The Annals of Applied Probability
2016, Vol. 26, No. 2, 691–721
DOI: 10.1214/15-AAP1098

DYNAMIC RANDOM NETWORKS AND THEIR GRAPH LIMITS

BY HARRY CRANE[1]

Rutgers University

We study a broad class of stochastic process models for dynamic networks that satisfy the minimal regularity conditions of (i) exchangeability and (ii) càdlàg sample paths. Our main theorems characterize these processes through their induced behavior in the space of graph limits. Under the assumption of time-homogeneous Markovian dependence, we classify the discontinuities of these processes into three types, prove bounded variation of the sample paths in graph limit space and express the process as a mixture of time-inhomogeneous, exchangeable Markov processes with càdlàg sample paths.

1. Introduction. Stochastic models for complex networks draw attention in many applications, including physics, epidemiology, sociology and national security. In some cases, for example, epidemiology, the network represents the environment in which a process of interest evolves, for example, the spread of an epidemic; in others, for example, sociology, the network structure is interesting in its own right. Either way, the structure of the network plays an important role in the scientific inquiry. In almost all real world applications, the underlying network changes with time, but so far only a scattering of articles in the machine learning [10, 14] and epidemiology [9] literature attempt to model these dynamics. Aside from our previous development of some basic theory for time-varying network models [6], we are not aware of any formal study of dynamic networks on a fixed population of vertices in the probability literature.

To address this issue, we analyze a broad class of dynamic network models whose minimal regularity properties should be appropriate for most conceivable applications. On the space of graphs with countably many vertices, we consider stochastic processes $\mathbf{\Gamma} = (\Gamma_t)_{t\geq 0}$ that are:

(i) *exchangeable*, that is, invariant under arbitrary relabeling of vertices by finite permutations, and

(ii) *càdlàg*, that is, the sample path $t \mapsto \Gamma_t$ is a càdlàg map in the product-discrete topology.

Received October 2014; revised December 2014.

[1]Supported in part by grants from the U.S. National Science Foundation and National Security Agency.

MSC2010 subject classifications. 60G05, 60G09, 60J25, 90B15.

Key words and phrases. Time-varying network, dynamic network, complex network, exchangeable random graph, partial exchangeability, graph limit, graphon, Markov process, Aldous–Hoover theorem, combinatorial stochastic process.

Time-homogeneous, exchangeable Markov processes are perhaps the most tractable within this class, and we specialize to this case throughout much of the article. Otherwise, assumptions (i) and (ii) are natural in both theoretical and applied study of random networks. Mathematically, exchangeability is a fundamental property of many combinatorial stochastic process models for partitions, trees and graphs, while in practice it reflects a natural indifference to arbitrary assignment of labels, a fundamental principle in Bayesian inference and statistical modeling. Càdlàg paths require that every edge remains in each state it visits for a positive amount of time. Taken together, exchangeability and càdlàg paths imply that the process does not depend on arbitrary assignments of labels to vertices and the edges do not behave erratically during infinitesimally small time intervals.

The *graph limit* of any graph is defined through the limiting homomorphism densities of finite subgraphs. If all of these limiting densities exist for a graph G, then they determine a unique graph limit, denoted $|G|$. As we see, the graph limit of an exchangeable random graph encodes much of its structural information.

We study processes $\mathbf{\Gamma}$ through their projection $|\mathbf{\Gamma}| := (|\Gamma_t|)_{t\geq 0}$ into the space of graph limits, which we show exists at all times almost surely. If, in addition, $\mathbf{\Gamma}$ is a Markov process, then so is $|\mathbf{\Gamma}|$. Thus, our main theorems establish a connection between graph-valued processes satisfying (i) and (ii) Aldous's and Hoover's theory of partially exchangeable arrays [1] and the Lovász–Szegedy theory of graph limits [12, 13]. We paraphrase three of our main theorems for exchangeable, continuous-time Markov processes $\mathbf{\Gamma}$:

- **Theorem 3.3**: With probability one, the projection $|\mathbf{\Gamma}| := (|\Gamma_t|)_{t\geq 0}$ of $\mathbf{\Gamma}$ exists and is a Markov process whose sample paths are càdlàg and have locally bounded variation.
- **Theorem 3.6**: The discontinuities of $\mathbf{\Gamma}$ can be characterized into three classes: at the time s of a discontinuity in $\mathbf{\Gamma}$, either:
  - (A) each edge has a positive probability of experiencing a discontinuity at time s or
  - (B) there exists a unique vertex $i \in \mathbb{N}$ such that either:
    - (B-1) a unique edge ij is discontinuous at time s or
    - (B-2) a positive proportion of edges incident to i is discontinuous at time s.
- **Theorem 3.10**: A Markov process on the space of graph limits whose sample paths are càdlàg and of locally bounded variation corresponds (in distribution) to the projection $|\mathbf{\Gamma}|$ of some exchangeable Markov process $\mathbf{\Gamma}$ satisfying (i) and (ii).

In addition to these theorems, we obtain various auxiliary results of their own interest. For example, in Proposition 4.8, we show a de Finetti-type theorem by which any time-homogeneous Markov process satisfying (i) and (ii) is a mixture of time-inhomogeneous Markov processes.

1.1. *Organization.* In Section 2, we introduce notation, definitions and assumptions. In Section 3, we formally state our main theorems and give illustrative examples. In Section 4, we characterize discrete-time Markov chains under assumption (i). In Section 5, we prove our main theorems for continuous-time graph-valued processes. In Section 6, we make concluding remarks and foreshadow future work.

2. Preliminaries.

2.1. *Graphs.* Throughout the paper, all graphs are *undirected* and have vertex set $[n] := \{1, \dots, n\}$, if finite with $n \geq 1$ vertices, or $\mathbb{N} := \{1, 2, \dots\}$, if infinite. For fixed $n \in \mathbb{N}$, a *graph* G with vertex set $[n]$ is a collection of *edges* $E_G \subseteq [n] \times [n]$ that has:

(1) no self loops, that is, $(i, i) \notin E_G$ for all $i \in [n]$, and
(2) undirected edges, that is, $(i, j) \in E_G$ if and only if $(j, i) \in E_G$.

Since G is undirected, we often write $i \sim_G j$ or $ij \in E_G$ to denote an edge between i and j in G. Alternatively, we can specify G by its *adjacency matrix* $(G^{ij})_{1 \leq i, j \leq n}$, for which

$$G^{ij} := \begin{cases} 1, & i \sim_G j, \\ 0, & \text{otherwise.} \end{cases}$$

We write $\mathcal{G}_n$ to denote the space of graphs with vertex set $[n]$.

REMARK 2.1. There is no substantive difference between the directed and undirected cases for the processes we study. Our analysis easily extends to processes on directed graphs as well as more general processes on exchangeable random arrays and hypergraphs.

For any $m \leq n$ and $G \in \mathcal{G}_n$, we define the *restriction* of G to $\mathcal{G}_m$ by $G|_{[m]} := (G^{ij})_{1 \leq i, j \leq m}$, the projection of the adjacency matrix of G to its leading $m \times m$ submatrix. Under this projection, $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is naturally embedded in the space $\mathcal{G}_\infty$ of *countably infinite graphs*, which we regard as *compatible sequences* $(G_n)_{n \in \mathbb{N}}$ of finite graphs, that is, $G_n \in \mathcal{G}_n$ for each $n \in \mathbb{N}$ and $G_n|_{[m]} = G_m$ for all $m \leq n$. We regard $G := (G_n)_{n \in \mathbb{N}}$ as an *infinite graph* with vertex set $\mathbb{N}$, from which the alternative specification as an infinite by infinite *adjacency array* is apparent. In particular, for each $n \geq 1$, the leading $n \times n$ submatrix $(G^{ij})_{1 \leq i, j \leq n}$ of the adjacency array $(G^{ij})_{i, j \geq 1}$ of $G \in \mathcal{G}_\infty$ coincides with the adjacency matrix of the restriction $G|_{[n]}$ of G to $\mathcal{G}_n$.

The limit space $\mathcal{G}_\infty$ is naturally equipped with the product-discrete topology induced by the ultrametric

$$d(G, G') := 1/\max\{n \in \mathbb{N} : G|_{[n]} = G'|_{[n]}\}, \qquad G, G' \in \mathcal{G}_\infty. \tag{1}$$

Under this topology, $\mathcal{G}_\infty$ is compact and Polish. We also equip $\mathcal{G}_\infty$ with the Borel σ-field $\mathcal{B} := \sigma\langle\bigcup_{n\in\mathbb{N}} \mathcal{G}_n\rangle$ generated by the finite restriction maps $\cdot|_{[n]} : \mathcal{G}_\infty \to \mathcal{G}_n$, for each $n \in \mathbb{N}$.

In Section 2.3, we introduce exchangeable random graphs and exchangeable Markov processes on $\mathcal{G}_\infty$, which we then study throughout the paper. Such processes exhibit invariance under arbitrary *relabeling* by finite permutations of $\mathbb{N}$. Specifically, we call a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ *finite* if it fixes all but finitely many $n \in \mathbb{N}$, and we write $\mathscr{S}_{\mathbb{N}}$ to denote the set of finite permutations of $\mathbb{N}$. Any $\sigma \in \mathscr{S}_{\mathbb{N}}$ acts naturally on $G \in \mathcal{G}_\infty$ by relabeling vertices, $G \mapsto G^\sigma$, where

$$G^\sigma := \left(G^{\sigma(i)\sigma(j)}\right)_{i,j\geq 1}. \tag{2}$$

More generally, given an injection $\psi : [m] \to [n]$, $m \leq n$, we define the projection $\psi^* : \mathcal{G}_n \to \mathcal{G}_m$, $G \mapsto \psi^*(G) =: G^\psi$, by

$$G^\psi := \left(G^{\psi(i)\psi(j)}\right)_{1\leq i,j\leq m}. \tag{3}$$

The projection operation in (3) is key to our definition of graph limits below.

2.2. *Graph limits.* Lovász and Szegedy [13] introduced graph limits while studying sequences of dense graphs. That work implicitly recalls the Aldous–Hoover theorem [1] for partially exchangeable arrays; see Theorem 2.4 below.

For fixed integers $m \leq n$ and fixed graphs $F \in \mathcal{G}_m$ and $G \in \mathcal{G}_n$, we define $\operatorname{ind}(F; G)$ as the number of injections $\psi : [m] \to [n]$ such that $G^\psi = F$, where G^ψ is defined in (3). Informally, $\operatorname{ind}(F; G)$ is the number of *induced copies* of F in G. We define the *density of F in G* by

$$t(F; G) := \frac{\operatorname{ind}(F; G)}{n^{\downarrow m}}, \qquad F \in \mathcal{G}_m, G \in \mathcal{G}_n, m \leq n, \tag{4}$$

where the denominator $n^{\downarrow m} := n(n-1)\cdots(n-m+1)$ is the number of injective maps $[m] \to [n]$. For an infinite graph $G \in \mathcal{G}_\infty$, we define the *limiting density of F in G* by

$$t(F; G) := \lim_{n\to\infty} t(F; G|_{[n]}), \qquad \text{if it exists.} \tag{5}$$

By now it is well known (cf. [1, 13]) that the limit in (5) exists almost surely whenever $G \in \mathcal{G}_\infty$ is the realization of an exchangeable random graph.

DEFINITION 2.2. The *graph limit* $|G|$ of $G \in \mathcal{G}_\infty$ is defined by

$$|G| := \left(t(F; G)\right)_{F\in\mathcal{G}^*}, \tag{6}$$

provided $t(F; G)$ exists for every $F \in \mathcal{G}^*$, where $\mathcal{G}^* := \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is the countable collection of all *finite* graphs.

REMARK 2.3 (Terminology). We use the term *graph limit* to distinguish from Lovász and Szegedy's term *graphon*, which is defined as a measurable function $W : [0, 1]^2 \to [0, 1]$. Because the graphon is unique only up to measure-preserving transformations of $[0, 1]^2$, it is not a convenient limiting object. On the other hand, the graph limit (as we define it) is unique whenever it exists.

The connection between a graphon W and the graph limit $(t(F; G))_{F \in \mathcal{G}^*}$ is as follows. Let $U_1, U_2, \ldots$ be independent, identically distributed Uniform[0, 1] random variables. Given $(U_1, U_2, \ldots)$, we define $\Gamma = (\Gamma^{ij})_{i,j \geq 1}$ to be conditionally independent Bernoulli random variables such that $\mathbb{P}\{\Gamma^{ij} = 1 | U_i, U_j\} = W(U_i, U_j)$. Thus, the graphon determines an exchangeable probability distribution on $\mathcal{G}_\infty$. Each of the limiting densities (5) of Γ exists, and the collection of these densities corresponds to the graph limit.

Assuming it exists, $|G|$ is an element of $[0, 1]^{\mathcal{G}^*} \cong [0, 1]^{\mathbb{N}}$, which we equip with the metric

$$d(x, x') := \sum_{n \in \mathbb{N}} 2^{-n} \sum_{F \in \mathcal{G}_n} |x_F - x'_F|, \qquad x, x' \in [0, 1]^{\mathcal{G}^*}. \tag{7}$$

Under (7), $[0, 1]^{\mathcal{G}^*}$ is compact, complete and separable. We write $\mathcal{D}^*$ to denote the closure of the subset of $[0, 1]^{\mathcal{G}^*}$ to which $\mathcal{G}_\infty$ projects under (6). As a closed subset of $[0, 1]^{\mathcal{G}^*}$, $\mathcal{D}^*$ is also compact and Polish.

2.3. *Exchangeable random graphs and the space of graph limits.* An *infinite exchangeable random graph* Γ is a random element of $\mathcal{G}_\infty$ that satisfies

$$\Gamma^\sigma =_{\mathcal{L}} \Gamma \qquad \text{for all } \sigma \in \mathscr{S}_{\mathbb{N}}, \tag{8}$$

where $=_{\mathcal{L}}$ denotes *equality in law*. In other words, the law of Γ is invariant under arbitrary relabeling of its vertices.

In general, we call a random $\mathcal{X}$-valued array $X := (X^{ij})_{i,j \geq 1}$ *weakly exchangeable* if X is symmetric and

$$X^\sigma =_{\mathcal{L}} X \qquad \text{for all } \sigma \in \mathscr{S}_{\mathbb{N}}, \tag{9}$$

where $X^\sigma := (X^{\sigma(i)\sigma(j)})_{i,j \geq 1}$. By the representation of Γ through its adjacency array $(\Gamma^{ij})_{i,j \geq 1}$, any exchangeable random graph corresponds to a weakly exchangeable $\{0, 1\}$-valued array. The Aldous–Hoover theorem makes explicit the connection between exchangeable random graphs and graph limits.

THEOREM 2.4 (Aldous–Hoover theorem: Aldous [1], Theorem 14.21, Hoover [11]). *Let $X := (X^{ij})_{i,j \geq 1}$ be a weakly exchangeable $\mathcal{X}$-valued array, where $\mathcal{X}$ is Polish. Then there exists a measurable function $f : [0, 1]^4 \to \mathcal{X}$ for which $f(\cdot, b, c, \cdot) = f(\cdot, c, b, \cdot)$ such that $X =_{\mathcal{L}} X_*$, where*

$$X_*^{ij} := f(\alpha, \xi_i, \xi_j, \eta_{\{i,j\}}), \qquad i, j \geq 1,$$

for $\{\alpha; (\xi_i)_{i \in \mathbb{N}}; (\eta_{\{i,j\}})_{i > j \geq 1}\}$ independent, identically distributed Uniform random variables on $[0, 1]$.

REMARK 2.5. As stated, Theorem 2.4 is a special case of the Aldous–Hoover theorem; see [1], Theorem 14.11, and the surrounding discussion.

The following corollary records some useful first properties of graph limits. For $D \in \mathcal{D}^*$, we write $D(F) = D_F$ to denote the component of D corresponding to $F \in \mathcal{G}^*$.

COROLLARY 2.6. *For any $D \in \mathcal{D}^*$ and any $n \geq 1$:*

(i) *the n-sector $D^{(n)} := (D_F)_{F\in\mathcal{G}_n}$ determines an exchangeable probability distribution on $\mathcal{G}_n$ through*

$$\mathbb{P}\{\Gamma_n = F\} = D^{(n)}(F), \qquad F \in \mathcal{G}_n; \quad \textit{and}$$

(ii) *the collection $(D^{(n)})_{n\geq 1}$ is a consistent family of exchangeable probability distributions on $(\mathcal{G}_n)_{n\geq 1}$, that is, for all $m \leq n$ and $F \in \mathcal{G}_m$,*

$$D^{(m)}(F) = \sum_{F^*\in\mathcal{G}_n : F^*|_{[m]}=F} D^{(n)}(F^*).$$

PROOF. Part (i) is immediate from the definition of the limit density (5). Part (ii) also follows from this definition since, for every $l \leq m$, each injection $\psi : [l] \to [n]$ corresponds to the collection of injections $\psi' : [m] \to [n]$ such that $\psi = \psi' \circ i_{l,m}$, where $i_{l,m} : [l] \to [m]$ is the *insertion map*, $j \mapsto j$. For $F \in \mathcal{G}_l$, $\operatorname{ind}(F; G|_{[n]})$ counts the number of $\psi : [l] \to [n]$ such that $G|_{[n]}^{\psi} = F$, and each of these maps corresponds to a collection of exactly $n^{\downarrow m}/n^{\downarrow l}$ injections $\psi' : [m] \to [n]$ by the above association. Consistency is an immediate byproduct. $\square$

From Corollary 2.6, we can regard each element of $\mathcal{D}^*$ as the projective limit of some family of exchangeable and consistent distributions on graphs with finitely many vertices. By Carathéodory's extension theorem, a graph limit $D \in \mathcal{D}^*$ corresponds to a unique exchangeable, dissociated probability measure γ_D on $\mathcal{G}_\infty$, where

$$\gamma_D\bigl(\{G \in \mathcal{G}_\infty : G|_{[n]} = F\}\bigr) = D^{(n)}(F), \qquad F \in \mathcal{G}_n,$$

for each $n \geq 1$. For any $D \in \mathcal{D}^*$, the marginals $\gamma_D^{(n)}$, $n \geq 1$, satisfy

$$\gamma_D^{(n)}(F) := D_F, \qquad F \in \mathcal{G}_n. \tag{10}$$

The following proposition summarizes the relationship between exchangeable random graphs and graph limits, which is a consequence of the Aldous–Hoover theorem (Theorem 2.4) and the Lovász–Szegedy graph limit theorem [13], Theorem 2.7.

PROPOSITION 2.7. *Let Γ be an infinite exchangeable random graph. Then, for every fixed graph $F \in \mathcal{G}_m$, $m \geq 1$, the limiting density $t(F;\Gamma)$ exists almost surely and, thus, the graph limit $|\Gamma|$ exists almost surely. Moreover, with $\gamma_D^{(n)}$ defined in* (10), *there exists a unique probability measure Δ on $\mathcal{D}^*$ so that $\Gamma \sim \gamma_\Delta$, where γ_Δ is characterized by its marginal distributions*

$$\gamma_\Delta^{(n)}(G) := \int_{\mathcal{D}^*} \gamma_D^{(n)}(G)\Delta(dD), \qquad G \in \mathcal{G}_n.$$

2.4. *Graph-valued processes.* We are interested in exchangeable processes $\mathbf{\Gamma} = (\Gamma_t)_{t\in T}$ on $\mathcal{G}_\infty$. Here, T denotes time and can be taken as either $T = \mathbb{Z}_+ := \{0, 1, \ldots\}$ (discrete-time) or $T = \mathbb{R}_+ := [0, \infty)$ (continuous-time). We consider both cases.

A graph-valued process $\mathbf{\Gamma}$ is:

- *exchangeable* if $\mathbf{\Gamma}^\sigma = (\Gamma_t^\sigma)_{t\in T}$ is a version of $\mathbf{\Gamma}$ for every $\sigma \in \mathscr{S}_\mathbb{N}$ and
- *càdlàg* if $t \mapsto \Gamma_t$ almost surely determines a mapping $T \to \mathcal{G}_\infty$ that is right-continuous and has left limits.

In particular, $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ has càdlàg sample paths if each finite restriction $\mathbf{\Gamma}^{[n]} := (\Gamma_t|_{[n]})_{t\geq 0}$ determines a càdlàg path in $\mathcal{G}_n$ with the discrete topology. Equivalently, $\mathbf{\Gamma}$ is càdlàg if, for every $i > j \geq 1$, the edge trajectory $(\Gamma_t^{ij})_{t\geq 0}$ is a càdlàg path in $\{0, 1\}$.

For any $\mathcal{G}_\infty$-valued process $\mathbf{\Gamma}$, we call the collection $(\mathbf{\Gamma}^{[n]})_{n\geq 1}$ of restrictions to $(\mathcal{G}_n)_{n\geq 1}$ its *finite state space sample paths*. Without further conditions, we characterize the behavior of $\mathbf{\Gamma}$ in terms of the finite-dimensional distributions, that is, the distribution of $(\Gamma_{t_j})_{j=1,\ldots,k}$ for arbitrary finite collections of times $0 \leq t_1 < \cdots < t_k < \infty$. We usually take càdlàg sample paths for granted and call an exchangeable, càdlàg process an *exchangeable graph-valued process*.

Any collection $\mathbf{\Gamma}_T := (\Gamma_t)_{t\in T}$ of graphs generates its *exchangeable σ-field $\mathcal{E}_T$*, which consists of events A that are measurable with respect to $\mathbf{\Gamma}_T$ and for which $A = A^\sigma := \{G^\sigma : G \in A\}$ for all $\sigma \in \mathscr{S}_\mathbb{N}$. These events are *exchangeable* in that they are invariant under relabeling. For example, the density $t(F;G)$ of a finite graph $F \in \mathcal{G}_m$ in $G \in \mathcal{G}_\infty$ is invariant under arbitrary relabeling of both F and G; see equation (4), its surrounding discussion, and also Corollary 2.6.

We pay special attention to time-homogeneous Markov processes.

DEFINITION 2.8. A Markov process $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ on $\mathcal{G}_\infty$ is *exchangeable* if:

(i) its initial state is an exchangeable random graph, that is, $\Gamma_0 =_\mathcal{L} \Gamma_0^\sigma$ for all $\sigma \in \mathscr{S}_\mathbb{N}$, and

(ii) its transition kernel

$$p_t(G, dG') := \mathbb{P}\{\Gamma_{s+t} \in dG' | \Gamma_s = G\}$$

is invariant under relabeling by $\mathscr{S}_{\mathbb{N}}$; that is, for every $\sigma \in \mathscr{S}_{\mathbb{N}}$ and every measurable subset $A \subseteq \mathcal{G}_\infty$,

$$p_t(G^\sigma, A^\sigma) = p_t(G, A) \qquad \text{for all } t \geq 0, \tag{11}$$

where $A^\sigma := \{G'^\sigma : G' \in A\}$ is the relabeling of A by σ.

We study exchangeable Markov processes $\mathbf{\Gamma}$ that possess càdlàg sample paths in the metric (1). For convenience, we call these processes *exchangeable Markov processes* on $\mathcal{G}_\infty$, or *exchangeable $\mathcal{G}_\infty$-valued Markov processes*.

REMARK 2.9. In many studies of Markov processes, the Feller property is the natural case of interest; however, under the assumed topology, the Feller property forbids nontrivial dependence on the exchangeable σ-algebra, a strong assumption in some applications. In this article, we do not assume the Feller property. We characterize further structural properties of exchangeable Feller processes in [5].

2.5. *Notation*. We use the Greek letter Γ to denote random graphs, the bold Greek letter $\mathbf{\Gamma}$ to denote graph-valued processes and the Roman letters F and G to denote fixed graphs.

To avoid confusion, we use superscripts to denote edges, for example, G^{ij} denotes the (i, j) entry of the adjacency array of $G \in \mathcal{G}_\infty$, and subscripts to denote time, for example, Γ_t^{ij} is the status of edge ij in $\mathbf{\Gamma}$ at time $t \geq 0$. For a subset $T' \subseteq T$, $\mathbf{\Gamma}_{T'} := (\Gamma_t)_{t\in T'}$ denotes the process $\mathbf{\Gamma}$ restricted to (or observed at) the time points in T' and $\mathbf{\Gamma}_{T'}^{ij} := (\Gamma_t^{ij})_{t\in T'}$ denotes the trajectory of edge ij at points in T'. Later, we introduce the notation $\mathbf{\Gamma}[T] := (\Gamma_T^{ij})_{i,j\geq 1}$ to denote an *array* whose (i, j) entry is the edge trajectory $(\Gamma_t^{ij})_{t\in T}$. When T is finite with cardinality n, the entries of $\mathbf{\Gamma}[T]$ are n-tuples in $\{0, 1\}^n$; when T is a subinterval, the entries are càdlàg paths $t \mapsto \Gamma_t^{ij}$ in $\{0, 1\}$.

3. Summary of main theorems and illustrative examples. Before summarizing the main theorems, we illustrate possible behaviors of some exchangeable graph-valued processes. For example, graph-valued Markov processes can exhibit various, seemingly pathological, behaviors (Example 3.1) and their projection into $\mathcal{D}^*$ need not possess the Feller property (Example 3.2).

EXAMPLE 3.1. Let $\{U_{\{i,j\}}\}_{i>j\geq 1}$ be a collection of independent and identically distributed Uniform random variables on $[0, 1]$ and let $h : [0, 1] \to [0, 1]$ be any increasing homeomorphism of the unit interval. We define $\mathbf{\Gamma} := (\Gamma_t)_{t\in[0,1]}$ on $\mathcal{G}_\infty$ by

$$\Gamma_t^{ij} = \begin{cases} 1, & \text{if } U_{\{i,j\}} > h(t), \\ 0, & \text{if } U_{\{i,j\}} \leq h(t). \end{cases}$$

Because $\{U_{\{i,j\}}\}_{i>j\geq 1}$ are independent and identically distributed, $\mathbf{\Gamma}$ must be exchangeable. Furthermore, $\mathbf{\Gamma}$ is Markovian and has càdlàg sample paths—each edge makes only a single jump at a random time that depends measurably on the overall edge density of the process. But if, for example, h is the Cantor function, then the jumps of $\mathbf{\Gamma}$ occur in a predictable set of measure zero. In fact, $\mathbf{\Gamma}$ behaves this way as long as the derivative of h is zero almost everywhere on $[0, 1]$.

Despite its bizarre behavior, $\mathbf{\Gamma}$ almost surely projects to a deterministic trajectory $|\mathbf{\Gamma}|$ in $\mathcal{D}^*$: at each $t \geq 0$, $|\Gamma_t|$ is the graph limit of an Erdős–Rényi random graph with parameter $1 - h(t)$. In general, our main theorems establish that the projection into $\mathcal{D}^*$ always exists and behaves regularly.

EXAMPLE 3.2. We define the *upper graph limit* of $G \in \mathcal{G}_\infty$ by

$$|G|^+ := \big(t^+(F; G)\big)_{F\in\mathcal{G}^*},$$

where

$$t^+(F; G) := \limsup_{n\to\infty} t(F; G|_{[n]}), \qquad F \in \mathcal{G}^*.$$

Since the limit superior always exists, we need not worry about existence of the graph limit for arbitrary initial states G_0.

Let $\{\tau_{\{i,j\}}\}_{i>j\geq 1}$ be independent, identically distributed standard exponential random variables and let $G_0 \in \mathcal{G}_\infty$ be any initial state with *upper edge density*

$$\delta^+ := \limsup_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \mathbf{1}\{i \sim_{G_0} j\}.$$

For $t > 0$, we define Γ_t by

$$\Gamma_t^{ij} = \begin{cases} \mathbf{1}_{[1/2,1]}(\delta^+), & \text{if } t \geq \tau_{\{i,j\}}, \\ G_0^{ij}, & \text{if } t < \tau_{\{i,j\}}. \end{cases}$$

If G_0 is the realization of an exchangeable random graph, then $\mathbf{\Gamma}$ is an exchangeable Markov process on $\mathcal{G}_\infty$ whose projection $|\mathbf{\Gamma}|$ into $\mathcal{D}^*$ is continuous and, in fact, deterministic. On the other hand, with $\mathbf{\Gamma}(\delta^+)$ denoting a Markov process whose initial state has upper edge density δ^+, the family $\{\mathbf{\Gamma}(\delta^+) : \delta^+ \in [0, 1]\}$ does not have the Feller property. Plainly, the Feller property requires that the Markov semigroup determines a continuous mapping from all initial states, but the above process is discontinuous at $\delta^+ = 1/2$. More succinctly, $|\mathbf{\Gamma}|$ enjoys the Feller property only if the trajectories $(t, \delta^+) \mapsto |\Gamma_t(\delta^+)|$ determine a jointly continuous flow.

Although the previous examples illustrate strange behaviors of exchangeable Markov processes $\mathbf{\Gamma}$ with càdlàg paths, our first theorem asserts that the projection to $\mathcal{D}^*$ is well behaved. In particular, $|\mathbf{\Gamma}|$ exists, is a Markov process, and has càdlàg sample paths with locally bounded variation.

THEOREM 3.3. *Let* $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ *be an exchangeable Markov process on* $\mathcal{G}_\infty$ *with càdlàg sample paths. Then the projection* $(|\Gamma_t|)_{t\geq 0}$ *into the space of graph limits exists with probability one. Moreover,* $|\mathbf{\Gamma}| := (|\Gamma_t|)_{t\geq 0}$ *is a Markov process on* $\mathcal{D}^*$ *whose sample paths are càdlàg and have locally bounded variation.*

REMARK 3.4. Because the paths of $|\mathbf{\Gamma}|$ have bounded variation, the continuous portion of $|\mathbf{\Gamma}|$ is purely deterministic, that is, there can be no Brownian-like component. Consequently, there is no analog to Brownian motion for exchangeable Markov processes on $\mathcal{G}_\infty$ under the product-discrete topology.

REMARK 3.5. Càdlàg sample paths for $\mathbf{\Gamma}$ means that $\mathbf{\Gamma}^{[n]}$ has a strictly positive hold time in every state it visits, for every $n \in \mathbb{N}$. On the other hand, the projection $|\mathbf{\Gamma}|$ tracks the flow of densities associated to the whole process $\mathbf{\Gamma}$ through its exchangeable σ-field. Conditional on $\mathcal{E}_{\mathbf{\Gamma}}$, $|\mathbf{\Gamma}|$ determines the transition probabilities of a *time-inhomogeneous* Markov process on $\mathcal{G}_n$, for every $n \in \mathbb{N}$. Thus, if $|\mathbf{\Gamma}|$ had unbounded variation, then there would be a positive probability that the sample paths of $\mathbf{\Gamma}$ are not càdlàg. We make this heuristic rigorous in Theorem 5.8.

Without the Feller property, the infinitesimal generator need not exist; therefore, we cannot describe the discontinuities directly through the jump rates. Also, the finite restrictions of $\mathbf{\Gamma}$ need not be Markovian, and so the common technique of discretization by projection to finite spaces has limited use. Nevertheless, we are able to characterize the discontinuities of $\mathbf{\Gamma}$ and $|\mathbf{\Gamma}|$.

THEOREM 3.6. *An exchangeable* $\mathcal{G}_\infty$*-valued Markov process* $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ *with càdlàg paths has at most countably many discontinuities almost surely. The discontinuities of* $\mathbf{\Gamma}$ *classify into three types: if* $\mathbf{\Gamma}$ *is discontinuous at* $t = s$, *then either*:

(A) $P(\mathbf{\Gamma}^{ij}$ *is discontinuous at* $s \mid \mathcal{E}_{[0,\infty)}) > 0$ *for every* $i \neq j \in \mathbb{N}$ *or*
(B) *there exists a unique* $i \in \mathbb{N}$ *such that either*:
  (B-1) $\mathbf{\Gamma}^{ij}$ *is discontinuous at* s *for some unique* $j \neq i$ *or*
  (B-2) *there exist constants* $0 \leq p_0, p_1 \leq 1$ (*possibly depending on* $\mathcal{E}_{[0,\infty)}$) *such that*

$$P\{\mathbf{\Gamma}^{ij} \text{ is discontinuous at } s \mid \mathcal{E}_{[0,\infty)}\} = p_k \qquad \text{on the event } \Gamma^{ij}_{s-} = k,$$

for all $j \neq i$;
and $P\{\mathbf{\Gamma}^{i'j'}$ *is discontinuous at* $s \mid \mathcal{E}_{[0,\infty)}\} = 0$ *if* $i \notin \{i', j'\}$.

We take a moment to discuss the intuition behind cases (A) and (B) and why they exhaust all possibilities for discontinuities in $\mathbf{\Gamma}$.

Broadly, we can classify discontinuities of $\mathbf{\Gamma}$ as one of two types: for a discontinuity time $t \geq 0$, let $S^{(n)}_t := \sum_{1\leq i<j\leq n} \mathbf{1}\{\Gamma^{ij}_{t-} \neq \Gamma^{ij}_t\}$ be the number of edges of

$\mathbf{\Gamma}^{[n]}$ with a discontinuity at time t and define $F_t := \lim_{n\to\infty} 2n^{-1}(n-1)^{-1}S_t^{(n)}$ as the limiting fraction of edges at which there is a discontinuity. By exchangeability, the limit F_t exists a.s. and satisfies either:

(A) $F_t > 0$ or
(B) $F_t = 0$.

Given a discontinuity at time t, (Γ_{t-}, Γ_t) is a jointly exchangeable pair of $\{0, 1\}$-valued arrays. Marginally, Γ_t is a weakly exchangeable array and, therefore, is itself an exchangeable random graph. By Aldous–Hoover in conjunction with Lovász–Szegedy, case (A) is covered by discontinuities of type-(A). Indeed, if a positive fraction of edges changes status, then exchangeability forces a positive probability that each edge has a discontinuity, conditional on the exchangeable σ-algebra. On the other hand, if a zero fraction of edges changes status, then either:

(B-1) $\lim_{n\to\infty} S_t^{(n)} < \infty$ or
(B-2) $\lim_{n\to\infty} S_t^{(n)} = \infty$.

In case (B-1), only finitely many edges are discontinuous at time t. But by exchangeability and the strong law of large numbers, if exactly two disjoint edges share a discontinuity at t, then there is positive probability that any two edges share a discontinuity at t, which would imply we are in case (A) above. In case (B-2), the discontinuous edges must have a vertex in common because case (B-1) precludes disjoint edges from jumping simultaneously.

The next example illustrates a process that evolves only by type-(B-2) discontinuities.

EXAMPLE 3.7. Let $N_i := \{N_i(t)\}_{t\geq 0}$ be a rate-1 Poisson process, for each $i \geq 1$. We assume that the processes $\{N_i\}_{i\geq 1}$ are mutually independent. As in Example 3.2, let δ_t^+ denote the upper edge density of $\mathbf{\Gamma}$ at time $t \geq 0$. We assume $\mathbf{\Gamma}$ starts at $\mathbf{0}_{\mathbb{N}}$, the *empty graph*. For each arrival time τ in N_i, $i \geq 1$, we change the edges incident to vertex i independently according to flips of a $(1 - \delta_{\tau-}^+)$-coin. The above process is Markov, exchangeable, and each of its discontinuities is type-(B-2).

COROLLARY 3.8. *The projection $|\mathbf{\Gamma}|$ of any exchangeable Markov process $\mathbf{\Gamma}$ has càdlàg paths with discontinuities only at the times of type-*(A) *discontinuities of $\mathbf{\Gamma}$.*

The following example illustrates that the projection $|\mathbf{\Gamma}|$ of $\mathbf{\Gamma}$ into $\mathcal{D}^*$ does not uniquely determine the law of $\mathbf{\Gamma}$. Theorem 3.10 gives a partial converse: any Markov process on $\mathcal{D}^*$ with càdlàg sample paths of locally bounded variation corresponds to the projection of some exchangeable $\mathcal{G}_\infty$-valued Markov process.

EXAMPLE 3.9. Let $\tau_1 < \tau_2 < \cdots$ be the occurrence times of a rate-1 Poisson process on $[0, \infty)$ and let Γ_0 be an Erdős–Rényi random graph with parameter $p = 1/3$. We construct processes $\mathbf{\Gamma}_1 := (\Gamma_1(t))_{t\geq 0}$ and $\mathbf{\Gamma}_2 := (\Gamma_2(t))_{t\geq 0}$ that have different transition laws and project to the same process in $\mathcal{D}^*$. For this example only, let $\delta(G)$ denote the *edge density* of G, that is, $\delta(G) := \lim_{n\to\infty} 2n^{-1}(n-1)^{-1}\sum_{1\leq i<j\leq n} \mathbf{1}\{ij \in G\}$.

(1) In $\mathbf{\Gamma}_1$:
   (a) if $\delta(\Gamma_1(\tau_n-)) = 2/3$, then all pairs ij for which $\Gamma_1^{ij}(\tau_n-) = 1$ simultaneously toss fair coins to determine whether there is an edge between i and j in $\Gamma_1(\tau_n)$ and
   (b) if $\delta(\Gamma_1(\tau_n-)) = 1/3$, then all pairs ij for which $\Gamma_1^{ij}(\tau_n-) = 0$ simultaneously toss fair coins to determine whether there is an edge between i and j in $\Gamma_1(\tau_n)$.
(2) In $\mathbf{\Gamma}_2$, all pairs ij, regardless of their status at time τ_n-, toss $1 - \delta(\Gamma_2(\tau_n-))$ coins to determine whether there is an edge between i and j at time τ_n.

The transition laws of $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ differ, but both projections $|\mathbf{\Gamma}_1|$ and $|\mathbf{\Gamma}_2|$ follow the same trajectory in $\mathcal{D}^*$—they alternate between the graph limits of Erdős–Rényi random graphs with $p = 2/3$ and $p = 1/3$.

THEOREM 3.10. *Let $\mathbf{D} := (D_t)_{t\geq 0}$ be a Markov process on $\mathcal{D}^*$ whose sample paths are càdlàg and have locally bounded variation. Then there exists an exchangeable Markov process $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ on $\mathcal{G}_\infty$ whose projection $|\mathbf{\Gamma}|$ into $\mathcal{D}^*$ has the same law as $\mathbf{D}$.*

We now move to our main discussion. In Section 4, we discuss discrete-time Markov chains on $\mathcal{G}_\infty$. The observations in Section 4 are important to some of our main theorems about continuous-time processes, which we prove throughout Section 5.

4. Discrete-time Markov chains.

Let $\mathbf{\Gamma} := (\Gamma_t)_{t\in T}$ be an exchangeable $\mathcal{G}_\infty$-valued Markov process observed at an arbitrary set of times T. For a finite subsequence of times $T' := \{0 \leq t_0 < t_1 < \cdots < t_n < \infty\} \subset T$, we define $\mathbf{\Gamma}[T'] = \mathbf{\Gamma}[t_0, t_1, \ldots, t_n] := (\Gamma_{T'}^{ij})_{i,j\geq 1}$, where

$$\Gamma_{T'}^{ij} := \big(\Gamma_{t_k}^{ij}\big)_{k=0}^{n}, \qquad i, j \geq 1,$$

is an $(n+1)$-tuple that records the status of the edge between i and j in T'. Thus, $\mathbf{\Gamma}[T']$ is a $\{0, 1\}^{n+1}$-valued array that tracks the status of edges at all times in T'. Regarding $\mathbf{\Gamma}[T']$ as an array, rather than a process, avails us of the Aldous–Hoover theorem for partially exchangeable arrays.

PROPOSITION 4.1. *For every finite sequence of times $T' \subset T$, the array $\mathbf{\Gamma}[T']$ is weakly exchangeable as in* (9).

PROOF. This follows directly from the exchangeability property of $\mathbf{\Gamma}$ (Definition 2.8). More explicitly, if $\mathbf{\Gamma}_T := (\Gamma_t)_{t\in T}$ is an exchangeable Markov process, then $(\Gamma_t)_{t\in T} =_{\mathcal{L}} (\Gamma_t^\sigma)_{t\in T}$ and $\mathbf{\Gamma}[T'] =_{\mathcal{L}} \mathbf{\Gamma}[T']^\sigma$ for all $\sigma \in \mathscr{S}_{\mathbb{N}}$, establishing weak exchangeability. $\square$

PROPOSITION 4.2. *For every $t \geq 0$, Γ_t determines a weakly exchangeable $\{0,1\}$-valued array $(\Gamma_t^{ij})_{i,j\geq 1}$ whose limit $|\Gamma_t|$ exists almost surely. Moreover, the projection $|\mathbf{\Gamma}| := (|\Gamma_t|)_{t\in T}$ is a Markov process on $\mathcal{D}^*$.*

PROOF. Weak exchangeability of Γ_t is a special case of Proposition 4.1 with $T' = \{t\}$. To prove existence of $|\Gamma_t|$ for fixed $t \in T$, we appeal to weak exchangeability and the Aldous–Hoover theorem [1], Theorem 14.21.

To show that $|\mathbf{\Gamma}|$ is a Markov process, we let $(\mathcal{F}_t)_{t\in T}$ denote the natural filtration of $\mathbf{\Gamma}$, that is, $\mathcal{F}_t := \sigma\langle \Gamma_s, s \leq t\rangle$ for each $t \in T$. By defining the *cemetery state* $|G| = \partial$ whenever $|G|$ does not exist, the map $|\cdot| : \mathcal{G}_\infty \to \mathcal{D}^* \cup \{\partial\}$ is measurable. Furthermore, since $|\Gamma_t^\sigma| = |\Gamma_t|$ for all $\sigma \in \mathscr{S}_{\mathbb{N}}$, $|\Gamma_t|$ is measurable with respect to the exchangeable σ-field $\mathcal{E}_t$ generated by Γ_t. The Markov property of $\mathbf{\Gamma}$ implies that the conditional law of $|\Gamma_{t'}|$, given $\mathcal{F}_t$, depends only on $(\Gamma_t, \Gamma_{t'})$, for all $t, t' \in T$ with $t' > t$. Exchangeability of $\mathbf{\Gamma}$ implies $(\Gamma_t, \Gamma_{t'}) =_{\mathcal{L}} (\Gamma_t^\sigma, \Gamma_{t'}^\sigma)$ for all $\sigma \in \mathscr{S}_{\mathbb{N}}$. Together, exchangeability and the Markov property imply that the conditional law of $|\Gamma_{t'}|$ given $|\Gamma_t|$ is the same as the conditional law of $|\Gamma_{t'}|$ given $|\Gamma_t^\sigma|$ and, therefore, $|\Gamma_{t'}|$ depends on Γ_t only through $\mathcal{E}_t$. Since $|\Gamma_t|$ generates $\mathcal{E}_t$, the conditional law of $|\Gamma_{t'}|$ given $\mathcal{F}_t$ is the same as the conditional law of $|\Gamma_{t'}|$ given $|\Gamma_t|$ and $|\mathbf{\Gamma}|$ is a Markov process. $\square$

REMARK 4.3. The above assertion that $|\mathbf{\Gamma}|$ is a Markov process only applies for conditional distributions at finitely many time points. In Section 5, we show that $|\Gamma_t|$ exists simultaneously for all t with probability one, from which we deduce that the projection $|\mathbf{\Gamma}|$ into $\mathcal{D}^*$ has bounded variation almost surely.

4.1. *Characterization of discrete-time Markov chains on $\mathcal{G}_\infty$.* We characterize the transition law of exchangeable discrete-time Markov chains $\mathbf{\Gamma}$ by extending the definition of subgraph density and graph limit, (4) and (6), respectively, to finite collections of graphs. For any finite $k, m, n \in \mathbb{N}$, let $F_1, \ldots, F_k \in \mathcal{G}_m$ and $G_1, \ldots, G_k \in \mathcal{G}_n$. We define

$$\mathrm{ind}(F_1, \ldots, F_k; G_1, \ldots, G_k) = \mathrm{ind}\big((F_i)_{1\leq i\leq k}; (G_i)_{1\leq i\leq k}\big)$$

as the number of injections $\psi : [m] \to [n]$ for which the subsequence that $G_1, \ldots, G_k$ induces on $\mathcal{G}_m$ through ψ coincides with $F_1, \ldots, F_k$, that is,

$$\mathrm{ind}\big((F_i)_{1\leq i\leq k}; (G_i)_{1\leq i\leq k}\big) = \#\big\{\psi : [m] \to [n] : \big(G_i^\psi\big)_{1\leq i\leq k} = (F_i)_{1\leq i\leq k}\big\}.$$

For $G_1, \dots, G_k \in \mathcal{G}_\infty$, we define the *limiting density of* $(F_i)_{1\le i\le k}$ *in* $(G_i)_{1\le i\le k}$ by

$$t\big((F_i)_{1\le i\le k}; (G_i)_{1\le i\le k}\big) \\ := \lim_{n\to\infty} t\big((F_i)_{1\le i\le k}; (G_i|_{[n]})_{1\le i\le k}\big), \qquad \text{if it exists,} \tag{12}$$

where

$$t\big((F_i)_{1\le i\le k}; (G_i|_{[n]})_{1\le i\le k}\big) := \frac{\mathrm{ind}((F_i)_{1\le i\le k}; (G_i|_{[n]})_{1\le i\le k})}{n^{\downarrow m}}$$

is the natural extension of (4).

PROPOSITION 4.4. *Let $\mathbf{\Gamma} = (\Gamma_t)_{t\ge 0}$ be an exchangeable Markov process on $\mathcal{G}_\infty$. For every $0 \le s < t < \infty$ and $F, F' \in \mathcal{G}_m$, $m \in \mathbb{N}$,*

$$Q_{s,t}^{(m)}(F, F') := \lim_{n\to\infty} \frac{\mathrm{ind}((F, F'); (\Gamma_s|_{[n]}, \Gamma_t|_{[n]}))}{\mathrm{ind}(F; \Gamma_s|_{[n]})} \\ = \frac{t((F, F'); (\Gamma_s, \Gamma_t))}{t(F; \Gamma_s)} \tag{13}$$

exists whenever $t(F;\Gamma_s) > 0$. By specifying $Q_{s,t}^{(m)}(F, F') = \delta_F(F')$, the point mass at F, whenever $t(F;\Gamma_s) = 0$, $Q_{s,t}^{(m)}$ determines a transition probability on $\mathcal{G}_m$ for every $m \in \mathbb{N}$. Moreover, for $m \le n$, the transition probability measures $Q_{s,t}^{(m)}$ and $Q_{s,t}^{(n)}$ are consistent *in the sense that*

$$Q_{s,t}^{(m)}(F, F') = \sum_{F'' \in \mathcal{G}_n : F''|_{[m]} = F'} Q_{s,t}^{(n)}(F^*, F''), \tag{14}$$

for all $F, F' \in \mathcal{G}_m$ and $F^* \in \{F'' \in \mathcal{G}_n : F''|_{[m]} = F\}$.

PROOF. By Proposition 4.1, $\mathbf{\Gamma}[s, t]$ is a weakly exchangeable $\{0, 1\} \times \{0, 1\}$-valued array. Therefore, by the Aldous–Hoover theorem and extension of the Lovász–Szegedy theorem [13], Theorem 2.7, the limiting density $t((F, F'); (\Gamma_s, \Gamma_t))$ exists for all $F, F' \in \mathcal{G}_m$. As long as $t(F;\Gamma_s) > 0$, the bounded convergence theorem implies

$$t(F;\Gamma_s) = \lim_{n\to\infty} \frac{\mathrm{ind}(F; \Gamma_s|_{[n]})}{n^{\downarrow m}} \\ = \lim_{n\to\infty} \sum_{F' \in \mathcal{G}_m} \frac{\mathrm{ind}((F, F'); (\Gamma_s|_{[n]}, \Gamma_t|_{[n]}))}{n^{\downarrow m}} \\ = \sum_{F' \in \mathcal{G}_m} t\big((F, F'); (\Gamma_s, \Gamma_t)\big),$$

so that $Q_{s,t}^{(m)}(F, F')$ determines a transition probability on $\mathcal{G}_m$ for every $m \in \mathbb{N}$. Consistency follows by a similar argument to Corollary 2.6(ii). $\square$

REMARK 4.5. By the same argument, the definition of limiting pairwise density in (13) extends to a limiting density of any k-tuple of finite graphs. These limiting densities determine the finite-dimensional distributions of $\mathbf{\Gamma}$.

For each $s \le t$, the transition probabilities $(Q_{s,t}^{(n)})_{n\ge 1}$ in (13) can be arranged in an infinite by infinite array $Q_{s,t}$ whose rows and columns are indexed by $\mathcal{G}^*$. We define $Q_{s,t}$ by

$$Q_{s,t}(F,F') := \begin{cases} Q_{s,t}^{(n)}(F,F'), & F,F' \in \mathcal{G}_n \text{ for some } n \ge 1,\\ 0, & \text{otherwise.}\end{cases}$$

In this way, $Q_{s,t}$ is a concatenation of random transition probability matrices on the finite state spaces $(\mathcal{G}_n)_{n\ge 1}$. By ordering the rows and columns so that each element of $\mathcal{G}_m$ occurs before each element of $\mathcal{G}_n$, for all $m \le n$, $Q_{s,t}$ has a block diagonal structure, that is, $Q_{s,t}$ has the form

$$Q_{s,t} := \begin{pmatrix} Q_{s,t}^{(1)} & 0 & 0 & \cdots \\ 0 & Q_{s,t}^{(2)} & 0 & \cdots \\ 0 & 0 & Q_{s,t}^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15}$$

The analog to Corollary 2.6 is now apparent for the collection $(Q_{s,t}^{(n)})_{n\ge 1}$.

PROPOSITION 4.6. *Let $Q_{t,t+1} = Q := (Q(F,F'))_{F,F'\in\mathcal{G}^*}$ be the block-diagonal matrix obtained by the limiting pairwise frequencies of two infinite graphs as in* (13). *Then*:

(i) *the n-sector $Q^{(n)} := (Q(F,F'))_{F,F'\in\mathcal{G}_n}$ determines an exchangeable transition probability measure on $\mathcal{G}_n$ through*

$$\mathbb{P}\{\Gamma_{t+1} = F' \mid \Gamma_t = F\} = Q^{(n)}(F,F'), \qquad F,F' \in \mathcal{G}_n, \quad \textit{and}$$

(ii) *the collection $(Q^{(n)})_{n\ge 1}$ is a family of exchangeable transition probability measures with the projective Markov property* (14).

4.2. *Discrete-time processes and weakly exchangeable arrays.* The following definition of dissociated array follows Aldous [1].

DEFINITION 4.7 (Dissociated random array). A random array $X := (X^{ij})_{i,j\ge 1}$ is *dissociated* if

$$(X^{ij})_{1\le i,j\le n} \quad \text{and} \quad (X^{ij})_{i,j\ge n+1} \qquad \text{are independent for every } n \ge 1. \tag{16}$$

By the Aldous–Hoover theorem, dissociated arrays are extreme in the space of weakly exchangeable arrays, in the same way that independent, identically distributed sequences are extreme in the space of exchangeable sequences; cf. de Finetti's theorem. In general, the law of any weakly exchangeable array can be expressed as a mixture of dissociated weakly exchangeable arrays.

PROPOSITION 4.8. *Let $\mathbf{\Gamma}$ be a time-homogeneous Markov process and let T' be a finite set of times. Conditional on $\mathcal{E}_{T'}$, $\mathbf{\Gamma}[T']$ is a dissociated weakly exchangeable array. Moreover, for each $n \in \mathbb{N}$ the restriction $\mathbf{\Gamma}_{T'}^{[n]}$ of $\mathbf{\Gamma}_{T'}$ to $\mathcal{G}_n$ is a conditionally time-inhomogeneous Markov chain with transition probabilities $Q_{s,t}^{(n)}$ as in (13).*

PROOF. Given $\mathcal{E}_{T'}$, $\mathbf{\Gamma}[T']$ is conditionally dissociated by Proposition 4.1 and [1], Proposition 14.6. We need only show that $\mathbf{\Gamma}_{T'}^{[n]} := (\Gamma_t^{[n]})_{t\in T'}$ is a conditionally time-inhomogeneous Markov chain with the appropriate transition probabilities.

We write $T' := \{0 \le s_0 < s_1 < \cdots < s_k < \infty\}$. By Proposition 4.1, $\mathbf{\Gamma}[T']$ is weakly exchangeable. The Aldous–Hoover theorem and the strong law of large numbers imply that the limiting density (12) exists almost surely for all collections $(F_t)_{t\in T'}$ in $\mathcal{G}_m$, $m \ge 1$. We need to show that these limiting densities satisfy

$$\lim_{n\to\infty} t\big((F_t)_{t\in T'}; \Gamma_{T'}^{[n]}\big) = P\{\Gamma_{s_0}|_{[n]} = F_{s_0} | \mathcal{E}_{s_0}\} \prod_{l=1}^{k} Q_{s_{l-1},s_l}^{(n)}(F_{s_{l-1}}, F_{s_l});$$

that is, we must show that the limiting density of F_{s_r} in Γ_{s_r} depends only on the limiting density of $\Gamma_{s_{r-1}}$.

To show this, we generate $\mathbf{\Gamma}_{T'}^{[n]}$ as follows. For each $1 \le r \le n-1$, the Markov property implies that the conditional law of $\Gamma_{s_{r+1}}^{[n]}$ given $\sigma\langle \Gamma_{s_j}, 0 \le j \le r\rangle$ is a measurable function of $|\Gamma_{s_r}|$. Moreover, by Proposition 4.4, $Q_{s_r,s_{r+1}}^{(n)}$ is a random transition probability on $\mathcal{G}_n$ whose distribution is a measurable function of $\mathcal{E}_{s_r}$. Therefore, given $\sigma\langle \Gamma_{s_j}, 0 \le j \le r\rangle$, we generate $\Gamma_{s_{r+1}}$ recursively by determining $\Gamma_{s_{r+1}}|_{[n+1]}$, for $n \in \mathbb{N}$, from the conditional law

$$\begin{aligned} &\mathbb{P}\{\Gamma_{s_{r+1}}|_{[n+1]} = G^* | \Gamma_{s_{r+1}}|_{[n]} = G, \Gamma_{s_r}|_{[n+1]} = G'\} \\ &\quad = \frac{Q_{s_r,s_{r+1}}^{(n+1)}(G', G^*)}{Q_{s_r,s_{r+1}}^{(n)}(G'|_{[n]}, G)}, \end{aligned} \tag{17}$$

where G^* is chosen from $\{G'' \in \mathcal{G}_{n+1} : G''|_{[n]} = G\}$. By the strong law of large numbers, $Q_{s_r,s_{r+1}}$ in (13) is identical to (17) for all pairs $(G'|_{[n]}, G)$ with $Q_{s_r,s_{r+1}}^{(n)}(G'|_{[n]}, G) > 0$. Moreover, $Q_{s_r,s_{r+1}}$ is measurable with respect to $\mathcal{E}_{(s_r,s_{r+1})} \subset \mathcal{E}_{T'}$. We conclude that, given $\mathcal{E}_{T'}$, $\mathbf{\Gamma}_{T'}^{[n]}$ is a conditionally time-inhomogeneous Markov chain with transition probabilities (13). $\square$

THEOREM 4.9. *Let $\mathbf{\Gamma} := (\Gamma_t)_{t\in\mathbb{Z}_+}$ be a discrete-time exchangeable Markov chain on $\mathcal{G}_\infty$ with projection $\mathbf{D} := (D_t)_{t\in\mathbb{Z}_+}$ into the space of graph limits, that is, $D_t := |\Gamma_t|$ for every $t \in \mathbb{Z}_+$. Then the conditional distribution of $Q_{t,t+1}$ in (15), given $\sigma\langle D_s, Q_{s,s+1}\rangle_{0\le s\le t-1}$, is a measurable function of D_t such that*

$$D_{t+1} =_{\mathcal{L}} D_t Q_{t,t+1} \tag{18}$$

and, for each $m \in \mathbb{N}$, *the conditional law of* $\Gamma_{t+1}|_{[m]}$ *given* $\sigma\langle\Gamma_s, 0 \le s \le t\rangle$ *is the* m*-sector* $Q^{(m)}_{t,t+1}$. *Conversely, for any measurable mapping* π *from* $\mathcal{D}^*$ *into the space of transition probability measures on* $\mathcal{G}_\infty$ *and any initial point* $D \in \mathcal{D}^*$, *there is a unique Markov chain* $(\Gamma_t)_{t\in\mathbb{Z}_+}$ *on* $\mathcal{G}_\infty$ *whose transition probability measure is governed by* π, *whose initial state has distribution* γ_D, *and whose projection into* $\mathcal{D}^*$ *satisfies* (18).

PROOF. The first half follows from Propositions 4.2, 4.4, 4.8 and the law of large numbers. The converse follows immediately, as we can explicitly construct a Markov chain on $\mathcal{G}_\infty$ that projects to $\mathbf{D}$ almost surely. $\square$

5. Continuous-time processes. The Aldous–Hoover theorem implies that Γ_t possesses a graph limit $|\Gamma_t|$ almost surely for any fixed t. In this section, we show that the graph limit $|\Gamma_t|$ exists simultaneously for all $t \ge 0$ with probability one. Moreover, if $\mathbf{\Gamma}$ is a Markov process, then so is its projection $|\mathbf{\Gamma}|$ into $\mathcal{D}^*$. To show these properties, we use weak exchangeability of $\mathbf{\Gamma}[T]$ from Proposition 4.1.

REMARK 5.1. Since $\mathbb{R}_+$ is covered by countably many intervals of unit length, it is sufficient to show that these properties hold with probability one for $\Gamma[T']$, where $T' = [0, 1]$.

The key to our study of $|\mathbf{\Gamma}|$ at *all time points* is the following description of $\mathbf{\Gamma}_{[0,1]}$ in terms of an array taking values in a Polish space $\mathcal{I}^*$. As we only consider undirected graphs, each edge trajectory $\mathbf{\Gamma}^{ij}_{[0,1]}$, $i \ne j$, is an alternating collection of 0s and 1s, called an *on-off cycle*. Formally, an *on-off cycle* y is a partition of $[0, 1]$ into finitely many non-overlapping intervals J_i along with an initial status $y_0 \in \{0, 1\}$. The starting condition y_0 is enough to determine the entire path $\{y_t\}_{t\in[0,1]}$, since the on-off cycle alternates between being on ($y_t = 1$) and off ($y_t = 0$) in successive subintervals J_i. We denote the space of on-off cycles by $\mathcal{I}$, which is a subset of the Skorokhod space of càdlàg functions $[0, 1] \to \{0, 1\}$. By letting $\mathcal{I}^*$ denote the closure of $\mathcal{I}$ in the Skorokhod space, we can partition $\mathcal{I}^* := \bigcup_{m\in\mathbb{N}} \mathcal{I}^*_m$, where $\mathcal{I}^*_m$ is the closure of on-off cycles with exactly m subintervals. Consequently, $\mathcal{I}^*$ is complete, separable and Polish. The fact that $\mathcal{I}^*$ is Polish allows us to apply the Aldous–Hoover theorem to characterize the behavior of $\mathbf{\Gamma}$ at an uncountable set of times.

PROPOSITION 5.2. *Let* $\mathbf{\Gamma}$ *be an exchangeable process on* $\mathcal{G}_\infty$ *with càdlàg paths. Then the collection* $(\Gamma^{ij}_{[0,1]})_{i,j\ge1}$ *of on-off cycles induced by* $\mathbf{\Gamma}$ *is a weakly exchangeable* $\mathcal{I}^*$*-valued array. Consequently, there is a random probability measure* Ψ *on* $\mathcal{I}^*$ *that generates the exchangeable* σ*-field* $\mathcal{E}_{[0,1]}$ *of* $(\Gamma^{ij}_{[0,1]})_{i,j\ge1}$ *and, conditional on* Ψ, $(\Gamma^{ij}_{[0,1]})_{i,j\ge1}$ *is distributed the same as* $(Y^{ij})_{i,j\ge1}$, *where*

$$Y^{ij} := F(U_i, U_j, V_{\{i,j\}}), \qquad i, j \ge 1,$$

for a measurable function $F : [0,1]^3 \to \mathcal{I}^*$ *satisfying* $F(a,b,\cdot) = F(b,a,\cdot)$ *and a collection* $\{(U_i)_{i\geq 1}; (V_{\{i,j\}})_{i>j\geq 1}\}$ *of independent, identically distributed Uniform random variables on* $[0,1]$.

PROOF. By assumption, each $\Gamma^{ij}_{[0,1]}$ is càdlàg and so $\Gamma^{ij}_{[0,1]} \in \mathcal{I} \subset \mathcal{I}^*$ for each $i, j \in \mathbb{N}$. Weak exchangeability of $(\Gamma^{ij}_{[0,1]})_{i,j\geq 1}$ follows from Proposition 4.1, because $\sigma\langle\Gamma^{ij}_{\mathbb{Q}\cap[0,1]}\rangle_{i,j\geq 1}$ generates the Borel σ-field on $\mathcal{I}$ and exchangeability does not rely on the Markov assumption. As $\mathcal{I}^*$ is Polish, the rest follows from the Aldous–Hoover theorem. □

COROLLARY 5.3. *If* $(\Gamma_t)_{t\geq 0}$ *is an exchangeable Markov process on* $\mathcal{G}_\infty$ *with càdlàg sample paths, then* $(\Gamma^{ij}_{[0,1]})_{i,j\geq 1}$ *is conditionally dissociated given* $\mathcal{E}_{[0,1]}$ *and each restriction* $(\Gamma^{ij}_{[0,1]})_{1\leq i,j\leq n}$ *has the same conditional law as the path of an inhomogeneous, continuous-time Markov chain on* $\mathcal{G}_n$ *with transition probabilities* $Q^{(n)}_{s,t}$ *defined in* (13); *that is, for all* $0 \leq s < t \leq 1$,

$$P\{\Gamma_t|_{[n]} = G' | \mathcal{E}_{[0,1]} \vee \sigma\langle\Gamma_r|_{[n]}\rangle_{0\leq r\leq s}, \Gamma_s|_{[n]} = G\} = Q^{(n)}_{s,t}(G, G'). \tag{19}$$

PROOF. By Propositions 4.8 and 5.2, $(\Gamma^{ij}_{[0,1]})_{i,j\geq 1}$ is conditionally dissociated given $\mathcal{E}_{[0,1]}$. That the finite restrictions of $(\Gamma^{ij}_{[0,1]})_{i,j\geq 1}$ behave like inhomogeneous continuous-time Markov chains with transition probabilities (19) follows routinely from Propositions 4.6 and 4.8, because $\mathcal{E}_{[0,1]}$ is generated by $\bigvee_{i=1}^{\infty} \mathcal{E}_{\mathcal{D}_n}$, where $\mathcal{D}_n$ is the set of nth-level dyadic rationals $m/2^n$ in $[0,1]$. □

5.1. *Existence of graph limits*. We now show that the projection $|\mathbf{\Gamma}|$ into $\mathcal{D}^*$ exists almost surely and that the Markov and càdlàg paths properties for $\mathbf{\Gamma}$ imply the same for $|\mathbf{\Gamma}|$.

REMARK 5.4. Since Y is an $\mathcal{I}^*$-valued array, we define $|Y|$ in the same way as for $\mathcal{G}_\infty$-valued processes $\mathbf{\Gamma}$. That is, for every $i > j \geq 1$, we regard Y^{ij} as an on-off cycle and we treat Y as a process $Y := (Y_t)_{t\in[0,1]}$ in $\mathcal{G}_\infty$. The limit $|Y|$ corresponds to $(|Y_t|)_{t\in[0,1]}$, with each Y_t regarded as an element of $\mathcal{G}_\infty$.

THEOREM 5.5. *Let* $Y := (Y^{ij})_{i,j\geq 1}$ *be a dissociated weakly exchangeable random array in* $\mathcal{I}^*$. *Then, with probability one, for every* $t \in [0,1]$, $(Y^{ij}_t)_{i,j\geq 1}$ *determines a deterministic graph limit, that is, for every* $F \in \mathcal{G}_m$, $m \in \mathbb{N}$, *and* $t \in [0,1]$,

$$\lim_{n\to\infty} t(F; Y^{[n]}_t) \qquad \textit{exists}, \tag{20}$$

where $t(F; Y^{[n]}_t)$ *has the same definition as in* (5) *by regarding* Y_t *as the adjacency array of an infinite graph.*

PROOF. For any $\varepsilon > 0$ and $F \in \mathcal{G}^*$, we show that the limits inferior and superior of the sequence of densities $(t(F; Y_t^{[n]}))_{n\in\mathbb{N}}$ are within ε of one another. To this end, we fix $F \in \mathcal{G}_m$, $m \in \mathbb{N}$, and for each $t \in [0, 1]$ we define

$$\delta_t^+(F) := \limsup_{n\to\infty} t\big(F; Y_t^{[n]}\big) \quad \text{and}$$
$$\delta_t^-(F) := \liminf_{n\to\infty} t\big(F; Y_t^{[n]}\big).$$

By Proposition 4.2, $\delta_t^+(F) = \delta_t^-(F)$ for all fixed $t \in [0, 1]$, but we wish to show that $\delta_t^+(F) = \delta_t^-(F)$ almost surely for all $t \in [0, 1]$ simultaneously, that is,

$$\sup_{t\in[0,1]} \big|\delta_t^+(F) - \delta_t^-(F)\big| = 0.$$

By assumption, Y^{ij} is an on-off cycle for every $i, j \geq 1$, so that the path $Y_{[0,1]}^{ij} = (Y_t^{ij})_{t\in[0,1]}$ in $\{0, 1\}$ has finitely many discontinuities almost surely and so must $Y_{[0,1]}^{[n]} := (Y_{[0,1]}^{ij})_{1\leq i,j\leq n}$, for every $n \in \mathbb{N}$. For every $\varepsilon > 0$, there is a finite subset $S_\varepsilon \subset [0, 1]$ and an at most countable partition $J_1, J_2, \ldots$ of the open set $[0, 1] \setminus S_\varepsilon$ such that

$$P\big\{Y_{[0,1]}^{ij} \text{ is discontinuous at } s \in S_\varepsilon\big\} \geq \varepsilon \quad \text{and}$$
$$P\big\{Y_{[0,1]}^{ij} \text{ is discontinuous in } J_l\big\} < \varepsilon, \qquad l = 1, 2, \ldots.$$

The existence of such a partition is guaranteed by the strong law of large numbers since, if such a partition did not exist, then there must be a sequence of intervals $(t - \rho_n, t + \rho_n)$ with $\rho_n \to 0$ that converges to $t \notin S_\varepsilon$ such that

$$P\big\{Y_{[0,1]}^{ij} \text{ is discontinuous in } (t - \rho_n, t + \rho_n)\big\} \geq \varepsilon \qquad \text{for every } n \geq 1.$$

Continuity from above implies

$$P\big\{Y_{[0,1]}^{ij} \text{ is discontinuous at } t \notin S_\varepsilon\big\} \geq \varepsilon > 0,$$

which contradicts the assumption $t \notin S_\varepsilon$.

The strong law of large numbers also implies that

$$\lim_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \mathbf{1}\big\{Y^{ij} \text{ is discontinuous in } J_l\big\} < \varepsilon$$

for each sub-interval J_l, $l = 1, 2, \ldots$. Thus, $\delta_t^+(F)$ and $\delta_t^-(F)$ cannot vary by more than ε over J_l and, since $\delta_t^+(F) = \delta_t^-(F)$ almost surely for each endpoint of J_l,

$$\sup_{t\in J_l} \big|\delta_t^+(F) - \delta_t^-(F)\big| \leq 2\varepsilon$$

for all $l = 1, 2, \ldots$ with probability one. Since $[0, 1]$ is covered by at most countably many subintervals J_l, $l = 1, 2, \ldots,$ and the nonrandom set $S := \bigcup_{\varepsilon>0} S_\varepsilon$, it follows that

$$\sup_{t\in[0,1]} \left|\delta_t^+(F) - \delta_t^-(F)\right| \leq 2\varepsilon \qquad \text{a.s.}$$

for every $\varepsilon > 0$ and every $F \in \mathcal{G}^*$. Thus, the limit (20) exists almost surely for every $F \in \bigcup_{m\in\mathbb{N}} \mathcal{G}_m$. Countable additivity of probability measures implies that the limit (20) exists almost surely for all $F \in \bigcup_{m\in\mathbb{N}} \mathcal{G}_m$ and, thus, $Y_{[0,1]}$ determines a process on $\mathcal{D}^*$ almost surely.

The fact that these limits are deterministic follows from the 0–1 law and our assumption that Y is dissociated: the homomorphism densities of any Γ_t depend only on the tail σ-field generated by $(\Gamma_t|_{\mathbb{N}\setminus[n]})_{n\geq 1}$. Almost sure existence of the limit follows by a straightforward martingale argument, as the sequence $(t(F; \Gamma_t|_{[n]}))_{n\geq 1}$ is a reverse martingale for every fixed $F \in \mathcal{G}^*$. $\square$

5.2. *Characterization of discontinuities.* We restate Theorem 3.6 for the reader's benefit.

Theorem 3.6. An exchangeable $\mathcal{G}_\infty$-valued Markov process $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ with càdlàg paths has at most countably many discontinuities almost surely. We can classify these discontinuities into three types: if $\mathbf{\Gamma}$ is discontinuous at $t = s$, then either:

(A) $P(\mathbf{\Gamma}^{ij}$ is discontinuous at $s|\mathcal{E}_{[0,\infty)}) > 0$ for every $i \neq j \in \mathbb{N}$ or

(B) there exists a unique $i \in \mathbb{N}$ such that either:

  (B-1) $\mathbf{\Gamma}^{ij}$ is discontinuous at s for some unique $j \neq i$ or

  (B-2) there exist constants $0 \leq p_0, p_1 \leq 1$ (possibly depending on $\mathcal{E}_{[0,\infty)}$) such that

$$P\{\mathbf{\Gamma}^{ij} \text{ is discontinuous at } s|\mathcal{E}_{[0,\infty)}\} = p_k \qquad \text{on the event } \Gamma_{s-}^{ij} = k,$$

  for all $j \neq i$;

  and $P\{\mathbf{\Gamma}^{i'j'}$ is discontinuous at $s|\mathcal{E}_{[0,\infty)}\} = 0$ if $i \notin \{i', j'\}$.

PROOF. By Theorem 2.4 and Proposition 5.2, it is enough to prove the above statement for a weakly exchangeable dissociated $\mathcal{I}^*$-valued array $Y := (Y^{ij})_{i,j\geq 1}$.

For $i > j \geq 1$, let S^{ij} denote the set of all $s \in [0, 1]$ such that

$$P\{Y^{ij} \text{ is discontinuous at } t = s\} > 0.$$

In Theorem 5.5, we showed that

$$S_\varepsilon^{ij} := \{s \in [0, 1] : P\{Y_{[0,1]}^{ij} \text{ is discontinuous at } s\} \geq \varepsilon\}$$

is a finite set and $S^{ij} := \bigcup_{\varepsilon>0} S_\varepsilon^{ij}$ is at most countable. We break down our argument into cases, depending on whether a discontinuity occurs inside or outside of $S := \bigcup_{i,j\geq 1} S^{ij}$.

CASE A: $s \in S$. For any fixed $s \in S$, the strong law of large numbers and weak exchangeability of Y imply that

$$\lim_{n\to\infty} \frac{\operatorname{ind}((F, F'); (Y_{s-}^{[n]}, Y_s^{[n]}))}{n^{\downarrow m}} = P\{Y_{s-}^{[n]} = F \text{ and } Y_s^{[n]} = F'\}$$

for every $F, F' \in \mathcal{G}_m$, $m \in \mathbb{N}$. Consequently, given $\mathcal{E}_{[0,1]}$, there is a measurable function that determines how Y behaves at s, for each $s \in S$. By weak exchangeability, either zero edges change or a positive fraction does. Since a positive fraction of edges can change, it follows that the projection of Y into $\mathcal{D}^*$ can have a discontinuity at time s as well.

CASE B: $s \notin S$. The next case involves a discontinuity of Y at $s \notin S$. First of all, if Y is discontinuous at $s \notin S$, then the overall proportion of pairs ij for which $Y_{[0,1]}^{ij}$ is discontinuous at s must be zero; otherwise, the strong law of large numbers implies

$$\begin{aligned} &P\{Y_{[0,1]}^{12} \text{ is discontinuous at } s \notin S\} \\ &\quad = \lim_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \mathbf{1}\{Y_{[0,1]}^{ij} \text{ is discontinuous at s}\} > 0, \end{aligned}$$

contradicting our assumption that $s \notin S$.

On the event that the proportion of pairs is 0, the total number of pairs that are discontinuous can be (B-1) finite or (B-2) infinite. Treating case (B-2) first, we define D^{ij} to be the set of discontinuity times of $Y_{[0,1]}^{ij}$, for any $i, j \in \mathbb{N}$. For distinct vertices $i, i', j, j' \in \mathbb{N}$, we must have

$$\begin{aligned} &P\{Y_{[0,1]}^{ij} \text{ and } Y_{[0,1]}^{i'j'} \text{ discontinuous at } t \in D^{ij} \setminus S | D^{ij}\} \\ &\quad \le P\{Y_{[0,1]}^{i'j'} \text{ discontinuous at } s \notin S\} = 0, \end{aligned}$$

by our definition of S. Furthermore, for all distinct pairs i, i', j, j', the unconditional probability that $Y_{[0,1]}^{ij}$ and $Y_{[0,1]}^{i'j'}$ have a common point of discontinuity outside S is 0. Since there are countably many disjoint pairs ij and $i'j'$, it easily follows that two disjoint edges (i.e., $\{i, j\} \cap \{i', j'\} = \varnothing$) cannot have a common discontinuity outside S.

The above argument forbids disjoint pairs ij and $i'j'$ from having a common discontinuity outside of S, but it does not restrict how a set of edges incident a given vertex $i \in \mathbb{N}$ can behave. In the restriction $\Gamma_t^{[n]}$, there are $n - 1$ edges/nonedges incident to each vertex $i = 1, \dots, n$; thus, the previous argument does not rule out any of the possible behaviors including a change of *all* edges incident vertex i, because

$$0 \le \lim_{n\to\infty} 2n^{-1}(n-1)^{-1} \sum_{1\le j\le n} \mathbf{1}\{\Gamma_{t-}^{ij} \ne \Gamma_t^{ij}\} \le \lim_{n\to\infty} 2n^{-1} = 0.$$

Our claim, however, is that either a single edge changes or a positive fraction of edges incident some fixed $i \in \mathbb{N}$ changes at any discontinuity time $s \notin S$. To show this, we define D^{ij} to be the set of discontinuity times of edge ij, $j \neq i$, and $D^i := \bigcup_{j\neq i} D^{ij}$ to be the set of discontinuities of all edges incident vertex i. For fixed $j \neq j' \neq i$, we have

$$\begin{aligned}\mathbb{P}\{&\Gamma^{ij}_{[0,1]} \text{ and } \Gamma^{ij'}_{[0,1]} \text{ both discontinuous at } t \in D^{ij} \setminus S | D^{ij}, \mathcal{E}_{[0,1]}\} \\ &= P\{\Gamma^{ij'}_{[0,1]} \text{ discontinuous at } t \in D^{ij} \setminus S | D^{ij}, \mathcal{E}_{[0,1]}\} \\ &= p^t_{jj'}.\end{aligned}$$

Either $p^t_{jj'} = 0$ or $p^t_{jj'} > 0$. If $p^t_{jj'} > 0$, then, by exchangeability,

$$P\{\Gamma^{ij}_{[0,1]} \text{ and } \Gamma^{ij''}_{[0,1]} \text{ discontinuous at } t \in D^{ij} \setminus S | D^{ij}, \mathcal{E}_{[0,1]}\} = p^t_{jj'} > 0,$$

for all $j'' \neq i$ with $\Gamma^{ij'}_{t-} = \Gamma^{ij''}_{i-}$. By the strong law of large numbers, a positive proportion of edges incident i is discontinuous at $t \notin S$. On the other hand, if $p^t_{jj'} = 0$, then the conditional probability that $\Gamma^{ij}_{[0,1]}$ and $\Gamma^{ij''}_{[0,1]}$ have a shared discontinuity at $t \in D^{ij} \setminus S$, given D^{ij} and $\mathcal{E}_{[0,1]}$, is zero for all $j'' \neq i$. Since the set of $j'' \neq i$ is countable, edge ij is the only discontinuity at $t \in D^{ij}$.

Cases (A) and (B) exhaust all possibilities. The proof is complete. $\square$

5.3. *Bounded variation of sample paths in $\mathcal{D}^*$.* Let $\mathbf{\Gamma} := (\Gamma_t)_{t\geq 0}$ be an exchangeable Markov process on $\mathcal{G}_\infty$ with càdlàg sample paths and let $\mathbf{D} := (D_t)_{t\geq 0}$ be its projection into $\mathcal{D}^*$. By Proposition 4.4, the limiting density $Q^{(m)}_{s,t}(F, F')$ exists for all $F, F' \in \mathcal{G}_m$ and all $0 \leq s \leq t < \infty$, and $D_t \in \mathcal{D}^*$ determines a unique probability measure on $\mathcal{G}_\infty$ for every fixed $t \geq 0$.

With $\mathcal{B}$ denoting the Borel σ-algebra on $\mathcal{G}_\infty$, we define the *variation* of $\mathbf{D}$ between times s and t by

$$V_{s,t} = \|D_s - D_t\|_{\mathrm{TV}} := \sup_{A\in\mathcal{B}} \big|D_s(A) - D_t(A)\big|. \tag{21}$$

For each $n \geq 1$, we write $D^{(n)}_t$ to denote the measure D_t induces on $\mathcal{G}_n$ by restriction. Because $\mathcal{G}_n$ is a finite state space, we can define the *variation* of $D^{(n)}$ between s and t by

$$V^{(n)}_{s,t} = \|D^{(n)}_s - D^{(n)}_t\|_{\mathrm{TV}} := \frac{1}{2} \sum_{G\in\mathcal{G}_n} \big|D^{(n)}_s(G) - D^{(n)}_t(G)\big|.$$

Conditional on $\mathcal{E}_{[0,1]}$, $Q_{s,t}$ is a transition probability measure on $\mathcal{G}_\infty$, which we can regard as an array for which only a finite number of entries in each row and column are nonzero, making the product $Q_{s,t} \circ Q_{t,u}$ well-defined and finite for $s \leq$

$t \le u$. Furthermore, $\{Q_{s,t}\}_{0 \le s \le t}$ determines a collection of transition probabilities for which

$$Q_{s,t} \circ Q_{t,u} = Q_{s,u} \quad \text{a.s.} \qquad \text{for all } s \le t \le u, \tag{22}$$

cf. the Chapman–Kolmogorov theorem for Markov chains. It follows that $\mathbf{D}$ satisfies

$$D_s Q_{s,t} = D_t \qquad \text{a.s.} \tag{23}$$

for all $0 \le s \le t$.

We define the *density transfer* $T_{s,t}$ of $\mathbf{D}$ between times s and t by

$$T_{s,t} := \int_{\mathcal{G}_\infty} Q_{s,t}(G, \mathcal{G}_\infty \setminus \{G\}) D_s(dG). \tag{24}$$

For each of the component processes $(D_t^{(n)})_{t \ge 0}$, the density transfer is

$$T_{s,t}^{(n)} := \sum_{F, F' \in \mathcal{G}_n : F' \ne F} D_s^{(n)}(F) Q_{s,t}^{(n)}(F, F'). \tag{25}$$

Note that the density transfer quantifies how much mass moves between times s and t.

PROPOSITION 5.6. *For every $n \ge 1$ and all $s \le u \le t$, the density transfer and variation of* $\mathbf{D}$ *satisfy*

$$V_{s,t}^{(n)} \le T_{s,t}^{(n)} \le T_{s,u}^{(n)} + T_{u,t}^{(n)}. \tag{26}$$

The analogous inequalities hold for the variation of the infinite process $\mathbf{D}$.

PROOF. For fixed $n \in \mathbb{N}$ and $F \in \mathcal{G}_n$, we write $D_s(F) := D_s^{(n)}(F) := t(F, \Gamma_s)$. Then

$$D_t(F) = \sum_{F' \in \mathcal{G}_n} D_s(F') Q_{s,t}^{(n)}(F', F);$$

whence,

$$\begin{aligned}
&\sum_{F \in \mathcal{G}_n} \bigl|D_s(F) - D_t(F)\bigr| \\
&\quad = \sum_{F \in \mathcal{G}_n} \left| D_s(F) - \sum_{F' \in \mathcal{G}_n} D_s(F') Q_{s,t}^{(n)}(F', F) \right| \\
&\quad = \sum_{F \in \mathcal{G}_n} \left| \sum_{F' \in \mathcal{G}_n} D_s(F) Q_{s,t}^{(n)}(F, F') - D_s(F') Q_{s,t}^{(n)}(F', F) \right| \\
&\quad \le \sum_{F \in \mathcal{G}_n} \sum_{F' \in \mathcal{G}_n : F' \ne F} \bigl| D_s(F) Q_{s,t}^{(n)}(F, F') - D_s(F') Q_{s,t}^{(n)}(F', F) \bigr| \\
&\quad \le 2 \sum_{F, F' \in \mathcal{G}_n : F \ne F'} D_s(F) Q_{s,t}^{(n)}(F, F').
\end{aligned}$$

The second inequality in (26) follows from Chapman–Kolmogorov. □

We write $T_{[a,b]} := (T_{s,t})_{a\le s\le t\le b}$ to be the *section* of the density flow restricted to the interval $[a,b] \subseteq [0,1]$. We then define the *total variation* of $T_{[a,b]}$ by

$$\|T_{[a,b]}\|_{\mathrm{TV}} := \sup \sum_{i=0}^{n-1} T_{s_i,s_{i+1}},$$

where the supremum is taken over all partitions of $[a,b]$ into finitely many subintervals with endpoints $a = s_0 < s_1 < \cdots < s_K = b$. Similarly, we define the *total variation* of $\mathbf{D}$ on $[a,b] \subseteq [0,1]$, respectively, $\mathbf{D}^{(n)}$, for $n \ge 1$, by

$$\|\mathbf{D}_{[a,b]}\|_{\mathrm{TV}} := \sup \sum_{j=0}^{K-1} V_{s_j,s_{j+1}},$$

respectively,

$$\big\|\mathbf{D}^{(n)}_{[a,b]}\big\|_{\mathrm{TV}} := \sup \sum_{j=0}^{K-1} V^{(n)}_{s_j,s_{j+1}},$$

where the supremum is taken over all partitions of $[a,b]$ into finitely many subintervals with endpoints $a \le s_0 < s_1 < \cdots < s_{K-1} < s_K = b$.

DEFINITION 5.7 (Locally bounded variation). For $n \ge 1$, we say the n-sector $\mathbf{D}^{(n)} := (D^{(n)}_t)_{t\in[0,1]}$ has *locally bounded variation* if

$$\big\|\mathbf{D}^{(n)}_{[a,b]}\big\|_{\mathrm{TV}} < \infty \qquad \text{for all } 0 \le a < b \le 1. \tag{27}$$

A path $\mathbf{D} := (D_t)_{t\in[0,1]}$ in $\mathcal{D}^*$ has *finite-dimensional locally bounded variation* if (27) holds for each of its n-sectors.

THEOREM 5.8. *For each $n \ge 1$, the density flow $(T^{(n)}_{s,t})_{0\le s\le t}$ has locally bounded variation almost surely. That is, with probability one, for all $0 \le a < b < \infty$,*

$$\big\|T^{(n)}_{[a,b]}\big\|_{\mathrm{TV}} < \infty \qquad \text{a.s.} \tag{28}$$

Before proving Theorem 5.8, we sketch the intuition. Under the assumption of càdlàg sample paths, the finite space sample paths of $\mathbf{\Gamma}$ can jump only a finite number of times in any bounded time interval. On the other hand, the projection of $\mathbf{\Gamma}$ into the space of graph limits has infinite variation if and only if there is a constant flow of mass into and out of every possible finite state. These two events are incompatible: a constant flow of mass in the space of graph limits forces the finite space paths of $\mathbf{\Gamma}$ to jump infinitely often with positive probability.

The following technical argument can obscure the above intuition. To reduce further technicalities, we prove the theorem in the special case when $\mathbf{\Gamma}$ is time-homogeneous, which allows us to restrict attention to its behavior on the unit interval.

PROOF OF THEOREM 5.8. Under the assumption that the covering process $\mathbf{\Gamma}$ is time-homogeneous, it is sufficient to show that (28) holds on $[0, 1]$, from which the case for arbitrary bounded intervals follows by scaling and shifting.

To summarize the argument: if (28) fails, then there must be a positive probability that the restriction $\mathbf{\Gamma}^{[n]}_{[0,1]}$ of $\mathbf{\Gamma}_{[0,1]}$ to $\mathcal{G}_n$ does not have càdlàg sample paths. Since we have endowed $\mathcal{G}_\infty$ with the product-discrete topology, the covering process $\mathbf{\Gamma}$ has càdlàg paths if and only if each of its component processes $\mathbf{\Gamma}^{[n]}$ does. In order for (28) to fail, there must be a constant flow of density in the component process $\mathbf{D}^{(n)}$, which implies that $\mathbf{\Gamma}^{[n]}_{[0,1]}$ has at least M discontinuities on $[0, 1]$ for every $M \geq 1$, contradicting the assumption that $\mathbf{\Gamma}$ has càdlàg paths.

To begin, we fix $n \geq 1$ and assume that there is a positive probability that $\|T^{(n)}_{[0,1]}\|_{\mathrm{TV}} = \infty$, that is, for every $L \geq 1$, there is positive probability that $\|T^{(n)}_{[0,1]}\|_{\mathrm{TV}} \geq L$. Thus, there must exist a partition of $[0, 1]$ into subintervals $J_1, \dots, J_K$ so that the total variation on each subinterval exceeds L. In particular, for $M \geq 1$ and $L = M^2 2^{6\binom{n}{2}}$, there is a sequence of times $0 = s_0 < s_1 < \cdots < s_K = 1$ so that

$$\sum_{i=0}^{K-1} T^{(n)}_{s_i, s_{i+1}} \geq M^2 2^{6\binom{n}{2}}.$$

We can, therefore, specify a collection $J_1, \dots, J_{M2^{3\binom{n}{2}}}$ of $M2^{3\binom{n}{2}}$ subintervals so that the total variation on each subinterval J_l exceeds $M2^{3\binom{n}{2}}$. Each J_l admits a further subpartition $J_{l1}, \dots, J_{l2^{\binom{n}{2}}}$ so that the total variation within each J_{li} exceeds $M2^{2\binom{n}{2}}$. We denote the endpoints of each subinterval J_l by $s_l < t_l$. For each $l = 1, \dots, M2^{3\binom{n}{2}}$, there must be at least one pair (F_l, F'_l), $F_l \neq F'_l$, for which

$$\sum_{j=1}^{2^{\binom{n}{2}}} D^{(n)}_{s_{lj}}(F_l) Q^{(n)}_{s_{lj}, t_{lj}}(F_l, F'_l) \geq M. \tag{29}$$

As there are $M2^{3\binom{n}{2}}$ subintervals J_l, each with at least one pair (F_l, F'_l) for which (29) holds, there must be some pair (F, F'), $F \neq F'$, that occurs as (F_l, F'_l) at least $M2^{\binom{n}{2}}$ times. We write $s^*_1 < s^*_2 < \cdots s^*_{M2^{\binom{n}{2}}}$ and $t^*_1 < \cdots < t^*_{M2^{\binom{n}{2}}}$ to be the left and right endpoints of the respective subintervals. For each $i = 1, \dots, M2^{\binom{n}{2}}$, there must be some state, or set of states, F_{0i} for which $D^{(n)}_0(F_{0i}) > 0$ and

$Q^{(n)}_{0,s^*_i}(F_{0i}, F) > 0$; thus,

$$
\begin{aligned}
\mathbb{P}\{\mathbf{\Gamma}^{[n]} &\text{ has no discontinuity in } [s^*_i, t^*_i] | \mathcal{E}_{[0,1]}, \Gamma^{[n]}_0 = F_{0i}\} \\
&\leq \prod_{j=1}^{2^{\binom{n}{2}}} \left(1 - Q^{(n)}_{0,s^*_{ij}}(F_{0i}, F) Q^{(n)}_{s^*_{ij},t^*_{ij}}(F, F')\right) \\
&\leq e^{-M}.
\end{aligned}
$$

By the pigeonhole principle, there must be some F_{0i}, $i = 1, \dots, M2^{\binom{n}{2}}$, that occurs at least M times. Writing F_0 to denote any such choice and $[s^*_j, t^*_j]$ to denote a collection of M subintervals for which $F_{0j} = F_0$, we have

$$
\begin{aligned}
&\mathbb{P}\{\mathbf{\Gamma}^{[n]}_{[0,1]} \text{ has fewer than } M \text{ discontinuities } | \mathcal{E}_{[0,1]}, \Gamma^{[n]}_0 = F_0\} \\
&\quad \leq \mathbb{P}\left\{\bigcup_{j=1}^{M} \{\mathbf{\Gamma}^{[n]}_{[0,1]} \text{ has no discontinuity in } [s^*_j, t^*_j]\} | \mathcal{E}_{[0,1]}, \Gamma^{[n]}_0 = F_0\right\} \\
&\quad \leq \sum_{j=1}^{M} \mathbb{P}\{\mathbf{\Gamma}^{[n]}_{[0,1]} \text{ has no discontinuity in } [s^*_j, t^*_j] | \mathcal{E}_{[0,1]}, \Gamma^{[n]}_0 = F_0\} \\
&\quad \leq M e^{-M}.
\end{aligned}
$$

We conclude that the conditional probability that $\mathbf{\Gamma}^{[n]}_{[0,1]}$ has at least M discontinuities, given $\mathcal{E}_{[0,1]}$ and $\Gamma^{[n]}_0 = F_0$, must exceed $1 - Me^{-M}$ for every $M \geq 1$; whence, there is positive probability that $\mathbf{\Gamma}^{[n]}_{[0,1]}$ has more than M discontinuities, for every $M \geq 1$, and a positive probability that $\mathbf{\Gamma}^{[n]}_{[0,1]}$ does not have càdlàg paths, a contradiction. □

COROLLARY 5.9. *For every $n \geq 1$, $\mathbf{D}^{(n)}$ has finite-dimensional locally bounded variation.*

PROOF. Follows by combining Proposition 5.6 and Theorem 5.8. □

Theorem 3.3 now follows from Theorems 5.5 and 5.8.

PROOF OF THEOREM 3.3. The existence of the graph limits for all $t \geq 0$ is immediate from Theorem 5.5 and Aldous–Hoover. The Markov property of $|\mathbf{\Gamma}|$ follows from Proposition 4.2. Finally, the sample paths of $|\mathbf{\Gamma}|$ are càdlàg since, by Theorem 5.5, $(|\Gamma_t|)_{t \geq 0}$ is continuous at every $t \notin S := \bigcup_{\varepsilon > 0} S_\varepsilon$ almost surely and S is an at most countable, nonrandom subset of $[0, 1]$. By Theorem 5.8, the paths of $|\mathbf{\Gamma}|$ have finite-dimensional locally bounded variation almost surely. □

5.4. *Construction of* $\mathbf{\Gamma}$ *from its projection to* $\mathcal{D}^*$. In this section, we prove Theorem 3.10, which establishes that any Markov process $\mathbf{D}$ on $\mathcal{D}^*$ whose paths are càdlàg and have locally bounded variation can be associated to an exchangeable Markov process on $\mathcal{G}_\infty$ with càdlàg sample paths.

PROPOSITION 5.10. *Let* $\mathbf{D} := (D_t)_{t\geq 0}$ *be a càdlàg path of finite-dimensional bounded variation in* $\mathcal{D}^*$. *Then, for each* $n \geq 1$, *there exists a two-parameter process* $(Q_{s,t}^{(n)})_{0\leq s\leq t\leq a}$ *of transition probability measures on* $\mathcal{G}_n$ *such that for every* $[c,d] \subseteq [0,a]$,

$$\sup_{c=s_0\leq s_1\leq\cdots\leq s_n=d} \sum_{i=0}^{n-1} T_{s_i,s_{i+1}}^{(n)} = \|\mathbf{D}_{[c,d]}^{(n)}\|_{\mathrm{TV}}, \tag{30}$$

where $\|\cdot\|_{\mathrm{TV}}$ *denotes the total variation norm on* $\mathcal{D}^*$ *and* $T_{s,t}$ *is the density transfer function. The mapping that sends* $\mathbf{D}^{(n)}$ *to* $(Q_{s,t}^{(n)})$ *is measurable and, for every* $c < d$, *the section* $(Q_{s,t}^{(n)})_{c\leq s\leq t\leq d}$ *depends only on* $(D_t^{(n)})_{c\leq t\leq d}$. *Moreover, for each* $m \leq n$, *the two-parameter processes* $(Q_{s,t}^{(m)})_{0\leq s\leq t}$ *and* $(Q_{s,t}^{(n)})_{0\leq s\leq t}$ *are consistent as in* (14). *It follows that there exists a two-parameter process* $(Q_{s,t})_{0\leq s\leq t}$ *of transition probability measures satisfying* (22), (23) *and* (30).

PROOF. In Theorem 3.6, we characterized the discontinuities of $\mathbf{D}$ and showed that with probability one $\mathbf{D}$ has at most countably many jumps, each of which occurs at the times of type-(A) discontinuities. Since, for each $n \geq 1$, the n-sector $\mathbf{D}^{(n)}$ is a projection of $\mathbf{D}$, its discontinuities must be in a subset of the discontinuities of $\mathbf{D}$. As a result, we can first establish the proposition in the special case in which $\mathbf{D}$ is continuously differentiable with respect to t on $\mathcal{D}^*$. In this case, we put $d_t^{(n)} := dD_t^{(n)}/dt$ so that $(d_t^{(n)}(F))_{F\in\mathcal{G}_n}$ sums to 0. We further define

$$S_t^{(n)} := \sum_{F\in\mathcal{G}_n : d_t^{(n)}(F)>0} d_t^{(n)}(F)$$

and

$$R_t^{(n)}(F, F') := \begin{cases} -d_t^{(n)}(F)d_t^{(n)}(F')/S_t^{(n)}, & d_t^{(n)}(F) < 0, d_t^{(n)}(F') > 0, \\ d_t^{(n)}(F)d_t^{(n)}(F')/S_t^{(n)}, & d_t^{(n)}(F) > 0, d_t^{(n)}(F') < 0, \\ d_t^{(n)}(F), & F = F', \\ 0, & \text{otherwise}, \end{cases}$$

for each $F, F' \in \mathcal{G}_n$. We can define the transition probability measure on $\mathcal{G}_n$ by

$$Q_{s,t}^{(n)}(F, F') := \exp\left\{\int_s^t R_u^{(n)}(F, F')\,du\right\}, \qquad F, F' \in \mathcal{G}_n.$$

On $\mathcal{G}_n$, $(Q_{s,t}^{(n)}(F, F'))_{F,F' \in \mathcal{G}_n}$ determines a stochastic matrix, because the rows of $R_u^{(n)}$ sum to zero. Since the total variation of any continuously differentiable function equals the integral of its absolute value, condition (30) holds.

Generalizing from the above case, we can now assume that $\mathbf{D}$ is continuous and has finite-dimensional locally bounded variation. In this case, although $\mathbf{D}$ is almost everywhere differentiable, the total variation of $\mathbf{D}$ between times s and t need not equal the integral of its derivative over $[s, t]$. If instead we change time and define $\hat{D}_{T_t} = D_t$, then $\hat{D}_t$ has differentiable paths and the above argument gives the two-parameter process $(\hat{Q}_{s,t}^{(n)})_{s \le t}$. We can reverse the time change to obtain $(Q_{s,t}^{(n)})_{s \le t}$ associated to the original process $\mathbf{D}^{(n)}$.

Finally, if $\mathbf{D}$ has jump discontinuities on a countable set S, then there must be some $N \in \mathbb{N}$ such that $\mathbf{D}^{(n)}$ has jump discontinuities for all $n \ge N$. In this case, we can make $\mathbf{D}^{(n)}$ continuous by the usual process of *stretching* $\mathbf{D}^{(n)}$ at each discontinuity point. In particular, suppose $\mathbf{D}^{(n)}$ is discontinuous at time t with a jump of size ΔT_t. Then we insert an interval of length ΔT_t between times $t-$ and t and let $\mathbf{D}^{(n)}$ evolve on $(t-, \Delta T_t + t-)$ by following a straight line path from $D_{t-}^{(n)}$ to $D_t^{(n)}$. The resulting path, denoted $\tilde{\mathbf{D}}^{(n)}$, has locally bounded variation and is continuous. Our previous argument gives a two-parameter process $(\tilde{Q}_{s,t}^{(n)})_{0 \le s \le t}$. We can then *contract* each of our stretched intervals to obtain the appropriate two-parameter process $(Q_{s,t}^{(n)})_{0 \le s \le t}$ that satisfies (30).

Conditions (22) and (23) are satisfied by construction. Furthermore, since the n-sectors $(\mathbf{D}^{(n)})_{n \ge 1}$ are compatible, the consistency condition (14) is satisfied for the finite-dimensional semigroups $(Q_{s,t}^{(n)})_{n \ge 1}$, from which the existence of $(Q_{s,t})_{0 \le s \le t}$ for the process $\mathbf{D}$ follows by standard measure theory; see, for example, [4]. □

We can now prove Theorem 3.10.

PROOF OF THEOREM 3.10. Assuming $\mathbf{D} := (D_t)_{t \ge 0}$ is a càdlàg Markov process in $\mathcal{D}^*$ whose sample paths have finite-dimensional locally bounded variation, we know from Proposition 5.10 that there is a random collection $(Q_{s,t})_{0 \le s \le t}$ such that, for any $c < d$, the section $(Q_{s,t})_{c \le s \le t \le d}$ is measurable with respect to $\mathbf{D}_{[c,d]}$. Given $(Q_{s,t})_{s \le t}$, we can construct a compatible collection $(\mathbf{\Gamma}^{(n)})_{n \ge 1}$ of finite space graph-valued *time-inhomogeneous* Markov processes from the consistent family of finite-dimensional semigroups $(Q_{s,t}^{(n)})_{n \ge 1}$. Put another way, given $(Q_{s,t})_{s \le t}$, we construct a conditionally dissociated $\mathcal{I}^*$-valued array corresponding to $Q_{s,t}$.

For each $n \ge 1$, $\mathbf{\Gamma}^{(n)} := (\Gamma_t^{(n)})_{t \ge 0}$ is exchangeable and has càdlàg sample paths and the collection $(\mathbf{\Gamma}^{(n)})_{n \ge 1}$ can be constructed to be compatible, that is, $\mathbf{\Gamma}^{(m)} = (\Gamma_t^{(n)}|_{[m]})_{t \ge 0}$ for all $m \le n$, by the consistency condition (14). By (30) and the assumption that each $\mathbf{D}^{(n)}$ has bounded variation, $\mathbf{\Gamma}$ must have càdlàg paths. Therefore, there exists an exchangeable process $\mathbf{\Gamma}$ on $\mathcal{G}_\infty$ whose finite restrictions are càdlàg and of locally bounded variation. By the Aldous–Hoover theorem, the projection of the limiting process $\mathbf{\Gamma}$ into $\mathcal{D}^*$ is almost surely equal to $\mathbf{D}$. In particular, for $t \geq 0$ and $F \in \mathcal{G}_m$,

$$\begin{aligned} t(F; \Gamma_t) &= \lim_{n\to\infty} \frac{\operatorname{ind}(F; \Gamma_t|_{[n]})}{n^{\downarrow m}} \\ &= \lim_{n\to\infty} \frac{1}{n^{\downarrow m}} \sum_{F' \in \mathcal{G}_m} D_0(F') \operatorname{ind}\big((F', F); (\Gamma_0|_{[n]}, \Gamma_t|_{[n]})\big) \\ &= \sum_{F' \in \mathcal{G}_m} D_0(F') Q_{0,t}^{(m)}(F', F) \\ &= D_t(F). \end{aligned}$$

This completes the proof. $\square$

6. Concluding remarks and future directions. We have described the evolution of dynamic random networks that satisfy the minimal assumptions of exchangeability and càdlàg sample paths. Under the further assumption of Markovian dependence, we obtain a precise description of sample path properties. Our main theorems are closely related to several current and future directions in the study of large graphs.

6.1. *Modeling sparse and/or scale-free networks.* The assumptions of exchangeability and càdlàg paths impose no obviously unwanted restrictions on sample path and/or model properties; however, we must comment on an inherent discrepancy between the statistical assumption of exchangeability and the empirical observation of sparsity in real world networks. Many real world networks have few edges relative to the number of vertices, which practitioners commonly interpret to mean that real world networks are *sparse*, that is, the number of edges grows on the order of the number of vertices *as the size of the graph grows to infinity*. On the other hand, exchangeable networks are dense almost surely, that is, the number of edges grows as the square of the number of vertices as the number of vertices increases to infinity. Since sparsity is defined asymptotically, no amount of finite data could ever contradict the exchangeability assumption. On the other hand, sparsity is believed to be widespread and we would like a framework in which exchangeable models can coexist with empirical observations. In concurrent work, we investigate these foundational questions more closely and devise a general framework for network modeling [8]. We also provide a construction of scale-free networks that are exchangeable with respect to labeling of its edges [7].

6.2. *Limits of sparse graph sequences.* The Lovász–Szegedy graph limit theory only applies to dense graph sequences. Recently, Borgs et al. [2, 3] have announced an extension of the theory of graph limits to sparse graphs via the theory of L^p *graphons*; however, this notion is decidedly less natural than the Lovász–Szegedy theory for dense graphs. In our preliminary analysis, we see no interpretation of an L^p graphon in terms of a proper probability measure on countable graphs. Finding such an interpretation, or developing a more natural graph limit theory for sparse graphs, is a topic for future study.

6.3. *Processes on dynamic networks*. Our general results about exchangeable graph-valued Markov processes should equip statisticians with some understanding of what common sample paths assumptions imply about network behavior. In the case we studied, the combination of exchangeability and càdlàg sample paths seems relatively tame, and we suspect that specific classes of models can be developed to treat relevant problems in epidemiology and physics.

6.4. *Hypergraphs*. The above arguments apply, without change, to the analysis of exchangeable, càdlàg Markov processes on spaces of hypergraphs, asymmetric arrays and finite-dimensional arrays valued in a finite space.

REFERENCES

[1] ALDOUS, D. J. (1985). Exchangeability and related topics. In *École D'été de Probabilités de Saint-Flour, XIII—1983*. *Lecture Notes in Math*. **1117** 1–198. Springer, Berlin. MR0883646

[2] BORGS, C., CHAYES, J. T., COHN, H. and ZHAO, Y. (2014). An L^p theory of sparse graph convergence I: Limits, sparse random graph models, and power law distributions. Preprint. Available at arXiv:1401.2906.

[3] BORGS, C., CHAYES, J. T., COHN, H. and ZHAO, Y. (2014). An L^p theory of sparse graph convergence II: LD convergence, quotients, and right convergence. Preprint. Available at arXiv:1408.0744.

[4] BURKE, C. J. and ROSENBLATT, M. (1958). A Markovian function of a Markov chain. *Ann. Math. Stat*. **29** 1112–1122. MR0101557

[5] CRANE, H. (2015). Exchangeable graph-valued Markov processes: Feller case. Unpublished manuscript.

[6] CRANE, H. (2015). Time-varying network models. *Bernoulli* **21** 1670–1696. MR3352057

[7] CRANE, H. and DEMPSEY, W. (2015). Edge exchangeable network models and the power law. Unpublished manuscript.

[8] CRANE, H. and DEMPSEY, W. (2015). A framework for statistical network modeling. Unpublished manuscript.

[9] GROSS, T., D'LIMA, C. and BLASIUS, B. (2006). Epidemic dynamics on an adaptive network. *Phys. Rev. Lett*. **96** 208–701.

[10] HANNEKE, S., FU, W. and XING, E. P. (2010). Discrete temporal models of social networks. *Electron. J. Stat*. **4** 585–605. MR2660534

[11] HOOVER, D. (1979). Relations on probability spaces and arrays of random variables. Preprint. Institute for Advanced Studies.

[12] LOVÁSZ, L. (2012). *Large Networks and Graph Limits*. *American Mathematical Society Colloquium Publications* **60**. Amer. Math. Soc., Providence, RI. MR3012035

[13] LOVÁSZ, L. and SZEGEDY, B. (2006). Limits of dense graph sequences. *J. Combin. Theory Ser. B* **96** 933–957. MR2274085

[14] SNIJDERS, T. A. B. (2006). Statistical methods for network dynamics. In *Proceedings of the XLIII Scientific Meeting*, *Italian Statistical Society* 281–296.

DEPARTMENT OF STATISTICS & BIOSTATISTICS
RUTGERS UNIVERSITY
110 FRELINGHUYSEN ROAD
PISCATAWAY, NEW JERSEY 08854
USA
E-MAIL: hcrane@stat.rutgers.edu