Electronic Journal of Statistics
Vol. 8 (2014) 1996–2021
ISSN: 1935-7524
DOI: 10.1214/14-EJS942

Assessing relative volatility/ intermittency/energy dissipation

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Abstract: We introduce the notion of relative volatility/intermittency and demonstrate how relative volatility statistics can be used to estimate consistently the temporal variation of volatility/intermittency when the data of interest are generated by a non-semimartingale, or a Brownian semistationary process in particular. This estimation method is motivated by the assessment of relative energy dissipation in empirical data of turbulence, but it is also applicable in other areas. We develop a probabilistic asymptotic theory for realised relative power variations of Brownian semistationary processes, and introduce inference methods based on the theory. We also discuss how to extend the asymptotic theory to other classes of processes exhibiting stochastic volatility/intermittency. As an empirical application, we study relative energy dissipation in data of atmospheric turbulence.

MSC 2010 subject classifications: Primary 62M09; secondary 76F55.
Keywords and phrases: Brownian semistationary process, energy dissipation, intermittency, power variation, turbulence, volatility.

Received February 2014.

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1. Introduction

The concept of volatility expresses the ubiquitous phenomenon that observational fields exhibit more variation than expected; that is, more than the most basic type of random influence[1] envisaged.

Accordingly, volatility is a *relative* concept, and its meaning depends on the particular setting under investigation. Once that meaning is clarified the question is how to assess the volatility empirically and then to describe it in stochastic terms and incorporate it in a suitable probabilistic model.

The 'additional' random fluctuations denoted as volatility/intermittency, generally vary, in time and/or in space, in regard to *Intensity* (*activity rate* and *duration*) and *Amplitude*. Typically the volatility/intermittency may be further classified into continuous and discrete (i.e., jumps) elements, and long and short term effects.

In turbulence and certain other areas of study the phenomenon is refered to as *intermittency* (Frisch, 1995, Chapter 8) rather than volatility. Energy dissipation is a key concept in the statistical theory of turbulence, and is in the character of a specific type of intermittency.

In finance the investigation of volatility is well developed and many of the procedures of probabilistic and statistical analysis applied (Barndorff-Nielsen and Shephard, 2010) are similar to those of relevance in turbulence.

In this paper, we introduce the notion of *relative volatility/intermittency* and the closely related statistics, *realised relative power variations*. They pave the way for practical applications of some recent advances in the asymptotic theory of power variations of *non-semimartingales* (see, e.g., Corcuera, Nualart and

[1] Often thought of as Gaussian.

Woerner (2006) and Barndorff-Nielsen, Corcuera and Podolskij (2011, 2013)) to volatility/intermittency measurements and inference with empirical data.

In the non-semimartingale setting, realised power variations need to be scaled properly, in a way that depends on the smoothness of the process through unknown parameters, to ensure convergence. Realised relative power variations, however, are *self-scaling* and, moreover, admit a statistically feasible central limit theorem, which can be used, e.g., to construct confidence intervals for the realised relative volatility/intermittency. (Self-scaling statistics have also been recently used by Podolskij and Wasmuth (2013) to construct a goodness-of-fit test for the volatility coefficient of a fractional diffusion.)

This paper is organised as follows. Section 2 presents some results from the theory of *Brownian semistationary* processes that are pertinent to assessment of volatility/intermittency, and the definitions of relative volatility/intermittency and realised relative power variations are given in Section 3. A stable functional central limit theorem for realised relative power variations of Brownian semistationary processes is presented in Section 4. An application to empirical data on atmospheric turbulence is carried out in Section 5, and Section 6 concludes. Appendices contain a discussion of extending the theory beyond Brownian semistationary processes (Appendix A), an alternative method of assessing the volatility/intermittency of a Brownian semistationary process (Appendix B), and some supporting results (Appendix C).

2. Brownian semistationary processes and realised volatility/intermittency

2.1. Probabilistic setup

Brownian semistationary ($\mathcal{BSS}$) processes, introduced by Barndorff-Nielsen and Schmiegel (2009), may be used as models of timewise development of homogeneous and isotropic turbulent velocity fields. More concretely, a $\mathcal{BSS}$ process can be used to describe the velocity at a fixed point in space and in the main direction of the flow in a turbulent field. While the original motivation for $\mathcal{BSS}$ processes arose out of a study in turbulence, these processes have since found widespread interest in regard to their theoretical properties and to applications beyond physics, including, e.g., modelling of electricity price dynamics (Barndorff-Nielsen, Benth and Veraart, 2013).

A generic $\mathcal{BSS}$ process $Y = \{Y_t\}_{t\geq 0}$ is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$ via the decomposition

$$Y_t = X_t + A_t, \tag{2.1}$$

where the process

$$X_t = \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s, \quad t \geq 0, \tag{2.2}$$

is constructed from a standard Brownian motion $B = \{B_t\}_{t\in\mathbb{R}}$ and a non-zero[2] càglàd volatility/intermittency process $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$, both of which are adapted to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, and using a square integrable kernel $g : (0,\infty) \to \mathbb{R}$ such that

$$\int_{-\infty}^{t} g(t-s)^2 \sigma_s^2 \mathrm{d}s < \infty \quad \text{a.s.}$$

for all $t \geq 0$. This condition ensures the existence of the stochastic integral in (2.2). In the decomposition (2.1), $A = \{A_t\}_{t\geq 0}$ is a process that allows for skewness in the distribution of Y_t. The process A is assumed to fulfill one of two negligibility conditions, viz. (2.7) and (4.3) given below (Appendix C presents more concrete criteria that can be used to check these conditions).

Example 2.1. In the context of turbulence, the *gamma kernel*

$$g(t) = ct^{\nu-1}e^{-\lambda t}, \quad t > 0, \tag{2.3}$$

where $c > 0$, $\nu > \frac{1}{2}$, and $\lambda > 0$, has a special role. In particular, if $\nu = \frac{5}{6}$ and σ is stationary with $\mathrm{E}\{\sigma_0^2\} < \infty$, then the autocorrelation function of X is identical to von Kármán's autocorrelation function (von Kármán, 1948) for ideal turbulence and also belongs to the Whittle–Matérn family of correlation functions (Guttorp and Gneiting, 2005). The parameter value $\nu = \frac{5}{6}$ agrees with Kolmogorov's (K41) scaling law of turbulence (Kolmogorov, 1941a,b).

Example 2.2. The process A can be specified as

$$A_t = \mu + \int_{-\infty}^{t} q(t-s)\sigma_s^2 \mathrm{d}s, \quad t \geq 0, \tag{2.4}$$

where the kernel q belongs to $L^1((0,\infty))$, which makes the integral in (2.4) convergent under the assumption $\sup_{t\in\mathbb{R}} \mathrm{E}\{\sigma_t^2\} < \infty$. In particular, q can be chosen to be of the gamma form (2.3). Lemma C.1 in Appendix C provides sufficient conditions for the process A to be negligible in the sense of conditions (2.7) and (4.3) when q is a gamma kernel.

2.2. Assessing volatility/intermittency

In relation to the $\mathcal{BSS}$ process Y, a central question is that of determining the dynamics of volatility/intermittency σ from Y. If X were a semimartingale and A of finite variation, then the answer would be given by the quadratic variation $[Y]$ of Y. In fact, if

$$g(0+) < \infty \quad \text{and} \quad g' \in L^2\big((0,\infty)\big), \tag{2.5}$$

then X is a semimartingale with $[X]_t = g(0+)^2\sigma_t^{2+}$ for any $t \geq 0$, where

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 \mathrm{d}s$$

[2]More precisely, a.e. sample path is not equal to zero on a set with positive Lebesgue measure.

is the accumulated volatility/intermittency (Barndorff-Nielsen and Schmiegel, 2009). Assuming normalisation $|g(0+)| = 1$, given a set of equidistant discrete observations of Y at time points $0, \delta, \ldots, \lfloor t/\delta \rfloor \delta$, where $\delta > 0$, the accumulated volatility σ_t^{2+} can then be estimated consistently as the limit in probability for $\delta \to 0$ of the realised quadratic variation

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} (Y_{j\delta} - Y_{(j-1)\delta})^2.$$

More generally, the volatility/intermittency functional $\sigma_t^{p+} = \int_0^t |\sigma_s|^p \mathrm{d}s$ for any $p > 0$ can be estimated consistently as $\delta \to 0$ using the realised p-th order *power variation*

$$[Y_\delta]_t^{(p)} = \sum_{j=1}^{\lfloor t/\delta \rfloor} |Y_{j\delta} - Y_{(j-1)\delta}|^p \tag{2.6}$$

rescaled by $\frac{\delta^{1-p}}{m_p}$, where $m_p = \mathrm{E}\{|\xi|^p\}$ with $\xi \sim N(0,1)$, see Barndorff-Nielsen et al. (2006).

Whenever the process σ is not identically equal to zero, the condition (2.5) is both sufficient and necessary for X to be a semimartingale. However, in many interesting situations (2.5) does not hold and thus X is not a semimartingale. They include the case where g is a gamma kernel with $\nu \in \left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right)$, which is of interest for turbulence. Then, in order to determine σ_t^{2+} by a limiting procedure from the realised quadratic variation $[Y_\delta]_t$ the latter has to be rescaled by a factor depending on δ and the scaling properties of X. Specifically, as shown by Barndorff-Nielsen and Schmiegel (2009), the appropriate scaling can be described using the *second-order structure function* (or *variogram*)

$$R(t) = \mathrm{E}\{(G_t - G_0)^2\}, \quad t \geq 0,$$

of the Gaussian core G of X defined by $G_t = \int_{-\infty}^t g(t-s)\mathrm{d}W_s$, $t \geq 0$.

Let us now recall the general version of the law of large numbers for power variations of $\mathcal{BSS}$ processes, due to Barndorff-Nielsen, Corcuera and Podolskij (2011). To this end, we need to introduce some conditions concerning the kernel g and the volatility/intermittency process σ. Below $L_f : (0,\infty) \to \mathbb{R}$ stands for a function that is *slowly varying* at zero, indexed by a given function f. Recall that slow variation at zero requires that $\lim_{t\to 0+} L_f(ut)/L_f(t) = 1$ for any $u > 0$.

Assumption 2.3. *For some $\nu \in \left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right)$, the kernel g and the process σ satisfy:*

(i) $g(t) = x^{\nu-1} L_g(t)$.
(ii) $g'(t) = x^{\nu-2} L_{g'}(t)$ *and* $g' \in L^2\big((\varepsilon,\infty)\big)$ *for any* $\varepsilon > 0$*. Moreover,* $|g'|$ *is non-decreasing on* (a,∞) *for some* $a > 0$.
(iii) $\int_1^\infty g'(s)^2 \sigma_{t-s}^2 \mathrm{d}s < \infty$ *a.s. for any* $t > 0$.

Moreover, the second-order structure function R satisfies:

(iv) $R(t) = t^{2\nu-1} L_R(t)$.
(v) $R''(t) = t^{2\nu-3} L_{R''}(t)$.
(vi) *For some* $b \in (0, 1)$,

$$\limsup_{s \downarrow 0} \sup_{t \in [s, s^b]} \left| \frac{L_{R''}(t)}{L_R(s)} \right| < \infty.$$

Example 2.4. If g is the gamma kernel (2.3) with $\nu \in \left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right)$ and $\sup_{t \in \mathbb{R}} \mathrm{E}\{\sigma_t^2\} < \infty$, then Assumption 2.3 is in force, see Barndorff-Nielsen, Corcuera and Podolskij (2011, pp. 1173).

Remark 2.5. Under Assumption 2.3, the process X is not a semimartingale, unless σ is identically equal to zero. The parameter ν describes the smoothness of the process X and is analogous to the *Hurst parameter* of *fractional Brownian motion*. In fact, the increments of the Gaussian core G over short time intervals are 'close' to increments of fractional Brownian motion with Hurst parameter $\nu - \frac{1}{2}$, see Corcuera et al. (2013, p. 2557).

The following statement is a special case of Theorem 3 of Barndorff-Nielsen, Corcuera and Podolskij (2011) that provides a law of large numbers for *multi-power variations* of $\mathcal{BSS}$ processes.

Theorem 2.6. *Let $p > 0$. Suppose that Assumption 2.3 holds and that the process A satisfies the negligibility condition*

$$\frac{\delta}{R(\delta)^{\frac{p}{2}}} [A_\delta]_t^{(p)} \xrightarrow[\delta \to 0]{\mathrm{P}} 0 \quad \textit{for any } t \geq 0, \tag{2.7}$$

where $[A_\delta]_t^{(p)}$ is defined analogously to (2.6). Then,

$$\frac{\delta}{R(\delta)^{\frac{p}{2}}} [Y_\delta]_t^{(p)} \xrightarrow[\delta \to 0]{\mathrm{P}} m_p \sigma_t^{p+} \quad \textit{for any } t \geq 0.$$

Remark 2.7. Assumption 2.3 (iv) implies, by Potter's bounds for slowly varying functions (Bingham, Goldie and Teugels, 1987, Theorem 1.5.6), that for any $\varepsilon > 0$ and $t_0 \in (0, 1)$ there exist C, $C' > 0$ such that

$$C t^{2\nu-1+\varepsilon} \leq R(t) \leq C' t^{2\nu-1-\varepsilon} \tag{2.8}$$

for any $t \in [0, t_0)$. Then, the negligibility condition (2.7) holds if

$$[A_\delta]_t^{(p)} = O_{\mathrm{P}}(\delta^\gamma)$$

for any $\gamma > p(\nu - \frac{1}{2}) - 1$. Another consequence of (2.8) is that under the assumptions of Theorem 2.6 the 'raw' realised quadratic variation $[Y_\delta]_t$ satisfies

$$[Y_\delta]_t \xrightarrow[\delta \to 0]{\mathrm{P}} \begin{cases} 0, & \nu \in \left(1, \frac{3}{2}\right), \\ \infty, & \nu \in \left(\frac{1}{2}, 1\right). \end{cases}$$

(In the critical case $\nu = 1$ the limit of $[Y_\delta]_t$ is indeterminate, unless we have more information on the slowly varying part L_R of the structure function R near zero.)

3. Realised relative volatility/intermittency

3.1. *Consistent estimation of relative volatility/intermittency*

Using Theorem 2.6 for estimation of the accumulated volatility σ_t^{2+} requires knowledge of the scaling factor $\delta/R(\delta)^{\frac{p}{2}}$. More precisely, the behaviour of the second-order structure function R near zero should be known or determinable from data with sufficient accuracy. We discuss the viability of estimation of the scaling factor in Appendix B.

However, instead of the precise of value of σ_t^{2+} for fixed t, we are often more interested in measuring the dynamics of σ_t^{2+}, as a function of t, in *relative* terms. That is, for $T > 0$ we are interested in the *relative volatility/intermittency* process

$$\widetilde{\sigma}_{t,T}^{2+} = \frac{\sigma_t^{2+}}{\sigma_T^{2+}}, \quad 0 \leq t \leq T,$$

which captures the variation of σ_t^{2+} in t but loses the original scale of measurement. Clearly, under the assumptions of Theorem 2.6, we may estimate $\widetilde{\sigma}_{t,T}^{2+}$ consistently using the *realised relative quadratic variation* of Y,

$$\widetilde{[Y_\delta]}_{t,T} = \frac{[Y_\delta]_t}{[Y_\delta]_T},$$

that is, $\widetilde{[Y_\delta]}_{t,T} \xrightarrow{p} \widetilde{\sigma}_{t,T}^{2+}$ as $\delta \to 0$. The realised relative quadratic variation $\widetilde{[Y_\delta]}_{t,T}$ is entirely empirically determined, self-scaling, and its consistency does not require information on the second-order structure function R.

More generally, for any $p > 0$, the relative volatility/intermittency functionals

$$\widetilde{\sigma}_{t,T}^{p+} = \frac{\sigma_t^{p+}}{\sigma_T^{p+}}, \quad 0 \leq t \leq T, \tag{3.1}$$

can be estimated consistently using the realised p-th order *relative power variations*

$$\widetilde{[Y_\delta]}_{t,T}^{(p)} = \frac{[Y_\delta]_t^{(p)}}{[Y_\delta]_T^{(p)}}, \quad 0 \leq t \leq T,$$

as outlined in the following result.

Theorem 3.1. *Let $p > 0$. Suppose that Assumption 2.3 holds and that the process A satisfies (2.7). Then for any $T > 0$,*

$$\widetilde{[Y_\delta]}_{t,T}^{(p)} \xrightarrow[\delta \to 0]{\mathrm{P}} \widetilde{\sigma}_{t,T}^{p+} \tag{3.2}$$

uniformly in $t \in [0, T]$.

Proof. Pointwise convergence in (3.2) follows immediately from Theorem 2.6. It remains to note that the convergence is uniform since the sample paths of $\{\widetilde{[Y_\delta]}^{(p)}_{t,T}\}_{0\le t\le T}$ are non-decreasing and since $\{\widetilde{\sigma}^{p+}_{t,T}\}_{0\le t\le T}$ is a continuous process. □

3.2. Connection to relative energy dissipation in turbulence

Let us briefly consider the interpretation of relative volatility/intermittency from the point of view of physics. In the classical theory of turbulence (see, e.g., Frisch, 1995), velocity fields are assumed to be differentiable — that is, in place of a $\mathcal{BSS}$ process Y we would consider a differentiable function $y : [0, T] \to \mathbb{R}$ describing the velocity component in the main direction of the flow. Then, for $t \in [0, T]$, the *surrogate energy dissipation* of y at time t is defined as

$$\varepsilon(t) = y'(t)^2$$

and the *coarse-grained energy dissipation* of y over the interval $[0, t]$ as

$$\varepsilon^+(t) = \int_0^t y'(s)^2 \mathrm{d}s.$$

Using the mean value theorem, it is easy to show that the realised quadratic variation of y, viz. $[y_\delta]_t$, is connected to $\varepsilon^+(t)$ via the convergence

$$\frac{[y_\delta]_t}{\delta} \xrightarrow[\delta\to 0]{} \varepsilon^+(t).$$

Thus, we find that the realised relative quadratic variation $\widetilde{[y_\delta]}_{t,T}$ satisfies

$$\widetilde{[y_\delta]}_{t,T} \xrightarrow[\delta\to 0]{} \frac{\varepsilon^+(t)}{\varepsilon^+(T)},$$

where the limit is the *relative energy dissipation* of y over the subinterval $[0, t]$ of $[0, T]$. Within the turbulence literature, this definition of the relative energy dissipation is strongly related to the definition of a *multiplier* in the cascade picture of the transport of energy from large to small scales (see Cleve, Schmiegel and Greiner (2008) and references therein).

Motivated by this discussion, in the turbulence context we interpret $\widetilde{\sigma}^{2+}_{t,T}$ as the relative energy dissipation of Y over $[0, t] \subset [0, T]$.

4. Central limit theorem for realised relative power variations

4.1. Stable convergence

We are about to derive a *stable* central limit theorem for realised relative power variations of $\mathcal{BSS}$ processes. To this end, recall first that random elements

$U_1, U_2, \ldots$ in some metric space $\mathbb{U}$ *converge stably* (in law) to a random element U in $\mathbb{U}$, defined on an extension $(\Omega', \mathcal{F}', P')$ of the underlying probability space $(\Omega, \mathcal{F}, P)$, if

$$\mathrm{E}\{f(U_n)V\} \xrightarrow[n\to\infty]{} \mathrm{E}'\{f(U)V\}$$

for any bounded, continuous function $f : \mathbb{U} \to \mathbb{R}$ and bounded random variable V on $(\Omega, \mathcal{F}, P)$. We write then $U_n \xrightarrow{\text{st}} U$. Stable convergence, introduced by Rényi (1963), is stronger than ordinary convergence in law and weaker than convergence in probability. It is essential to note that the limiting random element U is defined on an *extension* of the original probability space. In fact, when U is $\mathcal{F}$-measurable, the convergence $U_n \xrightarrow{\text{st}} U$ is equivalent to $U_n \xrightarrow{\text{P}} U$ (Podolskij and Vetter, 2010, Lemma 1).

Remark 4.1. The usefulness of stable convergence can be illustrated by the following example that is pertinent to the asymptotic results below. Suppose that $U_n \xrightarrow{\text{st}} \theta\xi$ in $\mathbb{R}$, where $\xi \sim N(0,1)$ and θ is a positive random variable independent of ξ. In other words, U_n follows asymptotically a mixed Gaussian law with mean zero and conditional variance θ^2. If $\hat{\theta}_n$ is a positive, consistent estimator of θ, i.e., $\hat{\theta}_n \xrightarrow{\text{P}} \theta$, then the stable convergence of U_n allows us to deduce that $U_n/\hat{\theta}_n \xrightarrow{\text{d}} N(0,1)$. We refer to Rényi (1963), Aldous and Eagleson (1978), Jacod and Shiryaev (2003, pp. 512–518), and Podolskij and Vetter (2010, pp. 332–334) for more information on the properties of stable convergence.

4.2. Stable functional central limit theorem

As a preparation for the stable central limit theorem for realised relative power variations, we recall the stable central limit theorem for realised power variations of $\mathcal{BSS}$ processes, due to Barndorff-Nielsen, Corcuera and Podolskij (2011). As usual, the central limit theorem requires somewhat stronger assumptions than the corresponding law of large numbers (Theorem 2.6). In particular, we need to control the Hölder regularity of the volatility/intermittency process σ as follows.

Assumption 4.2. *There exists a constant $\gamma > \frac{1}{2}$ such that for any $q > 0$ and $T > 0$,*

$$\mathrm{E}\{|\sigma_t - \sigma_s|^q\} \le C_{q,T}|t-s|^{\gamma q}, \quad s,t \in [0,T],$$

where $C_{q,T} > 0$ is a constant that may depend on q and T.

In what follows, we write $D([0,T])$ for the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$, endowed with the usual Skorohod metric (Jacod and Shiryaev, 2003, Chapter V). (Recall, however, that convergence to a continuous function in this metric is equivalent to uniform convergence.) We also introduce a function $\lambda_p : \left(\frac{1}{2}, \frac{5}{4}\right) \to (0,\infty)$ given by

$$\lambda_p(\nu) = \sum_{l=2}^{\infty} l!a_l^2 \left(1 + 2\sum_{j=1}^{\infty} \rho_\nu(j)^l\right), \tag{4.1}$$

where $a_2, a_3, \ldots$ are the coefficients in the expansion of the function $u_p(x) = |x|^p - m_p$, $x \in \mathbb{R}$, in second and higher-order Hermite polynomials $x^2 - 1$, $x^3 - 3x$, $\ldots$, satisfying $\sum_{l=2}^{\infty} l! a_l^2 < \infty$ (in the case $p = 2$ we have, clearly, $a_2 = 1$ and $a_l = 0$ for all $l > 2$). The sequence $(\rho_\nu(j))_{j=1}^{\infty}$ is the correlation function of *fractional Gaussian noise* with Hurst parameter $\nu - \frac{1}{2}$, namely

$$\rho_\nu(j) = \frac{1}{2}\big((j+1)^{2\nu-1} - 2j^{2\nu-1} + (j-1)^{2\nu-1}\big), \quad j \geq 1. \tag{4.2}$$

Theorem 4.3. *Let $p \geq 1$. Suppose that Assumptions 2.3 and 4.2 hold, $\nu \in \left(\frac{1}{2}, 1\right)$, and that A satisfies*

$$\frac{\sqrt{\delta}}{R(\delta)^{\frac{p}{2}}}[A_\delta]_t^{(p)} \xrightarrow[\delta \to 0]{\mathrm{P}} 0 \quad \text{for any } t \geq 0. \tag{4.3}$$

Then for any $T > 0$,

$$\delta^{-1/2}\left(\frac{\delta}{R(\delta)^{\frac{p}{2}}}[Y_\delta]_t^{(p)} - m_p \sigma_t^{p+}\right) \xrightarrow[\delta \to 0]{\mathrm{st}} \sqrt{\lambda_p(\nu)} \int_0^t |\sigma_s|^p \mathrm{d}W_s \quad \text{in } D([0,T]),$$

where $\{W_t\}_{t \in [0,T]}$ is a standard Brownian motion, independent of the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$.

Remark 4.4. The restriction $p \geq 1$ is not necessary, but we introduce it for the sake of simpler exposition. See Theorem 4 of Barndorff-Nielsen, Corcuera and Podolskij (2011) or Theorem 3.2 of Corcuera et al. (2013) for more general versions of Theorem 4.3.

Remark 4.5. Using the bounds (2.8), we deduce that, under Assumption 2.3 (iv), the negligibility condition (4.3) holds if

$$[A_\delta]_t^{(p)} = O_{\mathrm{P}}(\delta^\gamma)$$

for any $\gamma > p(\nu - \frac{1}{2}) - \frac{1}{2}$.

Building on Theorem 4.3, we can prove the following stable central limit theorem for realised relative power variations of Y.

Theorem 4.6. *Let $p \geq 1$. Suppose that Assumptions 2.3 and 4.2 hold, $\nu \in \left(\frac{1}{2}, 1\right)$, and that A satisfies (4.3). Then for any $T > 0$,*

$$\delta^{-1/2}\left(\widetilde{[Y_\delta]}_{t,T}^{(p)} - \widetilde{\sigma}_{t,T}^{p+}\right) \xrightarrow[\delta \to 0]{\mathrm{st}} \frac{\sqrt{\lambda_p(\nu)}}{m_p \sigma_T^{p+}}\left(\int_0^t |\sigma_s|^p \mathrm{d}W_s - \widetilde{\sigma}_{t,T}^{p+} \int_0^T |\sigma_s|^p \mathrm{d}W_s\right) \tag{4.4}$$

in $D([0,T])$, where $\widetilde{\sigma}_{t,T}^{p+}$ is given by (3.1) and W is a standard Brownian motion as in Theorem 4.3.

Theorem 4.6 follows from Theorem 4.3 by invoking the following simple result concerning the stable convergence of a process that has been normalised by its terminal value.

Lemma 4.7. *Let $T > 0$ be fixed and suppose that:*

- $Z^n = \{Z_t^n\}_{0 \le t \le T}$, *for any $n \in \mathbb{N}$, is a process defined on $(\Omega, \mathcal{F}, P)$ with non-decreasing sample paths in $D([0,T])$ such that $Z_T^n \neq 0$ a.s.,*
- $Z = \{Z_t\}_{0 \le t \le T}$ *is a process defined on $(\Omega, \mathcal{F}, P)$ with non-decreasing sample paths in $C([0,T])$ such that $Z_T \neq 0$ a.s.,*
- $\xi = \{\xi_t\}_{0 \le t \le T}$ *is a process defined on an extension $(\Omega', \mathcal{F}', P')$ of $(\Omega, \mathcal{F}, P)$ with sample paths in $C([0,T])$.*

If

$$\sqrt{n}(Z_t^n - Z_t) \xrightarrow[n\to\infty]{\text{st}} \xi_t \quad \textit{in } D([0,T]), \tag{4.5}$$

then

$$\sqrt{n}\left(\frac{Z_t^n}{Z_T^n} - \frac{Z_t}{Z_T}\right) \xrightarrow[n\to\infty]{\text{st}} \frac{1}{Z_T}\left(\xi_t - \frac{Z_t}{Z_T}\xi_T\right) \quad \textit{in } D([0,T]).$$

Proof. Since Z^n and Z have non-decreasing sample paths and the sample paths of Z are continuous, we have

$$\sup_{0\le t\le T}\left|\frac{Z_t^n}{Z_T^n} - \frac{Z_t}{Z_T}\right| \le \frac{2}{|Z_T|}\sup_{0\le t\le T}|Z_t^n - Z_t| \xrightarrow[n\to\infty]{\text{P}} 0$$

by (4.5). Due to the properties of stable convergence, we obtain then

$$\left(\sqrt{n}(Z_t^n - Z_t),\, \frac{Z_t^n}{Z_T^n}\right) \xrightarrow[n\to\infty]{\text{st}} \left(\xi_t,\, \frac{Z_t}{Z_T}\right) \quad \text{in } D([0,T])^2. \tag{4.6}$$

Let us now consider the decomposition

$$\sqrt{n}\left(\frac{Z_t^n}{Z_T^n} - \frac{Z_t}{Z_T}\right) = \frac{1}{Z_T}\left(\sqrt{n}(Z_t^n - Z_t) - \sqrt{n}(Z_T^n - Z_T)\frac{Z_t^n}{Z_T^n}\right).$$

Using again the fact that convergence to a continuous function in $D([0,T])$ is equivalent to uniform convergence, it follows that the map $(x,y) \mapsto x - x(T)y$ from $D([0,T])^2$ to $D([0,T])$ is continuous on $C([0,T])^2$. Since ξ and Z have continuous sample paths, the assertion follows from (4.6) and the properties of stable convergence. □

For practical applications, we need a statistically feasible version of Theorem 4.6. Conditional on $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, the limiting process on the right-hand side of (4.4) is a Gaussian bridge. In particular, its (unconditional) marginal law at time $t \in [0,T]$ is mixed Gaussian with mean zero and conditional variance

$$\frac{\lambda_p(\nu)}{(m_p\sigma_T^{p+})^2}\left(\left(1 - \widetilde{\sigma}_{t,T}^{p+}\right)^2\sigma_t^{2p+} + \left(\widetilde{\sigma}_{t,T}^{p+}\right)^2(\sigma_T^{2p+} - \sigma_t^{2p+})\right). \tag{4.7}$$

To be able to estimate the conditional variance (4.7), we need a consistent estimator of the factor $\lambda_p(\nu)$ that depends on the smoothness parameter ν. To this end, the following fact is crucial.

Lemma 4.8. *The function $\nu \mapsto \lambda_p(\nu)$ is continuous.*

Proof. It suffices to show that $\nu \mapsto \lambda_p(\nu)$ is continuous on $\left(\frac{1}{2}, \overline{\nu}\right)$ for any $\overline{\nu} \in \left(\frac{1}{2}, \frac{5}{4}\right)$. Applying the mean value theorem twice to (4.2), we can show that there is a constant $C > 0$ such that $|\rho_\nu(j)| \leq C j^{2\overline{\nu}-3}$ for any $j \geq 1$ and $\nu \in \left(\frac{1}{2}, \overline{\nu}\right)$. Thus for any $l \geq 2$ the function $\nu \mapsto \sum_{j=1}^{\infty} \rho_\nu(j)^l$ is continuous on $\left(\frac{1}{2}, \overline{\nu}\right)$, by Lebesgue's dominated convergence theorem. Moreover, since $|\rho_\nu(j)| \leq 1$ and $6 - 4\overline{\nu} > 1$, we have for any $\nu \in \left(\frac{1}{2}, \overline{\nu}\right)$ and $l \geq 2$,

$$\left| \sum_{j=1}^{\infty} \rho_\nu(j)^l \right| \leq \sum_{j=1}^{\infty} \rho_\nu(j)^2 \leq C^2 \sum_{j=1}^{\infty} \frac{1}{j^{6-4\overline{\nu}}} < \infty.$$

The continuity of λ_p follows then by applying Lebesgue's dominated convergence theorem to the outer sum in (4.1) (recall that $\sum_{l=2}^{\infty} l! a_l^2 < \infty$). $\square$

Barndorff-Nielsen, Corcuera and Podolskij (2011, 2013) and Corcuera et al. (2013) have developed estimators $\hat{\nu}_\delta$ of ν, based on the observations $Y_0, Y_\delta, \ldots, Y_{\lfloor T/\delta \rfloor \delta}$, that are consistent as $\delta \to 0$. Using such an estimator, Lemma 4.8, and the properties of stable convergence, we obtain a feasible central limit theorem for realised relative power variations.

Proposition 4.9. *Suppose that $\hat{\nu}_\delta \xrightarrow{\mathrm{P}} \nu$ as $\delta \to 0$. Then under the assumptions of Theorem 4.6, we have for any $T > 0$ and $t \in (0, T)$,*

$$\frac{\delta^{-1/2}\left(\widetilde{[Y_\delta]}_{t,T}^{(p)} - \widetilde{\sigma}_{t,T}^{p+}\right)}{\sqrt{V_{t,T}(\delta)}} \xrightarrow[\delta \to 0]{\mathrm{d}} N(0,1),$$

where

$$V_{t,T}(\delta) = \frac{\lambda_p(\hat{\nu}_\delta)}{\delta m_{2p}\left([Y_\delta]_T^{(p)}\right)^2} \left(\left(1 - \widetilde{[Y_\delta]}_{t,T}^{(p)}\right)^2 [Y_\delta]_t^{(2p)} + \left(\widetilde{[Y_\delta]}_{t,T}^{(p)}\right)^2 \left([Y_\delta]_T^{(2p)} - [Y_\delta]_t^{(2p)}\right) \right).$$

4.3. Inference on relative volatility/intermittency

Proposition 4.9 can be used to construct approximative, pointwise confidence intervals for the relative volatility/intermittency $\widetilde{\sigma}_{t,T}^{p+}$. Since, by construction, $\widetilde{\sigma}_{t,T}^{p+}$ assumes values in $[0,1]$, it is natural to constrain the confidence interval to be a subset of $[0,1]$. Thus, we define for any $a \in (0,1)$ the corresponding $(1-a) \cdot 100\,\%$ confidence interval as

$$\left[\max\left\{ \widetilde{[Y_\delta]}_{t,T}^{(p)} - z_{1-a/2}\sqrt{\delta V_{t,T}(\delta)}, 0 \right\}, \min\left\{ \widetilde{[Y_\delta]}_{t,T}^{(p)} + z_{1-a/2}\sqrt{\delta V_{t,T}(\delta)}, 1 \right\} \right],$$

where $z_{1-a/2} > 0$ is the $1 - \frac{a}{2}$-quantile of the standard Gaussian distribution.

Another application of the central limit theory is a non-parametric *homoskedasticity test* that is similar in nature to the classical Kolmogorov–Smirnov and Cramér–von Mises goodness-of-fit tests for empirical distribution functions. This extends the homoskedasticity tests proposed by Dette, Podolskij and Vetter (2006) and Dette and Podolskij (2008) to a non-semimartingale setting. Another extension of these tests to non-semimartingales, namely fractional diffusions, is given by Podolskij and Wasmuth (2013). The approach is also similar to the *cumulative sum of squares* test (Brown, Durbin and Evans, 1975) of structural breaks studied in time series analysis literature. To formulate our test, we introduce the hypotheses

$$\begin{cases} \mathrm{H}_0 : \sigma_t = \sigma_0 \text{ for all } t \in [0,T], \\ \mathrm{H}_1 : \sigma_t \neq \sigma_0 \text{ for some } t \in [0,T]. \end{cases}$$

As mentioned above, Theorem 4.6 implies that under H_0,

$$\delta^{-1/2}\left(\widetilde{[Y_\delta]}^{(p)}_{t,T} - \frac{t}{T}\right) \xrightarrow[\delta\to 0]{\text{st}} \frac{\sqrt{\lambda_p(\nu)}}{m_p \cdot T}\left(W_t - \frac{t}{T}W_T\right). \tag{4.8}$$

The distance between the realised relative power variation and the linear function $t \mapsto \frac{t}{T}$ can be measured using various norms and metrics. Here, we consider the typical sup and L^2 norms that correspond to the Kolmogorov–Smirnov and Cramér–von Mises test statistics, respectively. More precisely, we define the statistics

$$S_\delta^{\mathrm{KS}} = \frac{m_p\sqrt{T}}{\sqrt{\delta\lambda_p(\hat{\nu}_\delta)}} \sup_{k=1,\ldots,\lfloor T/\delta\rfloor-1} \left|\widetilde{[Y_\delta]}^{(p)}_{k\delta,T} - \frac{k}{\lfloor T/\delta\rfloor}\right|,$$

$$S_\delta^{\mathrm{CvM}} = \frac{m_p^2}{\lambda_p(\hat{\nu}_\delta)} \sum_{k=1}^{\lfloor T/\delta\rfloor-1} \left(\widetilde{[Y_\delta]}^{(p)}_{k\delta,T} - \frac{k}{\lfloor T/\delta\rfloor}\right)^2,$$

where $\hat{\nu}_\delta$ is any consistent estimator of ν. Under H_0, these statistics have the classical Kolmogorov–Smirnov and Cramér–von Mises limiting distributions, respectively, as outlined in the following result.

Proposition 4.10. *Suppose that the assumptions of Theorem 4.6 hold. Then, under* H_0*,*

$$S_\delta^{\mathrm{KS}} \xrightarrow[\delta\to 0]{\text{st}} \sup_{0\le s\le 1} \left|\overline{W}_s\right|, \tag{4.9}$$

$$S_\delta^{\mathrm{CvM}} \xrightarrow[\delta\to 0]{\text{st}} \int_0^1 \overline{W}_s^2 \mathrm{d}s, \tag{4.10}$$

where $\{\overline{W}_t\}_{t\in[0,1]}$ *is a standard Brownian bridge, independent of the filtration* $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$*. Moreover, under* H_1*, both* S_δ^{KS} *and* S_δ^{CvM} *diverge to infinity as* $\delta \to 0$*.*

Proof. Under H_0, we have

$$S_\delta^{\mathrm{KS}} = \frac{m_p\sqrt{T}}{\sqrt{\delta\lambda_p(\hat{\nu}_\delta)}} \sup_{0\le t\le T}\left|\widetilde{[Y_\delta]}^{(p)}_{t,T} - \frac{t}{T}\right| + O_p(\delta^{1/2}) \xrightarrow[\delta\to 0]{\mathrm{st}} \sup_{0\le s\le 1}\left|\overline{W}_s\right|,$$
$$S_\delta^{\mathrm{CvM}} = \frac{m_p^2}{\delta\lambda_p(\hat{\nu}_\delta)} \int_0^T \left(\widetilde{[Y_\delta]}^{(p)}_{t,T} - \frac{t}{T}\right)^2 \mathrm{d}t + O_p(\delta^{1/2}) \xrightarrow[\delta\to 0]{\mathrm{st}} \int_0^1 \overline{W}_s^2 \mathrm{d}s,$$

by (4.8), Lemma 4.8, and the scaling properties of Brownian motion. The divergence of S_δ^{KS} and S_δ^{CvM} as $\delta \to 0$ under H_1 is a straightforward consequence of Theorem 3.1. □

Remark 4.11. Well-known series expansions for the cumulative distribution functions of the limiting functionals in (4.9) and (4.10) can be found, e.g., in Lehmann and Romano (2005, p. 585) and Anderson and Darling (1952, p. 202), respectively.

Remark 4.12. The finite-sample performance of the test statistics S_δ^{KS} and S_δ^{CvM} is explored in a separate paper (Bennedsen, Lunde and Pakkanen, 2014a).

5. Application to turbulence data

We apply the methodology developed above to empirical data of turbulence. The data consist of a time series of the main component of a turbulent velocity vector, measured at a fixed position in the atmospheric boundary layer using a hotwire anemometer, during an approximately 66 minutes long observation period at sampling frequency of 5 kHz (i.e. 5000 observations per second). The measurements were made at *Brookhaven National Laboratory* (Long Island, NY), and a comprehensive account of the data has been given by Drhuva (2000).

As a first illustration, we study the observations up to time horizon $T = 800$ milliseconds. Using the smallest possible lag, $\delta = 0.2$ ms, this amounts to 4000 observations. Figure 1(a) displays the squared increments corresponding to these observations. As a comparison, the same time horizon is captured in Figure 1(b) but with lag $\delta = 0.8$ ms. Figure 1(c) compares the associated accumulated realised relative energy dissipations/quadratic variations. The graphs for these two lags show very similar behaviour, exhibiting how the total time interval is divided into a sequence of intervals over which the slope of the energy dissipation is roughly constant. On the other hand, the amplitudes of the volatility/intermittency are of the same order in the whole observation interval.

To be able to draw inference on relative volatility/intermittency using the data, we need to address two issues. Firstly, for this time series, the lags $\delta = 0.2$ ms and $\delta = 0.8$ ms are *below* the so-called *inertial range* of turbulence, where a $\mathcal{BSS}$ process with a gamma kernel, a model of *ideal* turbulence, provides an accurate description of the data—see Corcuera et al. (2013), where the same data are analysed. Secondly, the data were digitised using a 12-bit analog-to-digital converter. Thus, the measurements can assume at most $2^{12} = 4096$ different values, and due to the resulting discretisation error, a non-negligible number

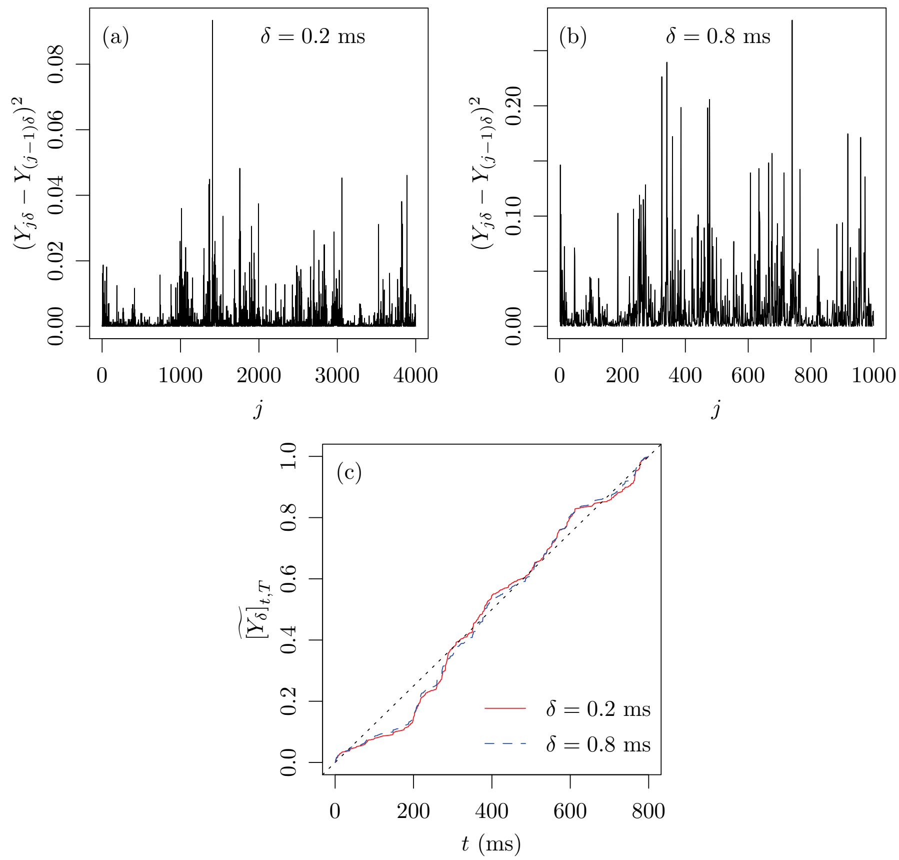

FIG 1. *Brookhaven turbulence data: (a) The squared increment process with lag $\delta = 0.2$ ms over the time horizon $T = 800$ ms. (b) The squared increment process with lag $\delta = 0.8$ ms over the same time horizon $T = 800$ ms. (c) The realised relative quadratic variations corresponding to $\delta = 0.2$ ms and $\delta = 0.8$ ms, and the same time horizon, $T = 800$ ms, as in plots (a) and (b).*

of increments are in fact equal to zero (roughly 20 % of all increments). These discretisation errors are bound to bias the estimation of the parameter ν, which is needed for the inference methods. We mitigate these issues by *subsampling*, namely, we apply the inference methods using a considerably longer lag, $\delta = 80$ ms, which is near the lower bound of the inertial range for this time series (Corcuera et al., 2013, Figure 1).

We divide the time series into 66 non-overlapping one-minute-long subperiods, testing the constancy of σ, i.e., the null hypothesis H_0, within each subperiod. Figure 2(a) displays the estimates of ν for each subperiod using the *change-of-frequency* method (Barndorff-Nielsen, Corcuera and Podolskij, 2013; Corcuera et al., 2013). All of the estimates belong to the interval $(\frac{1}{2}, 1)$ and they are scattered around the value $\nu = \frac{5}{6}$ predicted by Kolmogorov's (K41) scaling

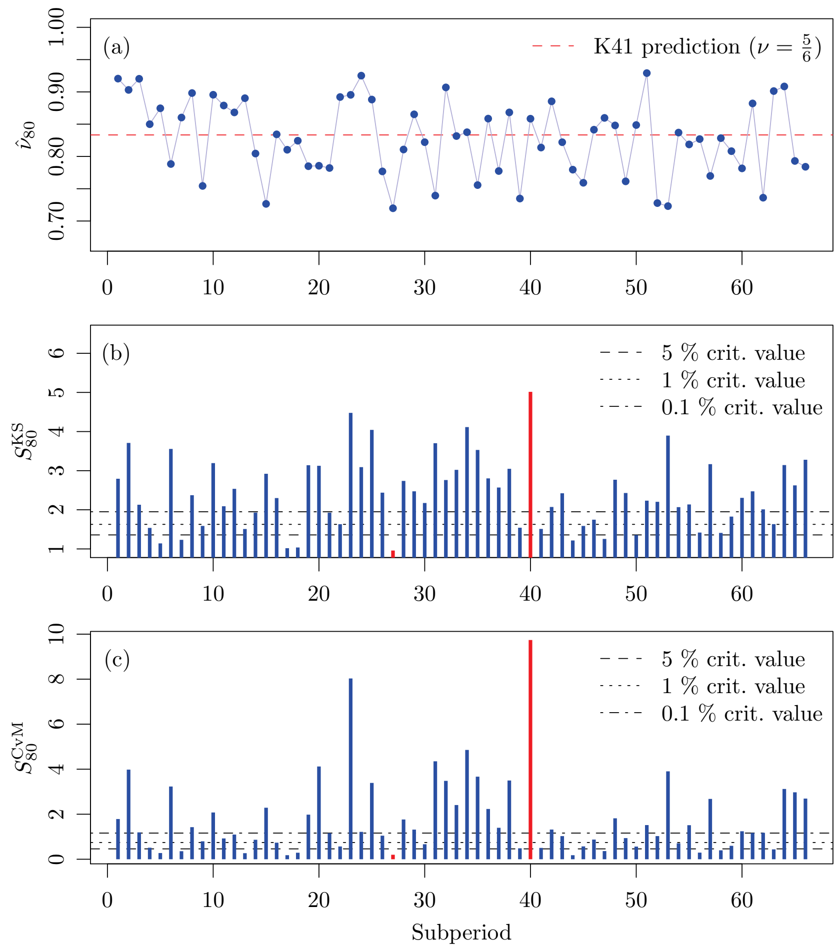

FIG 2. *Brookhaven turbulence data: (a) Estimates of ν, using the change-of-frequency method and lag $\delta = 80$ ms, for each one-minute subperiod and the value predicted by Kolmogorov's (K41) scaling law. (b) and (c) Kolmogorov–Smirnov and Cramér–von Mises-type test statistics and the corresponding critical values for the constancy of σ for each subperiod. The red bars indicate the 27th and 40th subperiods that are analysed in more detail in Figure 3.*

law of turbulence (Kolmogorov, 1941a,b). The homoskedasticity test statistics, for $p = 2$, and their critical values, derived using Proposition 4.10, in Figure 2(b) indicate that the null hypothesis of the constancy of σ is typically rejected. Moreover, the two variants, S_{80}^{KS} and S_{80}^{CvM} lead to rather similar results.

To understand what kind of intermittency the tests are detecting in the data, we look into two extremal cases, the 27th and 40th subperiods (the red bars in Figure 2(b) and (c)). To this end, we plot the realised relative energy dissipations, with $\delta = 80$ ms, during the 27th and 40th subperiods in Figure 3(a) and (b), respectively. We also include the pointwise confidence intervals, the p-values

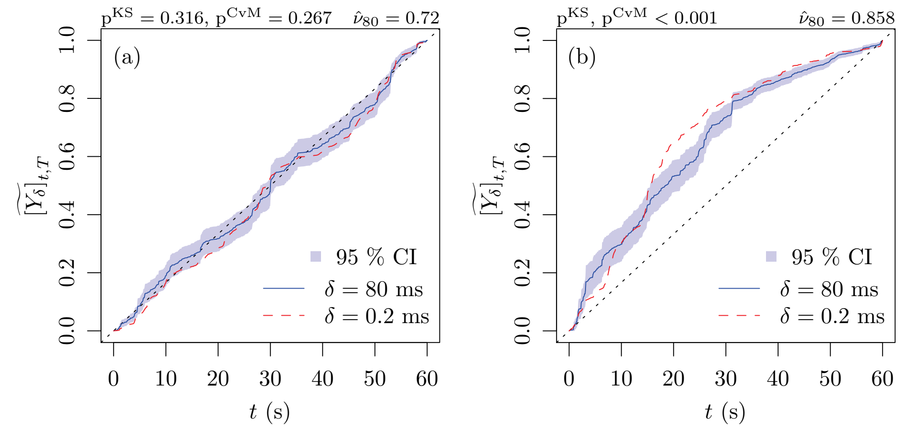

FIG 3. *Brookhaven turbulence data: Realised relative quadratic variations during the 27th (a) and 40th (b) subperiods with $\delta = 80$ ms and $\delta = 0.2$ ms. Additionally, p-values for the hypothesis* H_0*, estimates of ν using the change-of-frequency method, and* 95% *pointwise confidence intervals, all using the lag $\delta = 80$ ms.*

of the homoskedasticity tests, and as a reference, the realised relative quadratic variations using the smallest possible lag $\delta = 0.2$ ms. While the realised relative quadratic variations exhibit a slight discrepancy between the lags $\delta = 80$ ms and $\delta = 0.2$ ms, it is clear that 40th subperiod indeed contains significant intermittency, whereas during the 27th subperiod, the (accumulated) realised relative energy dissipation grows nearly linearly.

6. Conclusion

We have introduced the concept of relative volatility/intermittency and we have shown how relative volatility/intermittency can be assessed using realised relative quadratic variations in the context of non-semimartingale Brownian semistationary ($\mathcal{BSS}$) processes. (Straightforward extensions of the methodology beyond $\mathcal{BSS}$ processes are discussed in Appendix A.)

Realised relative quadratic variations are parameter-free statistics that provide estimates of the relative volatility/intermittency in subintervals of the full observation range, by relating the realised quadratic variation over each subinterval to the total realised quadratic variation for the entire range. They provide robust estimates of the relative accumulated volatility/intermittency as this develops over time and are intimately connected to the concept of relative energy dissipation in the statistical theory of turbulence. An extension to vector valued processes is an issue of interest, in particular in relation to the definition of the energy dissipation in three-dimensional turbulent fields.

Moreover, we have applied our estimation and inference methods to assess relative intermittency/energy dissipation in empirical data of atmospheric turbulence. In ongoing work (Bennedsen, Lunde and Pakkanen, 2014b), these methods are also being applied to volatility estimation with electricity price data, which exhibit non-negligible correlations in returns that can be successfully captured by models based on $\mathcal{BSS}$ processes (Barndorff-Nielsen, Benth and Veraart, 2013).

Acknowledgements

The authors would like to thank Mikkel Bennedsen and Mark Podolskij for valuable comments. O. E. Barndorff-Nielsen and M.S. Pakkanen were supported by CREATES (DNRF78), funded by the Danish National Research Foundation. Moreover, M.S. Pakkanen acknowledges support from from the Aarhus University Research Foundation regarding the project "Stochastic and Econometric Analysis of Commodity Markets" and from the Academy of Finland (project 258042).

Appendix A: Relative volatility/intermittency in the context of fractional processes and beyond

We have introduced relative volatility/intermittency in the context of $\mathcal{BSS}$ processes, but the concept has much wider applicability. The key asymptotic results for realised relative power variations, Theorems 3.1 and 4.6, can easily be generalised to other classes of processes. Indeed, Lemma 4.7 can take any stable[3] functional central limit theorem for power variations of some process (provided that the limiting process is continuous) as an 'input' to produce a 'relative' version of the result. As an example, we consider now briefly a generalisation to another class of non-semimartingales, namely fractional processes that are defined as integrals with respect to fractional Brownian motion. We also list below a number of other possible generalisations.

More concretely, let us consider a process $Y' = \{Y'_t\}_{t\geq 0}$ given by

$$Y'_t = \int_0^t u_s \mathrm{d}Z^H_s, \tag{A.1}$$

where $Z^H = \{Z^H_t\}_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $u = \{u_t\}_{t\geq 0}$ is a volatility/intermittency process with finite r-variation for some $r < \frac{1}{1-H}$ (we refer to Corcuera, Nualart and Woerner (2006) for the definition of r-variation). The integral in (A.1) is defined *pathwise*, in particular, it is not necessary to assume that u is adapted to the natural filtration of Z^H. We could also add to Y'_t a skewness term analogous to A_t of (2.1), but for simplicity it is eschewed here.

Corcuera, Nualart and Woerner (2006, Theorem 1) show that for any $p > 0$ and $t \geq 0$, the p-th power variation of Y' satisfies

$$\delta^{1-pH}[Y'_\delta]^{(p)}_t \xrightarrow[\delta\to 0]{\mathrm{P}} m_p u^{p+}_t,$$

[3] Stable convergence is crucial for the validity of Lemma 4.7.

where $u_t^{p+} = \int_0^t |u_s|^p \mathrm{d}s$. Thus, analogously to Theorem 3.1, we find that for any $T > 0$,

$$\widetilde{[Y'_\delta]}^{(p)}_{t,T} \xrightarrow[\delta\to 0]{\mathrm{P}} \widetilde{u}^{p+}_{t,T},$$

uniformly in $t \in [0, T]$, where

$$\widetilde{[Y'_\delta]}^{(p)}_{t,T} = \frac{[Y'_\delta]^{(p)}_t}{[Y'_\delta]^{(p)}_T}, \quad \widetilde{u}^{p+}_{t,T} = \frac{u^{p+}_t}{u^{p+}_T}.$$

Further, when $p \geq 1$, $H \in \big(0, \frac{3}{4}\big)$, and the sample paths of u are γ-Hölder continuous with $\gamma > \frac{1}{2}$, it holds that (Corcuera, Nualart and Woerner, 2006, Theorem 4) for any $T > 0$,

$$\delta^{-\frac{1}{2}}\big(\delta^{1-pH}[Y'_\delta]^{(p)}_t - m_p u^{p+}_t\big) \xrightarrow[\delta\to 0]{\mathrm{st}} \sqrt{\lambda_p\left(H+\frac{1}{2}\right)}\int_0^t |u_s|^p \mathrm{d}W_s \quad \text{in } D([0,T]),$$

where W is a standard Brownian motion independent of the natural filtration of Z^H. Using Lemma 4.7, we can then conclude that

$$\delta^{-\frac{1}{2}}\big(\widetilde{[Y'_\delta]}^{(p)}_{t,T} - \widetilde{u}^{p+}_{t,T}\big) \xrightarrow[\delta\to 0]{\mathrm{st}} \frac{\sqrt{\lambda_p\big(H+\frac{1}{2}\big)}}{m_p u^{p+}_T}\left(\int_0^t |u_s|^p \mathrm{d}W_s - \widetilde{u}^{p+}_{t,T}\int_0^T |u_s|^p \mathrm{d}W_s\right)$$

in $D([0, T])$.

In addition to $\mathcal{BSS}$ and fractional processes, relative volatility/intermittency statistics could be used in a similar vein at least in the following settings:

- Power and multipower variations of continuous *Itô semimartingales*, based on the asymptotic theory developed by Barndorff-Nielsen et al. (2006). Also, the consistency of realised relative power variations of certain *multifractal processes* (Duvernet, 2010; Duvernet, Robert and Rosenbaum, 2010; Ludeña and Soulier, 2014), which are *non-Itô* semimartingales, could be shown.
- Power variations of stochastic integrals with respect to symmetric *α-stable Lévy processes* (Corcuera and Farkas, 2010).
- Power variations of $\mathcal{BSS}$ processes using *higher-order increments* (Barndorff-Nielsen, Corcuera and Podolskij 2013; Corcuera et al., 2013). With second or higher order increments, the restriction $\nu < 1$ in Theorem 4.3 (and in its applications) can be lifted.
- Power variations of two-parameter *ambit fields* driven by white noise, observed on a line segment (Barndorff-Nielsen and Graversen, 2011) or on a square lattice (Pakkanen, 2014). However, in these settings only consistency of realised relative power variations can be established using the currently available asymptotic theory.

Appendix B: Estimating the scaling factor of realised quadratic variation

As seen in Sections 2 and 4, the asymptotic theory for power variations of the $\mathcal{BSS}$ process Y requires a suitable scaling of the realised power variation by a factor that depends on the second-order structure function R. We will now discuss whether the scaling factor can be estimated from the observed data, which would be an alternative to using relative volatility/intermittency statistics. For simplicity, we focus on quadratic variations, which are the most relevant in practical applications.

Assumption 2.3 postulates that $R(\delta)$ behaves like $\delta^{2\nu-1}$ as $\delta \to 0$, apart from a slowly varying factor $L_R(\delta)$. If $L_R(\delta)$ is 'well-behaved' and normalised in the sense that $\lim_{\delta\to 0} L_R(\delta) = 1$, then in Theorem 2.6 for the case $p = 2$ the scaling factor $\frac{\delta}{R(\delta)}$ can be replaced with $\delta^{2-2\nu}$, to wit,

$$\delta^{2-2\nu}[Y_\delta]_t \xrightarrow[\delta\to 0]{\mathrm{P}} \sigma_t^{2+} \tag{B.1}$$

for any $t \geq 0$. The condition $\lim_{\delta\to 0} L_R(\delta) = 1$ holds, e.g., when g is the gamma kernel (2.3) with $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and c is chosen in a suitable way (Barndorff-Nielsen, Corcuera and Podolskij, 2011, p. 1173). If, additionally, $L_R(\delta) = 1 + o(\delta^{\frac{1}{2}})$ as $\delta \to 0$, which is again true in the aforementioned situation with g of the gamma form, the convergence in the central limit theorem (Theorem 4.3) in the case $p = 2$ can be simplified to

$$\delta^{-\frac{1}{2}}\left(\delta^{2-2\nu}[Y_\delta]_t - \sigma_t^{2+}\right) \xrightarrow[\delta\to 0]{\mathrm{st}} \sqrt{2}\int_0^t \sigma_s^2 \mathrm{d}W_s \quad \text{in } D([0,T]). \tag{B.2}$$

As shown by Barndorff-Nielsen, Corcuera and Podolskij (2013) and Corcuera et al. (2013), the smoothness parameter ν can be estimated consistently in the infill asymptotic setting with an estimator $\hat{\nu}_\delta$ with the usual rate of convergence $\delta^{\frac{1}{2}}$. Then it is natural to ask, whether we can simply substitute ν with $\hat{\nu}_\delta$ in (B.1) and (B.2) without affecting the asymptotic behaviour of the scaled realised quadratic variation. From the following result we learn that $[Y_\delta]_t$ with the estimated scaling $\delta^{2-2\hat{\nu}_\delta}$ indeed attains consistency. However, the second-order behaviour is affected by the estimated scaling: the rate of convergence becomes slower and the asymptotic distribution is non-standard, due to the estimation error of ν. Similar results have been shown (under constant volatility) by Coeurjolly (2001, Proposition 4) in the context of fractional Brownian motion and by Brouste and Iacus (2013, Theorem 1) in the context of fractional Ornstein–Uhlenbeck processes.

Proposition B.1. *Let $\delta \in (0, 1)$ and let $\hat{\nu}_\delta$ be an estimator of the smoothness parameter ν such that*

$$\delta^{-\frac{1}{2}}(\hat{\nu}_\delta - \nu) \xrightarrow[\delta\to 0]{\mathrm{st}} \xi, \tag{B.3}$$

where ξ is an a.s. finite random variable.

(a) *If the assumptions of Theorem 2.6 hold and* $\lim_{\delta\to 0} L_R(\delta) = 1$, *then for any* $t \geq 0$,

$$\delta^{2-2\hat{\nu}_\delta}[Y_\delta]_t \xrightarrow[\delta\to 0]{\mathrm{P}} \sigma_t^{2+}.$$

(b) *If the assumptions of Theorem 4.3 hold and* $L_R(\delta) = 1 + o(\delta^{\frac{1}{2}})$ *as* $\delta \to 0$, *then*

$$\frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}\big(\delta^{2-2\hat{\nu}_\delta}[Y_\delta]_t - \sigma_t^{2+}\big) \xrightarrow[\delta\to 0]{\mathrm{st}} 2\xi\sigma_t^{2+} \quad \textit{in } D([0,T]).$$

Proof. (a) Let us write

$$\delta^{2-2\hat{\nu}_\delta}[Y_\delta]_t = \delta^{-2(\hat{\nu}_\delta-\nu)}\delta^{2-2\nu}[Y_\delta]_t = e^{Q_\delta}\delta^{2-2\nu}[Y_\delta]_t,$$

where $Q_\delta = 2\log(\delta^{-1})(\hat{\nu}_\delta - \nu)$. By the condition (B.3), we find that

$$Q_\delta = 2\delta^{\frac{1}{2}}\log(\delta^{-1})\delta^{-\frac{1}{2}}(\hat{\nu}_\delta - \nu) \xrightarrow[\delta\to 0]{\mathrm{P}} 0. \tag{B.4}$$

Thus, $e^{Q_\delta} \xrightarrow{\mathrm{P}} 1$ as $\delta \to 0$, and the assertion follows then from (B.1).

(b) Let us consider the decomposition

$$\frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}\big(\delta^{2-2\hat{\nu}_\delta}[Y_\delta]_t - \sigma_t^{2+}\big) = U_\delta\delta^{2-2\nu}[Y_\delta]_t + \frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}\big(\delta^{2-2\nu}[Y_\delta]_t - \sigma_t^{2+}\big),$$

where

$$U_\delta = \frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}\big(\delta^{-2(\hat{\nu}_\delta-\nu)} - 1\big) = \frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}\big(e^{Q_\delta} - 1\big).$$

By (B.2), we have clearly

$$\frac{\delta^{-\frac{1}{2}}}{\log(\delta^{-1})}\big(\delta^{2-2\nu}[Y_\delta]_t - \sigma_t^{2+}\big) \xrightarrow[\delta\to 0]{\mathrm{P}} 0 \quad \text{in } D([0,T]).$$

Due to (B.1) and the properties of stable convergence, it suffices now to show that $U_\delta \xrightarrow{\mathrm{st}} 2\xi$ as $\delta \to 0$. To this end, define $u(x) = e^x - 1 - x$, $x \in \mathbb{R}$. Observe that

$$U_\delta = 2\delta^{-\frac{1}{2}}(\hat{\nu}_\delta - \nu) + \frac{u(Q_\delta)}{\delta^{\frac{1}{2}}\log(\delta^{-1})}, \tag{B.5}$$

and in view of the condition (B.3) it remains to show that the second term on right-hand side of (B.5) converges to zero in probability as $\delta \to 0$. To this end, let $\eta > 0$ and consider

$$\mathrm{P}\left\{\left|\frac{u(Q_\delta)}{\delta^{\frac{1}{2}}\log(\delta^{-1})}\right| > \eta\right\} \leq \mathrm{P}\left\{\left|\frac{u(Q_\delta)}{\delta^{\frac{1}{2}}\log(\delta^{-1})}\right| > \eta,\, |Q_\delta| \leq 1\right\} + \mathrm{P}\{|Q_\delta| > 1\},$$

where $\lim_{\delta\to 0}\mathrm{P}\{|Q_\delta| > 1\} = 0$ by (B.4). Using the elementary inequality $|u(x)| \leq 3x^2$, valid when $|x| \leq 1$, we finally deduce that

$$\mathrm{P}\left\{\left|\frac{u(Q_\delta)}{\delta^{\frac{1}{2}}\log(\delta^{-1})}\right| > \eta,\, |Q_\delta| \leq 1\right\} \leq \mathrm{P}\left\{\left|\frac{3Q_\delta^2}{\delta^{\frac{1}{2}}\log(\delta^{-1})}\right| > \eta\right\} \xrightarrow[\delta\to 0]{} 0,$$

since

$$\frac{3Q_\delta^2}{\delta^{\frac{1}{2}}\log(\delta^{-1})} = 12\delta^{\frac{1}{2}}\log(\delta^{-1})\big(\delta^{-\frac{1}{2}}(\hat{\nu}_\delta - \nu)\big)^2 \xrightarrow[\delta\to 0]{\mathrm{P}} 0,$$

which in turn is a simple consequence of the condition (B.3). □

Appendix C: Sufficient conditions for the negligibility of the skewness term

This appendix provides some methods of checking the negligibility conditions (2.7) and (4.3) with some concrete specifications of the process $A = \{A_t\}_{t\geq 0}$.

Suppose first that the process A is given by

$$A_t = \mu + \int_0^t a_s \mathrm{d}s,$$

where $\mu \in \mathbb{R}$ is a constant and the process $\{a_t\}_{t\geq 0}$ is measurable and locally bounded. Then we can establish rather simple conditions for its negligibility in the asymptotic results for power variations. By Jensen's inequality, we have for any $p \geq 1$, $s \geq 0$, and $t \geq 0$,

$$|A_s - A_t|^p \leq C_a \cdot |s-t|^p,$$

where $C_a > 0$ is a random variable that depends locally on the path of a. Thus, we find that for any $t \geq 0$,

$$[A_\delta]_t^{(p)} = O_{\mathrm{a.s.}}(\delta^{p-1})$$

as $\delta \to 0$. Then, in view of Remarks 2.7 and 4.5 and the restriction $\nu < \frac{3}{2}$, the condition (2.7) holds always and (4.3) holds provided that $p > \frac{1}{3-2\nu}$ (which is always true if $p \geq 1$).

Suppose now, instead, that A follows

$$A_t = \mu + \int_{-\infty}^t q(t-s) a_s \mathrm{d}s, \tag{C.1}$$

where q is the gamma kernel

$$q(t) = c' t^{\eta-1} e^{-\rho t}$$

for some $c' > 0$, $\eta > 0$, and $\rho > 0$. We assume that the process $\{a_t\}_{t\in\mathbb{R}}$ is measurable, locally bounded, and satisfies

$$A_t^* = \sup_{0\leq u\leq t} \int_{-\infty}^u q(u-s)|a_s|\mathrm{d}s < \infty \quad \text{a.s.} \tag{C.2}$$

for any $t \geq 0$, which is true, e.g., when the auxiliary process $\int_{-\infty}^u q(u-s)|a_s|\mathrm{d}s$, $u \geq 0$, has a càdlàg or continuous modification.

Lemma C.1. *If A is given by (C.1), and (C.2) holds, then for any $p > 0$ and $t \geq 0$,*

$$[A_\delta]_t^{(p)} = O_{a.s.}(\delta^{p\min\{\eta,1\}-1}) \tag{C.3}$$

as $\delta \to 0$. Thus the condition (2.7) holds if $\min\{\eta, 1\} > \nu - \frac{1}{2}$ and (4.3) holds if $\min\{\eta, 1\} > \nu - \frac{p-1}{2p}$.

Proof. Let us first look into the properties of q. For the sake of simpler notation, we make the innocuous assumption that $c' = 1$. Since

$$q'(t) = \left(\frac{\eta - 1}{t} - \rho\right) q(t), \tag{C.4}$$

we find that q is decreasing when $\eta \leq 1$. When $\eta > 1$, q is increasing on $\left(0, \frac{\eta-1}{\rho}\right)$ and decreasing on $\left(\frac{\eta-1}{\rho}, \infty\right)$.

Let $t \geq 0$ be fixed, $\delta \in (0, 1)$, and let $j \geq 1$ be such that $j\delta \leq t$. Below, all big O estimates hold uniformly in such j. We consider the decomposition

$$\begin{aligned}
A_{j\delta} - A_{(j-1)\delta} = &\int_{(j-1)\delta}^{j\delta} q(j\delta - s) a_s \mathrm{d}s \\
&+ \int_{(j-2)\delta}^{(j-1)\delta} \big(q(j\delta - s) - q((j-1)\delta - s)\big) a_s \mathrm{d}s \\
&+ \int_{s^*}^{(j-2)\delta} \big(q(j\delta - s) - q((j-1)\delta - s)\big) a_s \mathrm{d}s \\
&+ \int_{-\infty}^{s^*} \big(q(j\delta - s) - q((j-1)\delta - s)\big) a_s \mathrm{d}s \\
= &\; I_\delta^1 + I_\delta^2 + I_\delta^3 + I_\delta^4,
\end{aligned}$$

where

$$s^* = -\max\left\{\frac{\eta - 1}{\rho}, 1\right\}.$$

When $\eta \geq 1$, q is bounded and we have $|I_\delta^1 + I_\delta^2| = a_t^* O(\delta)$, where

$$a_t^* = \sup_{s^* \leq s \leq t} |a_s| < \infty \quad \text{a.s.},$$

and when $\eta < 1$, we find that

$$|I_\delta^1 + I_\delta^2| \leq 2a_t^* \int_0^\delta q(s) \mathrm{d}s = a_t^* O(\delta^\eta).$$

Next, we want to show that

$$|I_\delta^3| = a_t^* O(\delta^{\min\{\eta,1\}}). \tag{C.5}$$

In the case $\eta \geq 2$ the derivative q' is bounded and (C.5) is immediate. Suppose that $\eta < 2$. Then, $|q'(t)| \leq Ct^{\eta-2}$ on any finite interval, where $C > 0$ depends on the interval. Using the mean value theorem, we obtain

$$|I^3_\delta| \leq Ca^*_t\delta \int_{s^*}^{(j-2)\delta} \big((j-1)\delta - s\big)^{\eta-2} \mathrm{d}s,$$

which implies (C.5). To bound $|I^4_\delta|$, note that, by (C.4), $|q'(t)| \leq C'q(t)$ for all $t \geq -s^*$, where $C' > 0$ is a constant. For any $s < s^*$, we have $(j-1)\delta - s > \frac{\eta-1}{\rho}$. Thus, by the mean value theorem,

$$\big|\big(q(j\delta - s) - q((j-1)\delta - s)\big)\big| \leq C'q\big((j-1)\delta - s\big)\delta$$

and, consequently,

$$|I^4_\delta| \leq C'\delta \int_{-\infty}^{(j-1)\delta} q\big((j-1)\delta - s\big)|a_s|\mathrm{d}s = A^*_t O(\delta).$$

Collecting the estimates, we have

$$|A_{j\delta} - A_{(j-1)\delta}| = \max\{a^*_t, A^*_t\}O(\delta^{\min\{\eta,1\}})$$

uniformly in j, whence (C.3) follows. Checking the sufficiency of the asserted criteria for (2.7) and (4.3) is now a straightforward task (based on Remarks 2.7 and 4.5). □

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