ (in s).*[4] *Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ with $|\zeta| < 1$ we have*

$$\sum_{n=1}^{\infty} g(\tau^n)\zeta^n = \frac{1}{2\pi \mathrm{i}} \int_{C_{1,2,\dots}} ds \frac{\pi}{\sin(-\pi s)} (-\zeta)^s g(\tau^s),$$

provided that the left-hand side converges and that there exist closed contours C_k, $k \in \mathbb{N}$ enclosing the positive integers from 1 to k and such that the integral of the integrand on the right-hand side over the symmetric difference of $C_{1,2,\dots}$ and C_k goes to zero as $k \to \infty$.

The statement follows easily from the fact that $\pi / \sin(-\pi s)$ has a pole at each $s = k \in \mathbb{Z}$ with residue equal to $(-1)^{k+1}$.

We will also need some precise estimates on $\mathfrak{h}$, which will be provided by the lemma that follows. These estimates will be valid when the relevant variables lie inside some carefully chosen contours, which we define next.

DEFINITION 4.2. Let $B(x, r) \subseteq \mathbb{C}$ denote the ball of radius r centered at x. For $x_1 < x_2$ and suitably small $r_1, r_2 > 0$, we define a positively oriented contour $\bar{\gamma}(x_1, r_1; x_2, r_2)$ consisting on the left half of $\partial B(x_1, r_1)$, the right half of $\partial B(x_2, r_2)$, and two lines connecting, respectively, the top and bottom ends of the two half circles. Additionally, for $\theta, M > 0$ we define a contour $D_{\theta,M}$ going by straight lines from $M - \mathrm{i}\infty$, to $M - \mathrm{i}\theta$, to $\frac{1}{2} - \mathrm{i}\theta$, to $\frac{1}{2} + \mathrm{i}\theta$, to $M + \mathrm{i}\theta$, to $M + \mathrm{i}\infty$. See Figure 2.

LEMMA 4.3. *Define the function*

$$\mathfrak{h}_0(z; s_1, s_2) = \frac{(z; \tau)_\infty (\tau^{s_1+s_2} z; \tau)_\infty}{(\tau^{s_1} z; \tau)_\infty (\tau^{s_2} z; \tau)_\infty}.$$

Then there exist constants $C > 0$ and $\rho \in (0, \min\{\frac{1}{2}(\tau^{-1/2} - 1), 1\})$ such that, given any $\delta \in (0, 1)$ there are $\theta, M > 0$ with the following property: if s_1, s_2 lie to the right of $D_{\theta,M}$ and z is inside $\bar{\gamma}(0, \delta; 1, \rho)$, then $|\mathfrak{h}_0(z; s_1, s_2)| < 1 + C\delta$.

[4] Here, $z \longmapsto z^s$ is defined by taking a branch cut along the negative real axis.

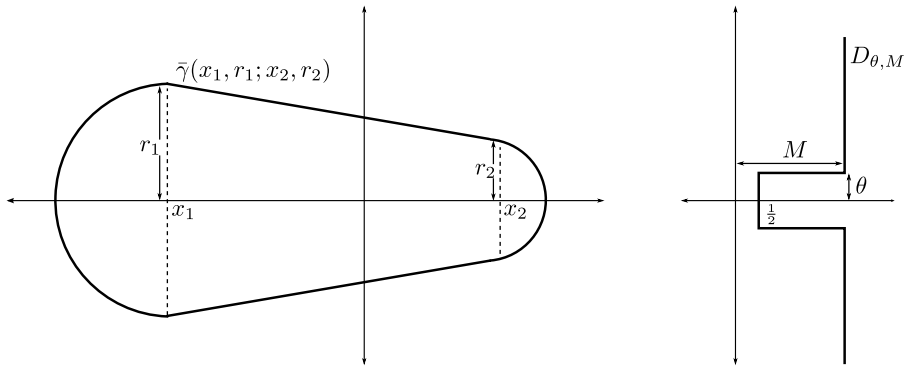

FIG. 2. *Contours in Definition 4.2.*

PROOF. Fix $\delta_0 \in (1, \tau^{-1/2})$ and $\rho_0 \in (0, \min\{\frac{1}{2}(\tau^{-1/2} - 1), 1\})$. For fixed s_1 and s_2, $\mathfrak{h}_0(z; s_1, s_2)$ is a meromorphic function of z, with poles at $z = \tau^{-s_1-\ell}$ and $z = \tau^{-s_2-\ell}$ for $\ell \geq 0$. Since we are interested only in $\Re(s_1) = \Re(s_2) \geq \frac{1}{2}$, all these poles lie outside of $B(0, \tau^{-1/2})$, and thus $\mathfrak{h}_0(z; s_1, s_2)$ is analytic in z inside $\bar\gamma(0, \delta_0; 1, \rho_0)$. Now, in general, if $D_1, \dots, D_m$ are bounded domains in $\mathbb{C}$ and f is a complex-valued function defined on $D = D_1 \times \cdots \times D_m$ which is analytic in each variable, then by the mean value theorem there exists a constant $C > 0$ such that for every $\vec w \in D$ and every $\vec w' \in B(\vec w, \delta) \cap D$ we have

$$|f(\vec w') - f(\vec w)| \leq C\delta. \tag{4.3}$$

We deduce that there is a $C_1 > 0$ such that if z, z' lie inside $\bar\gamma(0, \delta_0; 1, \rho_0)$ and $|z - z'| < r$, then

$$|\mathfrak{h}_0(z; s_1, s_2)| \leq |\mathfrak{h}_0(z'; s_1, s_2)| + C_1 r. \tag{4.4}$$

Now for $x, \alpha_1, \alpha_2, \in [0, 1]$ let

$$g(x; \alpha_1, \alpha_2) = \frac{(x; \tau)_\infty (\alpha_1\alpha_2 x; \tau)_\infty}{(\alpha_1 x; \tau)_\infty (\alpha_2 x; \tau)_\infty}.$$

A computation shows that $\partial_x g(x; \alpha_1, \alpha_2)|_{x=0} = (\tau - 1)^{-1}(1 - \alpha_1)(1 - \alpha_2)$. We deduce that

$$C_0 := - \sup_{s_1, s_2 \in [1/2, \infty)} \partial_x g(x; \tau^{s_1}, \tau^{s_2})\big|_{x=0} \in (0, \infty). \tag{4.5}$$

On the other hand, we claim that $g(x; \alpha_1, \alpha_2)$ is concave in $x \in [0, 1]$ for every fixed $\alpha_1, \alpha_2 \in (0, 1)$. To see this, write $g(x; \alpha_1, \alpha_2) = \prod_{\ell \geq 0} g_\ell(x; \alpha_1, \alpha_2)$ with $g_\ell(x; \alpha_1, \alpha_2) = \frac{(1-\tau^\ell x)(1-\tau^\ell \alpha_1\alpha_2 x)}{(1-\tau^\ell \alpha_1 x)(1-\tau^\ell \alpha_2 x)}$. Then it is enough to show that each g_ℓ is positive, decreasing, and concave. The positivity of g_ℓ is clear, while the decrease and concavity can be checked by computing $\partial_x g_\ell$ and $\partial_x^2 g_\ell$ (we leave the details to the

reader). As a consequence of this and (4.5), and since $\mathfrak{h}_0(x; s_1, s_2) = g(x; \tau^{s_1}, \tau^{s_2})$ and $g(0; s_1, s_2) = 1$, we deduce that

$$\mathfrak{h}_0(x; s_1, s_2) \le 1 - C_0 x \tag{4.6}$$

for all $s_1, s_2 \in [\frac{1}{2}, \infty)$ and $x \in [0, 1]$.

Choose $\rho < \min\{\rho_0, C_0/C_1\}$ and let $r(x) = (1-x)\delta + x\rho$. In order to prove the result it is enough to prove the following statement: there are $\theta, M > 0$ (depending on δ) and $C_2 > 0$ such that for all $x \in [0, 1]$, $z \in B(x, r(x))$ and s_1, s_2 lying to the right of $D_{\theta,M}$ we have

$$\bigl|\mathfrak{h}_0(z; s_1, s_2)\bigr| \le 1 + (C_1 + C_2)\delta. \tag{4.7}$$

Assume first that $s_1, s_2 \in [\frac{1}{2}, \infty)$. Fix $x \in [0, 1]$ and $z \in B(x, r(x))$. Then by (4.4) and (4.6), we have

$$\begin{aligned} \bigl|\mathfrak{h}_0(z; s_1, s_2)\bigr| &\le \bigl|\mathfrak{h}_0(x; s_1, s_2)\bigr| + r(x)C_1 \\ &\le 1 + C_1\delta + C_1(\rho - \delta)x - C_0 x < 1 + C_1\delta, \end{aligned} \tag{4.8}$$

so, in particular, (4.7) holds.

Now we want to extend this to all s_1, s_2 lying to the right of $D_{\theta,M}$. Write $s_a = \eta_a + \mathrm{i}\theta_a$. There are four cases to consider, depending on whether or not η_1 and η_2 are larger than M. Let us assume first that $\eta_1, \eta_2 \ge M$. Since $z \in B(0, 2)$ (because $\delta, \rho < 1$) we have that $\tau^s z \in B(0, 2\tau^M) \subseteq B(0, \frac{1}{2}\delta)$ for $\Re(s) \ge \frac{1}{2}$ and large enough M, and thus $|\tau^{s_1} z - \tau^{\eta_1} z| < \delta$, $|\tau^{s_2} z - \tau^{\eta_2} z| < \delta$ and $|\tau^{s_1+s_2} z - \tau^{\eta_1+\eta_2} z| < \delta$. An argument similar to the one above, based on (4.3), shows then that there is a constant $C_2 > 0$ such that

$$\bigl|\mathfrak{h}_0(z; s_1, s_2) - \mathfrak{h}_0(z; \eta_1, \eta_2)\bigr| < C_2\delta.$$

Using this together with the bound (4.8) for $\mathfrak{h}_0(z; \eta_1, \eta_2)$ yields (4.7).

The other three cases are similar. For example, if both s_1 and s_2 are in $[\frac{1}{2}, M] \times \mathrm{i}[-\theta, \theta]$ then, for M fixed as above, we can choose a small enough θ so that $|\tau^{s_1} z - \tau^{\eta_1} z| < \delta$, $|\tau^{s_2} z - \tau^{\eta_2} z| < \delta$ and $|\tau^{s_1+s_2} z - \tau^{\eta_1+\eta_2} z| < \delta$, and then the same argument works. The mixed case works similarly (although it may yield a different constant). □

PROOF OF THEOREM 1.4. We will prove this result in three steps. The first one will consist in showing that (4.2) holds when $|\zeta| < 1$. In the second step we will apply the Mellin–Barnes representation given by Lemma 4.1 to turn (4.2) into (1.14) for $|\zeta| < 1$, $\zeta \notin \mathbb{R}_{\ge 0}$. Finally we will analytically extend the resulting formula to all $\zeta \notin \mathbb{R}_{\ge 0}$.

Assume then that $|\zeta| < 1$, so that (4.1) holds. Using this formula together with (1.12) leads to

$$\mathbb{E}^{\text{h-fl}}\bigl[e_\tau\bigl(\zeta \tau^{N_x(t)}\bigr)\bigr] = \sum_{m \ge 0} \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1, \dots, n_k \ge 1 \\ n_1 + \cdots + n_k = m}} I_k(\vec{n}),$$

where

$$I_k(\vec{n}) = \frac{1}{(2\pi \mathrm{i})^k} \int_{\gamma_{-1,0}^k} d\vec{w} \det\Big[\frac{-1}{w_a \tau^{n_a} - w_b}\Big]_{a,b=1}^k \times \prod_a \zeta^{n_a} \mathfrak{f}(w_a; n_a) \mathfrak{g}(w_a; n_a) \prod_{a<b} \mathfrak{h}(w_a, w_b; n_a, n_b).$$

Interchanging the sums in k and m leads to

$$\begin{aligned} \mathbb{E}^{\text{h-fl}}\big[e_\tau\big(\zeta \tau^{N_x(t)}\big)\big] &= \sum_{k\geq 0} \sum_{m\geq k} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1 \\ n_1+\cdots+n_k=m}} I_k(\vec{n}) \\ &= \sum_{k\geq 0} \frac{1}{k!} \sum_{n_1,\dots,n_k\geq 1} I_k(\vec{n}). \end{aligned} \tag{4.9}$$

In order to justify the application of Fubini's theorem, it is enough to verify that the sum $\sum_{k\geq 0} \sum_{m\geq k} |\frac{1}{k!} \sum_{n_1+\cdots+n_k=m} I_k(\vec{n})|$ is finite, which by the triangle inequality, will follow if we verify that

$$\sum_{k\geq 0} \frac{1}{k!} \sum_{n_1,\dots,n_k\geq 1} \big|I_k(\vec{n})\big| < \infty. \tag{4.10}$$

The main difficulty we face at this point is the fact that the absolute value of $\mathfrak{h}(w_a, w_b; n_a, n_b)$ is in general not bounded by 1, which in principle introduces a factor of order c^{k^2} into our sum for some $c > 1$. To deal with this issue, we will have to choose the contour $\gamma_{-1,0}$ carefully, and moreover let it depend on k. Note, however, that this choice is made at this point only in order to obtain a suitable estimate, and does not fix the contour in the statement of the theorem.

Now fix $\rho > 0$ and $C > 0$ as in Lemma 4.3 and, for fixed k, let $\delta_k = C^{-1}(2^{1/k} - 1)$ and choose $\theta_k, M_k > 0$ as in Lemma 4.3 for $\delta = \delta_k$. Furthermore, let $\delta_k', \rho' > 0$, $\theta_k' < \theta_k$ and $M_k' > M_k$, and write $\bar{\gamma}_k = \bar{\gamma}(-1, \rho'; 0, \delta_k')$ and $\bar{D}_k = D_{\theta_k', M_k'}$ ($\bar{D}_k$ will be used in the second step). Note that $\bar{\gamma}_k$ is star-shaped with respect to the origin (i.e., any ray emanating from the origin intersects the contour in one and only one point). This implies, in particular, that the denominator inside the determinant appearing in $I(\vec{n})$ never vanishes. On the other hand, by choosing δ_k' and ρ' to be suitably small we may assume that $\bar{\gamma}_k$ is contained inside $B(0, \tau^{-1/2})$, in which case it is easy to check that there are not singularities of $\mathfrak{h}$ inside. Therefore, our choice of $\bar{\gamma}_k$ satisfies the requirements of Theorem 1.3.

Having made this choice of contour, we claim that we can choose an $\eta > 0$ such that if $\delta_k' = \eta \delta_k$ and ρ' is small enough then whenever $w_a, w_b \in \bar{\gamma}_k$ we have that $w_a w_b$ is contained inside $\bar{\gamma}(0, \delta_k; 1, \rho)$. To see this, observe that $\{ww' : w, w' \in [-1, 0]\} = [0, 1]$ and, therefore, given any open neighborhood U of $[0, 1]$ we can find an open neighborhood V of $[-1, 0]$ such that $\{ww' : w, w' \in V\}$ is contained

inside U. Our claim follows easily from this because given any such neighborhood V we can choose δ_k' and ρ small enough so that $\bar{\gamma}_k$ is contained inside V.

Making these choices, and thanks to our earlier choices of parameters and using Lemma 4.3, we get

$$|\mathfrak{h}(w_a, w_b; n_a, n_b)| = |\mathfrak{h}_0(w_a w_b; n_a, n_b)| \le 2^{1/k} \tag{4.11}$$

for $w_a, w_b \in \bar{\gamma}_k$ and $n_a, n_b \in \mathbb{Z}_{\ge 1}$ (since in this case n_a and n_b trivially lie to the right of $\bar{D}_k$). On the other hand, the only singularity of $\mathfrak{f}(w_a; n_a)$ occurs at $w_a = -1$, and since $\bar{\gamma}_k$ stays at distance at least ρ' from -1, this factor is uniformly bounded along the contour, say by some constant $c_1 > 0$ (independently of k). A similar argument shows that $|\mathfrak{g}(w_a; n_a)|$ is uniformly bounded (say by c_1 again), and we deduce that

$$\begin{aligned} |I_k(\vec{n})| &\le \frac{c_1^{2k} 2^{(1/2)(k-1)}}{(2\pi)^k} \int_{\bar{\gamma}_k^k} d\vec{w} \prod_a |\zeta|^{n_a} \left| \det\left[\frac{-1}{w_a \tau^{n_a} - w_b} \right]_{a,b=1}^k \right| \\ &\le c_2^k |\zeta|^{\sum_a n_a} \int_{\bar{\gamma}_k^k} d\vec{w} \prod_a \frac{1}{|w_a|} \left| \det\left[\frac{-w_a}{w_a \tau^{n_a} - w_b} \right]_{a,b=1}^k \right| \\ &\le c_2^k k^{k/2} |\zeta|^{\sum_a n_a} \int_{\bar{\gamma}_k^k} d\vec{w} \prod_a \frac{1}{|w_a|} \sup_{a,b=1,\dots,k} \left| \frac{w_a}{w_a \tau^{n_a} - w_b} \right|^k \end{aligned} \tag{4.12}$$

for some $c_2 > 0$, where in the last inequality we used Hadamard's bound. The supremum is clearly bounded by some constant $c_3 > 0$, uniformly in w_a, w_b and n_a. On the other hand, it is not hard to check that

$$\int_{\bar{\gamma}_k} dw_a \frac{1}{|w_a|} \le c_4 |\log(\delta_k')| = c_4 |\log(\eta \delta_k)| \le c_4' \log(k)$$

for some $c_4, c_4' > 0$ by our choice of δ_k and δ_k'. We deduce that

$$|I_k(\vec{n})| \le c^k \big(k^{1/2} \log(k)\big)^k |\zeta|^{\sum_a n_a}$$

for some $c > 0$ and thus, since we are taking $|\zeta| < 1$, (4.10) holds. Therefore, (4.9) holds for $|\zeta| < 1$.

As we mentioned at the beginning of the proof, the next step is to apply the Mellin–Barnes representation to (4.9). The idea is to focus on the kth term of the sum on the right-hand side of (4.9) for some fixed k, and then apply Lemma 4.1 one by one to each of the sums in $n_1, \dots, n_k$ with the contour $C_{1,2,\dots}$ taken as $\bar{D}_k = D_{\theta_k', M_k'}$ [and $\gamma_{-1,0}$ as $\bar{\gamma}_k = \bar{\gamma}(-1, \rho'; 0, \delta_k')$], which would prove the identity

$$\begin{aligned} \mathbb{E}^{\text{h-fl}}\big[e_\tau\big(\zeta \tau^{N_x(t)}\big)\big] = \sum_{k=0}^\infty \frac{1}{k!} \frac{1}{(2\pi \mathrm{i})^{2k}} \int_{\bar{D}_k^k} d\vec{s} \int_{\bar{\gamma}_k^k} d\vec{w} \det\left[\frac{-1}{w_a \tau^{s_a} - w_b} \right]_{a,b=1}^k \\ \times \prod_a (-\zeta)^{s_a} \mathfrak{f}(w_a; s_a) \mathfrak{g}(w_a; s_a) \prod_{a<b} \mathfrak{h}(w_a, w_b; s_a, s_b) \end{aligned} \tag{4.13}$$

for $\zeta \notin \mathbb{R}_{\geq 0}$ with $|\zeta| < (1-\tau)^{-1}$. To this end, we need to verify that the conditions of the lemma are satisfied. Note that, in view of the preceding argument, we are free to choose θ_k' and M_k' to be respectively even smaller and even larger than in our original choice. We start by observing that $w_a \tau^{s_a} - w_b$ never vanishes for s_a along this contour. To see this, note first that M_k' can be chosen to be sufficiently large so that if $\bar{\gamma}_k$ is scaled by $\tau^{M_k'}$ then any rotation of the resulting contour is contained inside $\bar{\gamma}_k$, which shows that $w_a \tau^{s_a} - w_b \neq 0$ for s_a with $\Re(s_a) \geq M_k'$. On the other hand, since $\bar{\gamma}_k$ is star-shaped, $w_a \tau^{s_a} - w_b \neq 0$ for $s_a \in [\frac{1}{2}, \infty)$, and thus the same holds in the strip $[\frac{1}{2}, M_k'] \times \mathrm{i}[-\theta_k', \theta_k']$ if θ_k' is small enough. This shows that there are no singularities of the determinant in the integrand in (4.13) for s_a lying to the right of $\bar{D}_k$. The singularities of the remaining factors are all avoided in this region for similar reasons.

What is left to check is that there are closed contours $C_{k,m}$ enclosing $1, \dots, m$ [and contained in $\{s : \Re(s) \geq \frac{1}{2}\}$] such that the integral on the symmetric difference of $\bar{D}_k$ and $C_{k,m}$ goes to 0 as $m \to \infty$. We choose $C_{k,m}$ to be union of the piece of $\bar{D}_k$ lying inside $B(0, m + \frac{1}{2})$ and the arc on the boundary of this ball lying to the right of $\bar{D}_k$. But this is actually not hard to see. We have already checked that $\mathfrak{f}(w_a; s_a)$, $\mathfrak{g}(w_a; s_a)$, $\mathfrak{h}(w_a, w_b; s_a, s_b)$ and the determinant have no singularities for s_a, s_b lying to the right of $\bar{D}_k$, and since these factors depend on s_a, s_b only through τ^{s_a}, τ^{s_b}, which live in a compact set, they are bounded uniformly. The necessary decay is going to come from the product $|\pi / \sin(\pi s_a)||\zeta^{s_a}|$. In fact, as $|\Im(s_a)| \to \infty$ with $\Re(s_a) = \frac{1}{2}$ we have that $|\pi / \sin(\pi s_a)|$ decays exponentially while $|\zeta^{s_a}|$ stays bounded. The same exponential decay applies in the circular part of $C_{k,m}$ restricted to $|\arg(s_a)| > \frac{\pi}{4}$ [since here $|\Im(s_a)| \to \infty$ as before]. Finally, note that on the circular piece of $C_{k,m}$ with $|\arg(s_a)| > \frac{\pi}{4}$ we have that s_a stays bounded away from all integers, so that $|\pi / \sin(\pi s_a)|$ is uniformly bounded, while $\Re(s_a) \to \infty$, so that $|\zeta^{s_a}|$ decays exponentially. Putting these facts together shows that the integrand has the right decay, and gives (4.13).

Our third step is to analytically extend (4.13) to all $\zeta \notin \mathbb{R}_{\geq 0}$, for which we need to show that both sides are analytic in ζ in that region. Observe first that the left-hand side is given by

$$\mathbb{E}^{\text{h-fl}}\big[e_\tau\big(\zeta \tau^{N_x(t)}\big)\big] = \sum_{n \geq 0} \frac{\mathbb{P}^{\text{h-fl}}(N_x(t) = n)}{((1-\tau)\zeta\tau^n; \tau)_\infty}.$$

For each $\zeta \notin \{(1-\tau)^{-1}\tau^{-m}\}_{m \in \mathbb{Z}_{\geq 0}}$, this series is uniformly convergent on a neighborhood of ζ, and thus the left-hand side is analytic for $\zeta \notin \mathbb{R}_{\geq 0}$.

Turning to the right-hand side of (4.13), observe that each summand in the series is clearly analytic in $\zeta \notin \mathbb{R}_{\geq 0}$. We will use now the fact that the limit of a uniformly absolutely convergent series of analytic functions is analytic to show that the right-hand side of (4.13) is analytic in ζ in any fixed neighborhood which avoids $\mathbb{R}_{\geq 0}$. Consider the kth term of our series and recall that we have chosen δ_k' and ρ' so

that $w_a w_b$ is inside $\bar{\gamma}(0, \delta_k; 1, \rho)$ for $w_a, w_b \in \bar{\gamma}_k$, while on the other hand $\theta_k' < \theta_k$ and $M_k' > M_k$. As a consequence, and thanks to Lemma 4.3 and our choice of parameters, we deduce as in (4.11) that $|\mathfrak{h}(w_a, w_b; s_a, s_b)| \le 2^{1/k}$ for $w_a, w_b \in \bar{\gamma}_k$ and $s_a, s_b \in \bar{D}_k$. As in the previous step, we have that $\mathfrak{f}(w_a; s_a)$, $\mathfrak{g}(w_a; s_a)$, $\mathfrak{h}(w_a, w_b; s_a, s_b)$ are uniformly bounded and proceeding as in (4.12) we deduce that the kth term of the series on the right-hand side of (4.13) is bounded in absolute value by

$$\frac{c_1^k}{k!}\frac{1}{(2\pi \mathrm{i})^{2k}}\int_{\bar{D}_k^k} d\vec{s}\int_{\bar{\gamma}_k^k} d\vec{w}\prod_a \left|\frac{\pi}{\sin(\pi s_a)}\right|\frac{|\zeta^{s_a}|}{|w_a|}\sup_{a,b=1,\dots,k}\left|\frac{w_a}{w_a\tau^{s_a}-w_b}\right|^k$$
$$\le \frac{c_2^k(k^{1/2}\log(k))^k}{k!}\frac{1}{(2\pi \mathrm{i})^k}\int_{D_k^k} d\vec{s}\prod_a\left|\frac{\pi}{\sin(\pi s_a)}\right||\zeta^{s_a}| \le \frac{c_3^k(k^{1/2}\log(k))^k}{k!}$$

for some constants $c_1, c_2, c_3 > 0$ which are uniform in ζ in a compact subset of $\mathbb{C}$ [here we have used again the fact that $|\pi/\sin(\pi s_a)|$ decays exponentially as $\Im(s_a) \to \infty$]. This shows that the right-hand side of (4.13) is absolutely summable, uniformly in ζ on a fixed neighborhood away from $\mathbb{R}_{\ge 0}$ as required, and thus finishes the analytic extension of (4.13) to all $\zeta \notin \mathbb{R}_{\ge 0}$.

At this point, we have proved (4.13). We may now deform the contours $\bar{D}_k$ and $\bar{\gamma}_k$ in each of the summands to the contours $\frac{1}{2} + \mathrm{i}\mathbb{R}$ and $\gamma_{-1,0}$ by appealing to Cauchy's theorem, thus finishing the proof. □

5. Formulas for the KPZ/stochastic heat equation. The one-dimensional Kardar–Parisi–Zhang (KPZ) "equation" is given by

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h - \tfrac{1}{2}\big[(\partial_x h)^2 - \infty\big] + \xi,$$

where ξ is a space–time white noise. This SPDE is ill-posed as written but, at least on the torus, it can be made sense of by a renormalization procedure introduced by Hairer in [17, 18]. His solutions coincide with the Cole–Hopf solution (which is known to be the physically relevant solution; see, e.g., the review [31]) obtained by setting

$$h(t, x) = -\log Z(t, x), \tag{5.1}$$

where Z is the unique solution to the (well-posed) stochastic heat equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \xi Z. \tag{5.2}$$

We will now give a contour integral ansatz for the moments of Z with the "tilted" half-flat initial data defined by $Z(0, x) = e^{-\theta x}\mathbf{1}_{x\ge 0}$.

To be more precise, we will provide a solution for the *delta Bose gas* with this initial data, which we interpret as the solution $v(t, x)$ to the following system of equations, where we write $W_k = \{\vec{x} \in \mathbb{R}^k : x_1 < x_2 < \cdots < x_k\}$ (see [6] for more details):

(1) For $\vec{x} \in W_k$,

$$\partial_t v(t, \vec{x}) = \tfrac{1}{2} \Delta v(t, \vec{x}),$$

where the Laplacian acts on $\vec{x}$.

(2) For $\vec{x}$ on the boundary of W_k, with $x_a = x_{a+1}$,

$$(\partial_{x_a} - \partial_{x_{a+1}} - 1) v(t, \vec{x}) = 0.$$

(3) For $\vec{x} \in W_k$,

$$\lim_{t \to 0} v(t, \vec{x}) = v_0(\vec{x}).$$

In the (tilted) half-flat case, we take $v_0(\vec{x}) = \prod_a e^{-\theta x_a} \mathbf{1}_{x_a \geq 0}$.

It is widely accepted in the physics literature that, if $Z(t, x)$ is a solution of the SHE, then $v(t; \vec{x}) = \mathbb{E}[Z(t, x_1) \cdots Z(t, x_k)]$ is a solution of the delta Bose gas. This fact is proved in [29], where it is also shown that there is at most one solution. Therefore, our formulas below for the solution of the delta Bose gas are indeed identifying the $\mathbb{E}[Z(t, x_1) \cdots Z(t, x_k)]$. In any case, in the last result of this section (Proposition 5.3) we will state a formula both for the delta Bose gas and for the moments of the solution of the SHE, with a proof for the second part which is independent of this correspondence.

Given $\alpha \in \mathbb{R}^k$, we will write $\vec{\alpha} + (\mathrm{i}\mathbb{R})^k = (\alpha_1 + \mathrm{i}\mathbb{R}) \times \cdots \times (\alpha_k + \mathrm{i}\mathbb{R})$.

PROPOSITION 5.1. *The delta Bose gas with tilted half-flat initial condition given by* $v_0(\vec{x}) = \prod_a e^{-\theta x_a} \mathbf{1}_{x_a \geq 0}$, $\theta \geq 0$, *is solved by*

$$v(t, \vec{x}) = \frac{1}{(2\pi \mathrm{i})^k} \int_{\vec{\alpha} + (\mathrm{i}\mathbb{R})^k} d\vec{z} \prod_{a<b} \left(\frac{z_a - z_b}{z_a - z_b - 1} \frac{z_a + z_b - 1}{z_a + z_b} \right) \times \prod_{a=1}^{k} \frac{1}{z_a} e^{(t/2) \sum_{a=1}^k (z_a - \theta)^2 + \sum_{a=1}^k (z_a - \theta) x_a}, \tag{5.3}$$

where $\alpha_1 > \alpha_2 + 1 > \cdots > \alpha_k + k - 1 > k - 1$ *and* $x_1 < \cdots < x_k$.

PROOF. We only verify that (3) is satisfied, the rest follows as in the case of δ_0 initial condition [6]. We need to show that

$$\lim_{t \to 0} v(t, \vec{x}) = \prod_{a-1}^{k} e^{-\theta x_a} \mathbf{1}_{x_a \geq 0}.$$

We will denote the integrand by $I_k(z_1, \ldots, z_k)$. Assume first that $x_1 < 0$. Thanks to the factor $e^{(1/2)t(z_1 - \alpha)^2}$ we may move the z_1 contour to $\alpha_1 + R + \mathrm{i}\mathbb{R}$, $R > 0$. Note that we do not cross any poles. Changing variables $z_1 \mapsto z_1 + R$ gives

$$v(t, \vec{x}) = \frac{1}{(2\pi \mathrm{i})^k} \int_{\vec{\alpha} + \mathrm{i}\mathbb{R}} d\vec{z} I_k(z_1 + R, z_2, \ldots, z_k).$$

Now me may compute the limit $t \to 0$, which removes the quadratic term in the exponential. The resulting integrand in $\lim_{t\to 0} v(t, \vec{x})$ contains a factor $\frac{1}{z_1+R} e^{x_1(z_1+R)}$, and since $x_1 < 0$, we may take $R \to \infty$ to deduce without difficulty that the integral vanishes in this case.

So we assume now that $x_1 \geq 0$ (and so $x_a \geq 0$ for all $a = 1, \ldots, k$). Our goal is to move the z_k contour to $-M + \mathrm{i}\mathbb{R}$ (with $M > \alpha_1$). We may do this thanks to the Gaussian factor as before. Observe that the poles for z_k on $\{-M \leq \Re(z_k) \leq \alpha_k\}$ are 0 and $-z_a$ for $a < k$. We begin with the second type of pole. We have, for $\ell < k$,

$$\begin{aligned}
&\operatorname*{Res}_{z_k=-z_\ell} I_k(z_1, \ldots, z_k) \\
&\quad = \int d\vec{z}\, I_{k-1}(z_1, \ldots, z_{k-1}) \frac{2z_\ell e^{-(z_\ell+\alpha)x_k+(1/2)t(z_k-\alpha)^2}}{2z_\ell - 1} \frac{1}{-z_\ell} \\
&\qquad \times \prod_{\substack{a=1 \\ a\neq\ell}}^{k-1} \frac{z_a + z_\ell}{z_a + z_\ell - 1} \frac{z_a - z_\ell - 1}{z_a - z_\ell} \\
&\quad = \int d\vec{z}\, I_{k-2}(z_1, \ldots, z_{\ell-1}, z_{\ell+1}, \ldots, z_{k-1}) \\
&\qquad \times \frac{-2e^{-z_\ell(x_k-x_\ell)-\alpha(x_\ell+x_k)+(1/2)t(z_\ell-\alpha)^2+(1/2)t(z_k-\alpha)^2}}{z_\ell(2z_\ell - 1)} \\
&\qquad \times \prod_{b=\ell+1}^{k-1} \frac{1 + z_\ell - z_b}{z_\ell - z_b - 1}.
\end{aligned}$$

Observe that, due to the cancellation leading to the second line, the z_ℓ integral has no poles on $\{\Re(z_\ell) > \alpha_\ell\}$. As before we may freely move the z_ℓ contour to $\alpha_\ell + R + \mathrm{i}\mathbb{R}$, $R > 0$. Changing variables $z_\ell \mapsto z_\ell + R$ and taking $t \to 0$ yields an integral over the original $z_1, \ldots, z_{k-1}$ contours and containing a factor $e^{-(z_\ell+R)(x_k-x_\ell)-\alpha(x_\ell+x_k)}$ and no quadratic term in the exponent. Since $x_k > x_\ell$, taking $R \to \infty$ shows that this term vanishes.

We still need to compute the pole at $z_k = 0$, but let us first observe that the z_k integral over the new contour $-M + \mathrm{i}\mathbb{R}$ also vanishes after taking the limit $t \to 0$. In fact, proceeding as above, now changing variables $z_k \to z_k - M$, the resulting k-fold integral equals

$$v(t, \vec{x}) = \frac{1}{(2\pi\,\mathrm{i})^k} \int_{\vec{\alpha}+\mathrm{i}\mathbb{R}} d\vec{z}\, I_k(z_1, z_2, \ldots, z_k - M).$$

In the limit $t \to 0$, the integrand contains a factor of the form $e^{x_k(z_k-M)}$, and since we are assuming $x_k > 0$ we may take $M \to \infty$ to deduce that the whole integral goes to 0.

So the only term left in the limit $t \to 0$ is the one corresponding to the pole at $z_k = 0$. We have

$$\operatorname*{Res}_{z_k=0} I_k(z_1, \dots, z_k)$$
$$= \int_{\alpha_j + \mathrm{i}\mathbb{R}} d\vec{z}\, I_{k-1}(z_1, \dots, z_{k-1}) e^{(1/2)t\alpha^2 - \alpha x_k} \prod_{a=1}^{k-1} \Big(\frac{z_a}{z_a - 1} \frac{z_a - 1}{z_a} \Big).$$

The last product is obviously 1, so we have proved that

$$\lim_{t\to 0} v(t, \vec{x}) = \lim_{t\to 0} \mathbf{1}_{x_k \geq 0} \int_{\alpha_j + \mathrm{i}\mathbb{R}} d\vec{z}\, I_{k-1}(z_1, \dots, z_{k-1}) e^{(1/2)t\alpha^2 - \alpha x_k}$$
$$= \mathbf{1}_{x_k \geq 0} e^{-\alpha x_k} \lim_{t\to 0} v\big(t, (x_1, \dots, x_{k-1})\big).$$

The result follows by induction. □

Observe that, as should be expected, multiplying (5.3) by θ^k and letting $\theta \to \infty$ yields (after shifting contours by θ and changing variables $z_a \mapsto z_a + \theta$) the solution of [6] for the delta Bose gas with *narrow wedge* initial condition [which corresponds to $Z(0, x) = \delta_0(x)$ at the level of the SHE], given by

$$v_0(t, \vec{x}) = \frac{1}{(2\pi\mathrm{i})^k} \int_{\vec{\alpha} + (\mathrm{i}\mathbb{R})^k} d\vec{z} \prod_{a<b} \frac{z_a - z_b}{z_a - z_b - 1} \prod_{a=1}^{k} e^{(t/2)\sum_{a=1}^k z_a^2 + \sum_{a=1}^k z_a x_a} \tag{5.4}$$

for $x_1 < \cdots < x_k$.

When $\theta = 0$, (5.3) gives the solution for the half-flat initial condition $Z(0, x) = \mathbf{1}_{x \geq 0}$, which can also be obtained by taking the weakly asymmetric limit of (1.7) (see the proof of Proposition 5.3 for a similar computation).

By linearity of (5.2), we have that, if $Z(0, y; t, x)$ is the solution to the SHE with initial data $Z(0, y; 0, x) = \delta_y(x)$, then $Z(t, x) = \int_{-\infty}^{\infty} dy Z(0, y, t, x) f(y)$ solves the SHE with initial condition $Z(0, x) = f(x)$, and hence

$$\mathbb{E}_f\big[Z(t, x_1) \cdots Z(t, x_k)\big] = \int_{\mathbb{R}^k} d\vec{y}\, \mathbb{E}\Big[\prod_a Z(0, y_a; t, x_a)\Big] \prod_a f(y_a)$$

(with the subscript in $\mathbb{E}_f$ denoting the initial condition for the SHE). Although we do not have a formula for the integrand on the right-hand side in general note that, by statistical time reversal invariance, we do have

$$\mathbb{E}\Big[\prod_a Z(0, y_a; t, x_a)\Big] = \mathbb{E}\Big[\prod_a Z(0, x_a; t, y_a)\Big].$$

Now if all the x_a's are the same, we can use the spatial statistical invariance and symmetry to see that $\mathbb{E}[\prod_a Z(0, x_a; t, y_a)] = \mathbb{E}[\prod_a Z(t, x - y_a)]$. Finally, changing variables and then restricting to the *Weyl chamber* $W_k = \{\vec{x} \in \mathbb{R}^k : x_1 < \cdots <$

$x_k\}$, we obtain

$$\mathbb{E}_f\big[Z(t,x)^k\big] = k! \int_{W_k} d\vec{y}\, \mathbb{E}\Big[\prod_a Z(t, y_k - x)\Big] \prod_a f(x - y_a).$$

Specializing to the tilted half-flat initial condition given by $f = f_\theta$ with $f_\theta(x) = e^{-\theta x}\mathbf{1}_{x\geq 0}$, and in view of the relation between the delta Bose gas and the moments of the SHE discussed above, this suggests an alternative route for obtaining a formula for $v(t; x, \dots, x)$ in this case, namely

$$v(t; x, \dots, x) = k! \int_{W_k} d\vec{y}\, v_0(t; \vec{y}) \prod_a e^{-\theta(x-y_a)} \mathbf{1}_{y_a \leq x} \tag{5.5}$$

with v_0 as in (5.4). Although this identity can be justified directly from the linearity of the delta Bose gas itself, it is not at all clear at a first look that this alternative computation would lead to the same formula as the one in Proposition 5.1.

To see directly why the above formula holds, we start by using the explicit formula for $v_0(t; \vec{y})$ and computing the y_a integrals over W_k, which yield

$$\frac{k!}{(2\pi \mathrm{i})^k} \int_{\vec{\alpha}+(\mathrm{i}\mathbb{R})^k} d\vec{z} \prod_{a<b} \frac{z_a - z_b}{z_a - z_b - 1} \prod_{a=1}^k \frac{1}{z_1 + \cdots + z_a} e^{(t/2)\sum_{a=1}^k z_a^2 + \sum_{a=1}^k z_a x}.$$

Now deform the z_a contours one by one so that they all coincide with the leftmost one. The answer is obtained by an argument analogous to the proof of Proposition 3.3, and is given by

$$\sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \dots}} \frac{k!}{m_1! m_2! \cdots} \frac{1}{(2\pi \mathrm{i})^{\ell(\lambda)}} \int_{(\alpha+\mathrm{i}\mathbb{R})^{\ell(\lambda)}} d\vec{w} \det\Big[\frac{1}{w_a + \lambda_a - w_b}\Big]_{\lambda_1, \lambda_b = 1}^{\ell(\lambda)}$$
$$\times H(w_1, w_1 + 1, \dots, w_1 + \lambda_1 - 1, \dots, w_{\ell(\lambda)}, w_{\ell(\lambda)+1}, \dots, w_{\ell(\lambda)+\lambda_{\ell(\lambda)}-1})$$

with

$$H(z_1, \dots, z_\ell) = \prod_{a=1}^\ell e^{(t/2)z_a^2 + z_a x} \sum_{\sigma \in S_\ell} \prod_a \frac{1}{z_{\sigma(1)} + \cdots + z_{\sigma(a)}} \prod_{a>b} \frac{z_{\sigma(a)} - z_{\sigma(b)} - 1}{z_{\sigma(a)} - z_{\sigma(b)}}.$$

Using the same procedure as in (3.9) to get rid of the multinomial coefficient $\frac{k!}{m_1! m_2! \cdots}$, the above turns into

$$\sum_{\ell=0}^k \frac{1}{\ell!} \sum_{\substack{m_1, \dots, m_\ell \geq 1 \\ m_1 + \cdots + m_\ell = k}} \frac{1}{(2\pi \mathrm{i})^\ell} \int_{(\alpha+\mathrm{i}\mathbb{R})^\ell} d\vec{w} \det\Big[\frac{1}{w_a + m_a - w_b}\Big]_{\lambda_a, \lambda_b = 1}^{\ell}$$
$$\times \prod_{a=1}^\ell e^{(t/2)z_a^2 + z_a x}$$
$$\times H(w_1, w_1 + 1, \dots, w_1 + m_1 - 1, \dots, w_\ell, w_{\ell+1}, \dots, w_\ell + m_\ell - 1).$$

In order to compute the sum over the symmetric group appearing in the definition of H, we will appeal to the following summation formula, which was used in [24].

LEMMA 5.2. *For $q_1, \dots, q_N, \kappa \in \mathbb{C}$,*

$$\sum_{\sigma \in S_N} \mu_{\vec{q}}(\sigma) \prod_{a<b} \frac{q_{\sigma(a)} - q_{\sigma(b)} - i\kappa}{q_{\sigma(a)} - q_{\sigma(b)}} = \prod_{a<b} \frac{q_a + q_b + i\kappa}{q_a + q_b},$$

where

$$\mu_{\vec{q}}(\sigma) := q_{\sigma(1)}^{-1}(q_{\sigma(1)} + q_{\sigma(2)})^{-1} \cdots (q_{\sigma(1)} + \cdots + q_{\sigma(N)})^{-1} \prod_a q_a.$$

This identity was discovered and checked for small values of N on Mathematica by Le Doussal and Calabrese. The formula can, in fact, be derived as a suitable limit of an analogous symmetrization identity proved in [22] in the context of ASEP with flat initial condition (see Lemma 2 in that paper).

Using the lemma, we obtain

$$H(z_1, \dots, z_\ell) = \prod_{a=1}^{\ell} e^{(t/2)z_a^2 + z_a x} \prod_{a<b} \frac{z_a + z_b - 1}{z_a + z_b}.$$

Replacing this formula above and doing some algebra leads directly to (5.9) below, which as we will see next is another way of writing the solution given in Proposition 5.1 when all the x_a's are the same, thus proving (5.5).

In what follows, we will turn our formula for the tilted half-flat delta Bose gas (with all x_a's the same) into one in which all the integration contours coincide. As we will see, this alternative version of our half-flat formula is essentially equivalent to the formulas given in [23, 24] [see (5.9) and the discussion that follows it]. We will argue afterwards (see Proposition 5.3), based on the convergence of ASEP to KPZ, that this formula does indeed give the half-flat SHE moments.

The first step is to deform the z_a contours in (5.3) one by one so that they all coincide with $\alpha_k + i\mathbb{R}$. The arguments are similar to the ones we used for ASEP in Section 3, so we only sketch them. We proceed similarly to the proof of Proposition 3.3, now accounting for poles of the form $z_a = z_b + 1$ for $a > b$ and computing the corresponding residues. Doing this in the case that all x_a's are equal, using the symmetrization identity

$$\sum_{\sigma \in S_k} \prod_{a>b} \frac{z_{\sigma(a)} - z_{\sigma(b)} - 1}{z_{\sigma(a)} - z_{\sigma(b)}} = k!,$$

which plays the role of (3.6) (and follows from suitably rescaling it[5]), and rewriting the sum over partitions as in (3.9) yields the following formula for the moments of the delta Bose gas with initial condition $v(0; x, \dots, x) = e^{-\theta x}\mathbf{1}_{x \geq 0}$ (here, and

[5]It also corresponds to a certain degeneration of the special case of the Hall–Littlewood polynomial normalization given in Section III.1 of [27].

below, x is repeated k times in the argument of v):

$$v(t;x,\dots,x) = k!\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{\substack{n_1,\dots,n_\ell,\\ n_1+\cdots+n_\ell=k}}\frac{1}{(2\pi\mathrm{i})^\ell}\int_{(\alpha+\mathrm{i}\mathbb{R})^\ell} d\vec{w}\,\det\Bigl[\frac{1}{w_a+n_a-w_b}\Bigr]_{a,b=1}^{\ell}$$
$$\times \bar{H}(w_1,\dots,w_1+n_1-1,\dots,w_\ell,\dots,w_\ell+n_\ell-1)$$

with

$$\bar{H}(z_1,\dots,z_m) = \prod_{a<b}\frac{z_a+z_b-1}{z_a+z_b}\prod_{a=1}^{m}\frac{1}{z_a}e^{(t/2)(z_a-\theta)^2+x(z_a-\theta)},$$

where $\alpha>0$. Rewriting the result as in the proof of Theorem 1.3 yields (after some simplification)

$$\begin{aligned} &v(t;x,\dots,x)\\ &\quad= 2^k k!\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{\substack{n_1,\dots,n_\ell,\\ n_1+\cdots+n_\ell=k}}\frac{1}{(2\pi\mathrm{i})^\ell}\int_{(\alpha+\mathrm{i}\mathbb{R})^\ell} d\vec{w}\,\det\Bigl[\frac{1}{w_a+n_a-w_b}\Bigr]_{a,b=1}^{\ell}\\ &\qquad\times\prod_a\frac{\Gamma(2w_a+n_a-1)}{\Gamma(2w_a+2n_a-1)}\\ &\qquad\times\exp\Bigl\{\frac{1}{2}t\Bigl[\frac{1}{3}n_a^3-\frac{1}{2}n_a^2+\frac{1}{6}n_a+n_a(w_a-\theta)^2+n_a(n_a-1)w_a\Bigr]\\ &\qquad\qquad+x\Bigl[\frac{1}{2}n_a^2-\frac{1}{2}n_a+n_a(w_a-\theta)\Bigr]\Bigr\}\\ &\qquad\times\prod_{a<b}\frac{\Gamma(w_a+w_b+n_a-1)\Gamma(w_a+w_b+n_b-1)}{\Gamma(w_a+w_b-1)\Gamma(w_a+w_b+n_a+n_b-1)}. \end{aligned}\tag{5.6}$$

Now we change variables $w_a\mapsto w_a-\frac{1}{2}(n_a-1)$ to obtain

$$\begin{aligned} &v(t;x,\dots,x)\\ &\quad= 2^k k!\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{\substack{n_1,\dots,n_\ell\ge 1\\ n_1+\cdots+n_\ell=k}}\frac{1}{(2\pi\mathrm{i})^\ell}\int_{\alpha+(1/2)(n_1-1)+\mathrm{i}\mathbb{R}} dw_1\cdots\\ &\qquad\qquad\times\int_{\alpha+(1/2)(n_\ell-1)+\mathrm{i}\mathbb{R}} dw_\ell\, I_\theta(\vec{w},\vec{n}) \end{aligned}\tag{5.7}$$

with

$$\begin{aligned} &I_\theta(\vec{w},\vec{n})\\ &\quad=\det\Bigl[\frac{1}{w_a-w_b+(1/2)n_a+(1/2)n_b}\Bigr]_{a,b=1}^{\ell} \end{aligned}\tag{5.8}$$

$$\times \prod_a \exp\Big\{ t\Big[\frac{1}{24} n_a^3 - \frac{1}{24} n_a + \frac{1}{2} n_a (w_a - \theta)^2 \Big] + x n_a (w_a - \theta) \Big\}$$
$$\times \prod_a \frac{\Gamma(2w_a)}{\Gamma(2w_a + n_a)}$$
$$\times \prod_{a<b} \frac{\Gamma(w_a + w_b + (1/2)(n_a - n_b))\Gamma(w_a + w_b - (1/2)(n_a - n_b))}{\Gamma(w_a + w_b - (1/2)(n_a + n_b))\Gamma(w_a + w_b + (1/2)(n_a + n_b))}.$$

The last step is to shift back the w_a contours from $\alpha + \frac{1}{2}(n_a - 1) + \mathrm{i}\mathbb{R}$ to $\alpha + \mathrm{i}\mathbb{R}$. As we will see, we will not cross any poles as we do this. To be more precise, we begin by moving the w_1 contour from $\alpha + \frac{n_1 - 1}{2} + \mathrm{i}\mathbb{R}$ to $\alpha + \mathrm{i}\mathbb{R}$. There are three types of possible singularities, the first from the Cauchy determinant and the other two from the Gamma functions:

(1) $w_1 = w_b - \frac{1}{2}(n_1 + n_b)$ for $b > 1$.
(2) $w_1 = -\ell$ for $\ell \in \mathbb{Z}_{\geq 0}$.
(3) $w_1 = -w_b \pm \frac{1}{2}(n_1 - n_b) - \ell$ for $\ell \in \mathbb{Z}_{\geq 0}$ and $b > 1$.

The first two types of singularity lie to the left of the origin, whereas our deformation region lies entirely to the right of the origin. Turning to (3), both singularities may or may not lie inside the deformation region, but in any case the singularity is removable: the simple pole coming from the numerator cancels with the zero of the denominator since $w_1 = -w_b \pm \frac{1}{2}(n_1 - n_b) - \ell$ implies $w_1 + w_b \mp \frac{1}{2}(n_1 - n_b) = -\ell \in \mathbb{Z}_{<0}$, which is a zero of $\frac{1}{\Gamma(\cdot)}$.

It remains to show that, having moved $w_1, \ldots, w_{j-1}$ from their respective starting points to $\alpha + \mathrm{i}\mathbb{R}$, we do not incur any residues when moving w_j from $\alpha + \frac{n_j - 1}{2} + \mathrm{i}\mathbb{R}$ to $\alpha + \mathrm{i}\mathbb{R}$. The argument is analogous to the case w_1 and is left to the reader.

This leads to the following.

PROPOSITION 5.3. *For the delta Bose gas with tilted half-flat initial condition* $v_0(t; \vec{x}) = \prod_a e^{-\theta x_a} \mathbf{1}_{x_a \geq 0}$ *we have*

$$v(t; x, \ldots, x) = 2^k k! \sum_{\ell=0}^{k} \frac{1}{\ell!} \sum_{\substack{n_1, \ldots, n_\ell \geq 1 \\ n_1 + \cdots + n_\ell = k}} \frac{1}{(2\pi \mathrm{i})^\ell} \int_{(\alpha + \mathrm{i}\mathbb{R})^k} d\vec{w}\, I_\theta(\vec{w}; \vec{n}) \tag{5.9}$$

with I_θ *given by* (5.8). *Moreover, in the pure half-flat initial condition corresponding to* $\theta = 0$, *the same identity holds for the moments of the SHE, that is,*

$$\mathbb{E}^{\text{h-fl}}\big[Z(t, x)^k\big] = 2^k k! \sum_{\ell=0}^{k} \frac{1}{\ell!} \sum_{\substack{n_1, \ldots, n_\ell \geq 1 \\ n_1 + \cdots + n_\ell = k}} \frac{1}{(2\pi \mathrm{i})^\ell} \int_{(\alpha + \mathrm{i}\mathbb{R})^k} d\vec{w}\, I_0(\vec{w}; \vec{n}).$$

In [24], the authors compute a formal series[6] for the generating function of $Z(t,x)$ using the explicit basis of eigenfunctions of the delta Bose gas [25, 28]. The generating function is expanded in the "number of strings", which essentially corresponds to the parameter ℓ in (5.9) (the "strings" essentially correspond to $n_1,\dots,n_\ell$, and index the eigenfunctions). The coefficients in this expansion are given in their formula (88), and one can check that, as expected, that formula coincides essentially with (5.9). By this we mean that, for fixed $n_1,\dots,n_\ell$, the summand in (5.9) coincides[7] with the summand on the right-hand side of (88) in [24] with $n_s=\ell$ and $m_a=n_a$ for $a=1,\dots,n_s$. This correspondence is consistent with (39) in their paper. See also [23].

PROOF OF PROPOSITION 5.3. The delta Bose gas case follows directly from the above discussion. The formula in the case of the moments of the half-flat SHE can be recovered directly as a weakly asymmetric limit of the half-flat ASEP moment formula given in Theorem 1.3. Let us briefly sketch how this is done.

Recall from (1.6) that (for the half-flat case) $h(t,x)=2N_x(t)-x$. According to the WASEP scaling theory (see [5]), if $\gamma=q-p=\varepsilon^{1/2}$ and we let

$$\begin{aligned}\nu_\varepsilon &= 1-2\sqrt{pq}=\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2+\mathcal{O}\bigl(\varepsilon^3\bigr),\\ \lambda_\varepsilon &= \frac{1}{2}\log\biggl(\frac{q}{p}\biggr)=\varepsilon^{1/2}+\frac{1}{3}\varepsilon^{3/2}+\mathcal{O}\bigl(\varepsilon^{5/2}\bigr),\end{aligned}\tag{5.10}$$

then

$$h_\varepsilon(t,x):=\lambda_\varepsilon h\bigl(\varepsilon^{-3/2}t/\gamma,\varepsilon^{-1}x\bigr)-\nu_\varepsilon\varepsilon^{-2}t$$

converges to the Cole–Hopf solution $h(t,x)$ of KPZ starting with $h(0,x)=0$ for $x>0$ and $h(0,x)=\infty$ for $x<0$, which in view of (5.1) corresponds to $Z(0,x)=\mathbf{1}_{x\geq 0}$. Translating back to $N_x(t)$, and in view of (5.10), we have

$$\begin{aligned}N_{\varepsilon^{-1}x}\bigl(\varepsilon^{-2}t\bigr)&=\frac{1}{2\lambda_\varepsilon}\bigl[h_\varepsilon(t,x)+\nu_\varepsilon\varepsilon^{-2}t\bigr]+\frac{1}{2}\varepsilon^{-1}x\\ &\approx\frac{1}{2}\varepsilon^{-1/2}h(t,x)-\frac{1}{48}\varepsilon^{-1/2}t+\frac{1}{4}\varepsilon^{-3/2}t+\frac{1}{2}\varepsilon^{-1}x.\end{aligned}$$

[6] As written, the computation in [24] is only formal, since in view of (88) in their paper the series given in their formula (40) is clearly divergent. Nevertheless, their computation implicitly leads to a formula like (5.9) in view of their formula (39).

[7] A diligent reader will notice two minor differences between (5.9) and the formula in [24]. First, the formulas differ by a factor of $\prod_a(-1)^{n_a}$ in the integrand, reflecting the fact that their generating function computation is implicitly calculating $\mathbb{E}[(-Z)^k]$, as opposed to $\mathbb{E}[Z^k]$ [see (40) in their paper]. Second, one needs to replace t by $2t$ in our formula to recover theirs. This is because in their definition of the SHE the Laplacian term lacks the prefactor $\frac{1}{2}$ [see (8) in their paper and compare with (5.2)].

Therefore, since $\log(\tau) \approx -2\varepsilon^{1/2}$, we deduce that

$$\tau^{N_{\varepsilon^{-1}x}(\varepsilon^{-2}t)-(1/4)\varepsilon^{-3/2}t-(1/2)\varepsilon^{-1}x} \approx e^{-h(t,x)-(1/24)t} = e^{-(1/24)t}Z(t,x).$$

This, together with tightness of the moments (which, e.g., can be obtained by adapting the arguments in Section 2.15 of [31]), gives

$$\mathbb{E}^{\text{h-fl}}\big[\tau^{k(N_{\varepsilon^{-1}x}(\varepsilon^{-2}t)-(1/4)\varepsilon^{-3/2}t-(1/2)\varepsilon^{-1}x+(1/24)t)}\big] \underset{\varepsilon\to 0}{\to} \mathbb{E}^{\text{h-fl}}\big[Z(t,x)^k\big]. \tag{5.11}$$

Now in view of (1.12), the left-hand side of (5.11) is given by

$$\begin{aligned} &k_\tau! \sum_{\ell=0}^{k} \frac{1}{\ell!} \sum_{\substack{n_1,\dots,n_\ell\ge 1\\ n_1+\cdots+n_\ell=k}} \frac{1}{(2\pi \mathrm{i})^\ell} \int_{\gamma_{-1,0}^\ell} d\vec{w} \det\Big[\frac{-1}{w_a\tau^{n_a}-w_b}\Big]_{a,b=1}^{\ell} \\ &\quad\times \prod_a e^{(1/24)n_a t}\tau^{-((1/4)\varepsilon^{-3/2}t-(1/2)\varepsilon^{-1}x)n_a}\mathfrak{f}(w_a;n_a)\mathfrak{g}(w_a;n_a) \\ &\quad\times \prod_{a<b}\mathfrak{h}(w_a,w_b;n_a,n_b). \end{aligned} \tag{5.12}$$

Observe that the two sums are finite, so in order to obtain (5.9) it is enough to show that the multiple integral converges to $I_0(\vec{w};\vec{n})$. This results in a relatively simple problem in asymptotic analysis. The starting point for the critical point analysis is to consider the product $\tau^{-((1/4)\varepsilon^{-3/2}t-(1/2)\varepsilon^{-1}x)n_a}\mathfrak{f}(w_a;n_a)$, which is given by

$$\begin{aligned} (1-\tau)^{n_a}\exp\Big\{&\Big(\frac{1}{1+w_a}-\frac{1}{1+\tau^{n_a}w_a}-\frac{1}{4}\log(\tau)\Big)\varepsilon^{-3/2}t \\ &-\frac{1}{2}\varepsilon^{-1}x\log(\tau)n_a+\log\Big(\frac{1+\tau^{n_a}w_a}{1+w_a}\Big)(\varepsilon^{-1}x-1)\Big\}. \end{aligned}$$

Scaling w_a near 1 through the change of variables $w_a \mapsto 1-(1-\tau)\tilde{w}_a$, the exponent above can be written as $(\frac{1}{6}n_a^3+\frac{1}{2}n_a^2\tilde{w}_a+\frac{1}{2}n_a\tilde{w}_a^2)t+(\frac{1}{2}n_a^2+n_a\tilde{w}_a)x+\mathcal{O}(\varepsilon^{1/2})$. The change of variables (for all the ℓ variables) gives a prefactor of $(-1)^\ell(1-\tau)^\ell$, while the factor $\prod_a(1-\tau)^{n_a}$ coming from the above product turns into $(1-\tau)^k$. We leave it to the reader to verify that, with this scaling, $\det[\frac{-1}{w_a\tau^{n_a}-w_b}]_{a,b=1}^{\ell} \approx (1-\tau)^{-\ell}\det[\frac{1}{\tilde{w}_a+n_a-\tilde{w}_b}]_{a,b=1}^{\ell}$, $\prod_a \mathfrak{g}(w_a;n_a) \approx 2^k(1-\tau)^{-k}\prod_a \frac{\Gamma(2\tilde{w}_a+n_a)}{\Gamma(2\tilde{w}_a+2n_a)}$ and

$$\prod_{a<b}\mathfrak{h}(w_a,w_b;n_a,n_b) \approx \prod_{a<b}\frac{\Gamma(\tilde{w}_a+\tilde{w}_b+n_a)\Gamma(\tilde{w}_a+\tilde{w}_b+n_b)}{\Gamma(\tilde{w}_a+\tilde{w}_b+n_a+n_b)\Gamma(\tilde{w}_a+\tilde{w}_b)}.$$

Note that near the critical point $w_a=1$ the contour $\gamma_{-1,0}$ turns into $\mathrm{i}\mathbb{R}$, negatively oriented. Introducing an additional factor $(-1)^\ell$ to flip the orientation of the re-

sulting contour, we deduce from the above estimates that (5.12) is approximately

$$
\begin{aligned}
2^k k! \sum_{\ell=0}^{k} \frac{1}{\ell!} \sum_{\substack{n_1,\dots,n_\ell,\\ n_1+\cdots+n_\ell=k}} \frac{1}{(2\pi \mathrm{i})^\ell} \int_{(\mathrm{i}\mathbb{R})^\ell} d\vec{\tilde{w}} \det\Big[\frac{1}{\tilde{w}_a + n_a - \tilde{w}_b}\Big]_{a,b=1}^{\ell} \\
\times \prod_a \frac{\Gamma(2\tilde{w}_a + n_a)}{\Gamma(2\tilde{w}_a + 2n_a)} \\
\times \prod_a \exp\Big\{ t\Big[\frac{1}{6}n_a^3 + \frac{1}{2}n_a^2\tilde{w}_a + \frac{1}{2}n_a\tilde{w}_a^2 + \frac{1}{24}n_a\Big] + x\Big[\frac{1}{2}n_a^2 + n_a\tilde{w}_a\Big]\Big\} \\
\times \prod_{a<b} \frac{\Gamma(\tilde{w}_a + \tilde{w}_b + n_a)\Gamma(\tilde{w}_a + \tilde{w}_b + n_b)}{\Gamma(\tilde{w}_a + \tilde{w}_b)\Gamma(\tilde{w}_a + \tilde{w}_b + n_a + n_b)}.
\end{aligned}
$$

Turning this into a rigorous proof involves estimating the integrand away from the critical point in order to show that the only contribution from the integral that survives in the limit is that near $w_a = 1$. This is not hard to do in this case because we do not need an estimate which is uniform in ℓ (which is the basic source of difficulty in turning the calculations of the Appendix into a rigorous proof), so we will leave the details to the reader. Now changing variables $\tilde{w}_a \mapsto \bar{w}_a - \frac{1}{2}$ turns the above formula into (5.6) (with $\alpha = \frac{1}{2}$), which by (5.7) gives the desired result. □

APPENDIX: ASYMPTOTICS FOR HALF-FLAT ASEP AND THE AIRY$_{2\to1}$ MARGINALS

In this section, we provide a formal critical point analysis of the long-time asymptotics of the τ-Laplace transform of $\tau^{N_x(t)}$ in the half-flat case which, in view of (1.15), gives the asymptotic distribution of the fluctuations of the height function $h(t,x)$.

More precisely, our derivation will provide a nonrigorous confirmation of the conjectured asymptotics

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}^{\text{h-fl}}\Big(\frac{h(t/(q-p), t^{2/3}x) - (1/2)t - t^{1/3}x^2\mathbf{1}_{x\le 0}}{t^{1/3}} \ge -r\Big) \\
&\qquad = \mathbb{P}\big(\mathcal{A}_{2\to1}(2^{-1/3}x) \le 2^{1/3}r\big),
\end{aligned} \tag{A.1}
$$

where $\mathcal{A}_{2\to1}$ is the Airy$_{2\to1}$ process. For background on this process and more details about this conjecture, see [32].

Our starting point is the formula for the e_τ-Laplace transform of $\tau^{N_x(t)}$ given in Theorem 1.4, where we take $\zeta = -\tau^{-(1/4)t - (1/2)t^{2/3}x + (1/2)t^{1/3}r - (1/4)t^{1/3}x^2\mathbf{1}_{x\le0}}$ and

let $\tilde{r} = r - \frac{1}{2}x^2\mathbf{1}_{x\le 0}$:

$$\begin{aligned}
&\mathbb{E}^{\text{h-fl}}\big[e_\tau\big(-\tau^{N_{t^{2/3}x}(t/\gamma)-(1/4)t-(1/2)t^{2/3}x+(1/2)t^{1/3}\tilde{r}}\big)\big] \\
&\quad= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(2\pi\mathrm{i})^{2k}}\int_{(\delta+\mathrm{i}\mathbb{R})^k} d\vec{s}\int_{\gamma^k_{-1,0}} d\vec{w}\,\det\Big[\frac{-1}{w_a\tau^{s_a}-w_b}\Big]_{a,b=1}^{k} \\
&\qquad\times\prod_a \tau^{-[(1/4)t+(1/2)t^{2/3}x-(1/2)t^{1/3}r]s_a}\tilde{\mathfrak{f}}(w_a;s_a)\mathfrak{g}(w_a;s_a) \\
&\qquad\times\prod_{a<b}\mathfrak{h}(w_a,w_b;s_a,s_b)
\end{aligned} \tag{A.2}$$

for $\delta\in(0,1)$, $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ as in (1.9), and with $\tilde{\mathfrak{f}}$ defined as $\mathfrak{f}$ with t replaced by t/γ (recall that $\gamma = q-p$).

We will perform a formal critical point analysis on the right-hand side. The reason the limit is not rigorous is that so far we have not been able to control the double product $\prod_{a<b}\mathfrak{h}(w_a,w_b;s_a,s_b)$ on the part of the contour away from the critical point, nor find an alternative contour where this can be done.[8] The derivation here is done to clarify the algebraic structure of the expansion around the critical point where one sees the Airy crossover distributions.

The leading order (in t) factor in the integrand comes $\tilde{\mathfrak{f}}(w_a,s_a)$ and the factor $\tau^{-(1/4)t}$, and can be written as $\prod_a \exp[t(\frac{1}{1+w_a}-\frac{1}{1+\tau^{s_a}w_a}-\frac{1}{4}s_a\log(\tau))]$. One can verify that the only critical point of $\frac{1}{1+w}-\frac{\tau}{1+\tau^s w}-\frac{1}{4}s\log(\tau)$ occurs at $(w,s)=(1,0)$. Moreover, the Hessian of this function vanishes at this point, while the third-order partial derivatives are not all 0, which suggests a $t^{1/3}$ scaling. On the other hand, this suggests that the w_a contour should be chosen to cross the line $\mathbb{R}_{\ge 0}$ at $w_a=1$. In view of this we change variables as follows:

$$w_a = 1+t^{-1/3}\widetilde{w}_a, \qquad s_a = -\frac{1}{\log(\tau)}t^{-1/3}\widetilde{s}_a. \tag{A.3}$$

We will need the following lemma.

[8] More precisely, $|\mathfrak{h}(w_a,w_b;s_a,s_b)|$ can be bounded uniformly by some constant C, but this constant is necessarily larger than one. This yields an estimate of the form C^{k^2} for some $C>1$, which is too big (note that $\sum_{k\ge 0}\frac{1}{k!}C^{k^2}$ is divergent). Therefore the rigorous asymptotics remains an interesting open problem. Observe that if the double product could be turned into a determinant [as happens for the first double product in (3.2), which turns into the determinant in (A.2)], then this problem would disappear, because by Hadamard's bound our estimate on $|\mathfrak{h}(w_a,w_b;s_a,s_b)|$ would essentially yield a factor $C^k k^{k/2}$, which is small enough for our purposes.

LEMMA A.1. *Let $a \in \mathbb{R}$, $\ell \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$. If $-\ell = kj$ for some $j \in \mathbb{Z}_{\geq 0}$ (i.e., $\ell = 0$ or k is a factor of $-\ell$) then, as $\varepsilon \to 0$,*

$$(\tau^{\ell}(1+\varepsilon a);\tau^{k})_{\infty} = -\varepsilon a \prod_{\substack{n=0\\ n\neq j}}^{\infty}(1-\tau^{kn+\ell}) + \mathcal{O}(\varepsilon^{2}).$$

On the other hand, if $\ell \neq 0$ and k is not a factor of $-\ell$ then, as $\varepsilon \to 0$,

$$(\tau^{\ell}(1+\varepsilon a);\tau^{k})_{\infty} = (\tau^{\ell};\tau^{k})_{\infty}\left[1 - \varepsilon a \sum_{m=0}^{\infty}\frac{\tau^{km+\ell}}{1-\tau^{km+\ell}}\right] + \mathcal{O}(\varepsilon^{2}).$$

PROOF. In the first case, we have $\tau^{kj+\ell} = 1$ so

$$\begin{aligned}(\tau^{\ell}(1+\varepsilon a);\tau^{k})_{\infty} &= \prod_{n=0}^{\infty}\left[1-(1+\varepsilon a)\tau^{kn+\ell}\right]\\ &= \left[1-(1+\varepsilon a)\right]\prod_{\substack{n=0\\ n\neq j}}^{\infty}\left[1-(1+\varepsilon a)\tau^{kn+\ell}\right]\\ &= -\varepsilon a \prod_{\substack{n=0\\ n\neq j}}^{\infty}(1-\tau^{kn+\ell}) + \mathcal{O}(\varepsilon^{2}).\end{aligned}$$

In the second case, we have

$$\begin{aligned}(\tau^{\ell}(1+\varepsilon a);\tau^{k})_{\infty} &= \prod_{n=0}^{\infty}\left[(1-\tau^{kn+\ell}) - \varepsilon a\tau^{kn+\ell}\right]\\ &= \prod_{n=0}^{\infty}\left[1-\tau^{kn+\ell}\right] - \varepsilon a\sum_{m=0}^{\infty}\tau^{km+\ell}\prod_{\substack{n=0\\ n\neq m}}^{\infty}\left[1-\tau^{kn+\ell}\right] + \mathcal{O}(\varepsilon^{2})\\ &\approx (\tau^{\ell};\tau^{k})_{\infty} - \varepsilon a(\tau^{\ell};\tau^{k})_{\infty}\sum_{m=0}^{\infty}\frac{\tau^{km+\ell}}{1-\tau^{km+\ell}}.\end{aligned}$$

□

The scaling (A.3) leads to the following asymptotics:

$$\begin{aligned}&\frac{\pi}{\sin(-\pi s_a)}\frac{1+w_a}{1+\tau^{s_a}w_a}(1-\tau)^{s_a} \approx \frac{\log(\tau)t^{1/3}}{\widetilde{s}_a},\\ &\frac{1}{w_a\tau^{s_a}-w_b} \approx \frac{\tau^{-1}t^{1/3}}{\widetilde{w}_a-\widetilde{w}_b-\widetilde{s}_a},\\ &t\left[\frac{1}{1+w_a}-\frac{1}{1+\tau^{s_a}w_a}-\frac{1}{4}s_a\log(\tau)+\frac{1}{2}t^{1/3}s_a r\log(\tau)\right]\end{aligned}$$

$$\approx \frac{1}{48}(\widetilde{s}_a^3 - 3\widetilde{s}_a^2\widetilde{w}_a + 3\widetilde{s}_a\widetilde{w}_a^2) - \tilde{r}\widetilde{s}_a,$$

$$\left(\frac{1+\tau^{s_a}w_a}{1+w_a}\tau^{-(1/2)s_a}\right)^{t^{2/3}x} \approx \left(1 - \frac{\widetilde{s}_a^2 - 2\widetilde{s}_a\widetilde{w}_a}{8t^{2/3}}\right)^{t^{2/3}x} \approx e^{-(1/8)(\widetilde{s}_a^2 - 2\widetilde{s}_a\widetilde{w}_a)x},$$

while, using Lemma A.1,

$$\frac{(-w_a;\tau)_\infty}{(-\tau^{s_a}w_a;\tau)_\infty}\frac{(\tau^{2s_a}w_a^2;\tau)_\infty}{(\tau^{s_a}w_a^2;\tau)_\infty} \approx \frac{(1+2(\widetilde{w}_a - \widetilde{s}_a)t^{-1/3};\tau)_\infty}{(1+(2\widetilde{w}_a - \widetilde{s}_a)t^{-1/3};\tau)_\infty} \approx \frac{2(\widetilde{s}_a - \widetilde{w}_a)}{\widetilde{s}_a - 2\widetilde{w}_a},$$

and similarly

$$\frac{(w_aw_b;\tau)_\infty(\tau^{s_a+s_b}w_aw_b;\tau)_\infty}{(\tau^{s_a}w_aw_b;\tau)_\infty(\tau^{s_b}w_aw_b;\tau)_\infty} \approx \frac{(\widetilde{w}_a+\widetilde{w}_b)(\widetilde{w}_a+\widetilde{w}_b-\widetilde{s}_a-\widetilde{s}_b)}{(\widetilde{w}_a+\widetilde{w}_b-\widetilde{s}_a)(\widetilde{w}_a+\widetilde{w}_b-\widetilde{s}_b)}.$$

Additionally, there is a factor of $(-1)^k t^{-2k/3}(\tau/\log(\tau))^k$ coming from the change of variables which, except for the $(-1)^k$, cancels exactly with factors coming out from the first line of the above list of asymptotics. To write the limit choose first $\delta = -t^{-1/3}/(2\log(\tau))$ in (A.2) and deform the s_a contour so that it departs the real axis at angles $\pm\pi/3$, and likewise deform the w_a contours so that they go through 1 and depart from that point at angles $\pm\pi/3$. The limiting contours then become $\frac{1}{2} + \langle$ for $\widetilde{s}_a$ and $\langle$ for $\widetilde{w}_a$, where $\langle$ consists on two infinite rays departing 0 at angles $\pm\pi/3$ (oriented with increasing imaginary part) and thus using the above asymptotics in (A.2) we obtain that the formal limit as $t \to \infty$ of $\mathbb{E}[e_\tau(-\tau^{N_{t^{2/3}x}(t/(q-p))-(1/4)t-(1/2)t^{2/3}x+(1/2)t^{1/3}\tilde{r}})]$ is given by

$$\begin{aligned} F_x(\tilde{r}) = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(2\pi\mathrm{i})^k}\int_{(1/2+\langle)^k} d\vec{\widetilde{s}}\frac{1}{(2\pi\mathrm{i})^k}\int_{(\langle)^k} d\vec{\widetilde{w}}\det\left[\frac{1}{\widetilde{w}_b - \widetilde{w}_a + \widetilde{s}_a}\right]_{a,b=1}^k \\ \times \prod_a \exp\left\{\frac{1}{48}(\widetilde{s}_a^3 - 3\widetilde{s}_a^2\widetilde{w}_a + 3\widetilde{s}_a\widetilde{w}_a^2) - \frac{1}{2}\tilde{r}\widetilde{s}_a - \frac{1}{8}(\widetilde{s}_a^2 - 2\widetilde{s}_a\widetilde{w}_a)x\right\} \\ \times \frac{2(\widetilde{s}_a - \widetilde{w}_a)}{\widetilde{s}_a(\widetilde{s}_a - 2\widetilde{w}_a)} \\ \times \prod_{a<b}\frac{(\widetilde{w}_a+\widetilde{w}_b)(\widetilde{w}_a+\widetilde{w}_b-\widetilde{s}_a-\widetilde{s}_b)}{(\widetilde{w}_a+\widetilde{w}_b-\widetilde{s}_a)(\widetilde{w}_a+\widetilde{w}_b-\widetilde{s}_b)}. \end{aligned}$$

Now we introduce the change of variables $\widetilde{w}_a = u_a$ and $\widetilde{s}_a = u_a - v_a$. The u_a contour is $\langle$, but we may freely deform it (thanks to the cubic terms in the exponent) to $1 + \langle$. A priori v_a depends on u_a, but again one can check that it can be deformed to $\rangle$, which is defined in the same way as $\langle$ but departing the origin at angles $\pm 2\pi/3$.

We obtain

$$F_x(\tilde{r}) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(2\pi \mathrm{i})^k} \int_{(1+\langle)^k} d\vec{u} \frac{1}{(2\pi \mathrm{i})^k} \int_{(\langle)^k} d\vec{v} \det\left[\frac{1}{u_b - v_a}\right]_{a,b=1}^{k} \times \prod_a \exp\left\{\frac{1}{48}(u_a^3 - v_a^3) + \frac{1}{8}(u_a^2 - v_a^2)x - \frac{1}{2}(u_a - v_a)\tilde{r}\right\} \frac{2v_a}{u_a^2 - v_a^2} \times \prod_{a<b} \frac{(u_a + u_b)(v_a + v_b)}{(u_a + v_b)(v_a + u_b)}.$$

Now we note that the determinant and the cross-product above simplify into a single determinant: using the Cauchy determinant formula

$$\det\left[\frac{1}{x_a - y_b}\right]_{a,b=1}^{k} = \frac{\prod_{a<b}(x_a - x_b)(y_b - y_a)}{\prod_{a,b}(x_a - y_b)},$$

we have

$$\begin{aligned} &\det\left[\frac{1}{u_b - v_a}\right]_{a,b=1}^{k} \prod_{a<b} \frac{(u_a + u_b)(v_a + v_b)}{(u_a + v_b)(v_a + u_b)} \\ &\quad = \frac{1}{\prod_a (u_a - v_a)} \prod_{a<b} \frac{(u_b^2 - u_a^2)(v_a^2 - v_b^2)}{(u_a^2 - v_b^2)(u_b^2 - v_a^2)} \\ &\quad = \frac{\prod_a (v_a^2 - u_a^2)}{\prod_a (v_a - u_a)} \det\left[\frac{1}{u_a^2 - v_b^2}\right]_{a,b=1}^{k} = \prod_a (u_a + v_a) \det\left[\frac{1}{u_b^2 - v_a^2}\right]_{a,b=1}^{k}. \end{aligned}$$

Using this above, we get

$$\begin{aligned} F_x(\tilde{r}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(2\pi \mathrm{i})^k} \int_{(1+\langle)^k} d\vec{u} \frac{1}{(2\pi \mathrm{i})^k} \int_{(\langle)^k} d\vec{v} \det\left[\frac{1}{u_b^2 - v_a^2}\right]_{a,b=1}^{k} \\ &\quad \times \prod_a \frac{2v_a}{u_a - v_a} e^{(1/48)(u_a^3 - v_a^3) - (1/8)(u_a^2 - v_a^2)x - (1/2)(u_a - v_a)\tilde{r}} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(2\pi \mathrm{i})^k} \int_{(1+\langle)^k} d\vec{u} \\ &\quad \times \det\left[\frac{1}{2\pi \mathrm{i}} \int_{\rangle} dv \frac{2v}{u_a - v} \frac{e^{(1/48)u_a^3 - (1/8)u_a^2 x - (1/2)u_a \tilde{r}}}{e^{(1/48)v^3 - (1/8)v^2 x - (1/2)v\tilde{r}}} \frac{1}{u_b^2 - v_a^2}\right]_{a,b=1}^{k}. \end{aligned}$$

This last expression is just the series expansion of a Fredholm determinant:

$$F_x(\tilde{r}) = \det(I - K)_{L^2(1+\langle)}$$

with

$$K(u,u') = \frac{1}{2\pi \mathrm{i}} \int_{\rangle} dv \frac{2v}{u-v} \frac{e^{(1/48)u^3+(1/8)u^2x-(1/2)u_a\tilde{r}}}{e^{(1/48)v^3-(1/8)v^2x-(1/2)v\tilde{r}}} \frac{1}{u'^2-v^2}$$
$$= \frac{1}{2\pi \mathrm{i}} \int_0^\infty d\lambda \int_{\rangle} dv \frac{2v}{u'^2-v^2} \frac{e^{(1/48)u^3+(1/8)u^2x-u(\lambda+(1/2)\tilde{r})}}{e^{(1/48)v^3+(1/8)v^2x-v(\lambda+(1/2)\tilde{r})}}.$$

For more details on Fredholm determinants, see Section 2 of [32]. Using the cyclic property of the Fredholm determinant, we deduce that $F_x(\tilde{r}) = \det(I-\widetilde{K})_{L^2([0,\infty))}$ with $\widetilde{K}(\lambda,\lambda') = \frac{1}{(2\pi \mathrm{i})^2} \int du \int dv \frac{2v}{u^2-v^2} \frac{e^{(1/48)u^3+(1/8)u^2x-u(\lambda+(1/2)\tilde{r})}}{e^{(1/48)v^3+(1/8)v_a^2x-v(\lambda'+(1/2)\tilde{r})}}$ and the same u and v contours. Scaling u and v by $2^{4/3}$ and changing variables $\lambda \mapsto 2^{-4/3}\lambda - \frac{1}{2}r$ and $\lambda' \mapsto 2^{-4/3}\lambda' - \frac{1}{2}r$ finally yields

$$\lim_{t\to\infty} \mathbb{E}\big[e_\tau\big(-\tau^{N_{t^{2/3}x}(t/\gamma)(-1/4)t-(1/2)t^{2/3}x+(1/2)t^{1/3}\tilde{r}}\big)\big]$$
$$= \det\big(I - K^{2\to 1}\big)_{L^2([2^{1/3}r,\infty))} \tag{A.4}$$

with

$$K^{2\to 1}(\lambda,\lambda') = \frac{1}{(2\pi \mathrm{i})^2} \int_{1+\langle} du \int_{\rangle} dv \frac{2v}{u^2-v^2} \frac{e^{(1/3)u^3+2^{-1/3}u^2x-u(\lambda-2^{-2/3}x^2\mathbf{1}_{x\le 0})}}{e^{(1/3)v^3+2^{-1/3}v^2x-v(\lambda'-2^{-2/3}x^2\mathbf{1}_{x\le 0})}}.$$

The u and v contours can be easily deformed to match those appearing in the kernel inside the Fredholm determinant which gives the finite dimensional distributions of the $\mathrm{Airy}_{2\to 1}$ process, see [10]. Comparing with that formula, we deduce that the right-hand side of (A.4) equals $\mathbb{P}(\mathcal{A}_{2\to 1}(2^{-1/3}x) \le 2^{1/3}r)$ which, in view of (1.15), finishes our formal derivation of (A.1).

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